First Midterm

Name: Solutions

October 9, 2013

1. Write parametric equations for the line along which the planes $x + 2y + 3z = 1$ and $2x + 5y - z = 2$ intersect.
Solution: The normal vectors to the two planes are $\langle 1, 2, 3\rangle$ and $\langle 2, 5, -1\rangle$, so the direction vector of the line is

$$\langle 1, 2, 3\rangle \times \langle 2, 5, -1\rangle = \langle -17, 7, 17\rangle.$$

To find a point on both planes, we set $z = 0$ in both equations and solve

$$x + 2y = 1 \qquad\qquad 2x + 5y = 2,$$

getting $x = 1$ and $y = 0$, so the point is $(1, 0, 0)$. Thus we can parametrize the line as

$$x = 1 - 17t \qquad\qquad y = 7t \qquad\qquad z = t.$$

2. Consider two lines, one passing through $(1, 0, 0)$ and $(0, 1, 0)$ and the other passing through $(0, -1, 0)$ and $(0, 0, 1)$. Find the minimum distance between them, as follows.

   (a) Write parametric equations for both lines, using s as the parameter for the first line and t as the parameter for the second.
   Solution:

$$\begin{aligned} x &= 1 - s \qquad\qquad & x &= 0 \\ y &= s & y &= t - 1 \\ z &= 0 & z &= t \end{aligned}$$

   (b) Write a function $f(s, t)$ that gives the distance *squared* between the point on the first line at time s and the point on the second line at time t.
   Solution: $f(s, t) = (1 - s)^2 + (s - t + 1)^2 + t^2$.

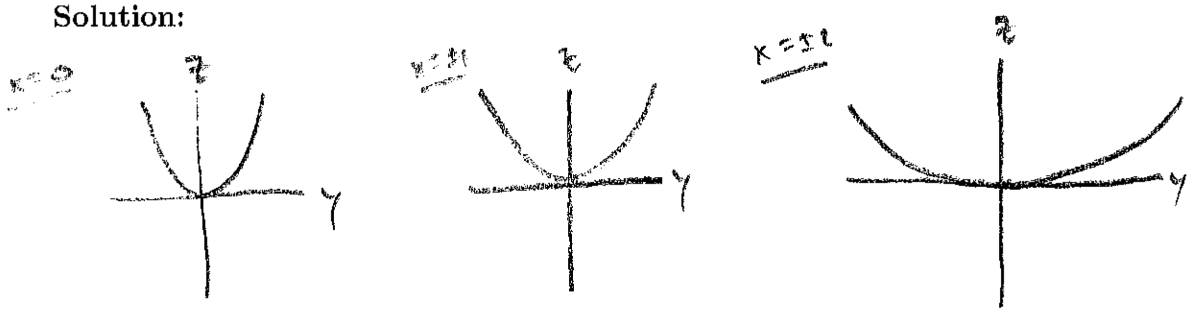
   (c) Find the minimum value of $f(s, t)$.
   Solution: We have

$$\begin{aligned} f_s(s, t) &= -2(1 - s) + 2(s - t + 1) = 4s - 2t \\ f_t(s, t) &= -2(s - t + 1) + 2t = -2s + 4t - 2. \end{aligned}$$

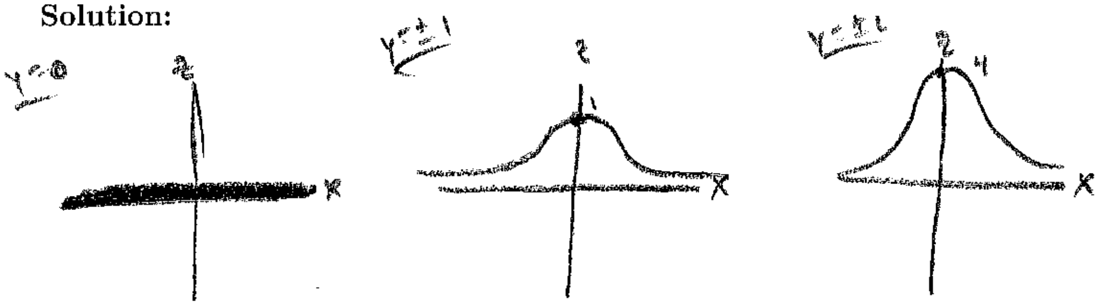
   Setting $f_s = 0$, we find that $t = 2s$. Setting $f_t = 0$ and substituting $t = 2s$, we find that $6s = 2$, so $s = \frac{1}{3}$, so $t = \frac{2}{3}$. Thus the minimum value is $f(\frac{1}{3}, \frac{2}{3}) = \frac{4}{3}$.

3. Consider the surface $z = \dfrac{y^2}{1+x^2}$.

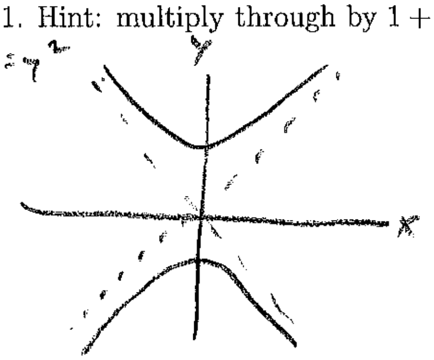
(a) Sketch the slices $x = 0$, $x = \pm 1$, and $x = \pm 2$.
Solution:



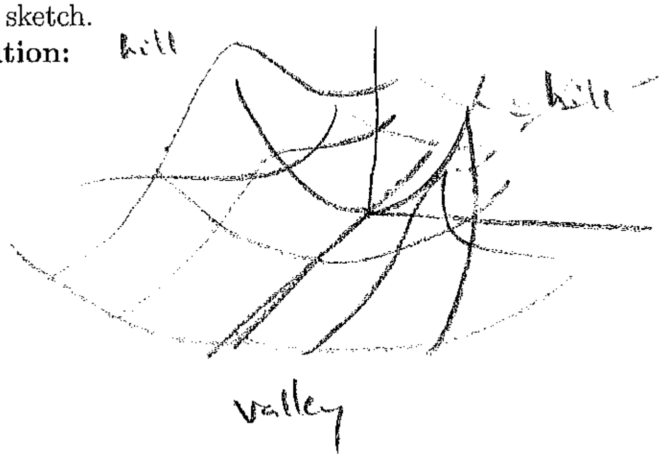
(b) Sketch the slices $y = 0$, $y = \pm 1$, and $y = \pm 2$.
Solution:



(c) Sketch the slice $z = 1$. Hint: multiply through by $1 + x^2$.
Solution: $1+x^2=y^2$



(d) Sketch the surface. The slices you drew in parts (a) through (c) should appear in your sketch.
Solution:



← valley behind, pass between the hills.

4. (a) In what direction is the function $f(x,y) = x^2 - y^2$ increasing most steeply at the point $(1,2)$? What is the slope in that direction? What is the slope in the direction $\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$?
Solution: The gradient $\nabla f(x,y) = \langle 2x, -2y \rangle$, so the direction of steepest increase at the point $(1,2)$ is $\nabla f(1,2) = \langle 2, -4 \rangle$. You can turn this into a unit vector $\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle$ if you like, but you don't have to. The slope in that direction is $|\langle 2, -4 \rangle| = \sqrt{20} = 2\sqrt{5}$. The slope in the direction $\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$ is $\langle 2, -4 \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = 1 - 2\sqrt{3}$.

(b) Write an equation for the tangent plane to the surface $y^2 + z^2 = x^3 - x$ at the point $(2, 2, \sqrt{2})$.
Solution: This is a level surface of the function $f(x,y,z) = y^2 + z^2 - x^3 + x$, whose gradient $\nabla f(x,y,z) = \langle -3x^2 + 1, 2y, 2z \rangle$. The tangent plane to the surface at the point $(2, 2, \sqrt{2})$ is perpendicular to $\nabla f(2, 2, \sqrt{2}) = \langle -11, 4, 2\sqrt{2} \rangle$, so the equation of the plane is

$$-11x + 4y + 2\sqrt{2}z = -10.$$

5. Consider the curve $\vec{r}(t) = \langle t^2 - 2t + 1, t^2, 2t^2 - 2t \rangle$.

(a) Find $\vec{v}$, $\vec{T}$, a_T, $\vec{a}_{\|}$, $\vec{a}_{\perp}$, and a_N at time $t = 1$. Circle your answers. As a sanity check, make sure that $\vec{a}_{\perp}$ is perpendicular to v.
Solution: We have $\vec{v}(t) = \langle 2t - 2, 2t, 4t - 2 \rangle$ and $\vec{a}(t) = \langle 2, 2, 4 \rangle$. Thus at time $t = 1$ we have

$$\begin{aligned}
\vec{v} &= \langle 0, 2, 2 \rangle \\
\vec{T} &= \frac{\vec{v}}{|\vec{v}|} = \langle 0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \rangle \\
a_T &= \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|} = 3\sqrt{2} \\
\vec{a}_{\|} &= a_T \vec{T} = \frac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \langle 0, 3, 3 \rangle \\
\vec{a}_{\perp} &= \vec{a} - \vec{a}_{\|} = \langle 2, -1, 1 \rangle \\
a_N &= |\vec{a}_{\perp}| = \sqrt{6}.
\end{aligned}$$

For the sanity check, we indeed have $\vec{a}_{\perp} \cdot v = 0$, and also $a_T^2 + a_N^2 = 24 = |\vec{a}|^2$.

(b) The curve meets the plane $x + y + 2z = 1$ at two points. One is $(1, 0, 0)$; what is the other?
Solution: We substitute $x = t^2 - 2t + 1$, $y = t^2$, and $z = 2t^2 - 2t$ into the equation of the plane $x + y + 2z = 1$ to get $4t^2 - 4t + 1 = 1$, so $t = 0$ or $t = 1$. In the first case we get $\vec{r}(0) = \langle 1, 0, 0 \rangle$; in the second case we get $\vec{r}(1) = \langle 0, 1, 0 \rangle$.

(c) Find the angle between the curve and the plane at those two points. Hint: First find the angle between the velocity vector to the curve and the normal vector to the plane; then think about what this has to do with the angle between the curve and the plane. Further hint: You can do this without a calculator, so the cosines in question must be among 0, $\pm\frac{1}{2}$, $\pm\frac{\sqrt{2}}{2}$, $\pm\frac{\sqrt{3}}{2}$, and ± 1.
Solution: The normal vector to the plane is $\vec{n} = \langle 1, 1, 2\rangle$. At $t = 0$, the angle between $\vec{n}$ and $\vec{v}(0) = \langle -2, 0, -2\rangle$ is

$$\cos^{-1}\left(\frac{\vec{n}\cdot\vec{v}(0)}{|\vec{n}|\,|\vec{v}(0)|}\right) = \cos^{-1}\left(\frac{-6}{\sqrt{6}\sqrt{8}}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = 150^\circ,$$

so the angle between the curve and the plane is 60°. At $t = 1$, the angle between $\vec{n}$ and $\vec{v}(1) = \langle 0, 2, 2\rangle$ satisfies

$$\cos^{-1}\left(\frac{\vec{n}\cdot\vec{v}(1)}{|\vec{n}|\,|\vec{v}(1)|}\right) = \cos^{-1}\left(\frac{6}{\sqrt{6}\sqrt{8}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ,$$

so the angle between the curve and the plane is 30°.