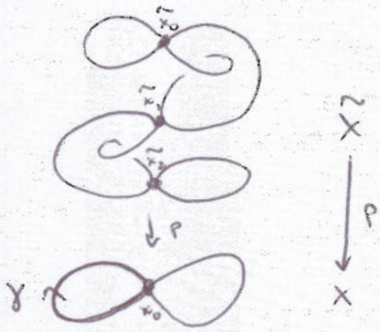
Homework 6

Due Monday, February 27, 2011

This homework looks long, but your solutions may well be shorter than the assignment.

1. Consider the following covering of $X = S^1 \vee S^1$:



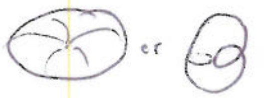
Let $x_0 \in X$ and $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ be as in the drawing. Show that $\mathrm{Aut}(\tilde{X} : X) = 1$, as follows.

   (a) Let γ be the loop based at x_0 shown in the diagram. Draw the unique lift $\tilde{\gamma}_0$ of γ that starts at $\tilde{x}_0$, and the unique lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ starting at $\tilde{x}_1$ and $\tilde{x}_2$, respectively.

   (b) Let $\varphi \in \mathrm{Aut}(\tilde{X} : X)$, that is $\varphi : \tilde{X} \to \tilde{X}$ is a homeomorphism with $p \circ \varphi = p$. If $\varphi(\tilde{x}_0) = \tilde{x}_1$, argue that $\varphi \circ \tilde{\gamma}_0$ is a lift of γ that starts at $\tilde{x}_1$, but that $\varphi \circ \tilde{\gamma}_0 \neq \tilde{\gamma}_1$, which is a contradiction.

   (c) Write "Similarly, we cannot have $\varphi(\tilde{x}_0) = \tilde{x}_2$, so $\varphi(\tilde{x}_0) = \tilde{x}_0$."

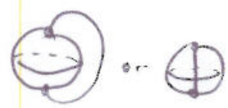
   (d) Conclude that φ is the identity.



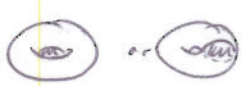
2. Last week we considered the following homotopy equivalent spaces:

   (a) The sphere with the north and south poles glued together.

   (b) The sphere with a line segment connecting the north and south poles.



   (c) The torus with a disc glued along the inner loop.



Draw a simply connected cover of each. Hint: We saw that they had $\pi_1 = \mathbb{Z}$, so the fibre will have to be $\mathbb{Z}$.

(Continued overleaf.)

3. (a) Let $p : \tilde{X} \to X$ be a covering, $x_0 \in X$ a basepoint, and $\tilde{x}_0 \in p^{-1}(x_0)$. Let Z be a simply-connected space, $z_0 \in Z$ a basepoint, $f_0, f_1 : Z \to X$ two maps with $f_0(z_0) = f_1(z_0) = x_0$, and $\tilde{f}_0, \tilde{f}_1$ the unique lifts with $\tilde{f}_0(z_0) = \tilde{f}_1(z_0) = \tilde{x}_0$. Draw a diagram. Show that if $f_0 \simeq f_1$ rel. basepoint then $\tilde{f}_0 \simeq \tilde{f}_1$ rel. basepoint.

   (b) Now suppose that $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ are coverings with $\tilde{X}$ and $\tilde{Y}$ simply connected. Let $x_0 \in X$ and $y_0 \in Y$ be basepoints, $\tilde{x}_0 \in p^{-1}(x_0)$, and $\tilde{y}_0 \in q^{-1}(y_0)$. Show that if $X \simeq Y$ rel. basepoint then $\tilde{X} \simeq \tilde{Y}$ rel. basepoint.

4. What is one question you have about last week's lectures?