Math 253
Homework 8
Due Friday, March 8, 2024

Here is the form of Taylor's theorem that we proved and have been using. Fix some $x>0$, and suppose we find some M such that $\left|f^{(d+1)}(t)\right| \leq M$ for all t between 0 and x. Then the difference between f(x) and the d^{th} Taylor polynomial

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(d)}(0)}{d!}x^d$$

is at most $\frac{Mx^{d+1}}{d!}$.

1. On homework 6 #10, you found that Taylor series for $\sin x$ is

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots.$$

a) Use a calculator to plug in $x=1$ to the 6^{th} Taylor polynomial for sin x. (Because the x^6 term is zero, the 5^{th} and 6^{th} Taylor polynomials are the same in this case.)

b) In lecture we saw that $M=1$ works for this function, so Taylor's theorem as stated above says that the number you found in part (a) is at most how far from the true value of $\sin(1)$?

c) Take your answer to part (a) plus your answer to part (b), and then your answer to part (a) minus your answer to part (b), to get upper and lower estimates on $\sin(1)$.

d) Use a calculator to get a more exact value of $\sin(1)$. (Make sure you're working in radians!) What is the difference between this and your answer to part (a)?

e) If you want Taylor's theorem to guarantee an error less than $10^{-7}=\frac{1}{10{,}000{,}000}$, what value of d should you take?

(Continued on the next page.)

2. On homework 6 #11 you found that the third Taylor polynomial for $f(x)=\sqrt{1+x}$ is

$$1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3.$$

For reference, the fourth derivative is

$$f^{(4)}(x)=-\frac{15}{16}(1+x)^{-7/2}.$$

Suppose we're interested in evaluating $\sqrt{1.5}$ and $\sqrt{2}$, which means taking $x=\frac{1}{2}$ and $x=1$.

a) Use a calculator to evaluate the Taylor polynomial above at $x=\frac{1}{2}$ and $x=1$.

b) Find an M such that $\left|f^{(4)}(t)\right|\leq M$ for all t between 0 and 1.
Hint: $(1+x)^{-7/2}$ is a decreasing function.

c) So Taylor's theorem says that the numbers you found in part (a) are at most how far from the true value of $\sqrt{1.5}$ and $\sqrt{2}$? (Two different answers.)

d) Take your answers to part (a) and add and subtract your answers to part (c) to get upper and lower estimates for $\sqrt{1.5}$ and $\sqrt{2}$.

e) Use a calculator to get more exact values for $\sqrt{1.5}$ and $\sqrt{2}$. What is the difference between these and your answers to part (a)?