Homework 5

Due Monday, November 4, 2019

Math 206

1. **The Monty Hall problem.** We discussed the following brain teaser, loosely based on the game show *Let's Make a Deal* hosted by Monty Hall and made famous by columnist Marilyn vos Savant in the 1990s.

   There are three doors. Behind one door is a car, and behind the other two doors are goats. You choose one door. The game show host chooses another door that has a goat behind it, opens it to show you the goat, and asks if you want to switch. You might think that your odds of winning the car are now 1/2, and it doesn't matter if you switch, but surprisingly, different strategies have different outcomes.

   Write some code to simulate playing the game many times, and see how often you win with the following strategies:

   (a) Never switch doors.

   (b) Always switch doors.

   (c) Randomly decide whether to switch doors.

   The actual game show was not really like this problem. Hall had a lot of leeway in how he ran the game, playing on the psychology of the contestant to make the show interesting. For example, he might only offer the chance to switch if he knew you'd picked the winning door, or he might offer $100 rather than the opportunity to switch doors.

   (d) Would you rather have $100, or a 1 in 3 chance of winning a car? If $100 isn't enough for you, how much would be enough? Exactly 1/3 of the value of the car? More? Less? Why? (This question is asking you to reflect on how a simple mathematical model compares to the real world.)

2. **A random walk on the line.** A bug starts at the origin on a number line. It flips a coin to decide whether to move one unit left or right. It does this many times.

   (a) What is the probability that after 20 steps, the bug is more than 10 units away from the origin? Write some code to approximate this.

   (b) What is the average distance that the bug will end up at after 20 steps? Write some code to approximate this.

   (c) If you study random walks seriously, you'll learn that the average distance from the origin after n steps is $\sqrt{n} \cdot \sqrt{2/\pi}$, at least when n is large. Does this agree with your findings for $n = 10$? For larger n?

3. **A random walk in the plane.** Now let the bug wander around the xy-plane, at each step deciding randomly whether to move one unit left, right, up, or down.

   (a) Write a formula for the distance from the origin to the point (x, y).

   (b) Again we could ask for the probability that after 20 steps, the bug is more than 10 units away from the origin. Do you expect the answer to be larger or smaller than when the bug was moving in the line? Or the same? Why?

   (c) Write some code to approximate the probability in (a). Does it agree with your prediction? If not, why do you think it happened this way?

   (d) Again we could ask for the average distance that the bug will end up at after 20 steps. Do you expect the answer to be larger or smaller than when the bug was moving in the line? Or the same? Why?

   (e) Write some code to approximate the average distance in (c). Does it agree with your prediction? If not, why do you think it happened this way?

4. If we two choose two numbers x_1, x_2 at random in the unit interval $[0, 1]$, what is the average distance between them? That is, if we choose randomly many times, calculate the distance each time, and take the average of all the distances we find, what is the average?

   (a) Write some code to approximate the answer.

   The python function `random.random()` returns a random floating-point number in the interval $[0, 1)$, which you could argue is good enough. Or you could use `random.uniform(0,1)`, which really chooses a random number in $[0, 1]$.

   (b) Set up a double integral to find the average exactly. If it looks feasible, evaluate the integral.

5. Consider the same question for two random points (x_1, y_1) and (x_2, y_2) in the unit rectangle – that is, all four numbers are randomly chosen in $[0, 1]$.

   (a) Do you expect the average distance to be less than or greater than what you found in #4? Or the same? Why?

   (b) Write a formula for the distance between (x_1, y_1) and (x_2, y_2).

   (c) Write some code to approximate the average distance.

   (d) Set up a quadruple integral to find the average exactly. It is probably too hard to evaluate.