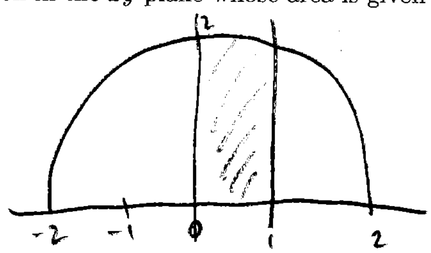
Solutions to Practice Midterm 1

1.

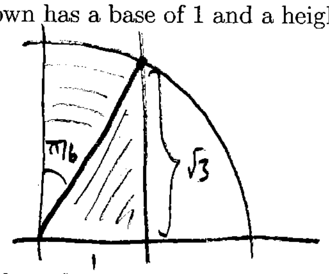
(a) Sketch the region in the xy-plane whose area is given by $\int_0^1 \sqrt{4-x^2}\,dx$.

Solution:



(b) Evaluate the integral geometrically by thinking about triangles and wedges of circles.

Solution: The area of a circle of radius 2 is 4π. The wedge shown is a twelfth of a circle. The triangle shown has a base of 1 and a height of $\sqrt{3}$. Thus the area is $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$.



(c) Evaluate the integral algebraically by substituting $x = 2\sin t$. Does your answer agree with your answer to part (b)? Hint: You will need the half-angle formula $\cos^2 t = \frac{1+\cos 2t}{2}$. Further hint: Don't forget to change bounds: if $2\sin t = 1$ then $\sin t = \frac{1}{2}$, so $t = \ldots$

Solution: If $x = 2\sin t$ then $dx = 2\cos t\,dt$. Then we have

$$4 - x^2 = 4 - 4\sin^2 t = 4(1-\sin^2 t) = 4\cos^2 t,$$

so $\sqrt{4-x^2} = 2\cos t$. To change bounds, write $\sin t = x/2$; if $x = 0$ then

$\sin t = 0$, so $t = 0$, and if $x = 1$ then $\sin t = 1/2$, so $t = \pi/6$. Thus

$$\begin{aligned}\int_0^1 \sqrt{4 - x^2}\, dx &= \int_0^{\pi/6} 2\cos t \cdot 2\cos t\, dt \\ &= \int_0^{\pi/6} 4\cos^2 t\, dt \\ &= \int_0^{\pi/6} (2 + 2\cos 2t)\, dt \\ &= \Big[2t + \sin 2t\Big]_0^{\pi/6} \\ &= \frac{\pi}{3} + \frac{\sqrt{3}}{2} - 0,\end{aligned}$$

where in the third line we have used the given half-angle formula.

2. Evaluate $\int x\sqrt{x-1}\, dx$ in two ways:

(a) By substituting $u = x - 1$. Hint: Then $x = u + 1$.
Solution: With this u we have $du = dx$.

$$\begin{aligned}\int x\sqrt{x-1}\, dx &= \int (u+1)\sqrt{u}\, du \\ &= \int (u^{3/2} + u^{1/2})\, du \\ &= \tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + C \\ &= \tfrac{2}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} + C\end{aligned}$$

(b) By parts, letting $u = x$ and $dv =$ the rest.
Solution: If $u = x$ and $dv = \sqrt{x-1}\, dx$ then $du = dx$ and $v = \frac{2}{3}(x-1)^{3/2}$.

$$\begin{aligned}\int x\sqrt{x-1}\, dx &= \tfrac{2}{3}x(x-1)^{3/2} - \int \tfrac{2}{3}(x-1)^{3/2}\, dx \\ &= \tfrac{2}{3}x(x-1)^{3/2} - \tfrac{2}{3}\cdot\tfrac{2}{5}(x-1)^{5/2} + C \\ &= \tfrac{2}{3}x(x-1)^{3/2} - \tfrac{4}{15}(x-1)^{5/2} + C.\end{aligned}$$

(c) Your answers to parts (a) and (b) look different. Show that they are the same. Hint: Factor out $(x-1)^{3/2}$ and clean up what's left. **Solution:** For part (a) we have

$$\begin{aligned}\tfrac{2}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} &= (x-1)^{3/2}\Big(\tfrac{2}{5}(x-1) + \tfrac{2}{3}\Big) \\ &= (x-1)^{3/2}\Big(\tfrac{2}{5}x + \tfrac{4}{15}\Big).\end{aligned}$$

For part (b) we have

$$\begin{aligned}\tfrac{2}{3}x(x-1)^{3/2} - \tfrac{4}{15}(x-1)^{5/2} &= (x-1)^{3/2}\Big(\tfrac{2}{3}x - \tfrac{4}{15}(x-1)\Big)\\ &= (x-1)^{3/2}\Big(\tfrac{2}{5}x + \tfrac{4}{15}\Big)\end{aligned}$$

3.

(a) What is $\lim_{t\to\infty} e^t$? What is $\lim_{t\to\infty} e^{-t}$? What is $\lim_{t\to\infty} te^{-t}$? Hint: Write the latter as $\lim_{t\to\infty} \frac{t}{e^t}$, which is of the form $\frac{\infty}{\infty}$, so you can use l'Hôpital's rule.

Solution: First we have $\lim_{t\to\infty} e^t = \infty$, so $\lim_{t\to\infty} e^{-t} = 0$. Then by l'Hôpital's rule we have

$$\lim_{t\to\infty} \frac{t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t} = 0.$$

(b) Find $\int te^{-t}\,dt$. Hint: Integrate by parts.

Solution: Let $u = t$ and $dv = e^{-t}\,dt$, so $du = dt$ and $v = -e^{-t}$.

$$\begin{aligned}\int te^{-t}\,dt &= -te^{-t} + \int e^{-t}\,dt\\ &= -te^{-t} - e^{-t} + C.\end{aligned}$$

(c) Find $\int_0^\infty te^{-t}\,dt$. Hint: Recycle your answer to parts (a), (b), and especially (c).

Solution:

$$\begin{aligned}\int_0^\infty te^{-t}\,dt &= \lim_{B\to\infty}\int_0^B te^{-t}\,dt\\ &= \lim_{B\to\infty}\Big[-te^{-t} - e^{-t}\Big]_0^B\\ &= \lim_{B\to\infty}\Big((-Be^{-B} - e^{-B}) - (-0\cdot 1 - 1)\Big)\\ &= (0-0)+1\\ &= 1.\end{aligned}$$