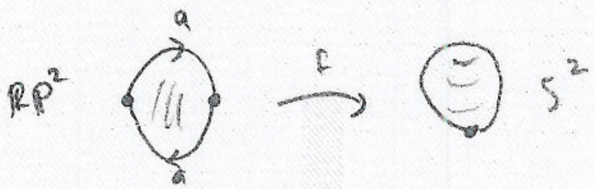
Enhanced Coursework

for Fourth Year and MSc Students

Due Friday, May 4, 2012 at 4 PM with Mary Harvey

1. Let $f : \mathbb{RP}^2 \to S^2$ be the map that collapses the loop labeled a below:



   (a) Show f induces the zero map on π_1 and on $H_i(-;\mathbb{Z})$ for $i > 0$. (Do not work hard.)

   (b) Let $A \subset \mathbb{RP}^2$ be the circle that f collapses. Write down the long exact sequence of the pair $(\mathbb{RP}^2, A)$ with $\mathbb{Z}_2$ coefficients. Indicate which groups are $\mathbb{Z}_2$ and which groups are 0.

   (c) Conclude that $f_* : H_2(\mathbb{RP}^2;\mathbb{Z}_2) \to H_2(S^2;\mathbb{Z}_2)$ is not zero. Thus f is not nullhomotopic, despite part (a).

2. Show that every finite cover of the torus is again a torus, as follows.

   (a) Optional: Show that if X is a surface and $p : \tilde{X} \to X$ is a covering then $\tilde{X}$ is a surface, and that if X is compact and $p : \tilde{X} \to X$ is a finite covering (that is, $p^{-1}(x)$ is finite for every $x \in X$) then $\tilde{X}$ is compact.

   (b) If X is a finite cell complex and $p : \tilde{X} \to X$ is an n-to-1 coveriing, argue that $\chi(\tilde{X}) = n \cdot \chi(X)$. (Some handwaving is acceptable here.)

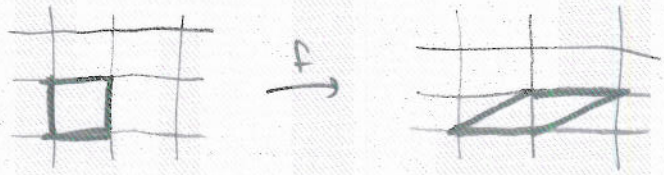
   (c) What is the Euler characteristic of the torus? What other compact surfaces have the same Euler characteristic?

   (d) Show that if $\tilde{X}$ is a cover of the torus then $\pi_1(\tilde{X})$ is abelian. Conclude that every finite cover of the torus is again a torus.

3. (a) Recall that we can decompose S^3 as the union of two solid tori A and B. Verify that $\pi_1(S^3) = 0$ by applying Van Kampen's theorem to this decomposition. (The issue of fattening A and B should be routine by now.)

(b) A *Dehn twist* is a homeomorphism of the torus obtained by cutting along a circle, giving it a twist, and it gluing back together:

Alternatively, if we view the torus as $\mathbb{R}^2/\mathbb{Z}^2$, it is this map:



Let $f: T \to T$ be a Dehn twist. What is $f_*: \pi_1(T) \to \pi_1(T)$?

(c) Now remove A from S^3 and glue it back in via a Dehn twist. Use Van Kampen's theorem to show that the resulting 3-fold is simply connected. Quote the Poincaré conjecture to conclude that it is again S^3 (of course you could also show this directly). Find this surprising, as I did.

4. Let $f: a \to b$ an arrow in a category $\mathcal{C}$.

(a) Which of the following are equivalent to which?

i. f is a monomorphism.

ii. f is an epimorphism.

iii. $f_*: \mathrm{Hom}_{\mathcal{C}}(c, a) \to \mathrm{Hom}_{\mathcal{C}}(c, b)$ is injective for every object c.

iv. $f^*: \mathrm{Hom}_{\mathcal{C}}(b, c) \to \mathrm{Hom}_{\mathcal{C}}(a, c)$ is injective for every object c.

(b) Show that the following are equivalent:

i. f is a split epimorphism.

ii. $f_*: \mathrm{Hom}_{\mathcal{C}}(c, a) \to \mathrm{Hom}_{\mathcal{C}}(c, b)$ is surjective for every object c.

(Hint: for (ii) implies (i), take $c = b$.) Similarly, f is a split monomorphism if and only if f^* is surjective for all c.

(c) Show that if f is a monomorphism and a split epimorphism then f is an isomorphism. (Similarly, if f is a split monomorphism and an epimorphism then f is an isomorphism. Thus f is an isomorphism iff f_* is bijective for all c iff f^* is bijective for all c.)