Math 253, Calculus III, Winter 2024

MIDTERM 2 STUDY GUIDE ANSWERS

3. Determine whether each of the following series is **absolutely convergent**, **conditionally convergent,** or **diverges**. State the test you are using and show all details of the test (GST= Geometric Series Test, CT=Comparison Test, LCT=Limit Comparison Test, IT=Integral Test, AST=Alternating Series Test, RT= Ratio Test).

a. $\sum_{k=1}^{\infty} \frac{2k^2-1}{k^2 3^k}$ Absolutely convergent by RT

b. $\sum_{k=0}^{\infty} \frac{2}{(-3)^k}$ Absolutely convergent by RT (or GST)

c. $\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k \ln k}$ Conditionally convergent by IT and AST

d. $\sum_{k=2}^{\infty} \frac{10k^2}{k^3-1}$ Divergent by CT (or LCT) and Harmonic Series Test

e. $\sum_{k=1}^{\infty} \frac{5}{4^k+3}$ Absolutely convergent by RT or (CT and GST)

f. $\sum_{k=1}^{\infty} \frac{k^{10} 2^k}{k!}$ Absolutely convergent by RT

g. $\frac{1}{4!} - \frac{4}{5!} + \frac{9}{6!} - \frac{16}{7!} + \ldots$ Absolutely convergent by RT

h. $\sum_{k=0}^{\infty} \left(\frac{3}{2}\right)^k$ Divergent by RT (or GST)

i. $\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$ Divergent by IT

j. $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ Conditionally convergent by IT (or CT) and AST

k. $\sum_{1}^{\infty} \frac{(-1)^n}{n!}$ Absolutely convergent by RT (and equals 1/e)

2. Determine the radius and the interval of convergence of the power series.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{n+1}$ R=1, (2,4]

b. $\sum_{k=0}^{\infty} \frac{(x+1)^k}{3^k}$ R=3, (-4,2)

c. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$ R=2, [-2,2)

d. $\sum_{n=1}^{\infty} \frac{x^n}{n\, 3^n}$ R=3, [-3,-3)

e. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}}$ R=1, [2,4)

f. $\sum_{n=0}^{\infty} \frac{3^n (x+5)^n}{(n+1)!}$ R= ∞, ($-\infty$,$+\infty$)

g. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x+1)^n}{2^n}$ R=2, (-3,1]

h. $\sum_{n=0}^{\infty} \frac{n!\cdot x^n}{e^{n+2}}$ R=0, the interval of convergence is empty

3. Write a power series representation for the function and determine the radius and the interval of convergence.

a. $f(x) = \frac{1}{1+2x} = \sum_{n=0}^{\infty} (-2)^n x^n$ R=$\frac{1}{2}$, ($-\frac{1}{2}$,$\frac{1}{2}$)

b. $f(x) = \frac{1}{(1+2x)^2} = \sum_{n=0}^{\infty} (n+1)(-2)^n x^n$ R=$\frac{1}{2}$, ($-\frac{1}{2}$,$\frac{1}{2}$)

c. $f(x) = \ln(1+2x) = \sum_{n=1}^{\infty} -\frac{(-2)^n x^n}{n}$ R=$\frac{1}{2}$, ($-\frac{1}{2}$,$\frac{1}{2}$]

d. $f(x) = \frac{x}{9-x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{9^{n+1}}$ R=3, (-3,3)

e. $f(x) = \ln(9-x^2) = \ln(9) + \ln(1-x^2/9) = \ln(9) + \sum_{n=1}^{\infty} -\frac{x^{2n}}{9^n (2n)}$ R=3, (3,3)

f. $f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ R= ∞, ($-\infty$,$+\infty$)

g. $f(x) = e^{-5x} = \sum_{n=0}^{\infty} \frac{(-5)^n x^n}{n!}$ R= ∞, ($-\infty$,$+\infty$)

h. $f(x) = e^{-5x^2} = \sum_{n=0}^{\infty} \frac{(-5)^n x^{2n}}{n!}$ R= ∞, ($-\infty$,$+\infty$)

i. $f(x) = \arctan(2x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{6n+3}}{2n+1}$ R=$\sqrt[3]{\frac{1}{2}}$, [$-\sqrt[3]{\frac{1}{2}}$,$\sqrt[3]{\frac{1}{2}}$]

4. Use a power series to calculate the indefinite and definite integrals

a. $\int \frac{x-\sin(x)}{x^3}dx = \int \frac{x-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}}{x^3}dx = \int \sum_{n=1}^{\infty}\frac{(-1)^{n+1} x^{2n-2}}{(2n+1)!}dx = \text{C} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{(2n+1)!(2n-1)}$

b. $\int_0^1 \frac{x-\sin(x)}{x^3}dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{(2n+1)!(2n-1)}\Big|_{x=0}^{x=1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!(2n-1)}$

c. $\int \frac{1-\cos(x)}{x^2}dx = \int \frac{1-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}}{x^2}dx = \int \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n)!}dx = \text{C} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{(2n)!(2n-1)}$

d. $\int_0^1 \frac{1-\cos(x)}{x^2}dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{(2n)!(2n-1)}\Big|_{x=0}^{x=1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!(2n-1)}$

e. $\int \frac{x-\arctan(x)}{x^2}dx = \int \frac{x-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}}{x^2}dx = \int \sum_{n=1}^{\infty}\frac{(-1)^{n+1} x^{2n-1}}{2n+1}dx = \text{C} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n+1)(2n)}$

f. $\int_0^1 \frac{x-\arctan(x)}{x^2}dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n+1)(2n)}\Big|_{x=0}^{x=1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)(2n)}$

g. $\int \sin(x^3)dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{(2n+1)!}dx = \text{C} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+4}}{(2n+1)!(6n+4)}$

h. $\int_0^1 \sin(x^3)dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+4}}{(2n+1)!(6n+4)}\Big|_{x=0}^{x=1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(6n+4)}$

i. $\int e^{-x^2}dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}dx = \text{C} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}$

k. $\int_0^1 e^{-x^2}dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}\Big|_{x=0}^{x=1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}$