Math 253
Homework 9
Due Friday, March 15, 2024

1. The function $\mathrm{Si}(x)=\int_0^x \frac{\sin t}{t}\,dt$, called the sine integral, has applications to signal processing.

   a) Try to evaluate this integral in the usual way, by making a u-substitution, integrating by parts, or whatever you can think of. Explain what you tried and where you got stuck. If you think you've succeeded, double-check by taking the derivative of your answer.

   b) We have seen that the Taylor series for $\sin(t)$ is

      $$t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots$$

      Divide by t and integrate from 0 to x to get the Taylor series for $\mathrm{Si}(x)$.

   c) Use this to get an approximate value for $\mathrm{Si}(1)$. The true value is $0.946083070367183\ldots$

2. Approximating π. In lecture we took the geometric series $\frac{1}{1-r}=1+r+r^2+r^3+\cdots$,

   substitued $r=-x^2$ to get $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$,

   and integrated to get the Taylor series for arctangent:

   $$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots.$$

   Then we noticed that $\tan\frac{\pi}{4}=1$, so we plugged in x=1 and multiplied by 4 to get an approximation to π. But the series converged very slowly: after 100 terms we only had π to one decimal place.

   We also noticed that $\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$ and wondered if this would yield a better approximation to π. Use a calculator to to plug $x=\frac{1}{\sqrt{3}}$ into the Taylor series for arctangent above, carry it out to 4 terms and then 5 terms, and multiply by 6. How good an approximation do you get to $\pi=3.14159265358989323846\ldots$?

   (Continued on the next page.)

3. This problem asks you to solve the differential equation $y' = -2ty$ using power series.

   a) Suppose that $y = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + c_6 t^6 + \cdots$
   Find y' and $-2ty$.

   b) By equating the constant terms, then the coefficients of t, then the coefficients of t^2 and so on, solve for c_1, c_2, c_3 and so on in terms of c_0.

   c) Take what you found in part (b) and write out the power series for y that solves the differential equation.

   d) By know you know the Taylor series for e^x:

   $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

   Substitute $x = -t^2$ and notice that your answer to part (c) is $c_0 e^{-t^2}$. Check that this satisfies $y' = -2ty$.

4. "Bessel functions" appear often in applied mathematics and physics, especially in analyzing the vibration of bells, cymbals, and drums. The Bessel function of order 0 is a solution to the differential equation $ty'' + y' + ty = 0$.

   a) Suppose that $y = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + c_6 t^6 + \cdots$
   Find ty, y', and ty''.

   b) Add them up to get $ty'' + y' + ty$, grouping together the constant terms, the terms involving t, the terms involving t^2, and so on; stop at t^5.

   c) Setting your answer to part (b) equal to zero determines the coefficients c_1, c_2, c_3 and so on in terms of c_0. Set the constant term equal to zero and solve for c_1. Set the the coefficient of t equal to zero and solve for c_2 in terms of c_0. Keep going up to t^5, which determines c_6.

   d) Suppose that $y(t)$ satisfies the differential equation $ty'' + y' + ty = 0$ and the initial condition $y(0) = 1$ (which determines c_0). Write out the sixth Taylor polynomial of $y(t)$. Give an approximation to $y(1)$.