CURRICULUM VITAE

ARKADIY BERENSTEIN

Education and academic degrees.

1996	Ph.D. in mathematics, Northeastern University. Thesis title: “Algebraic and combinatorial structure of quantum groups and their canonical bases”; thesis adviser Professor Andrei V. Zelevinsky.
1988	M.S. in Applied Mathematics, Moscow Railway Transportation Institute. Senior thesis: “Some estimates in queuing theory”; thesis adviser Professor Alexander Vainshtein.

Academic career and visiting positions.

07/2011–08/2011	Visitor, MIT.
04/2011–05/2011	Visitor, Hausdorff Research Institute, Bonn, Germany.
08/2010–09/2010	Visitor, Max Planck Institute, Bonn, Germany.
06/2010–07/2010	Visitor, MIT.
07/2010–08/2010	Visitor, IHES, France.
07/2009–08/2009	Visitor, MIT.
08/2007–09/2008	Visitor, MIT.
07/2007–08/2008	Visitor, University of Warwick, UK
06/2008–07/2008	Visitor, Mathematical Research Institute at Oberwolfach, Germany
12/2007–01/2008	Visitor, MIT.
06/2007–07/2007	Visitor, University of Warwick, UK
01/2007–03/2007	(Sabbatical leave) Visitor, Harvard.
10/2006–11/2006	(Sabbatical leave) Visitor, MIT.
06/2006–now	Associate Professor, University of Oregon.
08/2006–09/2006	Visitor, Max Planck Institute, Bonn, Germany.
07/2004–08/2004	Visitor, RIMS, Kyoto, Japan
07/2000–05/2006	Assistant Professor, University of Oregon.
07/1999–06/2000	Visiting Fellow, Harvard University.
07/1996–06/1999	Assistant Professor, Cornell University.
08/1992–09/1992	Visitor, RIMS, Kyoto, Japan
01/1992–06/1996	Graduate student, Northeastern University.

Other experience.

1992–1996	Research assistant for prof. David Kazhdan, Department of Mathematics, Harvard University.
1993	Consultant, Parametric Technology Corporation, Waltham, MA.

Principal research interests.

Representation theory, quantum groups, combinatorics, and birational algebraic geometry.

Research grants.

• June 2001–June 2004 National Science Foundation award DMS #0102382 for the research in *Representation Theory, Quantum Groups and Piecewise-Linear Combinatorics.*

• June 2005–June 2008 National Science Foundation award DMS #0501103 for the research in *Representation Theory, Quantum Groups and Birational Algebraic Geometry.*

• June 2008–June 2011 National Science Foundation award DMS #0800247 for the research in *Representation Theory, Quantum Groups and Canonical Bases.*

Other awards.

1993	Phi Kappa Phi (the National Honor Society)
1986	First prize, Moscow Mathematical Olympiad
1983	Second prize, the *Quantum* problem solving contest in mathematics

Publications.

Totally 44 items. See the list of publication for complete description.

Talks given.

More than 65 invited talks on various seminars, workshops, and conferences. See the list of talks for details.

Teaching experience.

Since 1992. Among the courses given recently, there are:

Graduate courses in Lie algebras, Lie groups, quantum groups, simple rings (University of Oregon)

Undergraduate courses in calculus 1–3, linear algebra (University of Oregon), applicable algebra, differential calculus, integral calculus, linear algebra (Cornell University).

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RESEARCH SUMMARY AND LIST OF PUBLICATIONS

Introduction

My research interests lie at the crossroads of Representation Theory, Quantum Groups, Combinatorics, and Rational and Noncommutative Algebraic Geometry.

Representation theory and *combinatorics* have always been deeply interconnected areas of mathematics. In a series of papers **[1, 4, 5, 6, 8, 9, 10]** we developed a new combinatorial approach to the representation theory of Lie algebras. We express various multiplicities in the theory (such as weight multiplicities and tensor product multiplicities) as numbers of lattice points in some convex polyhedra. This brings to life a new branch of combinatorics that we call *piecewise-linear combinatorics* which admits a reinterpretation of many classical combinatorial objects such as domino tableaux, Schützenberger involution, and Robinson-Schensted-Knuth correspondence (see also **[7, 13, 35]**). One of the most spectacular applications of the piecewise-linear combinatorics is a symmetric generalization of the classical Littlewood-Richardson rule by means of the *polytope of triangles* commonly known now as Berenstein-Zelevinsky, or BZ-triangles (**[9]**). Another result to be mentioned is a complete classification (**[5]**) of all 1-dimensional weight spaces for representations of semisimple Lie algebras.

Relative Schubert calculus developed in a joint work with R. Sjamaar **[14]** is another example of a link between combinatorics and representation theory. This new technique allows, among other things, to construct the moment cones for a large class of G-varieties in purely cohomological terms.

A study of *canonical bases* attracts my attention since 1989 when the work **[4]** was published. This area of research uses techniques from rational algebraic geometry as well as from representation theory. In a classical 1990 work by G. Lusztig the study of canonical bases was put in a natural framework of quantum groups. In the papers (**[8, 9]**) we explicitly constructed canonical bases for quantum groups of type $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ and formulated a general conjecture for all types.

The subsequent development of this conjecture lead us to an introduction of a concept of an *upper cluster algebra* and of its quantum counterpart (**[17, 18]**). The upper cluster algebras generalize the notion of a semisimple Lie group, while quantum cluster algebras contain, yet conjecturally, all quantum groups.

Kashiwara's *crystal bases* are a combinatorial model for canonical bases. In the papers **[15, 20]** we found a concept of a *geometric crystal* which essentially translates the concept of a Kashiwara crystal into the language of the algebraic geometry. The passage from one to the other includes, quite surprisingly, change of a group G to its Langlands dual, $G^\vee$.

In **[22]**, for any representation V of a quantum group, we constructed a quantum symmetric algebra $S_q(V)$ and a quantum exterior algebra $\Lambda_q(V)$. These algebras are, apparently, the most natural functors of the representation category, which (similarly to the ordinary symmetric and exterior algebras) turn direct sums into the tensor products. The flatness of some of those quantum algebras pours light on existence of geometric crystals playing an important role in the local Langlands program.

The *noncommutative algebraic geometry* was the topic of a joint work **[19]** (with V. Retakh). We found a class of rational factorizations of matrices over division rings of a very general nature (the division ring of a free algebra included). This is a generalization of the results of [**10**]; so, one might expect in the future an appearance of canonical bases in a purely noncommutative setting.

In the next section, I describe my results and ongoing projects in more detail.

Main results

The numbers refer to the list of publications given in the end.

In [**1**], we obtained piecewise-linear versions of the classical Young and Littlewood-Richardson rules that express weight and tensor product multiplicities for simple polynomial GL_n-modules (and more general modules corresponding to skew shapes). Our proof uses piecewise-linear involutive transformations and is much shorter than other known proofs.

In [**4**], a conjectural expression for the tensor product multiplicities for any classical Lie algebra $\mathfrak{g}$ was given. This expression, given in terms of partitions of weights into sums of positive roots, provides natural combinatorial labeling of "good bases" in simple $\mathfrak{g}$-modules (the concept of good bases was developed by I.M. Gelfand and A.V. Zelevinsky in 1985). We also generalized Gelfand-Tsetlin patterns from $\mathfrak{gl}_n$ to all classical Lie algebras. We were able to check the conjecture in various special cases. The general case is still open, but recent results by P. Littelmann seem to be very close to proving it.

The paper [**5**] was motivated by a combinatorial question of R. Stanley: when is there a unique (semistandard) Young tableau of a given shape and weight? Extending this question to an arbitrary root system, we described all pairs (highest weight, weight) such that the corresponding weight multiplicity is one.

In [**6**], one more version of the Littlewood-Richardson rule was found, which expresses the tensor product multiplicities in the most symmetric way. This expression (often referred to as the BZ-rule) attracted the attention of several researchers; in particular, R. Howe used it in his recent work on the invariants in the triple tensor product of simple $\mathfrak{gl}_n$-modules. As an application of this rule, we proved in [**6**] a conjecture by Kostant describing the spectrum of the exterior algebra of $\mathfrak{sl}_n$.

The original motivation of [**7**] was to understand a rather mysterious action of the symmetric group S_n on Young tableaux, discovered by Lascoux and Schützenberger. We introduced an action of S_n by piecewise-linear transformations on the space of Gelfand-Tsetlin patterns. In our approach, this group appears as a subgroup of the infinite group G_n, generated by some very simple piecewise-linear involutions (these involutions are continuous analogues of Bender-Knuth involutions acting on Young tableaux). The structure of G_n is not yet completely understood. Some relations were found in [**7**]; they involve the famous Schützenberger involution which also belongs to G_n. Another result of [**7**] is an explicit description of Kashiwara's crystal operators for type A, in terms of G_n.

A series of papers [**8, 9, 10, 11**] is devoted to the study of Lusztig's canonical bases for quantized enveloping algebras. More precisely, we concentrate on the dual canonical basis B in the q-deformed ring $\mathcal{A} = \mathcal{A}_r$ of polynomial functions on the group of $(r+1) \times (r+1)$ upper unitriangular matrices. In [**8**] we introduced and studied a family of *string bases* for $\mathcal{A}$ (this family includes the dual canonical basis). These bases are defined axiomatically and possess many interesting properties, e.g., they all are *good* in the sense of Gelfand and Zelevinsky. For every string basis, we construct a family of combinatorial labellings by *strings* (or, lattice points in some convex polyhedral cone). These labellings appeared, in a different context, in more recent works by M. Kashiwara and by P. Littelmann. We expect that the basis B has a nice multiplicative structure. Namely, we conjecture in **[8]** that B contains all the products of pairwise q-commuting elements of B. The conjecture was proved in **[8]** for $\mathcal{A}_2$ and $\mathcal{A}_3$. In fact, for $r \leq 3$ the dual canonical basis B is the only string basis existing; it consists of all q-commuting products of quantum minors (for r arbitrary, we proved that any string basis contains all the quantum minors).

In **[9]**, the structure theory of the dual canonical basis is used to obtain a direct representation-theoretic proof of the Littlewood-Richardson rule (or rather, its piecewise-linear versions discussed above). Another application of string technique is an explicit formula for the action of the longest element $w_0 \in S_{r+1}$ on the dual canonical basis in each simple $\mathfrak{sl}_{r+1}$-module. Translated into the language of Gelfand-Tsetlin patterns and Young tableaux, this involution coincides with the Schützenberger involution.

The papers **[10, 11]** are based on the remarkable parallelism, discovered by Lusztig, between the canonical basis (for type A_r) and the variety of totally positive unipotent upper triangular matrices. Using this parallelism, we obtained explicit formulas for piecewise-linear transformations that relate different combinatorial labellings (due to Lusztig) of the canonical basis to semisimlpe Lie groups. A whole new mechanism has been developed for the study of these piecewise-linear transformations: it can be called "subtraction-free algebraic geometry". Another new ingredient of our method is the use of *pseudo-line arrangements* (or wiring diagrams) representing elements of the Weyl group S_{r+1}, and of a special substitution which we call Chamber Ansatz. As a byproduct of this approach, a number of new criteria for total positivity is obtained.

In **[13]**, we define an action of the symmetric group on the set of domino tableaux, and prove that the number of domino tableaux of a given weight does not depend on the permutation of components of the weight. This allows us to give a direct bijective proof of the well-known result due to J. Stembridge that the number of self-evacuating tableaux of a given shape is equal to that of domino tableaux of the same shape. The main feature of our approach is that domino tableaux are defined as fixed points of a certain piecewise-linear involution acting on the set of Young tableaux.

In the paper **[14]**, we solved the following long-standing problem: Given a reductive group G and its reductive subgroup H, describe the *momentum cone* Δ_0, which is the rational polyhedral cone spanned by all the dominant G-weights λ such that the simple G-module V_λ contains a non-trivial H-invariant vector. Our result generalizes the result by Klyachko who solved this problem for the group $G = GL_n \times GL_n \times GL_n$ and the subgroup $H = GL_n$ embedded diagonally into G. We describe the facets of the cone Δ_0 in terms of the "relative" Schubert calculus on the flag varieties of the two groups.

In the paper **[15]** we associate to every reductive group G a category of *geometric crystals* which is a geometrization of Kashiwara's crystal bases. The main property of any geometric crystal is that the Weyl group of G acts rationally on the underlying variety. This property is very important for constructing γ-functions associated to irreducible representations of the Langlands dual group $G^\vee$ of G. It is also interesting that the Langlands dual group $G^\vee$ emerges when we reconstruct Kashiwara's crystals out of *positive geometric crystals.*

The paper [**16**] continues and to some extent concludes the project initiated twelve years before in [**4**]. Namely, we construct explicitly a family of polyhedral expressions for tensor product multiplicities for an arbitrary semisimple Lie algebra $\mathfrak{g}$. To be more precise, we give two such expressions for every reduced word representing w_0, the longest element of the Weyl group, and also produce two "universal" expressions which we call *tropical Plücker models*. It can be shown that these expressions include as special cases the expressions conjectures in [**4**]. As another application, we obtain a family of polyhedra-based formulas for the multiplicities occurring when one restricts a simple finite-dimensional $\mathfrak{g}$-module to the Levi subalgebra of some parabolic subalgebra in $\mathfrak{g}$.

In [**33**], we prove an analogue of the Gelfand-Kirillov conjecture for any simple quantum group G_q (here G_q is the q-deformed coordinate ring of a simple algebraic group G). Namely, it is established that the field of fractions of G_q is isomorphic to the field of fractions of a certain skew-polynomial ring. The proof is based on a construction of some group-like elements in G_q (which are q-analogs of elements in G).

The paper [**35**] is devoted to an explicit computation of the famous Robinson-Schensted-Knuth correspondence (RSK) between the set of the matrices with non-negative integer entries, and the set of plane partitions. More precisely, in suitable linear coordinates on both sets, the RSK is expressed via minima of linear forms, i.e, in the piecewise-linear terms. In particular, we answer the following question by C. Greene and G. Viennot: "What shape corresponds to a given matrix under the RSK?" Our main tool in establishing these formulas is the quantum matrices and crystal bases.

Papers [**3, 12**] deal with concavity condition for certain numerical sequences. In [**12**], we produce rather general sufficient conditions that together imply concavity of the following sequence of real numbers:

$$\left\{\gamma_n = \frac{\alpha_0\beta_0 + \dots \alpha_n\beta_n}{\alpha_0 + \cdots + \alpha_n}\right\},$$

where all α_i and β_i are positive real numbers. Namely, $\{\gamma_n\}$ is concave if:

(1) $\{\beta_n\}$ is concave,
(2) $\{\alpha_n\}$ is log-concave, and
(3) $\left\{\delta_n = \beta_n - \frac{\beta_{n-1}}{\alpha_{n-1}}/\alpha_n - \alpha_{n-2}/\alpha_{n-1}\right\}$ is non-increasing.

As a consequence, we get necessary and sufficient conditions for the concavity of the sequences $\{S_{n-1}(x)/S_n(x)\}$ and $\{S'_n(x)/S_n(x)\}$ for any nonnegative x, where $S_n(x)$ is the n-th partial sum of a power series with arbitrary positive coefficients $\{a_n\}$. This generalizes the main result of [**12**] which asserted that $\{S_{n-1}(x)/S_n(x)\}$ was concave, where S_n is the n-th partial sum of e^x (and, therefore, $S'_n(x) = S_{n-1}(x)$).

Cluster algebras. Cluster algebras were introduced in a recent work by S. Fomin and A. Zelevinsky as a general class of commutative rings attached to varieties (or, rather, schemes) covered by algebraic tori.

The paper [**17**] addresses and essentially solves the following problem: when is a *cluster algebra* $\mathcal{A}$ finitely generated? To solve the question we introduce a remarkable family of *acyclic* cluster algebras, and associate to every such algebra a generalized Cartan matrix A. Our main result is that each acyclic cluster algebra is finitely generated. We construct a monomial basis for each acyclic cluster algebra, and describe its spectrum as a certain *double Bruhat cell* connected with the corresponding Kac-Moody Lie algebra $\mathfrak{g}(A)$.

To explain emergence of double Bruhat cells in the "cluster realm" we introduced a notion of an *upper cluster algebra.* By definition, the upper cluster algebra $\overline{\mathcal{A}}$ contains the cluster algebra $\mathcal{A}$ and has the same field of fractions. The upper cluster algebra is in many respects better than the cluster algebra itself: in certain cases $\mathcal{A}$ is an infinitely generated algebra, while its upper cluster algebra is still finitely generated. Our main results in this direction are:

(1) If $\mathcal{A}$ is acyclic then $\overline{\mathcal{A}} = \mathcal{A}$,
(2) For each double Bruhat cell $BuB \cap B_-vB_-$ there exists an upper cluster algebra $\overline{\mathcal{A}}$ which is isomorphic to the coordinate ring of $BuB \cap B_-vB_-$ (in particular, this $\overline{\mathcal{A}}$ is finitely generated).

These results imply that such well-known objects as coordinate algebras of reductive groups, flag varieties and Grassmannians are, in fact, upper cluster algebras.

Ongoing projects

I am planning to continue research in Representation Theory and Combinatorics. One of my major goals is to understand the combinatorial, algebraic and geometric structure of canonical bases. The main tools for this study are the *Chamber Ansatz* (introduced in [**10, 11**] and *geometric crystals* (introduced in [**15**] and further studied in [**20, 43**]) which relate the structure of the canonical bases to the birational geometry of the Schubert varieties. A newly emerged tool is the *quantum cluster algebras* (introduced and studied in [**18**]), *quantum symmetric and exterior algebras* (introduced and studied in [**22**]), and *rational matrix factorizations* over division rings (introduced and studied in [**19**]). The pursuit of canonical bases in the *Haiman algebra* of double harmonics brought about a new direction of research: *quasiharmonic polynomials* for Coxeter groups and their impact on rational Cherednik algebras (these polynomials have been introduced and studied in [**27**]). A related work ([**24**]) brought about a vast generalization of rational Cherednik algebras to the *braided doubles* of modules in an arbitrary braided tensor category. In my work on all of these projects would like to bring together different approaches developed by Kashiwara, Kazhdan, Lusztig, Littelmann, Zelevinsky, Drinfel'd, Etingof, and myself.

Here are some of the results and projects in more detail.

Cluster algebras, continued. In the paper [**18**], we introduce and study quantum cluster algebras. These are the quantum deformations of the cluster algebras mentioned above. Among the examples of quantum cluster algebras are the quantum flag varieties, quantum Grassmannians and the quantized coordinate algebras of reductive groups, and quantum double Bruhat cells.

The quantum cluster algebras are related to their commutative counterparts in the same way the quantized enveloping algebras of Kac-Moody Lie algebras are related to their ordinary enveloping algebras. Our main results for quantum cluster algebras recover virtually all the properties of the ordinary cluster algebras and their upper versions: the Laurent phenomenon, finite generation, and the canonical basis.

The paper [**40**] is a continuation of [**17**]. We study and classify cluster algebras of rank 3. Our main result is that there are precisely two classes of cluster algebras of this rank: acyclic and *totally cyclic*. The acyclic ones (introduced in [**17**]) correspond to matrices of the form

$$B = \begin{pmatrix} 0 & a & c \\ -a' & 0 & b \\ -c' & -b' & 0 \end{pmatrix},$$

where a, b, c, a', b', c' are positive integers (or $c = c' = 0$, or $b = b' = 0$, or $a = a' = 0$).

We prove that this acyclic matrix B is the only (up to simultaneous permutations of rows and columns) acyclic matrix in its *mutation class*, and each mutation $\mu(B)$ of such B always has the following pattern of signs:

$$\begin{pmatrix} 0 & \varepsilon_1 & \varepsilon_3 \\ -\varepsilon_1 & 0 & \varepsilon_2 \\ -\varepsilon_3 & -\varepsilon_2 & 0 \end{pmatrix},$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 0, 1\}$ (i.e., $\mu(B)$ is always sign-skew-symmetric).

We also prove that a totally cyclic cluster algebra is completely determined by a matrix of the form

$$B = \begin{pmatrix} 0 & a & -c \\ -a' & 0 & b \\ c' & -b' & 0 \end{pmatrix},$$

where a, b, c, a', b', c' are positive integers satisfying $\frac{1}{abc} + \frac{1}{a'b'c'} \le \min\left(\frac{1}{aa'}, \frac{1}{bb'}, \frac{1}{cc'}\right)$. This matrix is also unique up to simultaneous permutations of rows and columns. Each mutation $\mu(B)$ of such B has a skew-symmetric *cyclic* pattern of signs:

$$\begin{pmatrix} 0 & \varepsilon & -\varepsilon \\ -\varepsilon & 0 & \varepsilon \\ \varepsilon & -\varepsilon & 0 \end{pmatrix},$$

where $\varepsilon \in \{-1, 1\}$.

Finally, we prove that a totally cyclic cluster algebra $\mathcal{A}$ corresponding to a *skew-symmetrizable* B (i.e., B as above such that $abc = a'b'c'$), is infinitely generated. This result is rather surprising because each acyclic cluster algebra is finitely generated (according to one of the main results of [**17**]).

Noncommutative double Bruhat cells and noncommutative loops over Lie algebras and Lie groups. In the first paper [**19**] we address and solve the following problem: given an $n \times n$ matrix X over a division ring $\mathcal{F}$, factor it (if possible) into *elementary matrices*, i.e., the matrices of the form $x_i(t) = I + tE_i$, where $t \in \mathcal{F}$ and E_i is the matrix unit in the intersection of the i-th row and the $(i+1)$-st column. Our main result is a non-commutative version of the properties of commutative matrix factorizations established in [**10**].

More precisely, let $\mathbf{i} = (i_1, \ldots, i_m)$ be a sequence of indices $i_k \in \{1, 2, \ldots, n-1\}$. And let $X = X_\mathbf{i} = (x_{ij})$ be an $n \times n$-matrix over a free field. For such an $\mathbf{i}$ and X let us write the formal factorization,

$$X_\mathbf{i} = (1 + t_1 E_{i_1})(1 + t_2 E_{i_2}) \ldots (1 + t_m E_{i_m}),$$

where all $t_k \in \mathcal{F}$.

The problem is to investigate for which sequences $\mathbf{i}$ the map $(t_1, \ldots, t_m) \mapsto X = X_{\mathbf{i}}$ is rationally invertible, i.e., each t_k is a non-commutative rational expression of the matrix coefficients x_{ij} of the matrix $X = X_{\mathbf{i}}$. We prove that it happens if and only if $\mathbf{i}$ is a reduced word for the longest element in the symmetric group S_n.

For a special reduced word $\mathbf{i} = (1, \ldots, n-1; 1, \ldots, n-2; \ldots; 1, 2; 1)$ we prove that each t_k is a ratio of certain quasi-minors of the matrix $X = X_{\mathbf{i}}$:

$$t_{ij} = |X_{i,j-1}|_{11} \cdot |X_{ij}|_{11}^{-1}$$

for $1 \le i < j \le n$. Here X_{ij} is the $i\times i$-submatrix of X with the rows $\{j-i+1, \ldots, j\}$ and the columns $\{n-i+1, \ldots, n\}$, and t_{ij} is an abbreviation for t_k with $k = n(i-1) - \binom{i+1}{2} + j)$.

These results are further generalized to *noncommutative double Bruhat cells* in the group $GL_n(\mathcal{F})$ for any (commutative or non-commutative) skew field $\mathcal{F}$. Each double Bruhat cell $G^{u,v}$ is parametrized by two permutations $u, v \in S_n$ and is merely the intersection of the ordinary Bruhat cells BuB and BvB, where B is the Borel subgroup of $GL_n(\mathcal{F})$. If we write a factorization of some $x \in G^{u,v}$ along some *double reduced word* for (u, v) then the factorization parameters are always ratios of exactly two *quasiminors* of the twisted matrix $x' = \psi^{u,v}(x)$.

The aim of the papers **[34]** and **[23]** is to introduce and study Lie algebras and Lie groups over noncommutative rings. For any Lie algebra $\mathfrak{g}$ sitting inside an associative algebra A and any associative algebra $\mathcal{F}$ we introduce and study the $\mathcal{F}$-loop algebra, which is the Lie subalgebra of $\mathcal{F} \otimes A$ generated by $\mathcal{F} \otimes \mathfrak{g}$. In most examples A is the universal enveloping algebra of $\mathfrak{g}$. Our description of the loop algebra has a striking resemblance to the commutator expansions of $\mathcal{F}$ used by M. Kapranov in his approach to noncommutative geometry. To each $\mathcal{F}$-loop algebra $(\mathfrak{g}, A)(\mathcal{F})$ we associate a "noncommutative algebraic" group which naturally acts on $(\mathfrak{g}, A)(\mathcal{F})$ by conjugations and conclude the paper with a number of examples of such groups.

Braided symmetric and exterior algebras. The aim the paper **[22]** is to the study symmetric algebras of modules over a quantized enveloping algebra $\mathbf{U} = U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a semisimple Lie algebra. We define a symmetric algebra $S_q(V)$ of an integrable $\mathbf{U}$-module V as the quotient of the tensor algebra $T(V)$ by the ideal generated by $\Lambda_q^2(V) \subset V \otimes V$, where $\Lambda_q^2(V)$ is the linear span of all the eigenvectors of the braiding operator $\mathcal{R} : V \otimes V \to V \otimes V$ corresponding to *negative* eigenvalues of $\mathcal{R}$, i.e., the eigenvalues of the form $-q^r$, $r \in \mathbb{Q}$. One can easily show that the correspondence $V \mapsto S_q(V)$ is a functor from the category of integrable $\mathbf{U}$-modules to the category of integrable $\mathbf{U}$-module algebras.

Our first observation is a striking similarity between the quantum symmetric algebras and the ordinary symmetric algebras. We proved that each quantum symmetric algebra is a flat deformation of a certain quotient of the ordinary symmetric algebra of the same vector space. To emphasize this similarity we say that a $\mathbf{U}$ module V is *flat* if $S_q(V)$ is a flat deformation of the ordinary symmetric algebra $S(V)$. Our immediate goal is to classify all flat $\mathbf{U}$-modules. To our surprise, it turned out that the 4-dimensional $U_q(\mathfrak{sl}_2)$-module $V = V_3$ is *not* flat.

Using the techniques of quantum Schur-Weyl duality we prove the following classification result for $\mathbf{U} = U_q(\mathfrak{sl}_2)$: an irreducible $(l+1)$-dimensional $U_q(\mathfrak{sl}_2)$-module V_l the is flat if and only if $l \le 2$.

For a simple Lie algebra $\mathfrak{g}$ we conjecture that the adjoint $\mathbf{U} = U_q(\mathfrak{g})$-module $V = \mathfrak{g}_q$ is flat.

The concept of quantum symmetric algebra $S_q(V)$ is complemented by that of the quantum exterior algebra $\Lambda_q(V)$, and the latter is the Koszul dual of the former: $\Lambda_q(V) = S_q(V^*)^!$. Both concepts naturally generalize to any braided category of vector spaces, which fact prompted us to refer to these objects as *braided* symmetric and exterior algebra.

Geometric crystals. Let G be a reductive algebraic group, and $G^\vee$, its Langlands dual group. A category of geometric and unipotent G-crystals was introduced in **[15]**; they are geometric analogues of the Kashiwara's $G^\vee$-crystals.

The paper **[20]** is devoted to the study of combinatorial aspects of geometric crystals. The study began in **[15]**, where we provided an explicit description of a structure of a *positive* geometric G-crystal on a variety X of a *combinatorial* Kashiwara $G^\vee$-crystal in the lattice of the same dimension. Our main result in the present paper is twofold. First, we introduce and strudy properties of *unipotent bicrystals* which form a very important class of geometric and unipotent crystals (introduced earlier in **[15]**). And, second, we provide an explicit functorial construction of Kashiwara crystal bases out of *positive unipotent bicrystals.*

It is natural to conjecture that each of so constructed combinatorial crystals are *crystal basis* for the coordinate rings of a certain $G^\vee$-variety. One of the goals of the present paper is to prove this conjecture for $X = G/U$, where U is the unipotent radical of a Borel subgroup of G: the corresponding combinatorial $G^\vee$-crystal is the crystal basis $\mathcal{B}$ for the homogeneous coordinate ring of the flag variety of $G^\vee$. This gives a new construction for $\mathcal{B}$ and defines new structures on $\mathcal{B}$: a "crystal" multiplication and the "crystal" Casimir element. Similar results hold for the geometric L-crystal on the variety $X = G/P$, where P is a parabolic subgroup of G and L is the Levi factor of P: the corresponding combinatorial $L^\vee$-crystal is the crystal basis for the coordinate ring of the unipotent radical of $P^\vee$.

The paper **[43]** (in preparation) deals with G-crystals in a more geometric way. Given an algebraic group G, our aim is to construct *algebro-geometric distributions* on G associated with algebraic representations ρ of the Langlands dual group $G^\vee$. It was shown in a recent work by A. Braverman and D. Kazhdan that an existence of such a distribution for a given ρ implies a corollary of the local Langlands conjectures. In order to obtain a distribution on G it suffices to construct a *W-equivariant distribution* on T, where $T \subset G$ is the maximal torus, and $W = \mathrm{Norm}_G(T)$ is the Weyl group. The *convolution product* of distributions on G turns the category of W-equivariant distributions on T into a monoid. We describe explicitly the monoidal structure for the W-equivariant distributions on T which are *unipotent crystals*, or (which is almost the same) for the distributions admiting a *W-equivariant trivialization.* However, it turns out that the latter condition is nontrivial. For example, if $G = \mathrm{GL}_2$ and ρ is the symmetric cube of the standard representation of G, we prove that the corresponding 4-dimensional W-equivariant distribution on T. the has no W-equivariant trivialization.

Another result of this work is affirmative: let $G = \mathrm{GL}_2 * \mathrm{GL}_2 * \mathrm{GL}_2$ be a subgroup of $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ which consists of all triples of matrices with equal determinants, and let $\rho = \rho_1 \otimes \rho_2 \otimes \rho_3$, where ρ_i is the standard representation of the i-th factor. We construct a W-equivariant trivialization of the corresponding 8-dimensional distribution on T. This construction uses the Piatetski-Shapiro model of the distribution. We generalize this construction to $G = \mathrm{GL}_m * \mathrm{GL}_n * \mathrm{GL}_k$.

Dunkl operators, Quasiharmonic polynomials, and canonical invariants of reflection groups. The aim of the paper **[27]** is study of representations of rational Cherednik algebras $H_c(W)$, where c is a complex parameter and W is a finite reflection group of a space V (in fact, $H_c(W)$ can be thought of as a c-deformation of the the algebra of polynomial differential operators on V).

Our ultimate goal is to fully understand the structure of these representations and prepare a ground for introducing canonical bases for the representations. In its turn, this study will allow to construct a canonical basis in the Haiman algebra of *diagonal harmonics*.

This problem hard because for generic c the algebra $H_c(W)$ has no finite-dimensional representations, and only for certain rational values $c = \frac{r}{h}$ (where h is the Coxeter number of W and r is a certain natural number not divisible by h) the algebra $H_c(W)$ admits a (unique) finite-dimensional representation. This representation, as an algebra with W-action, is isomorphic to $A_r = S(V)/V^{(r)} \cdot S(V)$, where $S(V)$ is the symmetric algebra of V and $V^{(r)}$ is a W-submodule of $S^r(V)$ isomorphic to V.

We first propose a c-deformation $V^{(r;c)}$ of the defining space $V^{(r)}$ such that the algebra $A_{r;c} = S(V)/V^{(r;c)} \cdot S(V)$ is still finite-dimensional, and carries a W-action. Our main conjecture is that $A_{r;c}$ is a flat c-deformation of the algebra A_r, i.e., the specialization of $A_{r;c}|_{c=\frac{r}{h}}$ is the original algebra A_r. The advantage of the deformed algebra $A_{r,c}$ is in the additional structure of a flat family which, similarly to quantum groups, can help in understanding of a canonical basis for A_r.

It is remarkable that even for $c = 0$ our construction of $A_{r;c}$ is new. Our key concept here is *quasi-harmonic elements* $\mathcal{QH}^{(c)}$ in $S(V)$, i.e., those symmetric tensors which are killed by almost all W-invariant *Dunkl operators* on $S(V)$. It turns out that the deformed space $V^{(r;c)}$ is a unique quasi-harmonic W-module of degree r isomorphic to V. Therefore, we believe that $A_{r;c}$ is a canonical deformation of the irreducible representation A_r of the reational Cherednik algebra $H_{\frac{r}{h}}(W)$.

In the follow up paper **[26]**, using Dunkl operators, we introduce a continuous family of canonical invariants of finite reflection groups. We verify that the elementary canonical invariants of the symmetric group are deformations of the elementary symmetric polynomials. We also compute the canonical invariants for all dihedral groups as certain hypergeometric functions.

Braided doubles and rational Cherednik algebras. The goal of the paper **[24]** is to introduce and study a certain double algebra $D(V, \Psi)$ associated to any braided vector space (V, Ψ) in a functorial way.

Each of our algebras has a triangular decomposition $A \cong A_- \otimes A_0 \otimes A_+$, where A_- (resp. A_+) is a sub-algebra generated by V (resp. by V^*), and A_0 is a certain Hopf algebra associated to (V, Ψ). In particular, if V is a Yetter-Drinfel'd module over a Hopf algebra H and the braiding $\Psi : V \otimes V \to V \otimes V$ is a homomorphism of Yetter-Drinfel'd modules, then $A_0 = H$.

Another characteristic property of our double is that the Harish-Chandra pairing $(\cdot, \cdot) : A_+ \times A_- \to A_0$ (given by $(a_+, a_-) = pr_{A_0}(a_+ \cdot a_-)$) is non-degenerate, where $pr_{A_0} : A \to A_0$ is the projection to the middle factor.

Our main result is that when $H = \mathbb{C}[W]$ is the group algebra of a Coxeter group and $V \subset \mathbb{C}[W]$ is the span of the class of all reflections in W, the double $D(V, \Psi)$ contains a canonical sub-algebra isomorphic to the rational Cherednik algebra $H_c(W)$. This result allows us to extend the concept of Cherednik algebra $H_c(W)$ to any group G and any conjugacy class S in G.

1. **Quantum folding.** In the paper **[28]** we introduce a quantum analogue of the classical folding of simply-laced Lie algebra $\mathfrak{g}$ to the non-simply-laced algebra $\mathfrak{g}^\sigma$ along a Dynkin diagram automorphism σ of $\mathfrak{g}$. For each quantum folding we replace $\mathfrak{g}^\sigma$ by its Langlands dual $\mathfrak{g}^{\sigma\vee}$ and construct a nilpotent Lie algebra $\mathfrak{n}$ which interpolates between the nilpotnent parts of $\mathfrak{g}$ and $\mathfrak{g}^{\sigma\vee}$, together with its quantized enveloping algebra $U_q(\mathfrak{n})$ and a Poisson structure on $S(\mathfrak{n})$. Remarkably, for the pair $(\mathfrak{g}, \mathfrak{g}^{\sigma\vee}) = (so_{2n+2}, sp_{2n})$, the algebra $U_q(\mathfrak{n})$ admits an action of the Artin braid group Br_n and contains a new algebra of quantum $n \times n$ matrices with an adjoint action of $U_q(sl_n)$, which generalizes the algebras constructed by K. Goodearl and M. Yakimov. The hardest case of quantum folding is, quite expectably, the pair (so_8, G_2) for which the PBW presentation of $U_q(\mathfrak{n})$ and the corresponding Poisson bracket on $S(\mathfrak{n})$ contain more than 700 terms each.

2. **Rank 2 buildings and Schubert Calculus.** The goal of the series of papers **[30,31]** is to introduce a version of Schubert calculus for each dihedral reflection group W. That is, to each "sufficiently rich" spherical building Y of type W we associate a certain cohomology theory $H^*(Y)$ and verify that, first, it depends only on W (i.e., all such buildings are "homotopy equivalent") and second, $H^*(Y)$ is the associated graded of the coinvariant algebra of W under certain filtration. We also construct the dual homology "pre-ring" on Y. The convex "stability" cones in $(\mathbb{R}^2)^m$ defined via these (co)homology theories of Y are then shown to solve the problem of classifying weighted semistable m-tuples on Y; equivalently, they are cut out by the *generalized triangle inequalities* for thick Euclidean buildings with the Tits boundary Y. The independence of the (co)homology theory of Y refines the well-known result that the stability cone depends on W rather than on Y. Quite remarkably, the cohomology ring $H^*(Y)$ is obtained from a certain universal algebra A_t by a kind of "crystal limit" that has been previously introduced by Belkale-Kumar for the cohomology of flag varieties and Grassmannians. Another degeneration of A_t leads to the homology theory $H_*(Y)$.

Littlewood-Richardson coefficients for reflection groups. In the paper **[32]** we explicitly compute all Littlewood-Richardson coefficients for semisimple and Kac-Moody groups G, that is, the structure constants (also known as the *Schubert structure constants*) of the cohomology algebra $H^*(G/P, \mathbb{C})$, where P is a parabolic subgroup of G. These coefficients are of importance in enumerative geometry, algebraic combinatorics and representation theory. Our formula for the Littlewood-Richardson coefficients is purely combinatorial and is given in terms of the Cartan matrix and the Weyl group of G. However, if some off-diagonal entries of the Cartan matrix are 0 or -1, the formula may contain negative summands. On the other hand, if the Cartan matrix satisfies $a_{ij}a_{ji} \geq 4$ for all i, j, then each summand in our formula is nonnegative that implies nonnegativity of all Littlewood-Richardson coefficients. We extend this and other results to the structure coefficients of the T-equivariant cohomology of flag varieties G/P and Bott-Samelson varieties $\Gamma_{\mathbf{i}}(G)$.

LIST OF PUBLICATIONS

Published papers.

1. A. Berenstein, A. Zelevinsky, Involutions on Gelfand-Tsetlin patterns and multiplicities in skew $\mathfrak{gl}_n$-modules, *Soviet Math. Dokl.*, vol. 300, **6** (1988).

2. A. Berenstein, A. Vainstein, A multiplicative analogue of Bergström's inequality for the Hadamard matrix product, *Russian Math. Surveys*, vol. 42, **6** (1988) p. 225.

3. A. Berenstein, A. Kreinin, A. Vainstein, A convexity property of the Poisson distribution and its application in queuing theory, *Journal of Soviet Mathematics*, vol. 47, **1** (1989).

4. A. Berenstein, A. Zelevinsky, Tensor product multiplicities and convex polytopes in partition space, *Journal of Geometry and Physics*, **5** (1989).

5. A. Berenstein, A. Zelevinsky, When the weight multiplicity is 1, *Funct. Anal. and applications*, vol. 24, **4** (1990), pp. 1–13.

6. A. Berenstein, A. Zelevinsky, Triple multiplicities for $\mathfrak{sl}_{r+1}$ and the spectrum of the exterior algebra of the adjoint representation, *Journal of Algebraic Combinatorics*, vol. 1, **1** (1992), pp. 7–22.

7. A. Berenstein, Anatol Kirillov, Groups generated by involutions, Gelfand-Tsetlin patterns and combinatorics of Young tableaux, *St. Petersburg Journal of Mathematics*, vol. 7, **1** (1995), pp. 92–152.

8. A. Berenstein, A. Zelevinsky, String bases for quantum groups of type A_r, *Advances in Soviet Mathematics*, vol. 16, part 1 (1993), pp. 51–89.

9. A. Berenstein, A. Zelevinsky, Canonical bases for the quantum group of type A_r and piecewise-linear combinatorics, *Duke Math. J.*, vol. 82, **3** (1996), pp. 473–502.

10. A. Berenstein, S. Fomin, A. Zelevinsky, Parametrizations of canonical bases and totally positive matrices, *Advances in Mathematics*, vol. 122, **1** (1996), pp. 49–149.

11. A. Berenstein, A. Zelevinsky, Total positivity in Schubert varieties, *Comment. Math. Helv.*, vol. 72, **1** (1997), pp. 128–166.

12. A. Berenstein, Alexander Vainstein, Concavity of weighted arithmetic means with applications, *Arch. Math.* (Basel), vol. 69, **2** (1997), pp. 120–126.

13. A. Berenstein, Anatol Kirillov, Domino tableaux, the Schützenberger involution and the symmetric group action, *Discrete Math.*, vol. 225, **1–3** (2000), pp. 15–24.

14. A. Berenstein, R. Sjamaar, Projections of Coadjoint Orbits and the Hilbert-Mumford Criterion, *J. Amer. Math. Soc.*, vol. 13, **2** (2000), pp. 433–466.

15. A. Berenstein, D. Kazhdan, Geometric and unipotent crystals, *Geom. Funct. Anal.*, Special Volume, Part I (2000), pp. 188–236.

16. A. Berenstein, A. Zelevinsky, Tensor product multiplicities, canonical bases and totally positive varieties, *Invent. Math.*, vol. 143, **1** (2001), pp. 77–128.

17. A. Berenstein, S. Fomin, A. Zelevinsky, Cluster algebras III: Upper and lower bounds, *Duke Math. Journal*, vol. 126, **1** (2005), pp. 1–52.

18. A. Berenstein, A. Zelevinsky, Quantum cluster algebras, *Advances in Mathematics*, vol. 195, **2** (2005), pp. 405–455.

19. A. Berenstein, V. Retakh, Noncommutative double Bruhat cells and their factorization, *Int. Math. Res. Not.*, **8** (2005), pp. 477–516.

20. A. Berenstein, D. Kazhdan, Geometric and unipotent crystals II: from geometric crystals to crystal bases, *Contemp. Math.*, **433**, Amer. Math. Soc., Providence, RI, 2007, pp. 13–88.

21. A. Berenstein, D. Kazhdan, Lecture notes on geometric crystals and their combinatorial analogues, *Combinatorial aspect of integrable systems*, MSJ Memoirs, **17**, Mathematical Society of Japan, Tokyo, 2007.

22. A. Berenstein, S. Zwicknagl, Braided symmetric and exterior algebras, *Trans. Amer. Math. Soc.*, **360** (2008), pp. 3429–3472.

23. A. Berenstein, V. Retakh, Lie algebras and Lie groups over noncommutative rings, *Advances in Mathematics*, Vol. 218, **6**, (2008), pp. 1723–1758.

24. Y. Bazlov, A. Berenstein, Braided doubles and rational Cherednik algebras, *Advances in Mathematics*, Vol. 220 (2009) **5** pp. 1466-1530.

25. Y. Bazlov, A. Berenstein, Noncommutative Dunkl operators and Braided Cherednik algebras, *Selecta Mathematica*, **14**, (2009), pp. 325–372.

26. A. Berenstein, Yu. Burman, Dunkl operators and canonical invariants of reflection groups, *SIGMA*, **5** (2009), 057, 18 pages.

27. A. Berenstein, Yu. Burman, Quasiharmonic polynomials and representations of rational Cherednik algebras, *Trans. Amer. Math. Soc.*, **362** (2010), pp. 229–260.

28. A. Berenstein, J. Greenstein, Quantum folding, Int. Math. Res. Not., doi: 10.1093/imrn/rnq264, 2010.

29. A. Berenstein, V. Retakh, A short proof of Kontsevich cluster conjecture, *C.R. Acad. Sci.*, Paris, Ser. I, vol 349(2011), pp. 119–122.

Papers accepted for publication.

30. A. Berenstein, M. Kapovich, Affine buildings for dihedral groups, to appear in *Geometriae Dedicata.*

31. A. Berenstein, M. Kapovich, Stability inequalities and universal Schubert calculus of rank 2, to appear in *Transformation Groups.*

Submitted.

32. A. Berenstein, E. Richmond, Littlewood-Richardson coefficients for reflection groups, arXiv:1012.1714, submitted to *Invent. Math.*

Preprints.

33. A. Berenstein, Group-like elements in quantum groups and Feigin's conjecture, Arxiv:q-alg/9605016.

34. A. Berenstein, V. Retakh, Noncommutative loops over Lie algebras, MPIM2006-131.

35. A. Berenstein, Anatol Kirillov, The Robinson-Schensted-Knuth bijection, quantum matrices and piecewise-linear combinatorics, *preprint FPSAC01*, Arizona State University, May 20-26, 2001.

Papers in preparation.

36. A. Berenstein, D. Kazhdan, Quantum Hankel algebras, clusters, and canonical bases.

37. Y. Bazlov, A. Berenstein, Twisted Cherednik algebras.

38. Y. Bazlov, A. Berenstein, Generalized Braided doubles and 2-categories.

39. A. Berenstein, V. Retakh, Noncommutative cluster varieties and triangulations.

40. A. Berenstein, S. Fomin, A. Zelevinsky, Cluster algebras of rank 3.

41. A. Berenstein, A. Zelevinsky, Quantum cluster algebras II: Canonical bases.

42. A. Berenstein, Anatol Kirillov, Geometric Robinson-Schensted-Knuth bijection.

43. A. Berenstein, D. Kazhdan, Algebro-geometric distributions on reductive groups and geometric crystals.

44. A. Berenstein, Kostka polynomials at roots of unity and characters of Weyl groups.

INVITED TALKS

1. *Equivariant Littlewood-Richardson coefficients*, ICMS workshop “New developments in noncommutative algebra and its applications,” Sabhal Mr Ostaig, Isle of Skye, Schotland, June 2011.
2. *Littlewood-Richardson coefficients for reflection groups*, Combinatorics Seminar, Bar Ilan University, Tel Aviv, June 2011.
3. *Littlewood-Richardson coefficients for reflection groups*, Algebra Seminar, University of Haifa, May 2011.
4. *Littlewood-Richardson coefficients for reflection groups*, Enveloping algebras and Representation Theory Seminar, Jussieu Institute of Mathematics, Paris, April 2011.
5. *Littlewood-Richardson coefficients for reflection groups*, Workshop on the Interaction of Representation Theory with Geometry and Combinatorics, Hausdorff Research Institute, Bonn, March 2011.
6. *Littlewood-Richardson coefficients for reflection groups*, Algebra Seminar, University of Manchester, UK, March 2011.
7. *Littlewood-Richardson coefficients for reflection groups*, Colloquium, Wayne State University, March 2011.
8. *Littlewood-Richardson coefficients for reflection groups*, Algebra and Discrete Mathematics Seminar, UC Davis, December 2010.
9. *Matrix factorizations over noncommutative ring*, Noncommutative Algebra Seminar, Max Planck Institute, Bonn, Germany, August 2010.
10. *Geometric Crystals*, Workshop “Whittaker Functions, Crystal Bases, and Quantum Groups,” Banff, Canada, June 2010
11. *q-commuting Dunkl operators and braided Cherednik algebras*, AMS Sectional Meeting, UC Riverside, November 2009.
12. *Lie algebras and Lie groups over noncommutative rings*, AMS Sectional Meeting, UC Riverside, November 2009.
13. *Geometric crystals and tropical combinatorics*, Workshop “Tropical Geometry in Combinatorics and Algebra,” MSRI, October 2009.
14. *Universal rank 2 Schubert calculus*, Conference on Eigenvalue and Saturation Problems for Reductive Groups, University of North Carolina, Chapel Hill, May 2009.
15. *From geometric crystals to crystal bases*, GADUDIS Conference, University of Glasgow, UK, March 2009.
16. *Braided doubles and braided Cherednik algebras*, Conference on Enveloping Algebras and Geometric Representation Theory, Oberwolfach, Germany, March 2009.
17. *Quantum Hankel algebras, clusters, and canonical bases*, International Conference on Cluster Algebras and Related Topics, Mexico City, Mexico, December 2008.
18. *Quantum Hankel algebras, clusters, and canonical bases*, Algebra and Discrete Mathematics Seminar, UC Davis, December 2008.
19. *q-commuting Dunkl operators and Braided Cherednik algebras*, Meeting of the American Mathematical Society, Vancouver, Canada, October 2008.
20. *q-commuting Dunkl operators and Braided Cherednik algebras*, Algebra Seminar, University of Leeds, UK, July 2008.

21. *Lie algebras and Lie groups over noncommutative rings*, International Colloquium on Integrable Systems and Quantum symmetries, Prague, June 2008.
22. *Braided doubles and rational Cherednik algebras*, Representation Theory Seminar, Northeastern University, December 2007.
23. *From geometric crystals to crystal bases*, Joint meeting of Lie Groups and Combinatorics seminars, MIT, December 2007.
24. *Braided doubles and rational Cherednik algebras*, Representation Theory Seminar, University of Minnesota, December 2007.
25. *From geometric crystals to crystal bases*, Combinatorics Seminar, University of Minnesota, December 2007.
26. *Braided doubles and rational Cherednik algebras*, Algebra Seminar, University of Oregon, November 2007.
27. *Lie algebras and Lie groups over noncommutative rings*, Lie Theory Seminar, UC Riverside, October 2007.
28. *Braided doubles and rational Cherednik algebras*, Colloquium, UC Riverside, October 2007.
29. *Lie algebras and Lie groups over noncommutative rings*, Conference *Group Representations and Combinatorics*, University of Florida, September 2007.
30. *Lie algebras and Lie groups over noncommutative rings*, Algebra Seminar, Warwick University, July 2007.
31. *Quasiharmonic polynomials for Coxeter groups and rational Cherednik algebras*, Workshop *Cherednik algebras*, ICMS, Edinburgh, UK, June 2007.
32. *Quasiharmonic polynomials for Coxeter groups and canonical elementary invariants*, Workshop *Interactions between Algebraic Combinatorics and Algebraic Geometry*, CRM, Montreal, May 2007.
33. *Polytopal models and tropical geometry*, Workshop *Buildings and combinatorial representation theory*, AIM, Palo Alto, March 2007.
34. *Noncommutative loops over Lie algebras*, Alumni Conference, Northeastern University, February 2007.
35. *Noncommutative loops over Lie algebras*, Infinite Dimensional Algebra Seminar, MIT, October 2006.
36. *Braided symmetric algebras*, Oberseminar, Max Planck Institute, Bonn, Germany, September 2006.
37. *Noncommutative double Bruhat cells*, Noncommutative Algebra Seminar, Max Planck Institute, Bonn, Germany, August 2006.
38. *Braided symmetric and exterior algebras*, Algebra Seminar, University of Washington, April 2006.
39. *Braided symmetric and exterior algebras*, Colloquium, Indiana University Purdue University of Indiana, September 2005.
40. *From geometric crystals to crystal bases*, Workshop *Generalized Kostka Polynomials*, AIM, Palo Alto, July 2005.
41. *From geometric crystals to crystal bases*, Lie Theory Seminar, UC Riverside, May 2005.
42. *From geometric crystals to crystal bases*, Winter Solstice Workshop, Weizmann Institute of Science, Israel, December 2004.

43. *From geometric crystals to crystal bases*, Workshops on Integrable Systems and Tropical Combinatorics, Research Institute for Mathematical Sciences, Kyoto, Japan, July-August 2004.
44. *Quantum cluster algebras*, International Conference on Quantum Groups, Technion, Haifa, July 2004.
45. *Crystal bases and Geometric crystals*, Combinatorics seminar, University of California, Davis, October 2003.
46. *Cluster algebras and canonical bases*, Algebra seminar, University of Oregon, March 2003.
47. *Macdonald's Identities*, Basic notions seminar, University of Oregon, October 2002.
48. *Cluster algebras and canonical bases*, Workshop on Representations of Lie Algebras, Rehovot, Israel, July 2002.
49. *Cluster algebras II: q-cluster algebras*, Algebra seminar, University of Oregon, June 2002.
50. *Cluster algebras I*, Algebra seminar, University of Oregon, May 2002.
51. *Geometric and combinatorial crystals*, AMS/SMF meeting, Lyon, France, July 2001.
52. *Linear Inequalities and the Schubert Calculus*, Basic notions seminar, University of Oregon, March 2001.
53. *Geometric and unipotent crystals*, Geometric Langlands seminar, University of Chicago, February 2001.
54. *Geometric crystals*, Algebra seminar, University of Oregon, October 2000.
55. *Geometric crystals*, Algebraic Groups Seminar, Ohio State University, February 2000.
56. *Tensor Product Multiplicities, Canonical Bases and Totally Positive Varieties*, Special Department of Mathematics Colloquium, Purdue University, February 2000.
57. *Geometric crystals*, Infinite Dimensional Algebra Seminar, MIT, December 1999.
58. *The Knuth bijection, quantum matrices and free crystals*, Combinatorics Seminar, MIT, February 1999.
59. *Total Positivity and Canonical Bases*, Lie Groups and Lie Algebras seminar, Cornell University, November 1998.
60. *Products of Schur polynomials, sums of Hermitian matrices and piecewise linear combinatorics*, Combinatorics and Algebraic Geometry seminar, Cornell University, September 1997.
61. *Group-like elements in quantum groups and Feigin's conjecture*, International Press conference, University of California at Irvine, January 1997.
62. *On Feigin's conjecture*, Lie Group Seminar, Cornell University, September–October 1996.
63. *Coordinatization of Quantum Groups and the Gelfand-Kirillov Conjecture*, Lie Algebras and Lie Groups Seminar, Yale University, February 1995.
64. *String bases in quantum groups*, Algebra seminar, University of Wisconsin at Madison, Madison, July 1995.
65. *Triple multiplicities in representations of* GL_{r+1}, Algebra Seminar, De Paul University, Chicago, March 1995.

66. *Canonical bases for the quantum group of type $\mathcal{A}_r$ and piecewise linear combinatorics*, Meeting of AMS, Chicago, March 1995.
67. *Gelfand-Tsetlin patterns and piecewise linear combinatorics*, Combinatorics Seminar, Northeastern University, October 1993.
68. *String bases for quantum groups*, Kobe University, Kyoto University, Japan, September 1992.
69. *When the weight multiplicity is one*, Algebra Seminar, Hebrew University, Jerusalem, Israel, October 1990.
70. *Tensor product multiplicities and convex polytopes*, Winter School on Representations Theory, Srni, Chekhoslovakia, January 1990.