Midterm 1
Math 253
February 9, 2024 Name: Solutions

Each part is worth 5 points, for a total of 60 points.
You may use a hand-written sheet of notes.
Show your work where appropriate.
No calculators or cheating.

1. Find a formula for the general term $a_n$ in the following sequences.
   Indicate whether you're starting from $n=1$ or $n=0$; either choice is ok.

   a) $5, -10, 15, -20, 25, \dots$

      If you start from $n=1$, you'll get $a_n=(-1)^{n-1}\cdot 5n$ or $(-1)^{n+1}\cdot 5n$.

      If you start from $n=0$, you'll get $a_n=(-1)^n\cdot 5(n+1)$ or $(-1)^n(5n+5)$.

   b) $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \dots$

      If you start from $n=1$, you'll get $a_n=\frac{n}{(n+1)^2}$.

      If you start from $n=0$, you'll get $a_n=\frac{n+1}{(n+2)^2}$.

      You could even start from $n=2$ and get $a_n=\frac{n-1}{n^2}$.

2. Suppose that $a_1=2$, and for $n\geq 2$ we have $a_n=a_{n-1}+3$.

   a) Write out the first five terms of the sequence.

      2, 5, 8, 11, 14, 17

   b) Find an explicit formula for $a_n$.

      $a_n=3n-1$.

3. Evaluate the following limits:

a) $\lim_{n\to\infty}\frac{e^{n+1}}{e^n+1}$

We see that the limit is of the form $\frac{\infty}{\infty}$.

One possibility is to multiply the top and bottom by $\frac{1}{e^n}$, which gives

$$\lim_{n\to\infty}\frac{e^1}{1+1/e^n}=\frac{e}{1+0}=e\,.$$

The other possibility is to use L'Hôpital's rule:

$$\lim_{n\to\infty}\frac{e^{n+1}}{e^n+1}=\lim_{n\to\infty}\frac{e^{n+1}}{e^n}=\lim_{n\to\infty}e=e.$$

b) $\lim_{n\to\infty}\frac{n^2}{\ln n}$.

We see that the limit is of the form $\frac{\infty}{\infty}$.

Applying L'Hôpital's rule (just once), we get

$$\lim_{n\to\infty}\frac{n^2}{\ln n}=\lim_{n\to\infty}2\frac{n}{1/n}=\lim_{n\to\infty}2n^2=\infty.$$

4. Consider the series $\frac{1}{4}+\frac{2}{9}+\frac{3}{16}+\frac{4}{25}+\frac{5}{36}+\cdots$.

a) Write it in sigma notation, that is, as $\sum_{n=1}^{\infty}(\mathit{something})$ or $\sum_{n=0}^{\infty}(\mathit{something})$.

Reusing the answer to problem 1a, we get

$$\sum_{n=1}^{\infty}\frac{n}{(n+1)^2}\text{ or }\sum_{n=0}^{\infty}\frac{n+1}{(n+2)^2}\text{ or even }\sum_{n=2}^{\infty}\frac{n-1}{n^2}.$$

b) Find the first three partial sums $S_1, S_2, S_3$.

$$S_1=\frac{1}{4},\qquad S_2=\frac{1}{4}+\frac{2}{9}=\frac{17}{36},\qquad S_3=\frac{1}{4}+\frac{2}{9}+\frac{3}{16}=\frac{95}{144}.$$

5. Geometric series:

a) Fill in the blank: if $|r|<1$, then

$$1+r+r^2+r^3+\cdots=\frac{1}{1-r}.$$

b) Does the geometric series $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\cdots$ converge or diverge? If it converges, find the limit.

We can write this as $\frac{3}{10}\left(1+\frac{1}{10}+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\cdots\right)$. We have $r=\frac{1}{10}<1$,

so the series converges to $\frac{3}{10}\cdot\frac{1}{1-1/10}$, which simplifies to $\frac{3}{10}\cdot\frac{1}{9/10}=\frac{3}{10}\cdot\frac{10}{9}=\frac{3}{9}=\frac{1}{3}$.

This agrees with something we already knew: $0.3333\ldots=\frac{1}{3}$.

6. Use the integral test to decide whether the following series converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n+5}}$

To evaluate $\int_1^{\infty}\frac{1}{\sqrt{2x+5}}\,dx$, which we saw on a quiz the first week, we substitute $u=2x+5$,

so $du=2\,dx$, so $dx=\frac{1}{2}\,du$, so the integral becomes

$$\int_7^{\infty}\frac{1}{2\sqrt{u}}\,du=\int_7^{\infty}\frac{1}{2}u^{-1/2}\,du=u^{1/2}\Big|_7^{\infty}=\infty-\sqrt{7}=\infty,$$ so the sum diverges.

b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

This appeared on the homework as §5.2 #164. To evaluate $\int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx$, we substitute

$u=\ln x$, so $du=\frac{1}{x}\,dx$, so the integral becomes

$$\int_{\ln 2}^{\infty}\frac{1}{u^2}\,du=\int_{\ln 2}^{\infty}u^{-2}\,du=\frac{u^{-3}}{-3}\Bigg|_{\ln 2}^{\infty}=0-\frac{(\ln 2)^{-3}}{-3}=\frac{1}{3(\ln 2)^3}.$$

This is finite, so the sum converges.