Solutions to Practice Final

1.

(a) $\displaystyle\int \frac{1}{2-3x}\,dx$ is which of the following?

$\ln(2-3x)+C$	$-\ln(2-3x)+C$
$3\ln(2-3x)+C$	$-3\ln(2-3x)+C$
$\frac{1}{3}\ln(2-3x)+C$	$-\frac{1}{3}\ln(2-3x)+C$
$2\ln(2-3x)+C$	$-2\ln(2-3x)+C$
$\frac{1}{2}\ln(2-3x)+C$	$-\frac{1}{2}\ln(2-3x)+C$

Solution: It is $-\frac{1}{3}\ln(2-3x)+C$. By the chain rule, the derivative of $\ln(2-3x)$ is $\frac{1}{2-3x}\cdot(-3)$, so we need the $-\frac{1}{3}$ out front to get $\frac{1}{2-3x}$. Alternatively, if we let $u=2-3x$ then $du=-3\,dx$, so $dx=-\frac{1}{3}\,du$.

(b) $\displaystyle\int \frac{1}{(2-3x)^2}\,dx$ is which of the following? Why?

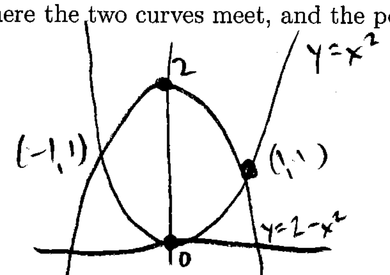
$\ln\left((2-3x)^2\right)+C$	$-\ln\left((2-3x)^2\right)+C$	$\frac{1}{2-3x}+C$	$-\frac{1}{2-3x}+C$
$3\ln\left((2-3x)^2\right)+C$	$-3\ln\left((2-3x)^2\right)+C$	$3\cdot\frac{1}{2-3x}+C$	$-3\cdot\frac{1}{2-3x}+C$
$\frac{1}{3}\ln\left((2-3x)^2\right)+C$	$-\frac{1}{3}\ln\left((2-3x)^2\right)+C$	$\frac{1}{3}\cdot\frac{1}{2-3x}+C$	$-\frac{1}{3}\cdot\frac{1}{2-3x}+C$
$2\ln\left((2-3x)^2\right)+C$	$-2\ln\left((2-3x)^2\right)+C$	$2\cdot\frac{1}{2-3x}+C$	$-2\cdot\frac{1}{2-3x}+C$
$\frac{1}{2}\ln\left((2-3x)^2\right)+C$	$-\frac{1}{2}\ln\left((2-3x)^2\right)+C$	$\frac{1}{2}\cdot\frac{1}{2-3x}+C$	$-\frac{1}{2}\cdot\frac{1}{2-3x}+C$

Solution: It is $\frac{1}{3}\cdot\frac{1}{2-3x}+C$. By the chain rule, the derivative of $(2-3x)^{-1}$ is $-(2-3x)^{-2}\cdot(-3)$, and the minus signs cancel, so we need the $\frac{1}{3}$ out front to get $\frac{1}{(2-3x)^2}$. Alternatively, if we let $u=2-3x$ then $du=-3\,dx$, so $dx=-\frac{1}{3}\,du$; then we also need to note that $\int u^{-2}\,du=-u^{-1}+C$.

2. (Based on §6.3 #16.)

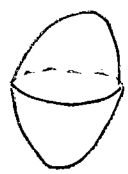
(a) Sketch the region in the plane bounded by the parabolas $y = x^2$ and $y = 2 - x^2$. Find the coordinates of the points where the two curves meet, and the points where they meet the y-axis.

Solution:



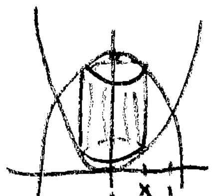
(b) Sketch the solid obtained by revolving the region around the y-axis.

Solution:



(c) Find the volume of the solid using the shell method.

Solution: A typical shell looks like this:



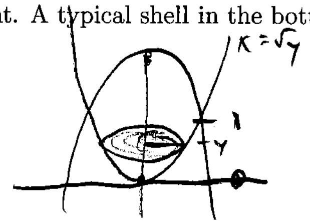
Its radius is x and its height is $(2 - x^2) - (x^2) = 2 - 2x^2$, so its area is $2\pi x \cdot (2 - 2x^2) = 4\pi(x - x^3)$, so the volume of the solid is

$$4\pi \int_0^1 (x - x^3)\, dx = 4\pi \left[\frac{x^2}{2} - \frac{x^2}{4} \right]_0^1 = 4\pi \left(\frac{1}{2} - \frac{1}{4} - 0 \right) = \pi.$$

Note that the integral needs to go from 0 to 1, not from -1 to 1, so that we only get each shell once.

(d) Find the volume of the solid using the disc method. Hint: It may be convenient to find the volume of just the top half or the bottom half, and then double it.

Solution: We take the hint. A typical shell in the bottom half looks like this:



It radius is $\sqrt{y}$, so its area is πy, so the volume of the bottom half of the solid is

$$\pi \int_0^1 y\, dy = \pi \left[\frac{y^2}{2} \right]_0^1 = \pi \left(\frac{1}{2} - 0 \right) = \frac{\pi}{2}.$$

Doubling this we get π again.

3. (§7.4 #13. Seasonal.) A turkey is taken out of the oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. Newton's law of cooling states that the turkey cools at a rate proportional to the difference between its temperature and the temperature of the room.

(a) If the temperature of the turkey is 150°F after half an hour, what will it be after 45 minutes?

Solution: Let t be the time in minutes and y the temperature of the turkey. By Newton's law of cooling,

$$\frac{dy}{dt} = k(y - 75)$$

for some number k. Separating variables, we get

$$\frac{1}{y-75}\,dy = k\,dt.$$

Integrating, we get

$$\ln(y - 75) = kt + C$$

for some number C. Exponentiating, we get

$$y - 75 = e^{kt}e^{C}.$$

We rename e^C to A and move the 75 to the other side to get

$$y = 75 + Ae^{kt}.$$

Next, we know that when $t = 0$, $y = 185$, so we find that $A = 110$, so

$$y = 75 + 110e^{kt}.$$

Next, we know that when $t = 30$, $y = 150$, so we find that $k = \frac{1}{30}\ln(\frac{75}{110})$, so

$$y = 75 + 110e^{t/30\cdot\ln(75/110)}.$$

Finally, we plug in $t = 45$ and find that $y \approx 137°$ F.

(b) When will the turkey have cooled to 100°F?

Solution: Letting $y = 100$ in the last equation, we find that

$$t = 30 \cdot \frac{\ln(25/110)}{\ln(75/110)} \approx 116 \text{ minutes.}$$

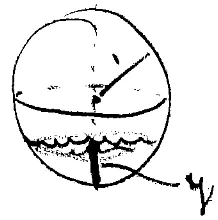
So you have plenty of time to make the gravy before the bird goes cold.

4. (Based on an example we did in lecture.) According to Toricelli's law, water drains from a tank at a rate proportional to the square root of the depth. A spherical tank of radius 1 meter was initially full of water and took an hour to drain. At what time was the tank half full?

Let's all choose the same variables: let t be the time in minutes, y the depth of the water in meters, and V the volume of water in cubic meters.

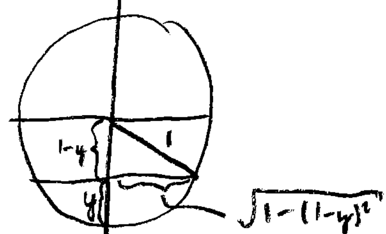
(a) Draw a picture of the tank partly filled with water, but not exactly half full. Label y.

Solution:



(b) What is the area of the surface of the water when the depth is y? Hint: The surface is a circle; start by finding its radius. It may help to draw a cross-section of the tank and some kind of triangle.

Solution: Here is the cross-section:



When the depth is y, the height of the right triangle is $(1-y)$, and the hypotenuse is 1, so the base is

$$\sqrt{1-(1-y)^2} = \sqrt{1-(1-2y+y^2)} = \sqrt{2y-y^2}.$$

This is the radius of the surface of the water. Thus the area is $\pi(\sqrt{2y-y^2})^2 = \pi(2y-y^2)$.

(c) Write a differential equation expressing Toricelli's law. Hint: The left-hand side is dV/dt; the right-hand side involves y and a constant k.

Solution: The rate at which the volume of water is changing is proportional to the square root of the depth, so

$$\frac{dV}{dt} = k\sqrt{y}$$

for some constant k.

(d) You could find V as a function of y, or y as a function of V, but it would be a lot of work. Instead, observe that dV/dy is the area you found in part (b) – why is this true? Then observe that

$$\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt}$$

by the chain rule, so you can replace dV/dt in your equation from part (c) with your new expression for dV/dy, times dy/dt, to get a differential equation involving only y.

Solution: dV/dy is the area of the surface of the water, because if the depth y increases a tiny bit dy then V increases by the area times dy. Alternatively you can say it's the fundamental theorem of calculus: the volume is the integral of the areas of the slices, so the derivative of the volume is the area. Using the chain rule observation, our differential equation becomes

$$\pi(2y-y^2)\frac{dy}{dt} = k\sqrt{y}.$$

(e) Take your equation from part (d), separate variables, and integrate. But do not try to solve for y.
Solution: First we separate variables:

$$\pi \frac{2y - y^2}{\sqrt{y}}\, dy = k\, dt.$$

We clean this up:

$$\pi(2y^{1/2} - y^{3/2})\, dy = k\, dt.$$

Then we integrate:

$$\pi \left(\tfrac{4}{3} y^{3/2} - \tfrac{2}{5} y^{5/2}\right) = kt + C.$$

(f) What is y when $t = 0$? When $t = 60$? Use these facts to find k and C. They will be a little messy, but not horribly so.

Solution: When $t = 0$, $y = 2$. We plug this into our equation, noting that $2^{3/2} = 2\sqrt{2}$ and $2^{5/2} = 4\sqrt{2}$, to get

$$C = \frac{16\pi\sqrt{2}}{15}.$$

When $t = 60$, $y = 0$. We plug this into our equation to get

$$k = -\frac{1}{60} \frac{16\pi\sqrt{2}}{15}.$$

(g) What is y when the tank is half full? Find t corresponding to that value y. It should be a little less than 30 minutes, because the water flows faster at first and slower later on.

Solution: When the tank is half full, $y = 1$. Plugging this into our equation, we get

$$\frac{14\pi}{15} = -\frac{1}{60} \frac{16\pi\sqrt{2}}{15} t + \frac{16\pi\sqrt{2}}{15}.$$

Dividing through by $\frac{16\pi\sqrt{2}}{15}$, this becomes

$$\frac{14}{16\sqrt{2}} = -\frac{1}{60} t + 1.$$

We can clean up the left-hand side slightly to get $\frac{7\sqrt{2}}{16}$. We solve for t to get

$$t = 60 \cdot \left(1 - \tfrac{7\sqrt{2}}{16}\right) \approx 23 \text{ minutes},$$

which is indeed a little less than 30.