Solutions to selected homework problems. 1.

1.4.22. This gives an equation $4.98x + 5.98y = 100.64$, or $498x + 598y = 10064$. Reduce by 2: $249x + 299y = 5032$. Now apply Euclidian algorithm to 249 and 299: $299 = 249 + 50$, $249 = 4 \cdot 50 + 49$, $50 = 49 + 1$, so

$$1 = 50 - 49 = 5 \cdot 50 - 249 = 5 \cdot 299 - 6 \cdot 249.$$ 

The general solution of our equation is $x = x_0 + 299t$, $y = y_0 - 249t$, where $(x_0, y_0)$ is a particular solution. One way to get a particular solution is to set $x_0 = -6 \cdot 5032$, $y_0 = 5 \cdot 5032$. To work with a bit smaller numbers we can observe that $5032 = 20 \cdot 249 + 52 = 19 \cdot 249 + 299 + 2$. Thus, we can get a solution of the form $x_0 = 19 - 6 \cdot 2 = 7$, $y_0 = 1 + 5 \cdot 2 = 11$. The general solution of the linear equation is $x = 7 + 299t$, $y = 11 - 249t$. Since we want $x$ and $y$ to be positive, we have to set $t = 0$. The answer: $x = 7$, $y = 11$.

1.4.36. Set $d = (a, b)$. Then the condition $a | bc$ is equivalent to $a | d \cdot b | c$. Since $(a, d) = 1$ (by Theorem 1.14), Theorem 1.13 implies that $a | c$. Hence, $a | dc$ as required.

1.4.37. Suppose $f$ is a common positive divisor of $\frac{a}{d}$ and $\frac{b}{d}$, so $f | \frac{a}{d}$ and $f | \frac{b}{d}$. This implies that $fd | a$ and $fd | b$. Thus, $fd$ is a common divisor of $a$ and $b$. Since $d$ is the greatest common divisor, it follows that $fd \leq d$, so $f \leq 1$, which implies that $f = 1$.

1.4.38. By Theorem 1.15, the general solution of this linear equation has form $x = x_0 + \frac{b}{d}t$, $y = y_0 - \frac{a}{d}t$, where $(x_0, y_0)$ is a particular solution. Since $a$ and $b$ have opposite signs, if we choose $t$ to be of the same sign as $b$ and with very large $|t|$ we will get positive $x$ and $y$.

1.5.28. $49 \equiv 3 \pmod{23}$, hence, $49^4 \equiv 3^4 \equiv 81 \equiv 12 \pmod{23}$.

1.5.32. $50 \equiv -1 \pmod{17}$, hence, $50^{69} \equiv -1 \equiv 16 \pmod{17}$.

1.5.42. This is not true: take $c = 2$, $b = 1$, $a = -1$.

1.5.46. We have $a^n b^n \equiv (ab)^n \equiv 1 \equiv a^n \pmod{m}$. Since $(a^n, m) = 1$, by Theorem 1.18, we get $b^n \equiv 1 \pmod{m}$.

4.1.14. This is not a complete solution: $x \equiv 11 \pmod{12}$ is missing.

4.1.16. This is a complete solution.