Practice Final Exam: Solutions
Math 253
March 20, 2024

1. (10 points) Does $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ converge or diverge, and why?

This is section 5.6 #321, which was on homework 5.

Applying the ratio test, we have

$$\lim_{n\to\infty} \frac{((n+1)!)^3/(3n+3)!}{(n!)^3/(3n)!}$$
$$= \lim_{n\to\infty} \left(\frac{(n+1)!}{n!}\right)^3 \cdot \frac{(3n)!}{(3n+3)!}$$
$$= \lim_{n\to\infty} \left(\frac{(n+1)\cdot n!}{n!}\right)^3 \cdot \frac{(3n)!}{(3n+3)(3n+2)(3n+1)\cdot(3n)!}$$
$$= \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}$$
$$= \lim_{n\to\infty} \frac{n^3+\text{lower terms}}{27n^3+\text{lower terms}}$$

Now we either use L’Hôpital’s rule three times or divide the top and bottom of the fraction by n^3, and either way we find that the limit is 1/27. Since this is less than 1, the series converges.

2. (10 points) For what values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{2n^3}$ converge?

First we take absolute values and apply the ratio test, which gives

$$\lim_{n\to\infty} \frac{|x|^{n+1}/2(n+1)^3}{|x|^n/2n^3}$$
$$= \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{2(n+1)^3}{2n^3}$$
$$= \lim_{n\to\infty} |x| \cdot \frac{2n^3 + \text{lower terms}}{2n^3}$$

We see that the part involving n goes to 1, so the whole limit goes to $|x|$. Thus the series converges if $|x|<1$, diverges if $|x|>1$, and if $|x|=1$ then we need to do more work.

If $x=1$ then we're talking about $\sum_{n=1}^{\infty} \frac{1}{2n^3}$, which converges by the integral test (since $3>1$).
If $x=-1$ then we're talking about $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^3}$, which converges absolutely, because if we throw away the signs then we get the thing that we just said converges.

Thus the series converges for $-1 \leq x \leq 1$.

Here is the form of Taylor's theorem that we proved and have been using. Fix some $x>0$, and suppose we find some M such that $\left|f^{(d+1)}(t)\right|\leq M$ for all t between 0 and x. Then the difference between f(x) and the dth Taylor polynomial

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(d)}(0)}{d!}x^d$$

is at most $\dfrac{Mx^{d+1}}{d!}$.

3. On the last midterm you computed several derivatives of $f(x)=\sin(2x)+\cos(x)$:

$$f'(x)=2\cos(2x)-\sin(x)$$
$$f''(x)=-4\sin(2x)-\cos(x)$$
$$f'''(x)=-8\cos(2x)+\sin(x)$$

Then you found that the third Taylor polynomial was $1+2x-\frac{1}{2}x^2-\frac{4}{3}x^3$.

a) (5 points) Use a calculator to evaluate the third Taylor polynomial at $x=0.1$.

$$1+2\cdot 0.1-\frac{1}{2}\cdot 0.001-\frac{4}{3}\cdot 0.0001=1.1936666\ldots$$

b) (5 points) Because $\sin t$ and $\cos t$ stay between -1 and 1 for all t, we see that the first term of $f^{(d+1)}(t)$ stays between -2^{d+1} and 2^{d+1}, and the second term stays between -1 and 1; so which of the following is a good choice for the M that appears in Taylor's theorem?

(i) 2^{d+1} (ii) 1 (iii) $2^{d+1}+1$ (iv) $2^{d+1}-1$ (v) 2^d+1 (vi) 2 (vii) -1

Choice (iv): $M=2^{d+1}+1$.

c) (5 points) So Taylor's theorem as stated above says that the number you found in part (a) is at most how far from the true value of $f(0.1)$?

$$\frac{Mx^{d+1}}{d!}=\frac{(2^4+1)(0.1)^4}{3!}=0.000283333\ldots.$$

d) (5 points) Take your answer to part (a) plus your answer to part (c), and then your answer to part (a) minus your answer to part (c), to get upper and lower estimates for $f(0.1)$.

Between 1.193383333… and 1.19395.

e) (5 points) Use a calculator to get a more exact value for $f(0.1)=\sin(0.2)+\cos(0.1)$. (Make sure you're working in radians!) If this isn't in the range that you found in part (d), go back and fix any mistakes.

I get 1.19367349607309…, which is in the right range.

4. The point of this problem is to approximate $\int_0^1 \frac{e^x - 1}{x} dx$, which cannot be found by the methods of math 252.

a) (5 points) We have seen that the Taylor series for e^x is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

Manipulate this to get the Taylor series for $\frac{e^x - 1}{x}$.

Subtracting 1 from the Taylor series for e^x, we get
$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

Dividing through by x, we get
$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{3^4}{4!} + \cdots$

b) (5 points) Use your answer to part (a) to find $\int_0^1 \frac{e^x - 1}{x} dx$.

(Your answer will be a series of numbers, not a power series.)

$$\left(x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \cdots \right) \Bigg|_{x=0}^{1} = 1 + \frac{1}{2 \cdot 2!} + \frac{1}{3 \cdot 3!} + \frac{1}{4 \cdot 4!} + \cdots.$$

c) (5 points) Use a calculator to get an approximate value for the series in part (b). The true value is 1.3179021514544…; if your answer is far from this, go back and fix any mistakes.

Going out to the $\frac{1}{4 \cdot 4!}$ term, I get 1.315972222… which agrees with the true value to four decimal places.

5. This problem asks you to solve the differential equation $y''=-ty$ using power series.

a) (5 points) Suppose that $y=c_0+c_1t+c_2t^2+c_3t^3+c_4t^4+c_5t^5+c_6t^6+\cdots$.

Find $-t\,y$, y', and y''.

$-ty=-c_0t-c_1t^2-c_2t^3-c_3t^4-c_4t^5-c_5t^6-c_6t^7+\cdots$

$y'=c_1+2c_2t+3c_3t^2+4c_4t^3+5c_5t^4+6c_6t^5+\cdots$

$y''=2c_2+3\cdot 2\cdot c_3t+4\cdot 3\cdot c_4t^2+5\cdot 4\cdot c_5t^3+6\cdot 5\cdot c_6t^4+\cdots$

b) (5 points) By equating the constant terms of y'' and $-ty$, then the coefficients of t, then the coefficients of t^2 and so on, solve for c_2, c_3, and so on up to c_6 in terms of c_0 and c_1.

From the constant terms we get $2c_2=0$, so $c_2=0$.

From the coefficients of t we get $3\cdot 2\cdot c_3=-c_0$, so $c_3=\frac{-c_0}{3\cdot 2}$.

From the coefficients of t^2 we get $4\cdot 3\cdot c_4=-c_1$, so $c_4=\frac{-c_1}{4\cdot 3}$.

From the coefficients of t^3 we get $5\cdot 4\cdot c_5=-c_2$, so $c_5=\frac{-c_2}{5\cdot 4}=0$.

From the coefficients of t^4 we get $6\cdot 5\cdot c_6=-c_3$, so $c_6=\frac{-c_3}{6\cdot 5}=\frac{c_0}{6\cdot 5\cdot 3\cdot 2}$.

6. (5 points) Write out the sixth Taylor polynomial of the particular solution that salsifies the initial conditions $y(0)=1$ and $y'(0)=-1$.
(The point is that these initial conditions determine c_0 and c_1, which determine the rest.)

We have $y(0)=c_0$ and $y'(0)=c_1$, so these initial conditions give $c_0=1$ and $c_1=-1$. Thus the sixth Taylor polynomial is

$y=1-t-\frac{1}{6}t^3+\frac{1}{12}t^5+\frac{1}{180}t^6$.