Solutions to Self-Quiz 3

a. True. This is just part of Theorem 2 in Section 7.1. The proof appears just before the statement of the theorem.

b. False. A counterexample is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

c. True. This is proved in the first part of the proof of Theorem 6 in Section 7.3. It is also a consequence of Theorem 7 in Section 6.2.

d. False. The principal axes of $\mathbf{x}^T A\mathbf{x}$ are the columns of any *orthogonal* matrix P that diagonalizes A. *Note*: When A has an eigenvalue whose eigenspace has dimension greater than 1, the principal axes are not uniquely determined.

e. False. A counterexample is $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The columns here are orthogonal but not orthonormal.

f. False. See Example 6 in Section 7.2.

g. False. A counterexample is $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $\mathbf{x}^T A\mathbf{x} = 2 > 0$, but $\mathbf{x}^T A\mathbf{x}$ is an indefinite quadratic form.

h. True. This is basically the Principal Axes Theorem from Section 7.2. Any quadratic form can be written as $\mathbf{x}^T A\mathbf{x}$ for some symmetric matrix A.

i. False. See Example 3 in Section 7.3.

j. False. The maximum value must be computed over the set of *unit* vectors. Without a restriction on the norm of **x**, the values of $\mathbf{x}^T A\mathbf{x}$ can be made as large as desired.

k. False. Any orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ changes a positive definite quadratic form into another positive definite quadratic form. Proof: By Theorem 5 of Section 7.2., the classification of a quadratic form is determined by the eigenvalues of the matrix of the form. Given a form $\mathbf{x}^T A\mathbf{x}$, the matrix of the new quadratic form is $P^{-1}AP$, which is similar to A and thus has the same eigenvalues as A.

l. False. The term "definite eigenvalue" is undefined and therefore meaningless.

m. True. If $\mathbf{x} = P\mathbf{y}$, then $\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T P^{-1} AP\mathbf{y}$.

n. False. A counterexample is $U = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. The columns of U must be *orthonormal* to make $UU^T\mathbf{x}$ the orthogonal projection of **x** onto Col U.

o. True. This follows from the discussion in Example 2 of Section 7.4., which refers to a proof given in Example 1.

p. True. Theorem 10 in Section 7.4 writes the decomposition in the form $U\Sigma V^T$, where U and V are orthogonal matrices. In this case, V^T is also an orthogonal matrix. Proof: Since V is orthogonal, V is invertible and $V^{-1} = V^T$. Then $(V^T)^{-1} = (V^{-1})^T = (V^T)^T$, and since V is square and invertible, V^T is an orthogonal matrix.

q. False. A counterexample is $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. The singular values of A are 2 and 1, but the singular values of $A^T A$ are 4 and 1.