Problem Set 8

March 28 and 29, in class

Recall some facts:

- If $f(x) \leq g(x)$ for all x in $[a, b]$ then $\displaystyle\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$.
- If k is a constant then $\displaystyle\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$.
- $\displaystyle\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$.

1. (a) Understand the following proof:

$$-|f(x)| \leq f(x) \leq |f(x)|$$

$$-\int_a^b |f(x)|\ dx \leq \int_a^b f(x)\ dx \leq \int_a^b |f(x)|\ dx$$

$$\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx$$

   (b) Explain why the last inequality makes sense. Hint: Find some functions for which the left-hand side and the right-hand side are not equal. Then explain why the left-hand side always has to be less.

2. Suppose that f is continuous, $a < b$, and $\displaystyle\int_a^b f(x)\ dx = 0$.

   (a) Does it necessarily follow that $f(x) = 0$ for all x in $[a, b]$?

   (b) Does it necessarily follow that $f(x) = 0$ for *some* x in $[a, b]$?

   (c) Does it necessarily follow that $\int_a^b |f(x)|\ dx = 0$?

   (d) Does it necessarily follow that $\left|\int_a^b f(x)\ dx\right| = 0$?

   (e) Must all the upper sums $U(P)$ be nonnegative?

   (f) Must all the upper sums $U(P)$ be positive?

   (g) Can a lower sum $L(P)$ be positive?

3. Suppose that f and g are continuous, $a < b$, and $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$.

   (a) Does it necessarily follow that $\int_a^b [f(x) - g(x)]\ dx \geq 0$?

   (b) Does it necessarily follow that $f(x) \geq g(x)$ for all x in $[a, b]$?

   (c) Does it necessarily follow that $f(x) \geq g(x)$ for *some* x in $[a, b]$?

   (d) Does it necessarily follow that $\left|\int_a^b f(x)\ dx\right| \geq \left|\int_a^b g(x)\ dx\right|$?

   (e) Does it necessarily follow that $\int_a^b |f(x)|\ dx \geq \int_a^b |g(x)|\ dx$?

   (f) Does it necessarily follow that $\int_a^b |f(x)|\ dx \geq \int_a^b g(x)\ dx$?

4. While $\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$, in general it is **not true** that

$$\int_a^b f(x)g(x)\ dx = \left(\int_a^b f(x)\ dx\right)\left(\int_a^b g(x)\ dx\right).$$

   (a) Find two numbers a and b and two functions f and g for which the equation above fails—any random choice of a, b, f, and g should work.

   (b) Find two numbers a and b and two functions f and g for which the equation above holds—you will have to choose a, b, f, and g carefully.