MATH 253 FINAL EXAM STUDY GUIDE SOLUTIONS

3. Recall that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for $|x| < 1$. Use Taylor's Inequality to determine what degree Taylor polynomial should we use in order to guarantee that we are approximating $\ln(\frac{4}{3})$ to within $\frac{1}{100}$?

Solution: We use use Taylor's Inequality (the remainder estimate). For this, we need the derivatives:

$$\begin{aligned} f(x) &= \ln(1+x) \\ f'(x) &= (1+x)^{-1} \\ f''(x) &= -(1+x)^{-2} \\ f'''(x) &= 2(1+x)^{-3} \\ f^{(4)}(x) &= -3 \cdot 2(1+x)^{-4}, \end{aligned}$$

etc., and

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$$

for $n = 1, 2, 3, 4, \ldots$. We need a bound M_n, such that for all x in $\left[0, \frac{1}{3}\right]$ we have $|f^{(n+1)}(x)| \leq M_n$. For $n = 0, 1, 2, 3, \ldots$, $|f^{(n+1)}(x)|$ is decreasing on $\left[0, \frac{1}{3}\right]$, so we can take

$$M_n = |f^{(n+1)}(0)| = n!.$$

Therefore the error when using the degree n Taylor polynomial is at most

$$\left|R_{f,0,n}\left(\tfrac{1}{3}\right)\right| \leq \frac{M_n\left|\frac{1}{3}\right|^{n+1}}{n!} = \frac{x!}{3^{n+1}(n+1)!!} = \frac{1}{3^{n+1}(n+1)}.$$

So we want

$$\frac{1}{3^{n+1}(n+1)} < \frac{1}{100},$$

that is, $3^{n+1}(n+1) > 100$. If $n = 2$ then $3^{n+1} \cdot (n+1) = 81$, which is not good enough. If $n = 3$ then $3^{n+1} \cdot (n+1) = 81 \cdot 4 > 100$, so we take $n = 3$.

For reference, here is an easier solution using the Alternating Series Test.

The series for $\ln\left(1 + \frac{1}{3}\right)$ is

$$\ln\left(1+\tfrac{1}{3}\right) = \frac{1}{3} - \left(\frac{1}{2}\right)\left(\frac{1}{3^2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3^3}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{3^4}\right) + \cdots,$$

with general term $\dfrac{(-1)^{n-1}}{n \cdot 3^n}$ for $n = 1, 2, 3, 4, \ldots$. These terms alternate in sign, are clearly decreasing in absolute value, and clearly approach 0 as $n \to \infty$. So we can use the error estimate from the Alternating Series Test: the absolute value of the error is less than the absolute value of the first term not used. The degree 3 term has absolute value $\left(\frac{1}{3}\right)\left(\frac{1}{3^3}\right) = \frac{1}{81}$, which is not good enough. So $n = 2$ is not not good enough. The degree 4 term has absolute value $\left(\frac{1}{4}\right)\left(\frac{1}{3^4}\right) = \frac{1}{81 \cdot 4} < \frac{1}{100}$, so we can take $n = 3$.

It is just a coincidence that both methods gave the same error estimate.

4. Find a number n such that the approximation of $f(x) = e^x$ by its Taylor polynomial of degree n centered at $\frac{1}{2}$ gives an error of less than $\frac{1}{100}$ on the interval $[0, 1]$.

Solution: No matter what choice of n we use, $f^{(n+1)}(x)$ will be e^x. Since $x \mapsto e^x$ is increasing on $[0,1]$, for any n, for all x in $[0,1]$ we have $|f^{(n+1)}(x)| \leq e$. Now, by Taylor's Inequality (the remainder estimate), using e for the number M there, and using $\left|x - \frac{1}{2}\right| \leq \frac{1}{2}$ at the second step,

$$|R_{f,1/2,n}(x)| \leq \frac{e\left|x - \frac{1}{2}\right|^{n+1}}{(n+1)!} \leq \frac{e}{2^{n+1} \cdot (n+1)!}.$$

So we want

$$\frac{e}{2^{n+1} \cdot (n+1)!} \leq \frac{1}{100}.$$

Now $e < 3$, so taking $n = 3$ works:

$$|R_{f,1/2,3}(x)| \leq \frac{e}{2^4 \cdot 4!} < \frac{3}{16 \cdot 24} = \frac{1}{16 \cdot 8} = \frac{1}{128} < \frac{1}{100}.$$

Alternate solution: As in the first solution, start with

$$|R_{f,1/2,n}(x)| \leq \frac{e}{2^{n+1} \cdot (n+1)!}.$$

Since $e < 4$ and $(n+1)! > 1$, it is enough to choose n so that $2^{n+1} \geq 400$. Thus, taking $n = 8$ works:

$$|R_{f,1/2,8}(x)| \leq \frac{e}{2^9 \cdot 9!} < \frac{4}{2^9 \cdot 24} < \frac{4}{2^9} = \frac{1}{2^7} = \frac{1}{128} < \frac{1}{100}.$$

Second alternate solution: As in the first solution, start with

$$|R_{f,1/2,n}(x)| \leq \frac{e}{2^{n+1} \cdot (n+1)!}.$$

We will require $n \geq 1$, so that $e/2^{n+1} < 4/4 = 1$. So we need only choose $n \geq 1$ and so large that $(n+1)! > 100$. So taking $n = 4$ works:

$$|R_{f,1/2,4}(x)| \leq \frac{e}{2^5 \cdot 5!} < \frac{4}{2^5 \cdot 5!} < \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}.$$

5. Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n! x^n}{(3n)!}$.

Solution: We use the Ratio Test. For $x \neq 0$, we have

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}(n+1)!x^{n+1}}{[3(n+1)]!}\right|}{\left|\frac{(-1)^n n! x^n}{(3n)!}\right|} = \lim_{n\to\infty} \frac{(n+1)!x^{n+1}(3n)!}{n!x^n(3n+3)!} = \lim_{n\to\infty} \frac{(n+1)|x|}{(3n+1)(3n+2)(3n+3)}$$

$$= |x| \lim_{n\to\infty} \frac{1}{3(3n+1)(3n+2)} = |x| \cdot 0 = 0.$$

Since this limit is less than 1, the series converges for every $x \neq 0$ by the Ratio Test, and of course it also converges when $x = 0$. So the radius of convergence is ∞.

6. Define a function S by $S(x) = \begin{cases} \dfrac{\sin(2x)}{x} & x \neq 0 \\ 2 & x = 0. \end{cases}$ Find $S''(0)$.

Solution: We start with the Taylor series centered at 0 for $\sin(x)$:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

valid for all real numbers x. For all real numbers x, we therefore have

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!} = 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \cdots ,$$

so, if $x \neq 0$,

$$S(x) = \frac{\sin(2x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n}}{(2n+1)!} = 2 - \frac{2^3 x^2}{3!} + \frac{2^5 x^4}{5!} - \frac{2^7 x^6}{7!} + \cdots .$$

This equation is also true for $x = 0$, by the definition of S. Therefore $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n}}{(2n+1)!}$ is the Taylor series centered at 0 for $S(x)$. So $S''(0)$ is $2!$ times the coefficient of x^2, that is,

$$S''(0) = 2\left(\frac{2^3}{3!}\right) = \frac{2^4}{6} = \frac{8}{3}.$$

7. Define a function L by $L(x) = \begin{cases} \dfrac{\ln(x)}{x-1} & x \neq 1 \\ 1 & x = 1. \end{cases}$, Find a power series centered at 1 which converges to $L(x)$ for x in $(0, 2)$.

Solution: We start with the Taylor series centered at 0 for $\ln(x+1)$:

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots ,$$

valid when $|x| < 1$ (and also for $x = 1$, but we don't need this). Replacing x with $x - 1$, we find that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \cdots$$

converges to $\ln(x) = \ln\big((x-1)+1\big)$ when $|x-1| < 1$ (and also for $x = 2$, but we don't need this). For $|x-1| < 1$ but $x \neq 1$, we therefore have

$$\begin{aligned} L(x) = \frac{\ln(x)}{x} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^{n-1}}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{n+1} \\ &= 1 - \frac{x-1}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^3}{4} + \frac{(x-1)^4}{5} - \frac{(x-1)^5}{6} + \cdots . \end{aligned}$$

This equation is also true for $x = 1$, by the definition of L.

8. Define $f(x) = e^{2x^3}$ for all real x. Find $f^{(12)}(0)$.

Solution: Substitute $2x^3$ in the degree 4 Taylor polynomial for e^x. (This degree is chosen so that $4 \cdot 3 \geq 12$.) This gives the degree 12 Taylor polynomial for e^{2x^3}:

$$e^{2x^3} \approx 1 + 2x^3 + \frac{(2x^3)^2}{2!} + \frac{(2x^3)^3}{3!} + \frac{(2x^3)^4}{4!} = 1 + 2x^3 + \frac{2^2 x^6}{2!} + \frac{2^3 x^9}{3!} + \frac{2^4 x^{12}}{4!}.$$

Select the coefficient of x^{12} and multiply it by $12!$, getting $\dfrac{2^4 \cdot 12!}{4!}$. So $f^{(12)}(0) = \dfrac{2^4 \cdot 12!}{4!}$.

9. Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^{2n}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}$.

Solution: We use the Ratio Test. For $x \neq 0$, we have

$$\lim_{n\to\infty} \frac{\left| \frac{3^{n+1} x^{2(n+1)}}{1\cdot3\cdot5\cdot7\cdots(2n-1)\cdot(2(n+1)-1)} \right|}{\left| \frac{3^n x^{2n}}{1\cdot3\cdot5\cdot7\cdots(2n-1)} \right|} = \lim_{n\to\infty} \frac{3^{n+1} x^{2(n+1)} \big[1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)\big]}{3^n x^{2n} \big[1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1) \cdot (2(n+1)-1)\big]}$$

$$= \lim_{n\to\infty} \frac{3x^2}{2n+1} = 0.$$

Since $0 < 1$, the series

$$\sum_{n=0}^{\infty} \frac{3^n x^{2n}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}$$

is absolutely convergent, and hence convergent, for every real number $x \neq 0$. This series converges for $x = 0$ for trivial reasons. Therefore the radius of convergence is ∞.

10. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{7^{n-3} \sqrt[3]{n+1}}$. You are told that its radius of convergence is 7. Given this, find its interval of convergence.

Solution: The open interval of convergence is $(2-7,\ 2+7) = (-5,\ 9)$. So we must test convergence at the endpoints. For $x = -5$ we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-7)^n}{7^{n-3} \cdot \sqrt[3]{n+1}} = 7^3 \sum_{n=1}^{\infty} \frac{1}{(n+1)^{1/3}} = 7^3 \sum_{n=2}^{\infty} \frac{1}{n^{1/3}}.$$

This series is divergent because the exponent $1/3$ is less than 1.

For $x = 9$ we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 7^n}{7^{n-3} \sqrt[3]{n+1}} = 7^3 \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n+1}}.$$

We use the Alternating Series Test. The absolute values of the summands are decreasing and have limit zero (I must see from your solution that you know these things), and the signs alternate by definition. So the Alternating Series Test applies, and shows that this series is convergent.

We conclude that the interval of convergence is $(-5, 9]$.