Final Exam
Math 253
March 21, 2024 Name: Solutions

There are 75 points in total, plus 5 points extra credit.
You may use any calculator that cannot access the internet.
If you don't have such a calculator, I can lend you one.
You may use a hand-written sheet of notes.
Show your work where appropriate.
No cheating.

1. (10 points) Does $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ converge or diverge, and why?

Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{(2n+2)!/((n+1)!)^2}{(2n)!/(n!)^2}$$
$$= \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \left(\frac{n!}{(n+1)!}\right)^2$$
$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cdot(2n)!}{(2n)!} \cdot \left(\frac{n!}{(n+1)\cdot n!}\right)^2$$
$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2}$$
$$= \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1}$$

Now we either use L'Hôpital's rule twice or divide the top and bottom of the fraction by n^2, and either way we find that the limit is 4. Since this is greater than 1, the series diverges.

2. (10 points) For what values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{3n^2}$ converge?

First we take absolute values and apply the ratio test, which gives

$$\lim_{n \to \infty} \frac{|x|^{n+1}/3(n+1)^2}{|x|^n/3n^2}$$
$$= \lim_{n \to \infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{3n^2}{3(n+1)^2}$$
$$= \lim_{n \to \infty} |x| \cdot \frac{3n^2}{3n^2+6n+3}$$

We see that the part involving n goes to 1, so the whole limit goes to $|x|$. Thus the series converges if $|x|<1$, diverges if $|x|>1$, and if $|x|=1$ then we need to do more work.

If $x=1$ then we're talking about $\sum_{n=1}^{\infty} \frac{1}{3n^2}$, which converges by the integral test (since $2>1$).
If $x=-1$ then we're talking about $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n^2}$, which converges absolutely, because if we throw away the signs then we get the thing that we just said converges.

Thus the series converges for $-1 \leq x \leq 1$.

Here is the form of Taylor's theorem that we proved and have been using. Fix some $x>0$, and suppose we find some M such that $\left|f^{(d+1)}(t)\right| \leq M$ for all t between 0 and x. Then the difference between f(x) and the d^{th} Taylor polynomial

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(d)}(0)}{d!}x^d$$

is at most $\frac{Mx^{d+1}}{d!}$.

3. On the last practice midterm, you computed several derivatives of $f(x)=e^{-x}$:

$$f'(x)=-e^{-x} \quad f''(x)=e^{-x} \quad f'''(x)=-e^{-x} \quad f^{(4)}(x)=e^{-x} \quad f^{(5)}(x)=-e^{-x}$$

From this you found that the Taylor series was $1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\cdots$.

a) (5 points) Use a calculator to evaluate the fifth Taylor polynomial at $x=1$.

$$1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}=0.36666\ldots$$

b) (5 points) For every d we have $\left|f^{(d+1)}(t)\right|=e^{-t}$, and we see that if $t\geq 0$ then $e^{-t}\leq 1$, so in Taylor's theorem we can take $M=1$. How far, at most, does the theorem say that the number you found in part (a) can be from the true value of $f(1)$?

If $x=1$, $d=5$, and $M=1$, then $\frac{Mx^{d+1}}{d!}=\frac{1}{5!}=\frac{1}{120}=0.0083333\ldots$

c) (5 points) Take your answer to part (a) plus your answer to part (b), and then your answer to part (a) minus your answer to part (b), to get upper and lower estimates for $f(1)$.

Between $0.3583333\ldots$ and 0.375.

d) (5 points) Use a calculator to get a more exact value for $f(1)=e^{-1}=\frac{1}{e}$.
If this isn't in the range that you found in part (c), go back and fix any mistakes.

I get $0.367879441171442\ldots$, which is in the right range.

e) If you want Taylor's theorem to guarantee an error less than $10^{-4}=\frac{1}{10{,}000}$, what value of d should you take?

We want $\frac{1}{d!}\leq\frac{1}{10{,}000}$, so $d!\geq 10{,}000$.
We find that $7!=5{,}040$ but $8!=40{,}320$, so we take $d=8$.

4. The point of this problem is to approximate $\int_0^1 \frac{\ln(x+1)}{x}dx$, which cannot be found by the methods of math 252.

a) (5 points) We have seen that the Taylor series for $\ln(x+1)$ is $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots$. Manipulate this to get the Taylor series for $\frac{\ln(x+1)}{x}$.

Dividing by x, we get

$$1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\cdots$$

b) (5 points) Use your answer to part (a) to find $\int_0^1 \frac{\ln(x+1)}{x}dx$.
(Your answer will be a series of numbers, not a power series.)

$$\left(x-\frac{x^2}{4}+\frac{x^3}{9}-\frac{x^4}{16}+\frac{x^5}{25}-\cdots\right)\Bigg|_{x=0}^{1}=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\cdots$$

c) (5 points) Use a calculator to get an approximate value for the series in part (b). The true value is $\frac{\pi^2}{12}=0.822467033424113\ldots$; if your answer is far from this, go back and fix any mistakes.

Going out to the $\frac{1}{25}$ term, I get $0.83861111\ldots$ which is within 2% of the true answer.7

5. This problem asks you to solve the differential equation $y''=y$ using power series.

a) (5 points) Suppose that $y=c_0+c_1t+c_2t^2+c_3t^3+c_4t^4+c_5t^5+c_6t^6+\cdots$.

Find y' and y''.

$y'=c_1+2c_2t+3c_3t^2+4c_4t^3+5c_5t^4+6c_6t^5+\cdots$

$y''=2c_2+3\cdot 2\cdot c_3t+4\cdot 3\cdot c_4t^2+5\cdot 4\cdot c_5t^3+6\cdot 5\cdot c_6t^4+\cdots$

b) (5 points) By equating the constant terms of y'' and y, then the coefficients of t, then the coefficients of t^2 and so on, solve for c_2, c_3, and so on up to c_6 in terms of c_0 and c_1.

From the constant terms we get $2c_2=c_0$, so $c_2=\frac{c_0}{2}$.

From the coefficients of t we get $3\cdot 2\cdot c_3=c_1, so\, c_3=\frac{c_1}{3\cdot 2}$.

From the coefficients of t^2 we get $4\cdot 3\cdot c_4=c_2$, so $c_4=\frac{c_2}{4\cdot 3}=\frac{c_0}{4\cdot 3\cdot 2}$.

From the coefficients of t^3 we get $5\cdot 4\cdot c_5=c_3$, so $c_5=\frac{c_3}{5\cdot 4}=\frac{c_1}{5\cdot 4\cdot 3\cdot 2}$.

From the coefficients of t^4 we get $6\cdot 5\cdot c_6=c_4$, so $c_6=\frac{c_4}{6\cdot 5}=\frac{c_0}{6\cdot 5\cdot 4\cdot 3\cdot 2}$.

c) (5 points) Write out the sixth Taylor polynomial of the particular solution that salsifies the initial conditions $y(0)=1$ and $y'(0)=-1$.
(The point is that these initial conditions determine c_0 and c_1, which determine the rest.)

We have $y(0)=c_0$ and $y'(0)=c_1$, so these initial conditions give $c_0=1$ and $c_1=-1$. Thus the sixth Taylor polynomial is

$$y=1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\frac{t^6}{6!}.$$

d) Extra credit (5 points): Do you recognize your answer to part (c) as the Taylor series of a familiar function? Can you verify that it satisfies $y''=y$? What if we had taken $y(0)=1$ and $y'(0)=1$? What about $y(0)=1$ and $y'(0)=0$?

Looking back at problem 3, we see that $y=e^{-t}$, so $y'=-e^{-t}$, so $y''=e^{-t}=y$ as desired.
If $y(0)=1$ and $y'(0)=1$ then we find that $y=e^t$, which also satisfies $y''=y$.
To solve $y(0)=1$ and $y'(0)=0$ we can take the average of these:
$$y=\frac{e^t+e^{-t}}{2}=1+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\cdots$$
This is known as the hyperbolic cosine and is denoted $\cosh(t)$.