Math 253, Calculus III, Spring 2024

MIDTERM 1 STUDY GUIDE ANSWERS

3. Determine whether the sequence converges or diverges. If converges, compute the limit.

a. $a_n = \frac{3+n^3}{1+3n^3} = \frac{3/n^3+1}{1/n^3+3}$ hence $\lim_{n\to\infty} a_n = \frac{3/\infty+1}{1/\infty+3} = \frac{0+1}{0+3} = \frac{1}{3}$.

b. $a_n = \frac{n^3}{1+n^2} = \frac{n}{1/n^2+1}$ hence $\lim_{n\to\infty} a_n = \frac{\infty}{1/\infty+1} = \frac{\infty}{1} = \infty$ - diverges.

c. $a_n = \frac{2^{3n+1}}{3^{2n+5}} = \frac{2\cdot 2^{3n}}{3\cdot 3^{2n}} = \frac{2\cdot 8^n}{3\cdot 9^n} = \frac{2}{3}\cdot (8/9)^n$ hence $\lim_{n\to\infty} a_n = \frac{2}{3}\cdot 0 = 0$.

d. $a_n = \frac{\ln(n)}{\sqrt{n}}$. Use L'Hospital rule. Let $f(x) = \frac{\ln(x)}{\sqrt{x}} = \frac{\ln(x)}{x^{1/2}}$. Since $\ln(x) \to +\infty$ and $x^{1/2} \to +\infty$ as $x \to +\infty$, the L'Hospital rule is applicable to $f(x)$ and

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(\ln(x))'}{(x^{1/2})'} = \lim_{x\to+\infty} \frac{1/x}{(1/2)x^{-1/2}} = \lim_{x\to+\infty} \frac{1/x}{(1/2)x^{-1/2}} = \lim_{x\to+\infty} \frac{2}{x^{1/2}} = 0 .$$

Therefore, $\lim_{n\to\infty} a_n = 0$.

e. $a_n = \frac{n\cos(n)}{1+n^3}$. Use the Squeeze theorem with $b_n = -\frac{n}{1+n^3}$ and $c_n = \frac{n}{1+n^3} = -b_n$. Then $b_n \le a_n \le c_n$ for all n because $-1 \le \cos(n) \le 1$. It is easy to see that $\lim_{n\to\infty} c_n =$ $= \lim_{n\to\infty} \frac{1}{1/n+n^2} = 0$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} -c_n = 0$. Therefore, the Squeeze theorem is applicable and $\lim_{n\to\infty} a_n = 0$.

f. $a_n = \frac{(-1)^n 3^n}{2^{2n}} = (-3/4)^n$ hence $\lim_{n\to\infty} a_n = 0$.

g. $a_n = \ln(3n^3+1) - \ln(n^3+1) = \ln(\frac{3n^3+1}{n^3+1}) = \ln(\frac{3+1/n^3}{1+1/n^3})$ hence $\lim_{n\to\infty} a_n = \ln(3)$.

h. $a_n = \frac{1}{n(\sqrt{n+1}-\sqrt{n-1})} = \frac{\sqrt{n+1}+\sqrt{n-1}}{n((\sqrt{n+1})^2-(\sqrt{n-1})^2)} = \frac{\sqrt{n+1}+\sqrt{n-1}}{n((n+1)-(n-1))}$

$= \frac{\sqrt{n+1}+\sqrt{n-1}}{2n} = \frac{\sqrt{(n+1)/n^2}+\sqrt{(n-1)/n^2}}{2} = \frac{\sqrt{1/n+1/n^2}+\sqrt{1/n-1/n^2}}{2}$. Hence

$$\lim_{n\to\infty} a_n = \frac{\sqrt{1/\infty+1/\infty^2}+\sqrt{1/\infty-1/\infty^2}}{2} = 0 .$$

i. $a_n = \frac{1}{\sqrt{n^2+n}-\sqrt{n^2-n}} = \frac{\sqrt{n^2+n}+\sqrt{n^2-n}}{(\sqrt{n^2+n})^2-(\sqrt{n^2+n})^2} =$

$= \frac{\sqrt{n^2+n}+\sqrt{n^2-n}}{(n^2+n)-(n^2-n)} = \frac{\sqrt{n^2+n}+\sqrt{n^2-n}}{2n} = \frac{\sqrt{(n^2+n)/n^2}+\sqrt{(n^2-n)/n^2}}{2} =$

$= \frac{\sqrt{1+1/n}+\sqrt{1-1/n}}{2}$. Hence $\lim\limits_{n\to\infty} a_n = \frac{\sqrt{1+1/\infty}+\sqrt{1-1/\infty}}{2} = \frac{\sqrt{1+0}+\sqrt{1-0}}{2} = 1$.

4. Express the repeating decimal as a fraction.

a. $2.121212... = 2+\sum_{n=1}^{\infty}\frac{12}{100^n} = 2+\frac{12/100}{1-1/100} = 2+\frac{12}{99} = 2+\frac{4}{33}$.

b. $3.131313... = 3+\sum_{n=1}^{\infty}\frac{13}{100^n} = 3+\frac{13/100}{1-1/100} = 3+\frac{13}{99}$.

c. $3.141414... = 3+\sum_{n=1}^{\infty}\frac{14}{100^n} = 2+\frac{14/100}{1-1/100} = 3+\frac{14}{99}$.

d. $2.717171... = 2+\sum_{n=1}^{\infty}\frac{71}{100^n} = 2+\frac{71/100}{1-1/100} = 2+\frac{71}{99}$.

e. $9.999999... = 9+\sum_{n=1}^{\infty}\frac{9}{10^n} = 9+\frac{9/10}{1-1/10} = 9+1 = 10$.

5. Find the sum of the series.

a. $\sum_{n=1}^{\infty}\frac{2^{3n+1}}{3^{2n+1}} = \sum_{n=1}^{\infty}\frac{2\cdot 2^{3n}}{3\cdot 3^{2n}} = \sum_{n=1}^{\infty}\frac{2\cdot 8^n}{3\cdot 9^n} = \sum_{n=1}^{\infty}\frac{2}{3}\cdot(8/9)^n = \frac{2}{3}\cdot\sum_{n=1}^{\infty}(8/9)^n = \frac{2}{3}\cdot\frac{8/9}{1-8/9} = \frac{16}{3}$.

b. $\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{2^{2n+1}} = \sum_{n=1}^{\infty}\frac{(-3)\cdot(-3)^n}{2\cdot 2^{2n}} = -\frac{3}{2}\cdot\sum_{n=1}^{\infty}\frac{(-3)^n}{4^n} = -\frac{3}{2}\cdot\sum_{n=1}^{\infty}(-3/4)^n = -\frac{3}{2}\cdot\frac{(-3/4)}{1-(-3/4)} =$

$= -\frac{3}{2}\cdot\frac{(-3/4)}{7/4} = -\frac{3}{2}\cdot\frac{(-3)}{7} = \frac{9}{14}$.

c. $\sum_{n=1}^{\infty}\frac{(x-1)^n}{3^{2n+1}}$, where $-8 < x < 10$. $\sum_{n=1}^{\infty}\frac{(x-1)^n}{3^{2n+1}} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{x-1}{9}\right)^n = \frac{1}{3}\cdot\frac{(x-1)/9}{1-(x-1)/9} = \frac{1}{3}\cdot\frac{x-1}{10-x}$.

d. $\sum_{n=1}^{\infty}\frac{1}{n(n+3)}$. Denote $s_n = \sum_{i=1}^{n}\frac{1}{i(i+3)}$. Then $s_n = \sum_{i=1}^{n}\frac{1}{3}(\frac{1}{i}-\frac{1}{i+3}) = \frac{1}{3}\cdot\sum_{i=1}^{n}(\frac{1}{i}-\frac{1}{i+3}) =$

$= \frac{1}{3}(\frac{1}{1}-\frac{1}{4}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+...+\frac{1}{n-2}-\frac{1}{i+1}+\frac{1}{i-1}-\frac{1}{i+2}+\frac{1}{i}-\frac{1}{i+3}) =$

$= \frac{1}{3}(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3})$ .So $\lim\limits_{n\to\infty} s_n = \frac{1}{3}(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}-\frac{1}{\infty+1}-\frac{1}{\infty+2}-\frac{1}{\infty+3})$

$= \frac{1}{3}(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}) = \frac{1}{3}(\frac{6}{6}+\frac{3}{6}+\frac{2}{6}) = \frac{11}{18}$ and $\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \frac{11}{18}$.

e. $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$. Denote $s_n = \sum_{i=2}^{n}\frac{1}{i^2-1}$. Then $s_n = \sum_{i=2}^{n}\frac{1}{(i-1)(i+1)} = \sum_{i=2}^{n}\frac{1}{2}(\frac{1}{i-1}-\frac{1}{i+1})$

$= \frac{1}{2}\cdot\sum_{i=2}^{n}(\frac{1}{i-1}-\frac{1}{i+1}) = \frac{1}{2}(\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+...+\frac{1}{n-2}-\frac{1}{n}+\frac{1}{n-1}-\frac{1}{n+1}) =$

$= \frac{1}{2}(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1})$. Hence $\lim_{n\to\infty} s_n = \frac{1}{2}(\frac{1}{1}+\frac{1}{2}-\frac{1}{\infty}-\frac{1}{\infty+1}) = \frac{1}{2}(\frac{1}{1}+\frac{1}{2}) = \frac{1}{2}(\frac{3}{2}) = \frac{3}{4}$ and $\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \frac{3}{4}$.

6. Use **any** test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty}\frac{n}{1+2n^3} < \sum_{n=1}^{\infty}\frac{n}{2n^3} = \frac{1}{2}\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}$, the latter series converges because $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p>1$. Therefore, the series $\sum_{n=1}^{\infty}\frac{n}{1+2n^3}$ converges by the comparison test.

b. $\sum_{n=1}^{\infty}\frac{n^3}{1+2n^4} = \sum_{n=1}^{\infty}\frac{1}{1/n^3+2n} \geq \sum_{n=1}^{\infty}\frac{1}{2+2n} = \sum_{n=1}^{\infty}\frac{1}{2+2n} = \frac{1}{2}\cdot\sum_{n=1}^{\infty}\frac{1}{1+n} = \frac{1}{2}\cdot\sum_{m=2}^{\infty}\frac{1}{m}$, the latter series diverges. so the series $\sum_{n=1}^{\infty}\frac{n^3}{1+2n^4}$ diverges by the comparison test.

c. $\sum_{n=1}^{\infty}\frac{n^4}{4^n}$. Since exponential functions grow much faster than the polynomial ones, we have $4^n > n^6$ for some $n > n_0$. Therefore, $\sum_{n=n_0+1}^{\infty}\frac{n^4}{4^n} < \sum_{n=n_0+1}^{\infty}\frac{n^4}{n^6} = \sum_{n=n_0+1}^{\infty}\frac{1}{n^2}$, which converges. Hence $\sum_{n=1}^{\infty}\frac{n^4}{4^n}$ converges by the comparison test.

d. $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln(n)}}$. Use the integral test. Indeed, the function

$f(x) = \frac{1}{x\sqrt{\ln(x)}} = \frac{1}{x\cdot\ln(x)^{1/2}}$ is positive and decreasing for $x \geq 2$. Compute the integral $\int_2^{+\infty}\frac{dx}{x\cdot\ln(x)^{1/2}}$. Let us use the substitution $u = \ln(x)$ so $du = \frac{dx}{x}$. Then

$$\int_2^{+\infty}\frac{dx}{x\cdot\ln(x)^{1/2}} = \int_{\ln(2)}^{+\infty}\frac{du}{u^{1/2}} = \int_{\ln(2)}^{+\infty}u^{-1/2}du = \frac{u^{1-1/2}}{1-1/2}\Big|_{u=\ln(2)}^{u=+\infty} = 2\cdot u^{1/2}\Big|_{u=\ln(2)}^{u=+\infty} =$$

$2((+\infty)^{1/2}-(\ln(2))^{1/2}) = +\infty$. Therefore, $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln(n)}}$ diverges by the integral test.

e. $\sum_{n=1}^{\infty}\frac{2^n}{2^n+3^n} < \sum_{n=1}^{\infty}\frac{2^n}{3^n} = \sum_{n=1}^{\infty}(2/3)^n$, the latter series converges because it is a geometric series with $r = 2/3$. Therefore, the series $\sum_{n=1}^{\infty}\frac{2^n}{2^n+3^n}$ converges by the comparison test.

f. $\sum_{n=1}^{\infty}\frac{3^n}{2^n+3^n} > \sum_{n=1}^{\infty}\frac{3^n}{3^n+3^n} = \sum_{n=1}^{\infty}\frac{1}{2} = +\infty$. Therefore, the series $\sum_{n=1}^{\infty}\frac{2^n}{2^n+3^n}$ diverges by the comparison test.