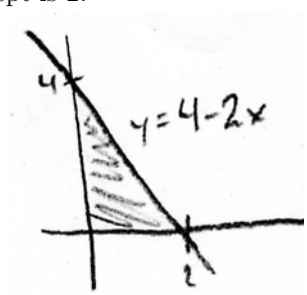
Solutions to Problem Set 11

I. Problems to be graded on completion.

1. a. Consider the line $2x + y = 4$. If $x = 0$ then $y = 4$, so the y-intercept is 4. If $y = 0$ then $2x = 4$, so $x = 2$, so the x-intercept is 2.



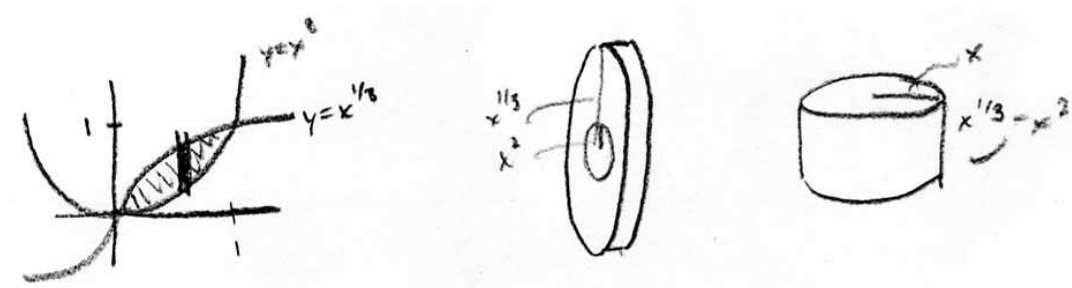
Around the x-axis, the radius of a slice is $4 - 2x$, so

$$\int_0^2 \pi(4x-2)^2\ dx = \pi \int_0^2 (16x^2 - 16x + 4)\ dx = \pi \left[\frac{16}{3}x^3 - 8x^2 + 4x\right]_0^2 = \frac{56\pi}{3} \approx 58.6.$$

Around the y-axis, the radius of a slice is $2 - \frac{1}{2}y$, so

$$\int_0^4 \pi\left(2 - \frac{1}{2}x\right)^2\ dx = \pi \int_0^4 \left(4 - 2x + \frac{1}{4}x^2\right)\ dx = \pi\left[4x - x^2 + \frac{1}{12}x^3\right]_0^4 = \frac{16\pi}{3} \approx 16.8.$$

b. These intersect when $x^2 = x^{1/3}$, so $x^6 = x$, so $x^6 - x = 0$. so $x(x^5 - 1) = 0$, so $x = 0$ or $x^5 = 1$, so $x = 1$. In between, which one is on top? If $x = \frac{1}{8}$, $x^2 = \frac{1}{64}$ while $x^{1/3} = \frac{1}{2}$, so $x^{1/3}$ is above x^2 on the interval $(0, 1)$.



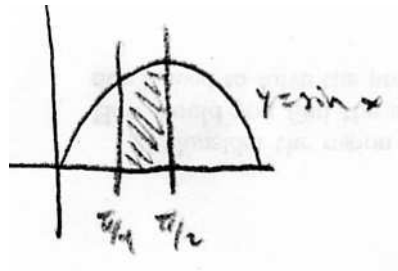
Around the x-axis, the outer radius is $x^{1/3}$ and the inner radius is x^2, so

$$\int_0^1 \pi x^{2/3}\ dx - \int_0^1 \pi x^4\ dx = \frac{3}{5}\pi x^{5/3}\Big|_0^1 - \frac{1}{5}\pi x^5\Big|_0^1 = \frac{2\pi}{5} \approx 1.26.$$

Around the y-axis, let's use the shell method. The radius of the shell is x and the height is $x^{1/3} - x^2$, so

$$\int_0^1 2\pi x(x^{1/3} - x^2)\ dx = 2\pi \int_0^1 (x^{4/3} - x^3)\ dx = 2\pi\left[\frac{3}{7}x^{7/3} - \frac{1}{4}x^4\right]_0^1 = \frac{5\pi}{14} \approx 1.12.$$

c. The region looks like this:



Around the x-axis, the radius of a slice is $\sin x$, so

$$\int_{\pi/4}^{\pi/2} \pi(\sin x)^2 \, dx = \frac{\pi}{2}\int_{\pi/4}^{\pi/2} (1-\cos 2x) \, dx = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/4}^{\pi/2} = \frac{\pi}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right) \approx 2.02.$$

Around the y-axis, let's use the shell method again. The radius is x and the height is $\sin x$, so

$$\int_{\pi/4}^{\pi/2} 2\pi x \sin x \, dx$$

would give us the number we wanted if we could do the integral.

2. a.

$$8\int_0^2 (\sqrt{4-x^2})^2 \, dx = 8\int_0^2 (4-x^2) \, dx = 8\left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{128}{3} = 42 + \frac{2}{3}$$

b. Consider half of an equilateral triangle. If the base is a, the hypotenuse is $2a$, so the height is $\sqrt{3}a$, so the area is $\frac{\sqrt{3}}{2}a^2$.

$$4\int_0^2 \frac{\sqrt{3}}{2}(\sqrt{4-x^2})^2 \, dx = 2\sqrt{3}\int_0^2 (4-x^2) \, dx = \frac{32}{\sqrt{3}} \approx 18.5$$

c. If the base of the triangle is $2\sqrt{4-x^2}$ and the height is 3, the area is $3\sqrt{4-x^2}$.

$$2\int_0^2 3\sqrt{4-x^2} \, dx = 6\int_0^2 \sqrt{4-x^2} \, dx = 6\pi$$

where we interpreted $\int_0^2 \sqrt{4-x^2} \, dx$ as the area of a quarter-circle as in problem set 9.

8.

$$\begin{aligned} y &= \tfrac{2}{3}(x^2+1)^{3/2} \\ y' &= (x^2+1)^{1/2}(2x) \\ (y')^2 &= (x^2+1)(2x)^2 = 4x^4+4x^2 \\ 1+(y')^2 &= 4x^4+4x^2+1 = (2x^2+1)^2 \end{aligned}$$

so

$$\int_1^2 \sqrt{1+(y')^2} \, dx = \int_1^2 (2x^2+1) \, dx = \left[\frac{2}{3}x^3 + x\right]_1^2 = \frac{17}{3} = 5 + \frac{2}{3}$$

9.

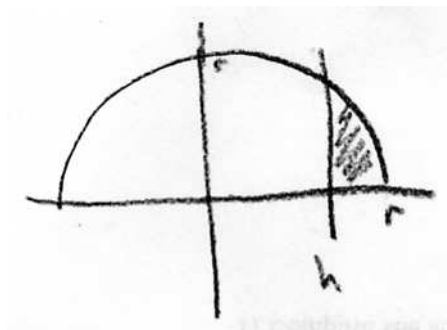
$$\begin{aligned} y &= (4-x^{2/3})^{3/2} \\ y' &= \tfrac{3}{2}(4-x^{2/3})^{1/2}(-\tfrac{2}{3}x^{-1/3}) = -x^{1/3}(4-x^{2/3})^{1/2} \\ (y')^2 &= x^{-2/3}(4-x^{2/3}) = 4x^{-2/3}-1 \\ 1+(y')^2 &= 4x^{-2/3} = (2x^{-1/3})^2 \end{aligned}$$

so

$$\int_1^8 \sqrt{1+(y')^2}\ dx = \int_1^8 2x^{-1/3}\ dx = 3x^{2/3}\Big|_1^8 = 9$$

II. Problems to be graded on correctness.

1.



The radius of a slice is $\sqrt{r^2-x^2}$, so

$$\int_h^r \pi(r^2-x^2)\ dx = \pi\left[r^2x - \frac{1}{3}x^3\right]_h^r = \pi\left(\frac{2}{3}r^3 - r^2h + \frac{1}{3}h^3\right).$$

27. The base is a quarter-circle $y=\sqrt{1-x^2}$ or $x=\sqrt{1-y^2}$. The cross-sections are squares.

$$\int_0^1 (\sqrt{1-y^2})^2\ dy = \int_0^1 (1-y^2)\ dy = \left[y - \frac{1}{3}y^3\right]_0^1 = \frac{2}{3}$$

2.

$$\begin{aligned} y &= \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x} \\ y' &= \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x} \\ (y')^2 &= \tfrac{1}{4}e^{2x} - \tfrac{1}{2} + \tfrac{1}{4}e^{-2x} \\ 1+(y')^2 &= \tfrac{1}{4}e^{2x} + \tfrac{1}{2} + \tfrac{1}{4}e^{-2x} = \left(\tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x}\right)^2 \end{aligned}$$

so

$$\int_{-1}^1 \sqrt{1+(y')^2}\ dx = \int_{-1}^1 \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right) dx = \left[\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right]_{-1}^1 = e - \frac{1}{e} \approx 2.35$$

3. a. $-3+1=-2$, not -4.

$$\frac{d}{dx}\left(-\tfrac{1}{4}x^{-4} + C\right) = x^{-5}$$

b. The integral of a product is not the product of the integrals.

$$\frac{d}{dx}\left(\tfrac{1}{3}x^3\tan x + C\right) = x^2\tan x + \tfrac{1}{3}x^3(\sec x)^2$$

c. There is no chain rule for integrals.

$$\frac{d}{dx}\left[\sin(x^3+1)\left(\frac{x^4}{4}+x\right)+C\right] = \cos(x^3+1)(3x^2)\left(\frac{x^4}{4}+x\right) + \sin(x^3+1)(x^3+1)$$

d. The integral of $\dfrac{1}{\text{whatever}}$ is not automatically $\log|\text{whatever}|+C$. This only works if whatever $=$ $x+a$ for some number a.

$$\frac{d}{dx}\left(\log|4x^2+1|+C\right) = \frac{8x}{4x^2+1}$$