Practice Midterm 1: Solutions
Math 253
February 8, 2024

1. Find a formula for the general term a_n in the following sequences.
   Indicate whether you're starting from $n=1$ or $n=0$; either choice is ok.

   a) 2, 5, 8, 11, 14, …

      If you start from $n=1$, you'll get $a_n=3n-1$.

      If you start from $n=0$, you'll get $a_n=3n+2$.

   b) $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, \dots$

      There are many equivalent ways to write the answer.

      If you start from $n=1$, you'll get $a_n=\frac{(-1)^{n-1}}{2^n}$ or something equivalent.

      If you start from $n=0$, you'll get $a_n=\frac{(-1)^n}{2^{n+1}}$ or something equivalent .

2. Suppose that $a_1=2$, and for $n\geq 2$ we have $a_n=3a_{n-1}$.

   a) Write out the first five terms of the sequence.

      2, 6, 18, 54, 162.

   b) Find an explicit formula for a_n.

      $a_n=2\cdot 3^{n-1}$

3. Evaluate the following limits:

a) $\lim_{n\to\infty} \frac{n^2+2n+3}{3n^2+4n+5}$

We see that the limit is of the form $\frac{\infty}{\infty}$.

One possibility is to multiply the top and bottom by $\frac{1}{n^2}$, which gives

$$\lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{3}{n^2}}{3+\frac{4}{n}+\frac{5}{n^2}} = \frac{1+0+0}{3+0+0} = \frac{1}{3}.$$

The other is to use L'Hôpital's rule twice:

$$\lim_{n\to\infty} \frac{n^2+2n+3}{3n^2+4n+5} = \lim_{n\to\infty} \frac{2n+2}{6n+4} = \lim_{n\to\infty} \frac{2}{6} = \frac{1}{3}$$

b) $\lim_{n\to\infty} \frac{n}{(\ln n)^2}$.

Again we see that the limit is of the form $\frac{\infty}{\infty}$.

Applying L'Hôpital's rule once, we get $\lim_{n\to\infty} \frac{1}{2(\ln n)\cdot\frac{1}{n}}$.

Simplifying, this becomes $\lim_{n\to\infty} \frac{n}{2\ln n}$, which is again of the form $\frac{\infty}{\infty}$.

Applying L'Hôpital's rule again, we get $\lim_{n\to\infty} \frac{1}{2/n} = \lim_{n\to\infty} \frac{n}{2} = \infty$.

4. Consider the series $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-\cdots$.

a) Write it in sigma notation, that is, as $\sum_{n=1}^{\infty}(something)$ or $\sum_{n=0}^{\infty}(something)$.

Reusing the answer to problem 1a, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^n} \text{ or } \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}.$$

b) Find the first three partial sums S_1, S_2, S_3.

$$S_1=\frac{1}{2},\ S_2=\frac{1}{2}-\frac{1}{4}=\frac{1}{4},\ S_3=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}=\frac{3}{8}.$$

c) Does the series converge or diverge? If it converges, find the sum.
Hint: It is a geometric series, although it doesn't start from 1.

We know that if $|r|<1$ then $1+r+r^2+r^3+\cdots=\frac{1}{1-r}$.
Thus $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots=\frac{1}{1+1/2}=\frac{1}{3/2}=\frac{2}{3}$.

Thus the series we're considering can either be found as $\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$, or as $1-\frac{2}{3}=\frac{1}{3}$.

5. Consider the telescoping series $\sum_{n=1}^{\infty}\left(\sqrt{n}-\sqrt{n-1}\right)$.

a) Find the first three partial sums S_1, S_2, S_3.

$S_1=\sqrt{1}-\sqrt{0}=1$

$S_2=\left(\sqrt{1}-\sqrt{0}\right)+\left(\sqrt{2}-\sqrt{1}\right)=\sqrt{2}$

$S_3=\left(\sqrt{1}-\sqrt{0}\right)+\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)=\sqrt{3}$

b) Give a formula for the n^th^ partial sum S_n.

$S_n=\sqrt{n}$

c) Does the series converge or diverge? If it converges, find the sum.

The limit of the partial sums is $\lim_{n\to\infty}\sqrt{n}=\infty$, so the series diverges.

6. Use the integral test to decide whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty}\frac{1}{(2n+5)^2}$

To evaluate $\int_1^{\infty}\frac{1}{(2x+5)^2}dx$, which we saw on the very first quiz, we substitute $u=2x+5$, so $du=2\,dx$, so $dx=\frac{1}{2}du$, so the integral becomes

$$\int_7^{\infty}\frac{1}{2u^2}du=\int_7^{\infty}\frac{1}{2}u^{-2}du=\frac{1}{2}\cdot\frac{u^{-1}}{-1}\Big|_7^{\infty}=0-\left(-\frac{1}{14}\right)=\frac{1}{14}.$$

This is finite, so the sum converges.

b) $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n}$

To evaluate $\int_1^{\infty}\frac{(\ln x)^2}{x}dx$, we substitute $u=\ln x$, so $du=\frac{1}{x}dx$, so the integral becomes

$$\int_0^{\infty}u^2du=\frac{u^3}{3}\Big|_0^{\infty}=\infty-0=\infty.$$

Thus the sum diverges.