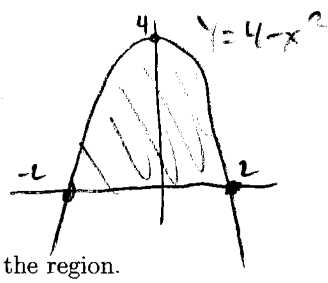
Solutions to Midterm Exam 1

Each part is worth 10 points.

1.

(a) Sketch the region in the xy-plane that lies above the x-axis and below the parabola $y = 4 - x^2$.

Solution:



(b) Find the area of the region.

Solution:

$$\int_{-2}^{2} (4 - x^2) dx = \left[4x - \tfrac{1}{3}x^3\right]_{-2}^{2}$$
$$= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)$$
$$= \tfrac{32}{3}.$$

Alternatively you can find $\int_0^2 (4 - x^2)\, dx$ and then double it. Or you can remember the formula for area under a parabola that appears in Simpson's rule.

2. **Evaluate** $\int \frac{x}{1-x^2}\,dx$ in three ways:

(a) By substituting $u = 1 - x^2$.

Solution: If $u = 1 - x^2$ then $du = -2x\,dx$. Thus

$$\begin{aligned}\int \frac{x}{1-x^2}\,dx &= -\frac{1}{2}\int \frac{1}{u}\,du \\ &= -\frac{1}{2}\ln u + C \\ &= -\frac{1}{2}\ln(1-x^2) + C.\end{aligned}$$

(b) By partial fractions: Factor $1 - x^2$ as $(1+x)(1+x)$, then find numbers A and B such that

$$\frac{x}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x}, \qquad (*)$$

then integrate the right-hand side.

Solution: Clearing denominators in the equation (1), we find that

$$x = A(1-x) + B(1+x).$$

Setting $x = 1$ we find that $B = \frac{1}{2}$. Setting $x = -1$ we find that $A = -\frac{1}{2}$. Thus

$$\begin{aligned}\int \frac{x}{1-x^2}\,dx &= -\frac{1}{2}\int \frac{1}{1+x}\,dx + \frac{1}{2}\int \frac{1}{1-x}\,dx \\ &= -\frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x) + C.\end{aligned}$$

(c) By substituting $x = \sin t$. Hint: You may need to recall that $\frac{\sin t}{\cos t} = \tan t$ and $\int \tan t\,dt = \ln(\sec t) + C = -\ln(\cos t) + C$.

Solution: If $x = \sin t$ then $dx = \cos t\,dt$. Thus

$$\begin{aligned}\int \frac{x}{1-x^2}\,dx &= \int \frac{\sin t}{\cos^2 t}\cos t\,dt \\ &= \int \frac{\sin t}{\cos t}\,dt \\ &= \int \tan t\,dt \\ &= -\ln(\cos t) + C \\ &= -\ln\sqrt{1-x^2} + C.\end{aligned}$$

(d) Your answers to parts (a), (b), and (c) look different. Show that they are the same.

Solution: For part (b) we can write

$$\begin{aligned} -\frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x) &= -\frac{1}{2}\Big(\ln(1+x) + \ln(1-x)\Big) \\ &= -\frac{1}{2}\ln\Big((1+x)(1-x)\Big) \\ &= -\frac{1}{2}\ln(1-x^2) \end{aligned}$$

which agrees with part (a). For part (c) we can write

$$\begin{aligned} -\ln\sqrt{1-x^2} &= -\ln\Big((1-x^2)^{1/2}\Big) \\ &= -\frac{1}{2}\ln(1-x^2) \end{aligned}$$

which agrees with part (a) again.

3.

(a) Evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$.

Hint: This is of the form $\frac{\infty}{\infty}$, so you can use l'Hôpital's rule.

Solution: By l'Hôpital's rule we have

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

(b) Evaluate $\int x^{-2} \ln x \, dx$.

Hint: Integrate by parts, letting $u = \ln x$ and $dv =$ the rest.

Solution: If $u = \ln x$ and $dv = x^{-2}\,dx$ then $du = x^{-1}\,dx$ and $v = -x^{-1}$.

$$\begin{aligned}\int x^{-2} \ln x \, dx &= -x^{-1} \ln x + \int x^{-2}\,dx \\ &= -x^{-1}\ln x - x^{-1} + C.\end{aligned}$$

(c) Evaluate $\int_1^\infty x^{-2} \ln x \, dx$.

Hint: Recycle your answers to parts (a) and (b).

Solution:

$$\begin{aligned}\int_1^\infty x^{-2} \ln x \, dx &= \lim_{B\to\infty} \int_1^B x^{-2} \ln x \, dx \\ &= \lim_{B\to\infty} \left[-x^{-1}\ln x - x^{-1}\right]_1^B \\ &= \lim_{B\to\infty} \left((-\tfrac{\ln B}{B} - \tfrac{1}{B}) - (-1\cdot 0 - 1)\right) \\ &= (0-0)+1 \\ &= 1.\end{aligned}$$

(d) Substitute $x = e^t$ into the integral from part (c) and clean it up to get $\int_0^\infty te^{-t}\,dt$. On the practice exam you found that the latter equals 1. Does this agree with your answer to part (c)?

Solution: If $x = e^t$ then $dx = e^t\,dt$. Solving $x = e^t$ for t we get $t = \ln x$; thus if $x = 1$ then $t = 0$, and as $x \to \infty$ we have $t \to \infty$.

$$\begin{aligned}\int_1^\infty x^{-2} \ln x \, dx &= \int_0^\infty e^{-2t}\cdot t \cdot e^t\,dt \\ &= \int_0^\infty te^{-t}\,dt.\end{aligned}$$

For part (c) we got 1, so this agrees with the practice exam.