Practice Midterm 2: Solutions
Math 253
February 29, 2024

1. Does $\sum_{n=1}^{\infty} \frac{1}{n^2 \ln n}$ converge or diverge, and why?

   We can apply the basic comparison test. If n is big then $\ln n \geq 1$, so $n^2 \ln n \geq n^2$, so $\frac{1}{n^2 \ln n} \leq \frac{1}{n^2}$. We know that $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by the integral test (because $2>1$), so $\sum_{n=1}^{\infty} \frac{1}{n^2 \ln n}$ converges as well.

2. Does $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}+1}$ converge or diverge, and why?

   We want to compare to $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$, which diverges by the integral test, but we cannot use the basic comparison test because our series is *less* than this one. But we can use the limit comparison test. The terms of the first series over those of the second are $\frac{1/(n^{2/3}+1)}{1/n^{2/3}} = \frac{n^{2/3}}{(n^{2/3}+1)} = \frac{1}{1+n^{3/2}}$, where the second equality came from multiplying top and bottom by $n^{3/2}$. Taking the limit as $n \to \infty$ we get $\frac{1}{1+0} = 1$, so the limit comparison test says that either both series converge or both diverge. We have already said that $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges, so $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}+1}$ diverges as well.

3. Does $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$ converge absolutely, conditionally, or not at all?

   Throwing away the signs, we get $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which converges by the integral test because $3/2>1$. Thus the original series converges absolutely.

4. Does $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ converge or diverge, and why?

We apply the ratio test, and consider

$$\frac{((n+1)!)^2/(2n+2)!}{(n!)^2/(2n)!}=\frac{((n+1)!)^2}{(n!)^2}\cdot\frac{(2n)!}{(2n+2)!}=\left(\frac{(n+1)!}{n!}\right)^2\cdot\frac{(2n)!}{(2n+2)!}.$$

We know that $(n+1)!=(n+1)\cdot n!$, and similarly $(2n+2)!=(2n+2)\cdot(2n+1)\cdot(2n)!$, so this cleans up to give

$$\frac{(n+1)^2}{(2n+2)(2n+1)}=\frac{n^2+2n+1}{4n^2+6n+2}.$$

As $n\to\infty$, this approaches $\frac{1}{4}$, either by applying L'Hôpital's rule twice or by multiplying top and bottom by $\frac{1}{n^2}$. Since $\frac{1}{4}<1$, the series converges.

5. Find the fourth Taylor polynomial of the function $f(x)=e^{-x}$.

Taking derivatives with the chain rule, we get
$f'(x)=-e^{-x}$, $f''(x)=e^{-x}$, $f'''(x)=-e^{-x}$, $f^{(4)}(x)=e^{-x}$.

Plugging in $x=0$ to f and its derivatives, we get
$f(0)=1$, $f'(0)=-1$, $f''(0)=1$, $f'''(0)=-1$, $f^{(4)}(0)=1$.

Thus the fourth Taylor polynomial is
$1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}$.

6. For which values of x does the series $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$ converge?

We take absolute values and apply the ratio test, then check the cases where the ratio test is inconclusive. If we say that $a_n=(-1)^n\frac{x^{2n}}{2n!}$, then we have
$$\frac{|a_{n+1}|}{|a_n|}=\frac{|x|^{2n+2}/(2n+2)!}{|x|^{2n}/(2n)!}=\frac{|x|^{2n+2}}{|x|^{2n}}\cdot\frac{(2n)!}{(2n+2)!}=\frac{|x|^2}{(2n+2)(2n+1)},$$
where in the last step we have used the fact that $(2n+2)!=(2n+2)\cdot(2n+1)\cdot(2n)!$. If x is fixed and $n\to\infty$ then this goes to zero, which is less than 1, so the series converges converges for *any* value of x.