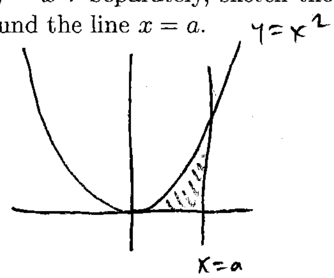
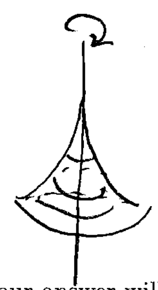
Solutions to Practice Midterm 2

1. Let a be a positive number.

(a) Sketch the region in the first quadrant bounded by the line $x = a$ and the parabola $y = x^2$. Separately, sketch the solid obtained by revolving this region around the line $x = a$.

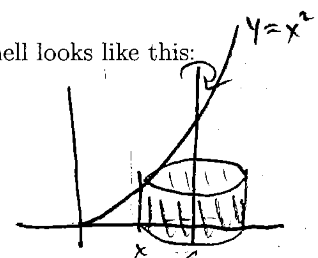
Solution:





(b) Find the volume of the solid using the shell method. Your answer will be a function of a.

Solution: A typical shell looks like this:



Its radius is $a - x$, and its height is x^2, so the area of its side is

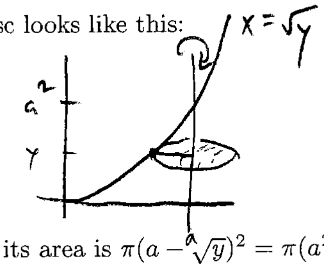
$$2\pi(a-x)\cdot x^2 = 2\pi(ax^2 - x^3).$$

Thus the volume of the solid is

$$\begin{aligned} 2\pi\int_0^a (ax^2 - x^3)\,dx &= 2\pi\left[\frac{a}{3}x^3 - \frac{1}{4}x^4\right]_0^a \\ &= 2\pi\left(\frac{a^4}{3} - \frac{a^4}{4} - 0\right) \\ &= 2\pi\cdot\frac{a^4}{12} \\ &= \frac{\pi}{6}a^4. \end{aligned}$$

(c) Find the volume of the solid using the disc method.

Solution: A typical disc looks like this:



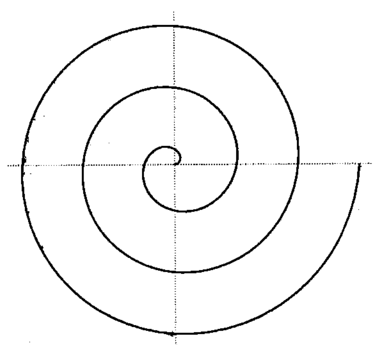
Its radius is $a - \sqrt{y}$, so its area is $\pi(a - \sqrt{y})^2 = \pi(a^2 - 2a\sqrt{y} + y)$. Thus the volume of the solid is

$$\begin{aligned}\pi \int_0^{a^2} (a^2 - 2ay^{1/2} + y)\,dy &= \pi \left[a^2 y - \frac{4}{3}ay^{3/2} + \frac{1}{2}y^2\right]_0^{a^2} \\ &= \pi\left(a^2 \cdot a^2 - \frac{4}{3}a \cdot a^3 + \frac{1}{2}a^4 - 0\right) \\ &= \pi\left(1 - \frac{4}{3} + \frac{1}{2}\right)a^4 \\ &= \frac{\pi}{6}a^4.\end{aligned}$$

2. In §6.4 #6 we encountered the parametric curve

$$x = t\cos t \qquad y = t\sin t,$$

called the *spiral of Archimedes*. It looks like this:



(a) Find $(dx/dt)^2$ and $(dy/dt)^2$. Simplify the expression $(dx/dt)^2 + (dy/dt)^2$. Hint: All the sines and cosines go away by the end.

Solution: We have

$$\begin{aligned}(dx/dt)^2 &= (\cos t - t\sin t)^2 \\ &= \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t\end{aligned}$$

$$\begin{aligned}(dy/dt)^2 &= (\sin t + t\cos t)^2 \\ &= \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t.\end{aligned}$$

Thus

$$\begin{aligned}
&(dx/dt)^2 + (dy/dt)^2 \\
&= (\cos^2 t + \sin^2 t) + (-2t \sin t \cos t + 2t \sin t \cos t) + t^2(\sin^2 t + \cos^2 t) \\
&= 1 + 0 + t^2.
\end{aligned}$$

(b) Find the length of the curve traced out as t goes from 0 to π. You will want to refer to #21 on the attached table of integrals; we did a similar integral by hand in lecture when we were finding the arc length of a parabola, but it was long.

Solution: The arc length is

$$\int_0^\pi \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^\pi \sqrt{1+t^2}\, dt.$$

Number 21 on the integral table says

$$\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln(u + \sqrt{a^2+u^2}) + C.$$

Taking $a = 1$ and $u = t$, we find that

$$\begin{aligned}
\int_0^\pi \sqrt{1+t^2}\, dt &= \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln(t+\sqrt{1+t^2})\right]_0^\pi \\
&= \left(\frac{\pi}{2}\sqrt{1+\pi^2} + \frac{1}{2}\ln(\pi + \sqrt{1+\pi^2})\right) - \left(0\sqrt{1} + \frac{1}{2}\ln(0+\sqrt{1})\right) \\
&= \frac{\pi}{2}\sqrt{1+\pi^2} + \frac{1}{2}\ln(\pi+\sqrt{1+\pi^2}),
\end{aligned}$$

where in the last line we have used the fact that $\ln(1) = 0$. If you plug this into a calculator you get about 6.1, so the scale on that picture is rather small.

3. You sit on the roof of a two-story house, which is 20 feet tall, holding a rope that is 30 feet long and weighs 3 pounds. Your friend on the ground holds a slinky, which we will model as a spring obeying Hooke's law with a natural length of 6 inches and a spring constant of 0.2 pounds per foot; somewhat unrealistically, we will neglect the weight of the spring. You throw one end of the rope down to your friend, who ties it to one end of the slinky and attaches the other end of the slinky to the ground. How much work will you do in pulling the rope back up to the roof and thereby stretching the slinky to the height of the house?

If you prefer SI units, make the house 6 m tall, the length of the rope 10 m, the mass of the rope 1.5 kg, the natural length of the slinky 15 cm, and the spring constant 3 N/m. Recall that the acceleration due to gravity is 9.8 m/s^2. Find the work in Newton-meters, i.e. in Joules.

Solution: The question is a little ambiguous; any reasonable interpretation of it will get full credit. I'll say that to start, the slinky sits on the ground, 6 inches tall, and the rope runs from the top of the slinky to the top of the house. So we didn't throw down any more rope than necessary, and we still have 10.5 ft of rope next to me, irrelevant to the problem. But if you prefer you could say that we threw all that rope down, and the excess 10.5 feet sits loose next to the slinky on the ground. This would be a different, slightly longer calculation, with a larger answer.

For the work done on the rope, consider a little piece of rope, dx feet long, which hangs x feet from the roof. The rope weighs 0.1 pounds per foot, so this piece weighs $0.1\,dx$ pounds, and we have to move it through x feet, thus doing $x \cdot 0.1\,dx$ foot-pounds of work. Adding up all these pieces we get

$$\int_0^{19.5} 0.1x\,dx = 0.05x^2\Big|_0^{19.5} = 0.05 \cdot 19.5^2 \approx 19 \text{ ft} \cdot \text{lbs}.$$

For the spring, suppose we've stretched the spring to a length of y feet, so the spring is exerting $0.2y$ pounds of force. Then we pull and increase the length by dy feet, thus doing $0.2y \cdot dy$ foot-pounds of work. Adding up all this work, we get

$$\int_0^{19.5} 0.2y\,dy$$

which is exactly twice as much as the previous integral, so about 38 ft·lbs.

Thus the answer is $19 + 38 =$ **57 ft · lbs**.

In SI units, the mass of the rope is .15 kg/m, so the weight of the rope is $9.8 \cdot .15 \approx 1.5$ N/m, and 6 meters minus 15 centimeters is 5.85 meters, so the work done in pulling up the rope is

$$\int_0^{5.85} x \cdot 1.5\,dx = 0.75x^2\Big|_0^{5.85} \approx 25.7 \text{ N} \cdot \text{m}.$$

The work done in stretching the spring is

$$\int_0^{5.85} 3y \cdot dy = 1.5y^2\Big|_0^{5.85} \approx 51.3 \text{ N} \cdot \text{m}.$$

The total is **77 N·m** or 77 J.