Midterm 2
Math 253
March 1, 2024 Name: Solutions

Each problem is worth 10 points, for a total of 60 points.
You may use a hand-written sheet of notes.
Show your work where appropriate.
No calculators or cheating.

1. Does $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ converge or diverge, and why?

   We can use the basic comparison test: we see that $\sqrt{n}-1<\sqrt{n}$, so $\frac{1}{\sqrt{n}-1}>\frac{1}{\sqrt{n}}$, and we know that $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the integral test (because $1/2 \leq 1$), so $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges as well.

   Alternatively we could use the limit comparison test.

2. Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converge or diverge, and why?

   This is similar to section 5.4 problem 207, which was the quiz on 2/14, but this one is a little simpler because the log isn't squared.
   We use the limit comparison test to compare it to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. We have

   $$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\ln n \cdot n^{3/2}}{n^2} = \lim_{n\to\infty} \frac{\ln n}{n^{1/2}},$$

   which is of the form $\frac{\infty}{\infty}$, so by L'Hôpital's rule it equals

   $$\lim_{n\to\infty} \frac{1/n}{\frac{1}{2}n^{-1/2}} = \lim_{n\to\infty} \frac{2}{n^{1/2}} = 0.$$

   Thus $\frac{\ln n}{n^2}$ is eventually smaller than $\frac{1}{n^{3/2}}$. Now $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the integral test (because 3/2 > 1), so $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges as well.

   Alternatively you could compare to $\frac{1}{n^{1.9}}$, or $\frac{1}{n^p}$ for any p between 1 and 2.

3. Does $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$ converge absolutely, conditionally, or not at all, and why?

Taking absolute values, we get $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which converges by the integral test (because $3/2>1$). So it converges absolutely.

4. Does $\sum_{n=0}^{\infty}\frac{2^n}{n!}$ converge or diverge, and why?

We use the ratio test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{n+1}/(n+1)!}{2^n/n!}=\lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{n!}{(n+1)!}=\lim_{n\to\infty}\frac{2}{n+1}=0,$$

where toward the end we used the fact that $(n+1)!=(n+1)\cdot n!$. Because $0<1$, the series converges.

5. Find the third Taylor polynomial of the function $f(x)=\sin 2x+\cos x$, that is, the polynomial of degree 3 whose value and first three derivatives at zero agree with those of f.

We can take three derivatives using the chain rule:

$f'(x)=2\cos 2x-\sin x$,
$f''(x)=-4\sin 2x-\cos x$,
$f'''(x)=-8\cos 2x+\sin x$.

Plugging in $x=0$ to f and its derivatives, we get

$f(0)=1$,
$f'(0)=2$,
$f''(0)=-1$,
$f'''(0)=-8$.

Thus the third Taylor polynomial is $\frac{1}{0!}+\frac{2}{1!}x-\frac{1}{2!}x^2-\frac{8}{3!}x^3$,

or if you want to clean it up, $1+2x-\frac{1}{2}x^2-\frac{4}{3}x^3$.

6. For which values of x does the series $\sum_{n=0}^{\infty} \frac{x^n}{n+1} = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$ converge?

This was problem 12 on homework 6.

First we take absolute values and apply the ratio test. We have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}/(n+2)}{|x|^n/(n+1)} = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n+1}{n+2} = \lim_{n\to\infty} |x| \cdot \frac{n+1}{n+2}.$$

As $n \to \infty$ we see that $\frac{n+1}{n+2} \to 1$, either using L'Hôpital's rule or by multiplying top and bottom by $1/n$, so the whole limit is $|x|$. Thus if $|x| < 1$ then the series converges absolutely, if $|x| > 1$ then the terms don't go to zero and the series diverges, and if $|x| = 1$ then the ratio test is inconclusive.

Looking closer at the last case, if $x = 1$ then we're talking about the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ which we know diverges, and if $x = -1$ then we're talking about the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which converges because the absolute values of the terms decrease to zero.