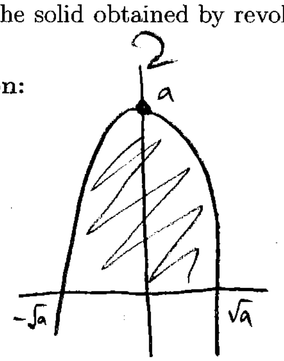
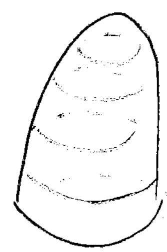
Solutions to Midterm Exam 2

1. Solids of revolution (30 points).

(a) Let a be a positive number. Sketch the region in the xy-plane that lies above the x-axis and below the parabola $y = a - x^2$. Indicate the coordinates of the points where the parabola meets the x- and y-axes. Separately, sketch the solid obtained by revolving this region around the y-axis. (10 points)

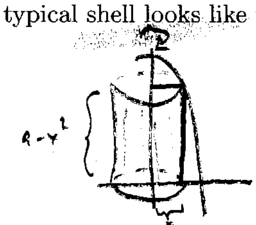
Solution:





(b) Find the volume of the solid using the shell method. Your answer will be a function of a. (10 points)

Solution: A typical shell looks like this:

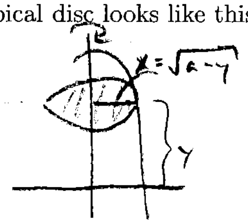


Its radius is x, and is height is $a - x^2$, so its area is $2\pi x(a - x^2) = 2\pi(ax - x^3)$. Thus the volume of the solid is

$$\begin{aligned} 2\pi \int_0^{\sqrt{a}} (ax - x^3)\, dx &= 2\pi \left[\frac{a}{2}x^2 - \frac{1}{4}x^4 \right]_0^{\sqrt{a}} \\ &= 2\pi \left[\frac{a^2}{2} - \frac{a^2}{4} - 0 \right] \\ &= \frac{\pi}{2} a^2. \end{aligned}$$

(c) Find the volume of the solid using the disc method. (10 points)

Solution: A typical disc looks like this:



Its radius is $\sqrt{a-y}$, so its area is $\pi(\sqrt{a-y})^2 = \pi(a-y)$. Thus the volume of the solid is

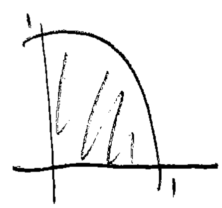
$$\pi \int_0^a (a-y)\,dy = \pi\left[ay - \frac{1}{2}y^2\right]_0^a$$
$$= \pi\left[a^2 - \frac{1}{2}a^2 - 0\right]$$
$$= \frac{\pi}{2}a^2.$$

Your answers to (b) and (c) should agree of course. Moreover, if you plug in $a = 4$ you should get 8π, as we saw on Quiz 5.

2. Center of mass (30 points).

(a) Sketch the part of the unit disc that lies in the first quadrant. What is its area? Hint: Don't use calculus. (10 points)

Solution:

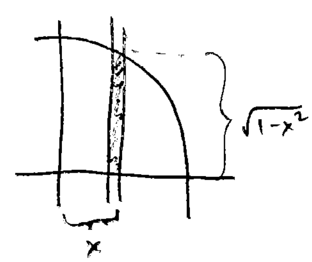


The area of the whole disc is $\pi \cdot 1^2 = \pi$, so the area of the part that lies in the first quadrant is $\pi/4$.

(b) Set up an integral to find either the x- or the y-coordinate of center of mass of this quarter-disc, assuming it has constant density. (By the symmetry of the region, the x- and y-coordinates of the center of mass are the same, so you only need to find one or the other.) (10 points)

Solution: For the x-coordinate, we take vertical slices as x goes from 0 to 1, add up x times the area of the slice, and divide by the area of the whole region. The area of a slice is $\sqrt{1-x^2}\,dx$, so the integral is

$$\frac{1}{\pi/4}\int_0^1 x\sqrt{1-x^2}\,dx.$$



Similarly, the y-coordinate is

$$\frac{1}{\pi/4}\int_0^1 y\sqrt{1-y^2}\,dy.$$

(c) Evaluate the integral you set up in part (b). (10 points)

Solution: We substitute $u = 1 - x^2$, so $du = -2x\,dx$. When $x = 0$ we have $u = 1$, and when $x = 1$ we have $u = 0$. Thus

$$\begin{aligned}\frac{4}{\pi}\int_0^1 x\sqrt{1-x^2}\,dx &= -\frac{2}{\pi}\int_1^0 \sqrt{u}\,du \\ &= \frac{2}{\pi}\int_0^1 u^{1/2}\,du \\ &= \frac{4}{3\pi}u^{3/2}\Big|_0^1 \\ &= \frac{4}{3\pi}.\end{aligned}$$

As a sanity check, you might want to make sure that your point lies inside the region, and in particular that the distance to the origin is less than 1.

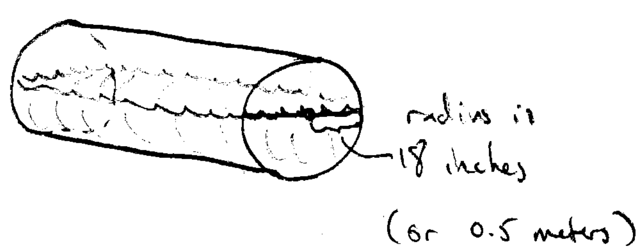
Let's check: The distance from $(\frac{4}{3\pi}, \frac{4}{3\pi})$ to the origin is $\frac{4}{3\pi}\sqrt{2}$. Is this less than 1? Equivalently, is $4\sqrt{2} < 3\pi$? Squaring both sides, is $32 < 9\pi^2$? Yes it is: $\pi > 3$, so $9\pi^2 > 9 \cdot 9 = 81$, which is greater than 32. Or if we had a calculator, we would find that $\frac{4\sqrt{2}}{3\pi} \approx 0.60$.

3. Hydrostatic force (40 points). A cylindrical drum, 3 feet in diameter, lies on its side, half-full of water. We are interested in the the hydrostatic force on one end of the drum. You will need to know that the pressure underwater equals the depth in feet times 0.43 pounds per square inch. Hint: I recommend working in inches rather than feet; one foot equals 12 inches.

If you prefer to use SI units, say that the drum is 1 meter in diameter, and recall that the pressure underwater is 9800 Pa per meter of depth.

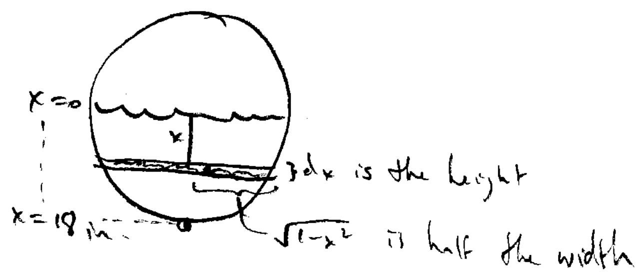
(a) (10 points) Draw a picture.

Solution:



(b) (30 points) Set up the integral, but do not evaluate it.

Solution: Consider the horizontal slice of the side of the drum that lies x inches below the water:



Thus x runs from 0, at the top of the water, in the middle of the drum, to 18 at the bottom. The pressure at this depth is $0.43 \cdot \frac{x}{12}$ pounds per square inch. The width of this slice is $2\sqrt{18^2 - x^2}$ inches, and the height is dx inches, so the area is $2\sqrt{18^2 - x^2}\, dx$ square inches, and the hydrostatic force on it is $\frac{0.43}{12}x \cdot 2\sqrt{18^2 - x^2}\, dx$ pounds. Adding up all these slices we get

$$\tfrac{0.43}{12} \int_0^{18} 2x\sqrt{18^2 - x^2}\, dx \qquad \text{pounds.}$$

If we were to evaluate this, we would start by letting $u = 18^2 - x^2$, so $du = -2x\, dx$, so the integral becomes

$$-\tfrac{0.43}{12} \int_{18^2}^{0} u^{1/2}\, du = \tfrac{0.43}{12} \int_0^{18^2} u^{1/2}\, du$$

which is not so bad.

If you use SI units, let x be measured in meters, and get

$$9800 \int_0^{0.5} 2x\sqrt{(0.5)^2 - x^2}\, dx \qquad \text{Newtons.}$$