Homework 8

Due Monday, November 25, 2019

Math 206

1. On Monday we discussed the fact that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n,$$

   and we messed around with the computer and figured out that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2.$$

   (a) Mess around in a similar way to find a formula for

$$1^p + 2^p + 3^p + \cdots + n^p$$

   with $p = 4, 5, 6$. Make sure that your answer works for some different values of n, not just for $n = 1000$ or $n = 10000$. Hint: $0.416666\ldots = 5/12$.

   (b) Describe at least three patterns that you observe.

   (c) Optional: Keep going if you want.

   (d) Optional: If you like proofs by induction, prove one of your formulas by induction.

2. On Wednesday we talked about how to fit a quadratic polynomial through three points: If you want $y = ax^2+bx+c$ to pass through three points (x_1, y_1) and (x_2, y_2) and (x_3, y_3), you should solve the matrix equation

$$\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

I emphasize that the x's and y's are numbers that are given to us, and a, b, and c are variables that we want to solve for.

That big square matrix is called a *Vandermonde* matrix, and we saw that we could save ourselves a lot of trouble by using the Python routines `numpy.vander` and `numpy.linalg.solve`.

(a) Take the three points we used, make up a fourth point, and fit a cubic polynomial

$$y = ax^3 + bx^2 + cx + d$$

that passes through all four. Graph the points and the polynomial using `desmos.com`.

(b) Make up a fifth point, fit a quartic polynomial

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

that passes through all five, and graph the points and the curve.

(c) You could rediscover our formula for

$$1^3 + 2^3 + 3^3 + \cdots + n^3,$$

by finding the coefficients of a quartic polynomial

$$f(n) = an^4 + bn^3 + cn^2 + dn + e$$

that satisfies

$$f(0) = 0 \qquad f(1) = 1 \qquad f(2) = 1^3 + 2^3 = 9$$

$$f(3) = 1^3 + 2^3 + 3^3 = 36 \qquad f(4) = 1^3 + 2^3 + 3^3 + 4^3 = 100.$$

Do you get the same coefficients that we found before?

(d) Do problem 1(a) this way. Is it easier than what you did on Monday?