GEOMETRIC AND UNIPOTENT CRYSTALS II: FROM UNIPOTENT BICRYSTALS TO CRYSTAL BASES

ARKADY BERENSTEIN AND DAVID KAZHDAN

To the memory of Joseph Donin

Abstract. For each reductive algebraic group G, we introduce and study *unipotent bicrystals* which serve as a regular version of birational *geometric and unipotent crystals* introduced earlier by the authors. The framework of unipotent bicrystals allows, on the one hand, to study systematically such varieties as Bruhat cells in G and their convolution products and, on the other hand, to give a new construction of many normal Kashiwara crystals including those for $G^\vee$-modules, where $G^\vee$ is the Langlands dual groups. In fact, our analogues of crystal bases (which we refer to as crystals *associated* to $G^\vee$-modules) are associated to $G^\vee$-modules directly, i.e., without quantum deformations.

One of the main results of the present paper is an explicit construction of the crystal $\mathcal{B}_0$ for the coordinate ring of the dual flag variety $X_0^\vee = G^\vee/U^\vee$ based on the *positive* unipotent bicrystal on the open Bruhat cell $X_0 = Bw_0B$. Our general *tropicalization* procedure assigns to each *strongly positive* unipotent bicrystal a normal Kashiwara crystal $\mathcal{B}$ equipped with the *multiplicity erasing* homomorphism $\mathcal{B} \to \mathcal{B}_0$ and the *combinatorial central charge* $\mathcal{B} \to \mathbb{Z}$ which is invariant under all crystal operators. Applying the construction to $\mathcal{B}_0 \times \mathcal{B}_0$ gives a crystal multiplication $\mathcal{B}_0 \times \mathcal{B}_0 \to \mathcal{B}_0$ and an invariant grading $\mathcal{B}_0 \times \mathcal{B}_0 \to \mathbb{Z}$.

0. Introduction

The present paper is a continuation of the study of geometric and unipotent crystals initiated in [2]. However, all necessary definitions and constructions are included, so that the paper can be read independently.

The aim of this paper is two-fold: first, to introduce *unipotent bicrystals* as regular versions of geometric and unipotent crystals from [2] and, second, to construct a large class of Kashiwara's crystal bases (or, rather, the combinatorial crystals directly *associated* to appropriate modules) via the *tropicalization* of *positive* unipotent bicrystals.

More precisely, let G be a split reductive algebraic group and $B \subset G$ be a Borel subgroup. Denote by $G^\vee$ the Langlands dual group of G, and by $\mathfrak{g}^\vee$ the Lie algebra of $G^\vee$. For each dominant integral weight λ of $\mathfrak{g}^\vee$, denote by V_λ the irreducible $\mathfrak{g}^\vee$-module and by $\mathcal{B}(V_\lambda)$ the corresponding normal Kashiwara crystal (see e.g., [14]). We will explicitly construct all $\mathcal{B}(V_\lambda)$ in terms of the positive unipotent bicrystal structure on the open Bruhat cell $X = Bw_0B$ or, which is the same, we construct

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the *associated crystal* $\mathcal{B}_0 = \bigsqcup_\lambda \mathcal{B}(V_\lambda)$ for the coordinate algebra of the basic affine space $X_0^\vee = G^\vee/U^\vee$ (Theorem 6.15).

More generally, for each positive unipotent bicrystal on a $U\times U$-variety X (where U is the unipotent radical of a Borel subgroup B of G), we construct an infinite normal Kashiwara crystal $\mathcal{B}$ which is a union of finite normal crystals $\mathcal{B}^\lambda$ (Section 6.2). If the unipotent bicrystal were *strongly positive*, then the resulting normal crystal gets equipped with *multiplicity erasing map* $\tilde{\mathbf{m}} : \mathcal{B} \to \mathcal{B}_0$ such that $\tilde{\mathbf{m}}(\mathcal{B}^\lambda) = \mathcal{B}(V_\lambda)$ if $\mathcal{B}^\lambda$ is not empty (Corollary 6.28) and with the *combinatorial central charge* $\tilde{\Delta} : \mathcal{B} \to \mathbb{Z}$ invariant under all crystal operators $\tilde{e}_i^n$ (Claim 6.12). In particular, if $\mathcal{B} = \mathcal{B}_0 \times \mathcal{B}_0$ is the crystal associated to the algebra $\mathbb{C}[X_0^\vee \times X_0^\vee] = \mathbb{C}[X_0^\vee] \otimes \mathbb{C}[X_0^\vee]$, we obtain:

• the associative *crystal multiplication* $\tilde{\mathbf{m}}_0 : \mathcal{B}_0 \times \mathcal{B}_0 \to \mathcal{B}_0$. This multiplication turns $\mathcal{B}_0$ into a monoid in the category of normal Kashiwara crystals.

• the *combinatorial central charge* $\tilde{\Delta}_0 : \mathcal{B}_0 \times \mathcal{B}_0 \to \mathbb{Z}$ invariant under all crystal operators $\tilde{e}_i^n$ acting on $\mathcal{B}_0 \times \mathcal{B}_0 \cong \oplus_{\lambda,\lambda'}\mathcal{B}(V_\lambda \otimes V_{\lambda'})$. It is a combinatorial analogue of a certain operator $\Delta_0 : \mathbb{C}[X_0^\vee] \otimes \mathbb{C}[X_0^\vee] \to \mathbb{C}[X_0^\vee] \otimes \mathbb{C}[X_0^\vee]$ commuting with the $\mathfrak{g}^\vee$-action. In turn, $\tilde{\Delta}_0$ provides a q-analog of the tensor product multiplicities:

$$[V_{\lambda''} : V_\lambda \otimes V_{\lambda'}]_q = \sum_{\tilde{b}} q^{\tilde{\Delta}_0(\tilde{b})} ,$$

where the summation is over all $\tilde{b} \in \mathcal{B}(V_\lambda \otimes V_{\lambda'})^{\lambda''} = (\mathcal{B}(V_\lambda) \times \mathcal{B}(V_{\lambda'}))^{\lambda''}$. Similarly, we define the q-tensor multiplicities for products of several irreducible $\mathfrak{g}^\vee$-modules (Section 7.3).

This shows that strongly positive unipotent bicrystals are the closest geometric "relatives" of Kashiwara crystal bases of $U_q(\mathfrak{g})$-modules. Moreover, we expect (Conjecture 7.3) that each $\mathcal{B}^\lambda$ obtained from a strongly positive unipotent bicrystal is always isomorphic to a union of copies of $\mathcal{B}(V_\lambda)$. To make the analogy between geometric and combinatorial objects more precise, we lift the Kashiwara construction of the product of Kashiwara crystals to the geometric level – unipotent bicrystals (similarly to the unipotent crystals introduced in [2]) form a nice monoidal category under the convolution product (Claims 2.1 and 2.29), and it turns out that strong positivity is preserved under this product (Theorem 3.37).

As an application of this method, for each parabolic subgroup P of G, we construct the unipotent bicrystal on the *reduced Bruhat cell* $U\overline{w_P}U$ which, in turn, produces the crystal associated to $\mathbb{C}[U_P]$, the coordinate algebra of the unipotent radical $U_P^\vee$ of the dual parabolic $P^\vee$ (Theorem 6.41).

One of the main tools of this paper is the transition from the unipotent bicrystals and geometric crystals to Kashiwara crystals which we refer to as the *tropicalization*. It is based on Theorem 4.12 (originally proved in our first paper [2]) which establishes the functoriality of the transition from the category of *positive algebraic tori* to the category of (marked) sets. In Section 4 we discuss in detail the semi-field of polytopes and related structures as a combinatorial foundation of the total positivity and the carrier of the tropicalization functor. To extend the total positivity to other rational varieties and unipotent bicrystals, we introduce in Section 3.1 a notion of a *positive variety* and discuss the properties of these new algebro-geometric objects.

In order to relate the combinatorial crystals with the actual bases for $\mathfrak{g}^\vee$-modules, we develop in Section 5 the theory of *perfect bases* of modules and their *associated*

crystals. The novelty of this approach is that our definition of the Kashiwara crystal *associated* to a $\mathfrak{g}^\vee$-module does not require any quantum deformation of $\mathfrak{g}^\vee$ or its modules. Our main results in this direction (Theorems 5.37 and 5.55) guarantee that the crystal associated to a module is well-defined and independent of the choice of an underlying perfect basis. Note, however, that our approach allows to prove uniqueness, rather than existence, of the appropriate Kashiwara crystals. The existence follows, for instance, from the fact that the canonical, global crystal, and semi-canonical bases are all perfect (see Remark 5.58).

A number of statements in the paper, which we refer to as *claims* and *corollaries*, are almost immediate. Their proofs are left to the reader.

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1. $U \times U$-varieties

1.1. **Definitions and notation.** Throughout the paper, $\mathbb{G}_m$ and $\mathbb{G}_a$ are respectively the multiplicative and the additive groups, and G is a split reductive algebraic group defined over $\mathbb{Q}$ (or over any other field of characteristic 0).

We fix a maximal torus $T \subset G$ and a Borel subgroup B of G containing T. Let $B^- \subset G$ be the Borel subgroup opposite to B, i.e., $B \cap B^- = T$. Denote by U and U^- respectively the unipotent radicals of B and B^-.

Let $X^\star(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_\star(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ be respectively the lattices of *characters* and *co-characters* of T. These lattices are also known as weight and the co-weight lattices of G. By definition, the lattices are dual to each other via the canonical pairing $\langle \cdot, \cdot \rangle : X^\star(T) \times X_\star(T) \to \mathbb{Z}$. Denote by I the set of vertices of the Dynkin diagram of G and for any $i \in I$ denote by $\alpha_i \in X^\star(T)$ the simple root, and by $\alpha_i^\vee \in X_\star(T)$ the corresponding simple coroot.

For each $i \in I$, we fix a group homomorphism $\phi_i : SL_2 \to G$ such that

$$\phi_i \begin{pmatrix} 1 & 0 \\ \mathbb{G}_a & 1 \end{pmatrix} \subset B^-, \ \phi_i \begin{pmatrix} 1 & \mathbb{G}_a \\ 0 & 1 \end{pmatrix} \subset B, \ \phi_i \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \alpha_i^\vee(c)$$

for $c \in \mathbb{G}_m$. Such a simultaneous choice of homomorphisms ϕ_i is also called a *splitting* of G. Note that any two splittings are conjugate by an element of $T \cap [G, G]$, where $[G, G]$ is the adjoint group of G.

Using a splitting $\{\phi_i\}$, $i \in I$, we define the co-characters $y_i : \mathbb{G}_a \to U^-$ and $x_i : \mathbb{G}_a \to U$ by

$$y_i(a) := \phi_i \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \in B^-, \ x_i(a) := \ \phi_i \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in B \ .$$

We denote by $U_i \subset U$ the image $x_i(\mathbb{G}_a)$, by $U_i^- \subset U^-$ the image $y_i(\mathbb{G}_a)$. Clearly, U (resp. U^-) is generated by U_i (resp. U_i^-), $i \in I$.

Denote by $\widehat{U}$ the set of all characters of U, (i.e., the set of group homomorphisms $\chi : U \to \mathbb{G}_a$). And for each $i \in I$, define the elementary character $\chi_i \in \widehat{U}$ by

$$\chi_i(x_j(a)) = \delta_{ij} \cdot a$$

for $a \in \mathbb{G}_a$. The family χ_i, $i \in I$, is a basis in the vector space $\widehat{U}$.

Following [5, Section 4.2], we define the "positive inverse" anti-automorphism $\iota : G \to G$ by:

$$\iota(y_i(a)) = y_i(a),\ \iota(t) = t^{-1}, \iota(x_i(a)) = x_i(a) \tag{1.1}$$

for $a \in \mathbb{G}_a$, $t \in T$.

Claim 1.1. *For each character $\chi : U \to \mathbb{G}_a$ and $u \in U$, one has*

$$\chi(\iota(u)) = \chi(u) = -\chi(u^{-1})\ .$$

Denote by W the Weyl group of G. By definition, W is generated by the simple reflections s_i, $i \in I$. Let $l : W \to \mathbb{Z}_{\geq 0}$ $(w \mapsto l(w))$ be the length function. For any sequence $\mathbf{i} = (i_1, \dots, i_\ell) \in I^\ell$, we write $w(\mathbf{i}) = s_{i_1} \cdots s_{i_\ell}$. A sequence $\mathbf{i} \in I^l$ is called a *reduced decomposition* of $w \in W$ if $w = w(\mathbf{i})$ and the length $l(w)$ of w is equal to ℓ. And let $R(w)$ be the set of all reduced decompositions of w. We denote by $w_0 \in W$ the element of the maximal length in W and refer to it as the *longest element* of W . For each $w \in W$, denote by

$$|w| \tag{1.2}$$

the smallest subset J of I such that w belongs to the sub-group of W generated by s_j, $j \in J$; and refer to $|w|$ as to the *support* of w. In other words, $|w|$ is the set of all $j \in I$ such that s_j occurs in each reduced decomposition of w. By definition $|w_0| = I$.

We say that a parabolic subgroup P is *standard* if $P \supset B$. Clearly, for each standard parabolic subgroup P, there exists a unique subset $J = J(P)$ of I such that P is generated by B and all U_j^-, $j \in J(P)$. Denote by L_P the Levi factor of P. It is clear that L_P is generated by T and all U_j^-, U_j, $j \in J$. Let $W_P := \mathrm{Norm}_{L_P}(T)/T$ be the Weyl group of L_P.

Claim 1.2. *For each standard parabolic subgroup P of G the Weyl group, W_P is a Coxeter subgroup of W generated by all s_j, $j \in J(P)$.*

Denote by $w_0^P \in W_P$ the longest element of W_P and define the *the parabolic element* $w_P \in W$ by

$$w_P = w_0^P \cdot w_0 \tag{1.3}$$

By definition, w_P is the minimal representative of w_0 in the set of right cosets $W_P \backslash W$. Also $|w_0^P| = J(P)$.

For $i \in I$, define $\overline{s_i} \in G$ by

$$\overline{s}_i = x_i(-1)y_i(1)x_i(-1) = \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ . \tag{1.4}$$

Each $\overline{s}_i$ belongs to $\mathrm{Norm}_G(T)$ and is a representative of $s_i \in W$. It is well-known ([6]) that the elements $\overline{s_i}$, $i \in I$, satisfy the braid relations. Therefore, we can

associate to each $w \in W$ its *standard representative* $\overline{w} \in \mathrm{Norm}_G(T)$ in such a way that for any $(i_1, \ldots, i_\ell) \in R(w)$, we have

$$\overline{w} = \overline{s_{i_1}} \cdots \overline{s_{i_\ell}} \ . \tag{1.5}$$

Claim 1.3. *For each $w \in W$, one has $\iota(\overline{w}) = \overline{w^{-1}}$.*

Define the monoid of dominant co-weights $X^\star(T)^+$ and the monoid of dominant weights $X_\star(T)^+$ by

$$X_\star(T)^+ = \{\lambda \in X_\star(T) : \langle \alpha_i, \lambda \rangle \geq 0 \ \forall i \in I\} \ ,$$
$$X^\star(T)^+ = \{\mu \in X^\star(T) : \langle \mu, \alpha_i^\vee \rangle \geq 0 \ \forall i \in I\} \ .$$

Following [5, Section 4.3], for each $\mu \in X^\star(T)^+$, define the *principal minor* Δ_μ to be the regular function $G \to \mathbb{A}^1$ uniquely determined by the property

$$\Delta_\mu(u_- t u_+) = \mu(t)$$

for all $u_- \in U^-$, $t \in T$, $u_+ \in U^+$. For any extremal weights $\gamma, \delta \in X^\star(T)$ of the form $\gamma = u\mu$, $\delta = v\mu$ for some $u, v \in W$ and a dominant weight $\mu \in X^\star(T)$, define the *generalized minor* $\Delta_{\gamma,\delta}$ to be the regular function on G given by

$$\Delta_{\gamma,\delta}(g) = \Delta_\mu(\overline{u}^{-1} g \overline{v}) \tag{1.6}$$

for all $g \in G$. Clearly, $\Delta_{\mu,\mu}(u) = 1$ and

$$\chi_i(u) = \Delta_{\mu, s_i\mu}(u) \tag{1.7}$$

for any $u \in U$, $i \in I$ and any $\mu \in X^\star(T)$ such that $\langle \mu, \alpha_i^\vee \rangle = 1$. If G is simply-connected, we will use the formula

$$\chi_i(u) = \Delta_{\omega_i, s_i\omega_i}(u) \tag{1.8}$$

where ω_i is a *fundamental weight*, i.e, $\left\langle \omega_i, \alpha_j^\vee \right\rangle = \delta_{i,j}$ for all $i, j \in I$.

1.2. **Basic facts on $U \times U$-varieties.** A $U \times U$-*variety* $\mathbf{X}$ is a pair (X, α), where X is an irreducible affine variety over $\mathbb{Q}$ and $\alpha : U \times X \times U \to X$ is a $U \times U$-action on X, where the first U-action is *left* and second is *right*, such that each group $e \times U$ and $U \times e$ acts freely on X. We will write the action as $\alpha(u, x, u') = uxu'$.

The $U \times U$-varieties form a category which morphisms are morphisms of underlying varieties commuting with the $U \times U$-action.

Define the *convolution product* $*$ of $U \times U$-varieties $\mathbf{X} = (X, \alpha)$ and $\mathbf{Y} = (Y, \alpha')$:

$$\mathbf{X} * \mathbf{Y} := (X * Y, \beta) \ ,$$

where the variety $X * Y$ is the quotient of $X \times Y$ by the following left action of U on $X \times Y$:

$$u(x, y) = (xu^{-1}, uy) \ .$$

For each $x \in X$, $y \in Y$, we denote by $x * y = \{(xu^{-1}, uy) | u \in U\} = U(x, y)$ the corresponding point of $X * Y$. Then the action $\beta : U \times X * Y \times U \to X$ is given by $u(x * y)u' = (ux) * (yu')$,

Clearly, both actions $U \times e$ and $e \times U$ on $X * Y$ are free. So $\mathbf{X} * \mathbf{Y}$ is, indeed, a $U \times U$-variety.

Claim 1.4. *The category of $U \times U$-varieties is naturally monoidal with respect to $*$ (and the unit object U).*

Remark 1.5. Even though the category of U-varieties is not strict, we can ignore it for all practical purposes (see e.g., [27]). The same applies to other monoidal categories (unipotent bicrystals, geometric crystals, Kashiwara crystals, etc.) which we will deal with in this paper.

Given a $U \times U$-variety $\mathbf{X}$, the right quotient X/U and the left quotient $U\backslash X$ are well-defined U-varieties. For each $x \in X$ define $U_{x\bullet} = \{u \in U : ux \in xU\}$, the stabilizer in U of the point xU in X/U, and $U_{\bullet x} = \{u \in U : xu \in Ux\}$ – the stabilizer in U of the point Ux in $U\backslash X$.

Claim 1.6. *Let $\mathbf{X}$ be a $U \times U$-variety, and $x \in X$ be a point. There is a unique group isomorphism $\varphi_x : U_{\bullet x} \to U_{x\bullet}$ such that $xu = \varphi_x(u)x$ for all $u \in U_{\bullet x}$. In particular, the orbit UxU is isomorphic to the quotient of $U \times U$ by the following action $\diamond$ of $U_{\bullet x}$:*

$$u_{\bullet x} \diamond (u, u') = (u\varphi_x(u_{\bullet x})^{-1}, u_{\bullet x}u') \ .$$

Example 1.7. For any $w \in W$ and a representative $\tilde{w} \in \mathrm{Norm}_G(T)$ of w, the *reduced Bruhat cell* $U\tilde{w}U$ in G is a $U \times U$ variety. Denote $U(w) := U \cap \tilde{w}U\tilde{w}^{-1}$ and $V(w) = U \cap \tilde{w}U^{-}\tilde{w}^{-1}$ (clearly, $U(w)$ and $V(w)$ depend only on w). Then $U_{\bullet\tilde{w}} = U(w^{-1}), U_{\tilde{w}\bullet} = U(w)$, the isomorphism $\varphi_{\tilde{w}} : U(w^{-1}) \to U(w)$ is given by $\varphi(u) = \tilde{w}u\tilde{w}^{-1}$, and $U\tilde{w}U$ has two unique factorizations

$$U\tilde{w}U = V(w)\tilde{w}U = U\tilde{w}V(w^{-1}) \ .$$

In particular, for $w = w_P$ as in (1.3), one has $U_{\bullet\tilde{w_P}} = U \cap L_P$ and $V(w_P) = U_P$, where L_P is the Levi factor of P and U_P is the unipotent radical of P, and one has two unique factorizations:

$$U\tilde{w}_P U = U_P\tilde{w}_P U = U\tilde{w}U_{P^{op}} \ , \tag{1.9}$$

where $P^{op} = (w_0 L_P w_0)B$ is the standard parabolic subgroup opposite of P.

Claim 1.8. *For each point $x * y$ of $X * Y$, one has*

$$U_{x*y\bullet} = \varphi_x(U_{\bullet x} \cap U_{y\bullet}), \ U_{\bullet x*y} = \varphi_y(U_{\bullet x} \cap U_{y\bullet}) \ .$$

Claim 1.9. *For any U-varieties $\mathbf{X}$ and $\mathbf{Y}$, the morphism*

$$X * Y/U \to X/U \times Y/U$$

given by $x * yU \mapsto (xU, yU)$ is surjective; and the fiber over any point (xU, yU) is isomorphic to $U/U_{y\bullet}$ via $xu * yU \mapsto u \cdot U_{y\bullet}$ for any $u \in U$.

We say that a $U \times U$-variety $\mathbf{X}$ is *regular* if the action of $U \times U$ on X is free or, equivalently, each $U \times U$-orbit in X is isomorphic to $U \times U$.

Claim 1.10. *The convolution product of regular $U \times U$-varieties is also regular.*

For any $U \times U$-variety $\mathbf{X}$, define $\mathbf{X}^{op} = (X, \alpha^{op})$, where α^{op} is the twisted $U \times U$-action given by

$$(u, x, u') \mapsto \iota(u') \cdot x \cdot \iota(u) \ ,$$

where $\iota : G \to G$ is the positive inverse anti-automorphism (defined in (1.1). Clearly, $\mathbf{X}^{op}$ is a well-defined $U \times U$-variety. We will refer to $\mathbf{X}^{op}$ as the *opposite* $U \times U$-variety of $\mathbf{X}$.

Claim 1.11. *The correspondence $\mathbf{X} \to \mathbf{X}^{op}$ defines an involutive covariant functor from the category of $U \times U$-varieties into itself. This functor reverses the convolution product, i.e., the $U \times U$-variety $(\mathbf{X} * \mathbf{Y})^{op}$ is naturally isomorphic to $\mathbf{Y}^{op} * \mathbf{X}^{op}$.*

1.3. **Standard $U \times U$-orbits and their convolution products.** We will consider here the $U \times U$-orbits in G of the form $X = U\overline{w}U$, where $w \in W$, and $\overline{w}$ is the standard representative of w in $\mathrm{Norm}_G(T)$ as defined in (1.5). We will refer to such a $U \times U$-orbit as *standard*.

In the this section we investigate convolution products of standard $U \times U$-varieties. The following facts are well-known.

Claim 1.12.

(a) *For any $w, w' \in W$, there is a unique element $w'' = w \star w' \in W$ such that $Bw''B$ is a dense open subset of the product $BwB \cdot Bw'B = BwBw'B$.*

(b) *The map $(w, w') \to w \star w' \in W$ defines a monoid structure on W such that and $s_i \star s_i = s_i$ for all $i \in I$ and*

$$w \star w' = ww'$$

whenever $l(w'w') = l(w) + l(w')$.

(c) *For any w, w', one has*

$$(w \star w')^{-1} = w'^{-1} \star w^{-1} \ .$$

(d) *For any $w, w' \in W$, there exists $w'' \in W$ such that $l(ww'') = l(w) + l(w'')$ and*

$$w \star w' = ww'' \ .$$

The following result describes the convolution product of standard $U \times U$-orbits.

Proposition 1.13. *For each $w, w' \in W$, all $U \times U$-orbits in the convolution product $(U\overline{w}U) * (U\overline{w'}U)$ are isomorphic to $U\overline{w \star w'}U$. In particular, if $w \star w' = ww'$ (i.e., if $l(ww') = l(w) + l(w')$), then*

$$(U\overline{w}U) * (U\overline{w'}U) \cong U\overline{ww'}U \ .$$

Proof. We will prove both parts by induction on $l(w)$. If $l(w) = 0$, i.e., w is the identity element of W, we have nothing to prove.

Now let $w = s_i$ for some $i \in I$. We have to show that each $U \times U$-orbit in $(U\overline{s_i}U) * (U\overline{w'}U)$ is isomorphic to $U\overline{s_i \star w'}U$ for any $w' \in W$. In the proof we will use the obvious equality $U\overline{s_i}U = U\overline{s}_iU_i$

We first consider the case when $l(s_iw') > l(w')$, i.e., $s_iw' = s_i \star w'$ (we will implicitly use here the obvious fact that $(U\overline{s_i}U)(U\overline{s_iw'}U) = U\overline{s_iw'}U$ in G). Since $U_i\overline{w'}U = \overline{w'}U$, we obtain

$$(U\overline{s_i}U) * (U\overline{w'}U) = (U\overline{s_i}U_i) * (\overline{w'}U) = (U\overline{s_i}) * (U_i\overline{w'}U) = (U\overline{s_i}) * (\overline{w'}U) = U\overline{s_i} * \overline{w'}U \ .$$

Then, taking into account that $U_{\overline{s_i}\bullet} = U(s_i) = U \cap \overline{s_i}U\overline{s_i}^{-1}$, in the notation of Example 1.7 and Claim 1.8, we have

$$U_{\overline{s_i} * \overline{w'}\bullet} = \overline{s}_i(U(s_i) \cap U(w'))\overline{s}_i^{-1} = U(s_i) \cap U(s_iw') = U(s_iw') = U_{\overline{s_iw'}\bullet} \ .$$

Therefore, $(U\overline{s_i}U) * (U\overline{w'}U) = U\overline{s_i} * \overline{w'}U$ is isomorphic to $U\overline{s_iw'}U = U\overline{s_i \star w'}U$ as a $U \times U$-orbit.

Second, consider the case when $l(s_i w') < l(w')$, that is, $w' = s_i \star w'$. Note that $U\overline{s_i}U_i = U\overline{s_i}U_i = U \cdot U_i^- \cdot \overline{s_i}$. Then we obtain the following decomposition into $U \times U$-orbits:

$$(U\overline{s_i}U) * (U\overline{w'}U) = \bigsqcup_{x \in U_i^- \cdot \overline{s_i}} Ux * \overline{w'}U \ .$$

Since $U(s_i)$ is the unipotent radical of that i-th minimal parabolic $P_i \supset B$ which has the Levi factor $L_i = T \cdot \phi_i(SL_2)$ and $U_i^- \cdot \overline{s_i} \subset L_i$, we obtain for each $x \in U_i^- \cdot \overline{s_i}$:

$$U(s_i) \cdot x = x \cdot U(s_i) \ .$$

This implies that for any $x \in U_i^- \cdot \overline{s_i}$ we have $U_{x\bullet} = U(s_i) = U \cap \overline{s_i}U\overline{s_i}^{-1}$.

Therefore, we obtain (again in the notation of Example 1.7 and Claim 1.8):

$$U_{x*\overline{w'}\bullet} = \overline{s}_i(U(s_i) \cap U(w'))\overline{s}_i^{-1} = U(s_i) \cap U(s_i w') = U(w') = U_{\overline{w'}\bullet} \ ,$$

that is, each orbit $Ux * \overline{w'}U$ of $(U\overline{s_i}U) * (U\overline{w'}U)$ is isomorphic to $U\overline{w'}U$.

This proves the assertion for $w = s_i$.

Furthermore, let $l(w) > 1$. Let us consider the convolution product

$$Z = (U\overline{w}U) * (U\overline{w'}U) \ .$$

Since $l(w) > 1$, there exists $i \in I$ such that $l(s_i w) < l(w)$. Based on the above, we have an isomorphism of $U \times U$-varieties $U\overline{w}U \cong (U\overline{s_i}U) * (U\overline{s_i w}U)$, which, in turn, using Claim 1.4, implies the isomorphism:

$$Z \cong ((U\overline{s_i}U) * (U\overline{s_i w}U)) * (U\overline{w'}U) = (U\overline{s_i}U) * ((U\overline{s_i w}U) * (U\overline{w'}U)) \ .$$

On the other hand, by the inductive hypothesis, each orbit $\mathcal{O}$ of the $U \times U$-variety $(U\overline{s_i w}U) * (U\overline{w'}U)$ is isomorphic to $U\overline{(s_i w) \star w'}U$. Therefore, each $U \times U$-orbit in Z is isomorphic to an orbit in

$$(U\overline{s_i}U) * (U\overline{(s_i w) \star w'}U) \ ,$$

and, by the already proved assertion, each $U \times U$-orbit in the latter $U \times U$-variety is isomorphic to

$$U\overline{s_i \star ((s_i w) \star w')}U = U\overline{(s_i \star (s_i w)) \star w'}U = U\overline{w \star w'}U \ .$$

This finishes the proof of the proposition. □

Remark 1.14. Unlike for the product of Bruhat cells BwB and $Bw'B$ in G (as in Claim 1.12), Proposition 1.13 guarantees that the convolution product $(BwB) * (Bw'B)$ contains only isomorphic $U \times U$-orbits.

Corollary 1.15. *For any sequence* $\mathbf{i} = (i_1, \dots, i_\ell) \in I^\ell$, *each orbit in the* ℓ*-fold convolution product* $(U\overline{s_{i_1}}U) * \cdots * (U\overline{s_{i_\ell}}U)$ *is isomorphic to* $U\overline{s_{i_1} \star \cdots \star s_{i_\ell}}U$. *In particular, if* $\mathbf{i}$ *is a reduced decomposition of* $w \in W$, *then*

$$(U\overline{s_{i_1}}U) * \cdots * (U\overline{s_{i_\ell}}U) \cong U\overline{w}U \ .$$

1.4. **Linear functions on $U \times U$-varieties.** Let $\mathbf{X}$ be a $U \times U$-variety and let $\chi : U \to \mathbb{A}^1$ be a character. We say that a function $f : X \to \mathbb{A}^1$ is $(U \times U, \chi)$-*linear* if

$$f(u \cdot x \cdot u') = \chi(u) + f(x) + \chi(u') \tag{1.10}$$

for any $x \in X, u, u' \in U$.

Denote by $\chi^{st} : U \to \mathbb{A}^1$ the *standard* regular character

$$\chi^{st} := \sum_{i \in I} \chi_i \ . \tag{1.11}$$

Lemma 1.16. *A $(U \times U, \chi^{st})$-linear function on $U \times U$-variety $\mathbf{X}$ is also $(U \times U, \chi^{st})$-linear on the $U \times U$-variety $\mathbf{X}^{op}$ (see Claim 1.11).*

Proof. Indeed, under the twisted action of $U \times U$ on X, we obtain for $u, u' \in U$, $x \in X$:

$$f(\iota(u') \cdot x \cdot \iota(u)) = \chi(\iota(u')) + f(x) + \chi(\iota(u)) = \chi(u') + f(x) + \chi(u)$$

because $\chi_i(\iota(u)) = \chi_i(u)$ for all $i \in I$ by by Claim 1.1. The lemma is proved. □

Claim 1.17. *Let $\mathbf{X}$ be a $U \times U$-variety, $x \in X$ be a point, and $\chi : U \to \mathbb{A}^1$ be a non-zero character of U. Then the orbit UxU admits a $(U \times U, \chi)$-linear function if and only if*

$$\chi(u) = \chi(\varphi_x(u)) \tag{1.12}$$

for each $u \in U_{\bullet x}$, where $\varphi_x : U_{\bullet x} \to U_{x\bullet}$ is as in Claim 1.6.

Clearly, any $(U \times U, \chi)$-linear function on a single orbit UxU is determined by its value at x.

Corollary 1.18. *Let $X = U\tilde{w}U$ be a $U \times U$-orbit in G, where $\tilde{w} \in \mathrm{Norm}(T)$ is a representative of $w \in W$. Let $\chi \neq 0$ be a character of U. Then X admits a $(U \times U, \chi)$-linear function $f : X \to \mathbb{A}^1$ if and only if*

$$\chi(u) = \chi(\tilde{w}^{-1} u \tilde{w}) \tag{1.13}$$

for any $u \in U(w) = \tilde{w}U\tilde{w}^{-1} \cap U$ and f is determined by $f(\tilde{w})$.

We say that a character $\chi : U \to \mathbb{A}^1$ is *regular* if $\chi(U_i) \neq 0$ for each $i \in I$.

The following result gives a surprisingly simple classification of standard $U \times U$-orbits that admit a $(U \times U, \chi)$-linear function.

Proposition 1.19. *Let χ be a regular character of U and let $\widetilde{w} \in \mathrm{Norm}_G(T)$ be a representative of $w \in W$. Assume that the $U \times U$-orbit $U\widetilde{w}U$ admits a $(U \times U, \chi)$-linear function. Then $w = w_P$ for some standard parabolic subgroup P of G (see (1.3)).*

Proof. Let $R^+ \subset X^\star(T)$ be the set of all positive roots of G. For each $\alpha \in R^+$, let U_α be the corresponding 1-dimensional subgroup of U. Note that $\tilde{w}U_\alpha\tilde{w}^{-1} = U_{w\alpha}$ for any $w \in W$; hence $U_\alpha \subset U(w) = U \cap \tilde{w}U\tilde{w}^{-1}$ if and only if $\alpha \in R^+ \cap w(R^+)$.

Therefore, if a regular character $\chi : U \to \mathbb{A}^1$ and $w \in W$ satisfy (1.13), then for each $i \in I$ one has: either $w^{-1}(\alpha_i) \notin R^+$ or $w^{-1}(\alpha_i) = \alpha_j$ for some $j \in I$.

Equivalently, if we denote $\sigma = w^{-1}w_0$, the latter condition reads: for each $i \in I$ either $\sigma(\alpha_i) \in R^+$ or $\sigma(\alpha_i) = -\alpha_j$ for some $j \in I$. Let us show that the latter condition holds if and only if σ is equal to the longest element w_0^P of some standard parabolic subgroup P of G. Indeed, denote by $\Pi = \{\alpha_i | i \in I\}$ the set of simple roots of G and define

$$\Pi_1 := \Pi \cap \sigma^{-1}(-\Pi),\ \Pi_2 := \Pi \cap \sigma(-\Pi)\ .$$

Obviously, $\sigma(\Pi_1) = -\Pi_2$, and $\sigma(\Pi \setminus \Pi_1) \subset R^+$.

Let $P = P(\Pi_1)$ be the corresponding standard parabolic and $\tau := \sigma w_0^P$. All we have to prove is that $\tau = e$. In order to do so it suffices to show that $\tau(\Pi) \subset R^+$. By definition of τ,

$$\tau(\Pi_1) = \sigma w_0^P(\Pi_1) = \sigma(-\Pi_1) \subset R^+\ .$$

So, it suffices to show that $\tau(\Pi \setminus \Pi_1) \subset R^+$ as well.

Assume, by contradiction, that this is false and there exists $\alpha_j \in \Pi \setminus \Pi_1$ such that $\tau(\alpha_j) \in -R^+$. First of all, since $\alpha_j \notin \Pi_1$, we obtain $w_0^P(\alpha_j) = \alpha_j + \beta$ where β is a non-negative combination of Π_1. This implies that

$$\tau(\alpha_j) = \sigma w_0^P(\alpha_j) = \sigma(\alpha_j + \beta) = \sigma(\alpha_j) + \sigma(\beta) = \sigma(\alpha_j) - \beta'\ ,$$

where β' is a non-negative combination of Π_2.

Therefore, $\sigma(\alpha_j) = \tau(\alpha_j) + \beta'$. Since $\tau(\alpha_j) \in -R^+$ and $\sigma(\alpha_j) \in R^+$, this immediately implies that both $-\tau(\alpha_j)$ and $\sigma(\alpha_j)$ are positive combinations of Π_2. In turn, this implies that α_j is a positive combination of $\sigma^{-1}(\Pi_2) = -\Pi_1$, i.e., α_j is a negative combination of Π_1. This contradiction proves the proposition. □

Now let us construct such a $(U \times U, \chi^{st})$-linear function f_P (where χ^{st} is defined in (1.11) on a $U \times U$-sub-variety of Bw_PB.

Claim 1.20. *Let $w \in W$ and let $i, j \in I$ be such that $w(\alpha_i) = \alpha_j$. Then*

$$\overline{w}^{-1} x_i(a) \overline{w} = x_j(a)$$

for all $a \in \mathbb{G}_a$, and, therefore,

$$\chi^{st}(\overline{w}^{-1} u \overline{w}) = \chi^{st}(u)$$

for any $u \in U_i$. In particular, for each standard parabolic P, one has

$$\chi^{st}(\overline{w_P}^{-1} u \overline{w_P}) = \chi^{st}(u)$$

for any $u \in L_P \cap U$, where L_P is the Levi factor of P.

Let P be a standard parabolic subgroup of G with the Levi factor L. Denote by $Z(L_P) \subset T$ the center of L_P. Proposition 1.19 and Claim 1.20 imply the following corollary.

Corollary 1.21. *A $U \times U$-orbit $U\tilde{w}U$ admits a χ^{st}-linear function, if and only if $\tilde{w} \in Z(L_P) \cdot \overline{w_P}$ for some standard parabolic subgroup $P \subset G$.*

Define a function $f_P : UZ(L_P)\overline{w_P}U \to \mathbb{A}^1$ by

$$f_P(ut\overline{w_P}u') = \chi^{st}(u) + \chi^{st}(u') \tag{1.14}$$

for $u, u' \in U$, $t \in Z(L_P)$.

In particular, taking $P = B$ (and $L_P = T$, $Z(L_P) = T$), we obtain a function $f_B : UT\overline{w_0}U \to \mathbb{A}^1$ given by

$$f_B(ut\overline{w_0}u') = \chi^{st}(u) + \chi^{st}(u') \tag{1.15}$$

for $u, u' \in U$, $t \in T$. The latter function, unlike the former, can be extended to any character $\chi : U \to \mathbb{A}^1$ via $f_{B,\chi} \to \mathbb{A}^1$:

$$f_{B,\chi}(ut\overline{w_0}u') = \chi(u) + \chi(u') \tag{1.16}$$

Using Claims 1.17 and 1.20, we obtain the following result.

Claim 1.22. *For any standard parabolic P, the function f_P is a $(U \times U, \chi^{st})$-linear function on $UZ(L_P)\overline{w_P}U$.*

Remark 1.23. Since each regular character $\chi : U \to \mathbb{A}^1$ belongs to the T-orbit of χ^{st} under the adjoint action, i.e., $\chi(u) = \chi^{st}(t_0 u t_0^{-1})$ for $t_0 \in T$, one can obtain the $(U \times U, \chi)$-linear function $f_{P,\chi}$ on $t_0 UZ(L_P)\overline{w_P}Ut_0^{-1} = UZ(L_P)t_0\overline{w_P}t_0^{-1}U$ by twisting (1.14) with this $t_0 \in T$.

Define a projection $\pi^+ : B^- \cdot U \to U$ by

$$\pi^+(bu) = u \tag{1.17}$$

for $b \in B^-$, $u \in U$.

The following result provides a very useful formula for computing f_P.

Lemma 1.24. *For any $g \in UZ(L_P)\overline{w_P}U$, one has*

$$f_P(g) = \chi^{st}(\pi^+(\overline{w_P}^{-1}g)) + \sum_{i \in I \setminus J(P)} \chi_i(\pi^+(\overline{w_P^{-1}}^{-1}\iota(g))) \ , \tag{1.18}$$

where $\iota : G \to G$ is defined in (1.1) *(and $J(P)$ is the set of all those $i \in I$ for which $U_i^- \in P$). In particular,*

$$f_B(g) = \chi^{st}(\pi^+(\overline{w_0}^{-1}g)) + \chi^{st}(\pi^+(\overline{w_0}^{-1}\iota(g)))$$

for $g \in Bw_0B$.

Proof. We proceed similarly to the proof of [2, Proposition 5.12]. Indeed, according to Example 1.7 for $\tilde{w} = t\overline{w}_P$, $t \in Z(L)$, one has a unique factorization $g = u_P t\overline{w_P}u$, where $t \in Z(L)$, $u_P \in U_P$, and $u \in U$. Clearly, $\pi^+(\overline{w_P}^{-1}g) = u$. Furthermore, let us write a unique factorization $u = u_L u'$, where $u_L \in U(w_P^{-1}) = U \cap w_0 L w_0$ and $u' \in V(w_P^{-1})$. Let $u_L := \overline{w_P}u'\overline{w_P}^{-1}$. Clearly, $u'_P \in U \cap L$, therefore $tu'_p = u'_P t$ and $g = u_P u_L t\overline{w_P}u'$. This implies that $\iota(g) = \iota(u')\iota(t\overline{w_P})\iota(u_P u_L) = \iota(u')\overline{w_P^{-1}}^{-1}t^{-1}\iota(u_P u_L)$. Therefore, $\pi^+(\overline{w_P^{-1}}^{-1}\iota(g)) = \iota(u_L)\iota(u_P)$. Note that $\chi_i(\iota(u_P u_L)) = \chi_i(u_P u_L)$ for all $i \in I$ by Claim 1.1 and $\chi_i(u_P u_L) = \chi_i(u_P)$ for all $i \in I \setminus J(P)$. Putting it together, the right hand side of (1.18) equals

$$\chi^{st}(u) + \sum_{i \in I \setminus J(P)} \chi_i(u_P) = f_P(g) \ .$$

The lemma is proved. □

Taking into account that for a simply-connected group G one has

$$\chi_i(\pi^+(g)) = \frac{\Delta_{\omega_i, s_i\omega_i}(g)}{\Delta_{\omega_i,\omega_i}(g)} ,$$

we obtain the following result.

Corollary 1.25. *For any $g \in UZ(L_P)\overline{w_P}U$, one has*

$$f_P(g) = \sum_{i\in I} \frac{\Delta_{w_P\omega_i, s_i\omega_i}(g)}{\Delta_{w_P\omega_i,\omega_i}(g)} + \sum_{i\in I\setminus J(P^{op})} \frac{\Delta_{w_0 s_i\omega_i,\omega_i}(g)}{\Delta_{w_0\omega_i,\omega_i}(g)} , \tag{1.19}$$

where P^{op} is the standard parabolic opposite of P (as in (1.9)). In particular,

$$f_B(g) = \sum_{i\in I} \frac{\Delta_{w_0\omega_i, s_i\omega_i}(g) + \Delta_{w_0 s_i\omega_i,\omega_i}(g)}{\Delta_{w_0\omega_i,\omega_i}(g)}$$

for $g \in Bw_0B$.

We finish the section with a technical result which will be used in Section 2.

For each $w \in W$, let π^w and ${}^w\pi$ be regular morphisms $BwB \to T$ given by

$$\pi^w(ut\overline{w}u') = t, {}^w\pi(u\overline{w}tu') = t \tag{1.20}$$

for $u, u' \in U$, $w \in W$, and $t \in T$.

Clearly, both π^w and ${}^w\pi$ are $U \times U$-invariant, and

$$w({}^w\pi(g)) = \pi^w(g)$$

for all $g \in BwB$, $w \in W$.

Lemma 1.26. *For any $w \in W$, we have*

$$\pi^w(\iota(g)^{-1}) = t_w\pi^w(g)$$

for any $g \in BwB$, where $t_w = \overline{w^{-1}}^{-1}\overline{w}^{-1} \in T$.

Proof. Indeed, let $g = ut\overline{w}u'$ for some $u, u' \in U$, $t \in T$. By definition, $\pi^w(g) = t$. Then

$$\iota(g)^{-1} = u_1 t\overline{w^{-1}}^{-1}u_1' ,$$

where $u_1 = \iota(u)^{-1} \in U, u_1 = \iota(u)^{-1} \in U$. Therefore, $\iota(g)^{-1} = u_1 t \cdot t_w\overline{w}u_1'$ and $\pi^w(\iota(g)^{-1}) = t \cdot t_w = t_w \cdot t$. □

2. Unipotent bicrystals and geometric crystals

2.1. **Unipotent bicrystals.** A *unipotent bicrystal*, or U-bicrystal is a pair $(\mathbf{X}, \mathbf{p})$, where $\mathbf{X}$ is a $U \times U$-variety, and $\mathbf{p} : X \to G$ is a $U \times U$-equivariant morphism, where the action $U \times G \times U \to G$ is given by $(u, g, u') \mapsto ugu'$.

A morphism of U-bicrystals is any structure-preserving morphism of underlying $U \times U$-varieties.

For U-bicrystals $(\mathbf{X}, \mathbf{p})$, $(\mathbf{Y}, \mathbf{p}')$, let us define the convolution product by

$$(\mathbf{X}, \mathbf{p}) * (\mathbf{Y}, \mathbf{p}') = (\mathbf{X} * \mathbf{Y}, \mathbf{p}'') ,$$

where $\mathbf{X} * \mathbf{Y}$ is the convolution product of $U \times U$-varieties as in Section 1.2 and $\mathbf{p}'' : X * Y \to G$ is defined by $\mathbf{p}''(x * y) = \mathbf{p}(x)\mathbf{p}'(y)$ (clearly, the morphism $\mathbf{p}''$ is well-defined).

Claim 2.1. *The category of U-bicrystals with the product $*$ and structure-preserving morphisms as arrows is naturally monoidal (where U is the unit object).*

For each unipotent bicrystal $(\mathbf{X}, \mathbf{p},)$, let us define the sub-variety $X^- \subset X$ by

$$X^- = \mathbf{p}^{-1}(B^-) \tag{2.1}$$

(i.e., $X^- = \mathbf{p}^{-1}(\mathbf{p}(X) \cap B^-)$).

We refer to the variety X^- as the *unipotent crystal* associated to $(\mathbf{X}, \mathbf{p})$. The variety X^- is never empty because any $U \times U$-orbit in G has a non-trivial intersection with B^-.

Remark 2.2. The above definition of a unipotent crystal is equivalent to the original definition in [2]: a rational U-action on X is given below by (2.2) and $\mathbf{f} : X^- \to B^-$ is the restriction of $\mathbf{p}$ to X^-. And this $\mathbf{f}$ is U-equivariant.

Next, we will list some obvious facts about unipotent crystals.

Claim 2.3. *For any unipotent bicrystal $(\mathbf{X}, \mathbf{p})$, the restriction to X^- of the quotient map $X \to X/U$ is an open inclusion (and, therefore, a birational isomorphism) $j_X : X^- \hookrightarrow X/U$. In particular, X^- possesses a rational U-action via*

$$u(x) := u \cdot x \cdot \pi^+(\mathbf{p}(u \cdot x))^{-1} \tag{2.2}$$

for $x \in X, u \in U$, where $\pi^+ : B^- \cdot U \to U$ is the projection to the second factor (defined in (1.17)*).*

For any unipotent bicrystals $(\mathbf{X}, \mathbf{p})$, $(\mathbf{Y}, \mathbf{p}')$ and any $x^- \in X^-$, $y^- \in Y^-$, we have $\mathbf{p}(x^-)\mathbf{p}'(y^-) \in B^-$. Therefore, the correspondence

$$(x, y) \mapsto j_{X,Y}(x^-, y^-) := x^- * y^- \tag{2.3}$$

for $x^- \in X^-, y^- \in Y^-$ is a well-defined morphism $j_{X,Y} : X^- \times Y^- \to (X * Y)^-$.

Lemma 2.4. *For any unipotent bicrystals $(\mathbf{X}, \mathbf{p})$ and $(\mathbf{Y}, \mathbf{p}')$, the morphism $j_{X,Y}$ is an open inclusion (hence a birational isomorphism) $X^- \times Y^- \xrightarrow{\sim} (X * Y)^-$.*

Proof. First, we need the following obvious fact.

Claim 2.5. *For any unipotent bicrystals $(\mathbf{X}, \mathbf{p})$ and $(\mathbf{Y}, \mathbf{p}')$, the restriction of the surjective morphism from Claim 1.9 to $X^- * Y^- \subset X^- * Y/U \subset X * Y/U$ defines a biregular isomorphism*

$$X^- * Y^- \xrightarrow{\sim} X^- \times Y^- \ ,$$

where we identified X^- (resp. Y^-) with its image in X/U (resp. in Y/U).

This guarantees that the morphism (2.3) is injective. To show that the image is dense it suffices to use the irreducibility of $(X * Y)^-$ and to count the dimension:

$$\dim(X * Y)^- = \dim X * Y - \dim U = \dim X + \dim Y - 2 \dim U = \dim X^- + \dim Y^- \ .$$

The lemma is proved. □

Claim 2.6. *For any unipotent bicrystal $(\mathbf{X}, \mathbf{p})$, there exists a unique element $w \in W$ such that the intersection $\mathbf{p}(X) \cap BwB$ is dense in $\mathbf{p}(X)$.*

We will refer to w as the *type* of $(\mathbf{X}, \mathbf{p})$.

Example 2.7. For each $w \in W$, let $X_w = U\overline{w}U$. Clearly, $X_w^- = B_w^-$ where we denote, following [2],

$$B_w^- = B^- \cap U\overline{w}U \ . \tag{2.4}$$

The pair $(\mathbf{X}_w, \mathrm{id})$, where id stands for the natural inclusion $\mathrm{id} : X_w \hookrightarrow G$ is a U-bicrystal. Also the pair (BwB, id) is a U-bicrystal such that $(BwB)^- = TB_w^-$. Each of these U-bicrystals has type w.

Claim 2.8. *Let $(\mathbf{X}, \mathbf{p})$ and $(\mathbf{Y}, \mathbf{p}')$ be unipotent bicrystals of types w and w', respectively. Then the product $(\mathbf{X}, \mathbf{p}) * (\mathbf{Y}, \mathbf{p}')$ is of type $w \star w'$ (see Claim 1.12).*

For any unipotent bicrystal $(\mathbf{X}, \mathbf{p})$ of type w, define rational morphisms $hw_X, lw_X : X \to T$ by the formula

$$hw_X(x) = \pi^w(\mathbf{p}(x)),\ lw_X(x) = {}^w\pi(\mathbf{p}(x)) \ . \tag{2.5}$$

for $x \in X$, where π^w and ${}^w\pi$ are defined in (1.20).

We will refer to these morphisms as the *highest weight* of $(\mathbf{X}, \mathbf{p})$ and the *lowest weight* of $(\mathbf{X}, \mathbf{p})$, respectively.

Claim 2.9. *For any unipotent bicrystal $(\mathbf{X}, \mathbf{p})$, both the highest weight and the lowest weights $hw_X, lw_X : X \to T$ are $U \times U$-invariant rational morphisms.*

In the notation of Claim 1.11, for each U-bicrystal $(\mathbf{X}, \mathbf{p})$ denote $(\mathbf{X}, \mathbf{p})^{op} := (\mathbf{X}^{op}, \mathbf{p} \circ \iota)$. It follows immediately from Claim 1.11 that $(\mathbf{X}, \mathbf{p})^{op}$ is also a U-bicrystal. We will refer to it as the *opposite* U-bicrystal of $(\mathbf{X}, \mathbf{p})$. Clearly, if $(\mathbf{X}, \mathbf{p})$ is of type w, then $(\mathbf{X}, \mathbf{p})^{op}$ is of type w^{-1}.

Claim 2.10. *The correspondence $(\mathbf{X}, \mathbf{p}) \mapsto (\mathbf{X}, \mathbf{p})^{op}$ defines an involutive covariant functor from the category of U-bicrystals into itself.*

2.2. **From unipotent bicrystals to geometric crystals.** We start with the recollection of some definitions and results of [2] on *geometric crystals* (in a slightly modified form).

Definition 2.11. Given varieties X and Y and a rational morphism $f : X \to Y$, denote by $\mathrm{dom}(f) \subset X$ the maximal open subset of X on which f is defined; denote by $f_{\mathrm{reg}} : \mathrm{dom}(f) \to Y$ the corresponding regular morphism. We denote by $\mathrm{ran}(f) \subset Y$ the closure of the constructible set $f_{\mathrm{reg}}(\mathrm{dom}(f))$ in Y.

It is easy to see that for any irreducible algebraic varieties X, Y, Z and rational morphisms $f : X \to Y,\ g : Y \to Z$ such that $\mathrm{dom}(g)$ intersects $\mathrm{ran}(f)$ non-trivially, the composition $(f, g) \mapsto g \circ f$ is well-defined and is a rational morphism $X \to Z$.

For any algebraic group H, we call a rational action $\alpha : H \times X \to X$ *unital* if $\mathrm{dom}(\alpha) \supset \{e\} \times X$.

Definition 2.12. A *geometric pre-crystal* is a 5-tuple $\mathcal{X} = (X, \gamma, \varphi_i, \varepsilon_i, e_i^{\cdot} | i \in I)$, where X is an irreducible algebraic variety, γ is rational morphism $X \to T$, $\varphi_i, \varepsilon_i : X \to \mathbb{A}^1$ are rational functions, and each $e_i^{\cdot} : \mathbb{G}_m \times X \to X$ is a unital rational action of the multiplicative group $\mathbb{G}_m$ (to be denoted by $(c, x) \mapsto e_i^c(x)$) such that

for each $i \in I$, one has either: $\varphi_i = \varepsilon_i = 0$ and the action $e_i^{\cdot}$ is trivial, or: $\varphi_i \neq 0$, $\varepsilon_i \neq 0$, and

$$\gamma(e_i^c(x)) = \alpha_i^\vee(c)\gamma(x), \varepsilon_i(x) = \alpha_i(\gamma(x))\varphi_i(x),$$
$$\varepsilon_i(e_i^c(x)) = c\varepsilon_i(x), \varphi_i(e_i^c(x)) = c^{-1}\varphi_i(x)$$

for $x \in X$, $c \in \mathbb{G}_m$.

We will refer to the set of all $i \in I$ such that $\varphi_i \neq 0$ as the *support* of $\mathcal{X}$ and denote it by $\operatorname{Supp}\mathcal{X}$.

Example 2.13. We consider T as a *trivial* geometric pre-crystal with $\varphi_i = \varepsilon_i = 0$ and the action $e_i^{\cdot}$ is trivial for all $i \in I$. That is, the support of this geometric pre-crystal is the empty set $\emptyset$. Another example of a trivial geometric pre-crystal is any sub-variety of T, in particular, a single point $e \in T$. Yet another example of a geometric pre-crystal on T is a 5-tuple $\mathcal{T} = (T, id_T, \varphi_i, \varepsilon_i, e_i^{\cdot}|i \in I)$, where $e_i^c(t) = \alpha_i^\vee(c) \cdot t$ for all $c \in \mathbb{G}_m$, $t \in T$, $i \in I$ and $\varepsilon_i, \varphi_i \in X^\star(T)$ are such that $\langle \varepsilon_i, \alpha_i^\vee \rangle = 1$, $\varphi_i = \varepsilon_i - \alpha_i^\vee$ for all $i \in I$ (e.g., $\varepsilon_i = \rho$, $\varphi_i = \rho - \alpha_i$ for all $i \in I$).

Definition 2.14. A *morphism* of geometric pre-crystals $\mathbf{f} : \mathcal{X} \to \mathcal{X}'$ is a pair (f, J), where f is a rational morphism of underlying varieties $X \to X'$ and $J \subset \operatorname{Supp}\mathcal{X} \cap \operatorname{Supp}\mathcal{X}'$ such that $e_i^c \circ f = f \circ e_i^c$ for all $i \in \operatorname{Supp}\mathcal{X}$, $c \in \mathbb{G}_m$ and:

$$\varepsilon_j \circ f = \varepsilon'_j, \ \varphi_j \circ f = \varphi'_j$$

for all $j \in J$ (we will refer to this J as the *support* of f and denote it by $\operatorname{Supp}\mathbf{f}$).

Clearly, there is a category whose objects are geometric pre-crystals and arrows are *dominant* morphisms of geometric pre-crystals, where the composition of morphisms $\mathbf{f} = (f, J)$ and $\mathbf{f}' = (f', J')$ is defined by $\mathbf{f}' \circ \mathbf{f} := (f' \circ f, J' \cap J)$ (i.e., $\operatorname{Supp}\mathbf{f}' \circ \mathbf{f} = \operatorname{Supp}\mathbf{f}' \cap \operatorname{Supp}\mathbf{f}$).

Now we define the product of geometric pre-crystals.

Definition 2.15. Given geometric pre-crystals $\mathcal{X} = (X, \gamma, \varphi_i, \varepsilon_i, e_i^{\cdot}|i \in I)$ and $\mathcal{Y} = (Y, \gamma', \varphi'_i, \varepsilon'_i, e_i^{\cdot}|i \in I)$, define the 5-tuple

$$\mathcal{X} \times \mathcal{Y} = (X \times Y, \gamma'', e_i^{\cdot}, \varphi''_i, \varepsilon''_i|i \in I) \ ,$$

where

• the morphism $\gamma'' : X \times Y \to T$ is given by

$$\gamma''(x, y) = \gamma(x)\gamma'(y) \tag{2.6}$$

for $x, y \in X \times Y$;

• the functions $\varphi''_i, \varepsilon''_i : X \times Y \to \mathbb{A}^1$ is given by

$$\varphi''_i(x, y) = \varphi_i(x) + \frac{\varphi'_i(y)}{\alpha_i(\gamma(x))}, \ \varepsilon''_i(x, y) = \varepsilon'_i(y) + \varepsilon_i(x) \cdot \alpha_i(\gamma'(y)) \tag{2.7}$$

for $x, y \in X \times Y$;

• the rational morphisms $e_i^{\cdot} : \mathbb{G}_m \times X \times Y \to X \times Y$ are given by the formula (for $i \in \operatorname{Supp}\mathcal{X} \cup \operatorname{Supp}\mathcal{Y}$)

$$e_i^c(x, y) = (e_i^{c_1}(x), e_i^{c_2}(y)) \ , \tag{2.8}$$

for $x, y \in X \times Y$, where

$$c_1 = \frac{c\varepsilon_i(x) + \varphi_i'(y)}{\varepsilon_i(x) + \varphi_i'(y)}, \ c_2 = \frac{\varepsilon_i(x) + \varphi_i'(y)}{\varepsilon_i(x) + c^{-1}\varphi_i'(y)} \ . \tag{2.9}$$

Claim 2.16. *For any geometric pre-crystals $\mathcal{X}$ and $\mathcal{Y}$, the 5-tuple $\mathcal{X} \times \mathcal{Y}$ is also a geometric pre-crystal (with $\operatorname{Supp} \mathcal{X} \times \mathcal{Y} = \operatorname{Supp} \mathcal{X} \cup \operatorname{Supp} \mathcal{Y}$). Moreover, the category of geometric pre-crystals is monoidal under the product $(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \times \mathcal{Y}$ (where the unit object is the single point $\{e\}$ as in Example 2.13).*

Remark 2.17. The formula (2.8) for the action of $e_i^{\cdot}$ on $X \times Y$ provides a geometric analogue of the tensor product of Kashiwara's crystals. See also Remark 5.7 below.

Example 2.18. For each geometric pre-crystal, $\mathcal{X}$ the product $T \times \mathcal{X}$ (where T is considered the trivial geometric pre-crystal as in Example 2.13) is a geometric pre-crystal with $\operatorname{Supp} T \times \mathcal{X} = \operatorname{Supp} \mathcal{X}$. The projection to the first factor is a morphism (with the empty support) of geometric pre-crystals $T \times \mathcal{X} \to T$.

Given a geometric pre-crystal $\mathcal{X} = (X, \gamma, \varphi_i, \varepsilon_i, e_i^{\cdot} | i \in I)$ denote by $\mathcal{X}^{op}$ the 5-tuple $(X, \gamma^{op}, \varphi_i^{op}, \varepsilon_i^{op}, (e_i^{\cdot})^{op} | i \in I)$, where $\gamma^{op}(x) = \gamma(x)^{-1}$, $\varphi_i^{op} = \varepsilon_i$, $\varepsilon_i^{op} = \varphi_i$, and $(e_i^c)^{op} = e_i^{c^{-1}}$. Clearly, $\mathcal{X}^{op}$ is also a geometric pre-crystal which we will refer to as the *opposite* of $\mathcal{X}$.

Claim 2.19. *The correspondence $\mathcal{X} \to \mathcal{X}^{op}$ defines an involutive covariant functor from the category of geometric pre-crystals into itself. This functor reverses the product, i.e., $(\mathcal{X} \times \mathcal{Y})^{op}$ is naturally isomorphic to $\mathcal{Y}^{op} * \mathcal{X}^{op}$. On the underlying varieties this isomorphism is the permutation of factors $X \times Y \xrightarrow{\sim} Y \times X$.*

For a geometric pre-crystal $\mathcal{X}$ and a sequence $\mathbf{i} = (i_1, \dots, i_\ell) \in I^\ell$ we define a rational morphism $e_{\mathbf{i}}^{\cdot} : T \times X \to X$ by

$$(t, x) \mapsto e_{\mathbf{i}}^t(x) = e_{i_1}^{\alpha^{(1)}(t)} \circ \dots \circ e_{i_\ell}^{\alpha^{(\ell)}(t)}(x) \ , \tag{2.10}$$

where $\alpha^{(k)} = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_{k+1}}(\alpha_{i_k})$, $k = 1, \dots, \ell$ are the associated roots.

Definition 2.20. A geometric pre-crystal $(X, \gamma, \varphi_i, \varepsilon_i, e_i^{\cdot} | i \in I)$ is called a *geometric crystal* if for any sequence $\mathbf{i} = (i_1, \dots, i_\ell) \in I^\ell$ such that $s_{i_1} \cdots s_{i_\ell} = 1$ one has

$$e_{\mathbf{i}} = \mathrm{id}_X \ . \tag{2.11}$$

It is easy to see that the relations (2.11) are equivalent to the following relations between $e_i^{\cdot}, e_j^{\cdot}$ for $i, j \in J$:

$$e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} \text{ if } \langle \alpha_i, \alpha_j^\vee \rangle = 0;$$
$$e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} \text{ if } \langle \alpha_j, \alpha_i^\vee \rangle = \langle \alpha_i, \alpha_j^\vee \rangle = -1;$$
$$e_i^{c_1} e_j^{c_1^2 c_2} e_i^{c_1 c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^2 c_2} e_i^{c_1} \text{ if } \langle \alpha_j, \alpha_i^\vee \rangle = -2\langle \alpha_i, \alpha_j^\vee \rangle = -2;$$
$$e_i^{c_1} e_j^{c_1^3 c_2} e_i^{c_1^2 c_2} e_j^{c_1^3 c_2^2} e_i^{c_1 c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^3 c_2^2} e_i^{c_1^2 c_2} e_j^{c_1^3 c_2} e_i^{c_1} \text{if } \langle \alpha_j, \alpha_i^\vee \rangle = -3\langle \alpha_i, \alpha_j^\vee \rangle = -3.$$

Since the above relations are invariant under taking the inverse of both hand sides, the opposite $\mathcal{X}^{op}$ of a geometric crystal $\mathcal{X}$ is always a geometric crystal. Therefore, the correspondence $\mathcal{X} \mapsto \mathcal{X}^{op}$ is a covariant functor from the category of geometric crystals into itself.

Remark 2.21. Unlike geometric pre-crystals, the category of geometric crystals is not monoidal. This was a main reason for introducing unipotent crystals in [2] and unipotent bicrystals in the present work.

Next, we will construct geometric crystals out of unipotent bicrystals. For a unipotent bicrystal $(\mathbf{X}, \mathbf{p})$, define a 5-tuple

$$\mathcal{F}(\mathbf{X}, \mathbf{p}) = (X^-, \gamma, \varphi_i, \varepsilon_i, e_i^{\cdot} | i \in I) \tag{2.12}$$

as follows:
• The variety X^- is given by $X^- := \mathbf{p}^{-1}(B^-)$ as in (2.1).
• A morphism $\gamma : X^- \to T$ is the composition of $\mathbf{p} : X^- \to B^-$ with the canonical projection $B^- \to B^-/U^- = T$.
• Regular functions $\varphi_i, \varepsilon_i : X^- \to \mathbb{A}^1$, $i \in I$ are as follows. Let pr_i be the natural projection $B^- \to B^- \cap \phi_i(SL_2)$ (where ϕ_i is a homomorphism $SL_2 \to G$ defined in Section 1.1). Using the fact that $x \in X^-$ if and only if $\mathbf{p}(x) \in B^-$, we set:

$$\varphi_i(x) := \frac{b_{21}}{b_{11}}, \ \varepsilon_i(x) := \frac{b_{21}}{b_{22}} = \varphi_i(x)\alpha_i(x) \tag{2.13}$$

for all $x \in X^-$, where $pr_i(\mathbf{p}(x)) = \phi_i \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$.
• A rational morphism $e_i^{\cdot} : \mathbb{G}_m \times X^- \to X$, $i \in I$ is given by ($x \in X^-$, $c \in \mathbb{G}_m$):

$$e_i^c(x) = x_i \left(\frac{c-1}{\varphi_i(x)} \right) \cdot x \cdot x_i \left(\frac{c^{-1}-1}{\varepsilon_i(x)} \right) \tag{2.14}$$

if $\varphi_i \neq 0$ and $e_i^c(x) = x$ if $\varphi_i = 0$.

Remark 2.22. Even in the case when the homomorphism $\phi_i : SL_2 \to G$ is not injective (i.e., when the fiber $\phi_i^{-1}(b) \subset SL_2$ consists of two points which differ by a sign) the functions φ_i and ε_i are well-defined.

Remark 2.23. It is easy to see that for any unipotent bicrystal $(\mathbf{X}, \mathbf{p})$, we have: $\varphi_i \neq 0$ if and only if $i \in |w|$ (see (1.2)), where w is the type of $(\mathbf{X}, \mathbf{p})$.

Remark 2.24. For each $i \in I$ such that $\varphi_i \neq 0$, the action $e_i^{\cdot}$ is, indeed, rational because it is undefined at the zero locus of φ_i (or, which is the same, at the zero locus of ε_i).

Proposition 2.25.

(a) *For any unipotent bicrystal* $(\mathbf{X}, \mathbf{p})$, *the* 5*-tuple* $\mathcal{F}(\mathbf{X}, \mathbf{p})$ *given by* (2.12) *is a geometric crystal.*
(b) *For any unipotent bicrystals* $(\mathbf{X}, \mathbf{p})$, $(\mathbf{Y}, \mathbf{p}')$, *the formula* (2.3) *defines an isomorphism (of full support* I*) of geometric crystals*

$$\mathcal{F}(\mathbf{X}, \mathbf{p}) \times \mathcal{F}(\mathbf{Y}, \mathbf{p}') \xrightarrow{\sim} \mathcal{F}((\mathbf{X}, \mathbf{p}) * (\mathbf{Y}, \mathbf{p}')) \ .$$

(c) *For any unipotent bicrystal* $(\mathbf{X}, \mathbf{p})$, *one has*

$$\mathcal{F}((\mathbf{X}, \mathbf{p})^{op}) = (\mathcal{F}(\mathbf{X}, \mathbf{p}))^{op}.$$

Proof. Prove (a). First, let us show that each e_i^c takes X^- to X^-. This immediately follows from the following fact.

Lemma 2.26. *For any $i \in I$ such that $\varphi_i \neq 0$, the rational action* (2.2) *is given on generators of U by*

$$(x_i(a))(x) = x_i(a) \cdot x \cdot x_i(a') \tag{2.15}$$

for $x \in X^-$ and $a \in \mathbb{G}_a$, where $a' = -\frac{a}{\alpha_i(\gamma(x))(1+a\varphi_i(x))}$.

Proof. Without loss of generality, it suffices to take $X = GL_2$, therefore, $X^- = \left\{ \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} : b_{11}b_{22} \neq 0 \right\}$. We have

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & a' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_{11} + ab_{21} & a'(b_{11} + ab_{21}) + ab_{22} \\ b_{21} & b_{22} + a'b_{21} \end{pmatrix} .$$

Clearly, this product belongs to X^- if and only if $a'(b_{11} + ab_{21}) + ab_{22} = 0$. Since $\alpha_i(\gamma(x)) = \frac{b_{11}}{b_{22}}$ and $\varphi_i(x) = \frac{b_{21}}{b_{11}}$, we obtain the desired result. □

Furthermore, taking $a = \frac{c-1}{\varphi_i(x)}$ in (2.15), we obtain (2.14). In turn, each action $e_i^{\cdot} : X^- \to X^-$ given by (2.14) coincides with that from [2, Formula (3.7)]. Since X^- is a unipotent crystal, [2, Theorem 3.8] implies that (2.12) defines a geometric crystal. This proves part (a).

Part (b) now. For $x \in X^-$, $y \in Y^-$, we have $\gamma(x * y) = pr(\mathbf{p}(x)\mathbf{p}(y)) = pr(\mathbf{p}(x))pr(\mathbf{p}(y)) = \gamma(x)\gamma(y)$ because the natural projection $pr : B^- \to T$ is a group homomorphism. This proves (2.6). Prove (2.7) now. Indeed, $pr_i(\mathbf{p}(x)) = b = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$, $pr_i(\mathbf{p}(y)) = b' = \begin{pmatrix} b'_{11} & 0 \\ b'_{21} & b'_{22} \end{pmatrix}$ (where $pr_i : B^- \to B^- \cap \phi_i(SL_2)$ is defined above in (2.12)) and

$$\varphi_i(x) = \frac{b_{21}}{b_{11}}, \varepsilon_i(x) = \frac{b_{21}}{b_{22}}, \varphi_i(y) = \frac{b'_{21}}{b'_{11}}, \varepsilon_i(y) = \frac{b'_{21}}{b'_{22}}, \alpha_i(\gamma(x)) = \frac{b_{11}}{b_{22}}, \alpha_i(\gamma(y)) = \frac{b'_{11}}{b'_{22}} .$$

Therefore,

$$\varphi_i(x * y) = \frac{(bb')_{21}}{(bb')_{11}} = \frac{b_{21}b'_{11} + b_{22}b'_{21}}{b_{11}b'_{11}} = \varphi_i(x) + \frac{\varphi'_i(y)}{\alpha_i(\gamma(x))}$$

$$\varepsilon_i(x * y) = \frac{(bb')_{21}}{(bb')_{22}} = \frac{b_{21}b'_{11} + b_{22}b'_{21}}{b_{22}b'_{22}} = \varepsilon_i(x) \cdot \alpha_i(\gamma'(y)) + \varepsilon'_i(y) .$$

This proves (2.7). It remains to prove (2.8). Indeed, by definition (2.14),

$$e_i^c(x * y) = x_i(a) \cdot x * y \cdot x_i(a') ,$$

where $a = \frac{c-1}{\varphi_i(x*y)}$, $a' = \frac{c^{-1}-1}{\varepsilon_i(x*y)}$. If we take c_1, c_2 as in (2.9), we easily see that $a = \frac{c_1-1}{\varphi_i(x)}$, $b = \frac{c_2^{-1}-1}{\varepsilon_i(y)}$. This in conjunction with the identity $\frac{c_1^{-1}-1}{\varepsilon_i(x)} = -\frac{c_2-1}{\varphi_i(y)}$ implies that

$$e_i^c(x * y) = x_i(a) \cdot x * y \cdot x_i(a') = x_i(a) \cdot x \cdot x_i(a'') * x_i(-a'')y \cdot x_i(a') = e_i^{c_1}(x) * e_i^{c_2}(y) ,$$

where $a'' = \frac{c_1^{-1}-1}{\varepsilon_i(x)} = -\frac{c_2-1}{\varphi_i(y)}$. This proves (2.8). Part (b) is proved.

Part (c) follows from the fact that $\varphi_i(\iota(b)) = \varepsilon_i(b)$ and $\varepsilon_i(\iota(b)) = \varphi_i(b)$ for all $b \in B^-$, $i \in I$.

The proposition is proved. □

Remark 2.27. As we argued above (Remark 2.21), the product of geometric crystals is not always a geometric crystal. However, Proposition 2.25(b) implies that if the factors are geometric crystals coming from unipotent bicrystals, then the product is always a geometric crystal.

Claim 2.28. *For any unipotent bicrystal* $(\mathbf{X}, \mathbf{p})$ *of type* $w \in W$*, one has:*

(a) *The restriction* $\mathbf{p}|_{X^-} : X^- \to TB_w^-$ *defines a morphism of geometric crystals*

$$\mathbf{f}_w : \mathcal{F}(\mathbf{X}, \mathbf{p}) \to \mathcal{F}(BwB, \mathrm{id}) \ , \tag{2.16}$$

where (BwB, id) *is the* U*-bicrystal defined in Example 2.7. The support of* $\mathbf{f}_w$ *is* $\mathrm{Supp}\,\mathcal{F}(\mathbf{X}, \mathbf{p}) = \mathrm{Supp}\,\mathcal{F}(BwB, \mathrm{id}) = |w|$ *(see* (1.2)*).*

(b) *The restriction of the highest weight morphism* hw_X *(given by* (2.5)*) to* X^- *defines a morphism of geometric crystals* $\mathcal{F}(\mathbf{X}, \mathbf{p}) \to T$ *(see Example 2.13). The support of this morphism is* $\emptyset$*.*

2.3. **Unipotent χ-linear bicrystals.** Let χ be a character of U. A *unipotent χ-linear bicrystal* or, $(U \times U, \chi)$-linear bicrystal is a triple $(\mathbf{X}, \mathbf{p}, f)$ where $(\mathbf{X}, \mathbf{p})$ is a unipotent bicrystal, and f is a $(U \times U, \chi)$-linear function on X (see Section 1.4).

A morphism of $(U \times U, \chi)$-linear bicrystals is any structure-preserving morphism of underlying U-bicrystals.

For any (U, χ)-linear bicrystals $(\mathbf{X}, \mathbf{p}, f)$ and $(\mathbf{Y}, \mathbf{p}', f')$, define the convolution product

$$(\mathbf{X}, \mathbf{p}, f) * (\mathbf{Y}, \mathbf{p}', f') := (\mathbf{Z}, \mathbf{p}'', f'') \ , \tag{2.17}$$

where $(\mathbf{Z}, \mathbf{p}'') = (\mathbf{Z}, \mathbf{p}) * (\mathbf{Y}, \mathbf{p}')$ is the product of U-bicrystals (see Section 2.1) and the function f'' on $Z = X * Y$ is defined by

$$f''(x * y) = f(x) + f'(y) \ .$$

For each $(U \times U, \chi)$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f)$ denote $(\mathbf{X}, \mathbf{p}, f)^{op} := ((\mathbf{X}, \mathbf{p})^{op}, f)$. Due to Lemma 1.16, this is also a $(U \times U, \chi)$-linear bicrystal.

Claim 2.29.

(a) *The category of unipotent χ-linear bicrystals is naturally monoidal with respect to the convolution product* $*$ *(where the unit object is* (U, id, χ)*).*

(b) *The correspondence* $(\mathbf{X}, \mathbf{p}, f) \mapsto (\mathbf{X}, \mathbf{p}, f)^{op}$ *is an involutive covariant functor from the category of unipotent χ-linear bicrystals into itself.*

We get a very surprising combinatorial corollary of the above arguments.

Proposition 2.30. *In the notation of Section 1.4, we have:*

(a) *For any standard parabolic subgroups* P, P'*, there is a standard parabolic subgroup* $P'' = P \star P'$ *such that*

$$w_P \star w_{P'} = w_{P''} \ .$$

(b) *The operation* $(P, P') \to P \star P'$ *defines a monoid structure on the set of standard parabolic subgroups.*

(c) *This monoid is commutative with the unit element* $P = G$*.*

(d) $P \star P' \subset P \cap P'$ *for any standard parabolic subgroups* P *and* P'*.*

Proof. Let f and f' be $(U\times U,\chi)$-linear functions respectively on Bw_PB and $Bw_{P'}B$ as in Proposition 1.19. Then the triples $(U\overline{w_P}U,\mathbf{p},f)$ and $(U\overline{w_{P'}}U,\mathbf{p}',f')$, where $\mathbf{p}: U\overline{w_P}U \hookrightarrow G$ and $\mathbf{p}': U\overline{w_{P'}}U \hookrightarrow G$ are natural inclusions, are $(U\times U,\chi)$-linear bicrystals. Let $Z := (U\overline{w_P}U)*(U\overline{w_{P'}}U)$. Consider the convolution product $(U\overline{w_P}U,\mathbf{p},f)*(U\overline{w_{P'}}U,\mathbf{p}',f') = (Z,\mathbf{p}'',f'')$. By definition, f'' is a $(U\times U,\chi^{st})$-linear function on Z.

By Proposition 1.13 any $U\times U$ orbit in Z is isomorphic to $U\overline{w_P\star w_{P'}}U$. Hence the restriction of f'' to any such orbit becomes a $(U\times U,\chi)$-linear function on this orbit, i.e., the standard $U\times U$-orbit $U\overline{w_P\star w_{P'}}U$ admits a $(U\times U,\chi)$-linear function. Therefore, according to Proposition 1.19, the element $w_P\star w_{P'}$ must be of the form $w_{P''}$ for some standard parabolic subgroup P''. This proves (a)

It is clear that the operation $\star$ is associative, which proves (b). To prove (c), consider a map $\varphi: W\to W$ given by $\varphi(w)=w_0w^{-1}w_0$. It is easy to see that φ is also an anti-automorphism of the monoid $(W,\star)$. On the other hand, each element of the form w_P is fixed by φ because

$$\varphi(w_P)=w_0(w_0^Pw_0)^{-1}w_0=w_0(w_0w_0^P)w_0=w_0^Pw_0=w_P\ .$$

Therefore, $w_P\star w_{P'}=w_{P'}\star w_P$. This proves the commutativity of the operation $P\star P'$. Part (c) is proved.

Prove (d) now. In order to prove the inclusion $P\star P'\subset P\cap P'$, we use Claim 1.12. Recall that the *right* Bruhat order on W is defined by: $w\preceq w'$ if there exists w'' such that $w''w=w'$ and $l(w')=l(w'')+l(w)$. One can easily show that $w\preceq w'$ implies $w'w_0\preceq ww_0$. Take $w=w_{P'}$, $w'=w_P\star w_{P'}=w_0^{P\star P'}w_0$. Then Claim 1.12 guarantees that $w\preceq w'$, hence, by above, $w'w_0\preceq ww_0$. Since $ww_0=w_0^{P'}$ and $w'w_0=w_0^{P\star P'}$, we obtain the inequality

$$w_0^{P\star P'}\preceq w_0^{P'}\ ,$$

which is equivalent to $P\star P'\subseteq P'$. By symmetry, we also obtain $P\star P'\subseteq P$. Part (d) is proved.

The proposition is proved. □

Remark 2.31. It would be interesting to find a combinatorial or geometric description of the operation $(P,P')\mapsto P\star P'$. For instance, the product of G-orbits $G/P\times G/P'$ contains a (unique) closed G-orbit isomorphic to $G/(P\cap P')$, which is a good approximation for $G/(P\star P')$.

2.4. **From unipotent χ-linear bicrystals to decorated geometric crystals.** We start the section with the useful notion of decorated geometric crystals.

Definition 2.32. A *decorated* geometric pre-crystal is a pair $(\mathcal{X},f)$, where $\mathcal{X}=(X,\gamma,\varphi_i,\varepsilon_i,e_i|i\in I)$ is a geometric pre-crystal (as defined in Section 2.2 above) with the support I, and f is a function on X such that

$$f(e_i^c(x))=f(x)+\frac{c-1}{\varphi_i(x)}+\frac{c^{-1}-1}{\varepsilon_i(x)} \tag{2.18}$$

for $x\in X$, $c\in\mathbb{G}_m$, $i\in I$.

We say that a decorated geometric pre-crystal $(\mathcal{X},f)$ is a decorated geometric crystal if $\mathcal{X}$ is a geometric crystal.

Definition 2.33. For decorated geometric pre-crystals $(\mathcal{X}, f)$ and $(\mathcal{Y}, f')$, we define the *product*

$$(\mathcal{X}, f) \times (\mathcal{Y}, f') := (\mathcal{X} \times \mathcal{Y}, f * f') \ , \tag{2.19}$$

where $f * f' : X \times Y \to \mathbb{A}^1$ is given by $(f * f')(x, y) = f(x) + f'(y)$.

Lemma 2.34. *The product* (2.19) *of decorated geometric pre-crystals is a well-defined decorated geometric pre-crystal.*

Proof. It suffices to show that the function $f * f'$ from (2.19) satisfies (2.18). Indeed, in the notation of Definition 2.15, we have

$$\begin{aligned}(f * f')(e^c(x, y)) &= (f * f')(e^{c_1}(x), e^{c_2}(y)) = f(e^{c_1}(x)) + f'(e^{c_2}(y)) \ . \\ &= f(x) + \frac{c_1 - 1}{\varphi_i(x)} + \frac{c_1^{-1} - 1}{\varepsilon_i(x)} + f'(y) + \frac{c_2 - 1}{\varphi_i'(y)} + \frac{c_2^{-1} - 1}{\varepsilon_i'(y)}.\end{aligned}$$

Taking into account that

$$\varphi_i''(x, y) = \frac{\varepsilon_i(x) + \varphi_i'(y)}{\alpha_i(\gamma(x))}, \ \varepsilon_i''(x, y) = (\varepsilon_i(x) + \varphi_i'(y)) \cdot \alpha_i(\gamma'(y)) \ ,$$

and

$$\frac{c_1 - 1}{\varphi_i(x)} = \frac{c - 1}{\varphi_i''(x, y)}, \frac{c_2^{-1} - 1}{\varepsilon_i'(y)} = \frac{c^{-1} - 1}{\varepsilon_i''(x, y)} \ ,$$

$$\frac{c_1^{-1} - 1}{\varepsilon_i(x)} = \frac{1 - c}{c\varepsilon_i(x) + \varphi_i'(y)}, \ \frac{c_2 - 1}{\varphi_i'(y)} = \frac{1 - c^{-1}}{\varepsilon_i(x) + c^{-1}\varphi_i'(y)} = -\frac{c_1^{-1} - 1}{\varepsilon_i(x)} \ ,$$

we obtain

$$(f * f')(e_i^c(x, y)) = (f * f')(x, y) + \frac{c - 1}{\varphi_i''(x, y)} + \frac{c^{-1} - 1}{\varepsilon_i''(x, y)} \ .$$

The lemma is proved. □

Clearly, the product of decorated geometric pre-crystals is associative. Therefore, the category of decorated geometric pre-crystals is monoidal.

It is easy to see that for a decorated geometric (pre)crystal $(\mathcal{X}, f)$, the pair $(\mathcal{X}^{op}, f)$ is also decorated geometric (pre)crystal so that we will denote the latter one by $(\mathcal{X}, f)^{op}$. Clearly, the association $(\mathcal{X}, f) \mapsto (\mathcal{X}, f)^{op}$ is a multiplication-reversing functor from the category of decorated geometric pre-crystals to itself.

For each $(U \times U, \chi)$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f)$, we abbreviate

$$\mathcal{F}(\mathbf{X}, \mathbf{p}, f) := (\mathcal{F}(\mathbf{X}, \mathbf{p}), f|_{X^-}) \ . \tag{2.20}$$

The following fact is a "decorated" version of Proposition 2.25.

Proposition 2.35.

(a) *In the notation of Proposition 2.25, $\mathcal{F}(\mathbf{X}, \mathbf{p}, f)$ is a decorated geometric crystal for any $(U \times U, \chi^{st})$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f)$.*
(b) *$\mathcal{F}((\mathbf{X}, \mathbf{p}, f) * (\mathbf{Y}, \mathbf{p}', f')) = \mathcal{F}(\mathbf{X}, \mathbf{p}, f) \times \mathcal{F}(\mathbf{Y}, \mathbf{p}', f')$ for any $(U \times U, \chi^{st})$-linear bicrystals $(\mathbf{X}, \mathbf{p}, f) * (\mathbf{Y}, \mathbf{p}', f')$.*
(c) *In the notation of Claim 2.29, we have $(\mathcal{F}((\mathbf{X}, \mathbf{p}, f)^{op})) = (\mathcal{F}(\mathbf{X}, \mathbf{p}, f)^{op}$.*

Proof. Prove (a). It suffices to show that $f|_{X^-}$ satisfies (2.18). Indeed, by (2.14), we have for each $x \in X^-$, $c \in \mathbb{G}_m$, $i \in I$,

$$f(e_i^c(x)) = f\left(x_i\left(\frac{c-1}{\varphi_i(x)}\right) \cdot x \cdot x_i\left(\frac{c^{-1}-1}{\varepsilon_i(x)}\right)\right)$$

$$= \chi^{st}\left(x_i\left(\frac{c-1}{\varphi_i(x)}\right)\right) + f(x) + \chi^{st}\left(x_i\left(\frac{c^{-1}-1}{\varepsilon_i(x)}\right)\right) = f(x) + \frac{c-1}{\varphi_i(x)} + \frac{c^{-1}-1}{\varepsilon_i(x)} .$$

Parts (b) and (c) easily follow. The proposition is proved. □

2.5. **Parabolic $(U \times U, \chi)$-linear bicrystals and the central charge.** We say that a $(U \times U, \chi)$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f)$ is *parabolic* if a dense $U \times U$-invariant subset of the image $\mathbf{p}(X)$ in G admits a $(U \times U, \chi)$-linear function.

Proposition 1.19 guarantees that the type of each parabolic $(U \times U, \chi)$-linear bicrystal is w_P for some standard parabolic subgroup $P \subset G$ (i.e., $P \supset B$).

Denote $X_P := UZ(L_P)\overline{w_P}U$ for each standard parabolic subgroup P, where L_P is the Levi factor of P.

Claim 2.36. *For each standard parabolic subgroup P of G, one has:*

(a) *$(\mathbf{X}_P, \mathrm{id}, f_P)$ is a parabolic $(U \times U, \chi^{st})$-linear bicrystal of type w_P, where f_P is the $(U \times U, \chi^{st})$-linear function on X_P defined in (1.14) and $\mathrm{id} : X_P \hookrightarrow G$ is the natural (i.e., identical) inclusion.*

(b) *The morphism $\iota : G \to G$ (see (1.1)) defines an isomorphism of $(U \times U, \chi^{st})$-linear bicrystals (see Claim 2.29):*

$$(\mathbf{X}_P, \mathrm{id}, f_P)^{op} \xrightarrow{\sim} (\mathbf{X}_{P^{op}}, \mathrm{id}, f_{P^{op}}) ,$$

where P^{op} is the only standard parabolic which Levi factor $L_{P^{op}}$ is related to the Levi factor L_P of P by $L_{P^{op}} = w_0 L_P w_0$. In particular, $(\mathbf{X}_B, \mathrm{id}, f_B)$ is self-opposite.

(c) *If $P = G$, i.e., $(\mathbf{X}, \mathbf{p}, f)$ is a parabolic $(U \times U, \chi^{st})$-linear bicrystal of type w_0, then the pre-image $X_0 = \mathbf{p}^{-1}(Bw_0B)$ is a dense $U \times U$-invariant subset of X and the $U \times U$-action on X_0 is free and, therefore, restriction of $\mathbf{p}$ to each $U \times U$-orbit $\mathcal{O} \subset X_0$ is an inclusion $\mathcal{O} \hookrightarrow Bw_0B$.*

According to Proposition 1.19, for any parabolic $(U \times U, \chi^{st})$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f)$ of type w_P the intersection of X_P with $\mathbf{p}(X)$ is dense in $\mathbf{p}(X)$.

For any parabolic $(U \times U, \chi^{st})$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f)$ of type w_P, define a rational function Δ_X on X by

$$\Delta_X(x) := f(x) - f_P(\mathbf{p}(x)) \tag{2.21}$$

for any $x \in X$.

We will call this function the *central charge* of $(\mathbf{X}, \mathbf{p}, f)$.

Lemma 2.37. *For any parabolic $(U \times U, \chi^{st})$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f)$, the central charge Δ_X is a $U \times U$-invariant rational function on X.*

Proof. By definition,

$$\Delta_X(uxu') = f(uxu') - f_P(\mathbf{p}(uxu')) = f(x) + \chi^{st}(u) + \chi^{st}(u') - f_P(u\mathbf{p}(x)u')$$
$$= f(x) - f_P(\mathbf{p}(x)) = \Delta_X(x)$$

for any $x \in X$, $u, u' \in U$. This proves the lemma. $\square$

Similarly to [5, Formula (4.4)], define an automorphism $\sigma : G \to G$ by

$$\sigma(g) = \overline{w_0} \cdot \iota\,(g)^{-1} \cdot \overline{w_0}^{\,-1}\ , \tag{2.22}$$

where the positive inverse $\iota : G \to G$ is given by (1.1). By definition, σ is an involution such that $\sigma(B) = B^-$. According to [5, Formula (4.5)], this involution satisfies

$$\Delta_{w_1\lambda, w_2\lambda}(\sigma(g)) = \Delta_{w_0 w_1\lambda, w_0 w_2\lambda}(g) \tag{2.23}$$

for any $g \in G$, $w_1, w_2 \in W$, and $\lambda \in X^\star(T)^+$.

Next, define a morphism $\pi : (Bw_0B) * (Bw_0B) \to U$ by

$$\pi(g * g') = uu'\ , \tag{2.24}$$

where $g = u_0 t \overline{w_0} u,\ g' = u' t' \overline{w_0} u'_0\ , u, u', u_0, u'_0 \in U, t, t' \in T$.

Claim 2.38. *The morphism π is $U \times U$-invariant, i.e.,*

$$\pi(ug * g'u') = \pi(g * g')$$

for any $g \in Bw_0B$, $g' \in Bw_0B$,$u, u' \in U$.

Define a morphism $\mathbf{m} : U \times T \times T \to B^-$ by

$$\mathbf{m}(u, t, t') = t \cdot \sigma\,(u \cdot t')\ , \tag{2.25}$$

for $u \in U$, $t, t' \in T$.

Proposition 2.39. *Let $(\mathbf{X}, \mathbf{p}, f)$ and $(\mathbf{Y}, \mathbf{p}', f')$ be parabolic $(U \times U, \chi^{st})$-linear bicrystals of type w_0 each. Then the central charge of the convolution product $(\mathbf{X}, \mathbf{p}, f)*$ $(\mathbf{Y}, \mathbf{p}', f')$ is given by the formula*

$$\Delta_{X*Y}(x * y) = \Delta_X(x) + \Delta_Y(y) + \chi^{st}(\pi(g, g')) + f_B(\mathbf{m}(\pi(g * g'), t, t')), \tag{2.26}$$

where $g = \mathbf{p}(x)$, $g' = \mathbf{p}'(y)$, $t = \pi^{w_0}(g), t' = \pi^{w_0}(g')$ (and π^{w_0} was defined in (1.20)*).*

Proof of Proposition 2.39. We need the following fact.

Lemma 2.40. *For any standard parabolic subgroup P with the Levi factor L_P, the automorphism $g \mapsto (\iota(g))^{-1}$ preserves $X_P = UZ(L_P)\overline{w_P}U$, and for any $g \in X_P$, we have $f_P(\iota(g)^{-1}) = -f_P(g)$.*

Proof. Note that $\iota(U)^{-1} = U$ and $\iota(Z(L_P))^{-1} = Z(L_P)$. Therefore, in order to show that $\iota(X_P)^{-1} = X_P$ it suffices to prove that $\iota\,(\,\overline{w_P}\,)^{-1} \in Z(L_P)\overline{w_P}$. Using Claim 1.3 and $w_0^{-1} = w_0$, $(w_0^P)^{-1} = w_0^P$, we obtain

$$\iota\,(\,\overline{w_P}\,)^{-1} = \left(\overline{(w_P)^{-1}}\right)^{-1} = \overline{w_0 w_0^P}^{\,-1} = \left(\overline{w_0}\,\overline{w_0^P}^{\,-1}\right)^{-1} = \overline{w_0^P}\,\overline{w_0}^{\,-1}\ .$$

Therefore,

$$\iota\,(\,\overline{w_P}\,)^{-1} = \overline{w_0^P}^{\,2}\left(\overline{w_0^P}^{\,-1}\overline{w_0}\right)\overline{w_0}^{\,-2} = \overline{w_0^P}^{\,2}\ \overline{w_P}\ \overline{w_0}^{\,-2}\ . \tag{2.27}$$

Finally, note that $\overline{w_0}^{\,-2} \in Z(G)$, $\overline{w_0^P}^{\,2} \in Z(L_P)$. Therefore,

$$\iota(\overline{w_P})^{-1} \in Z(L_P)\overline{w_P}Z(G) = Z(L_P)\overline{w_P}\ .$$

This proves that $\iota(X_P)^{-1} = X_P$. Next, prove that $f_P(\iota(g)^{-1}) = -f_P(g)$ for any $g \in X_P$. Indeed, for any $g \in Z(L_P)\overline{w_P}$ and $u, u' \in U$ we obtain using Claim 1.1:

$$f_P(\iota(ugu')^{-1}) = f_P(\iota(u)^{-1}\iota(g)^{-1}\iota(u')^{-1}) =$$
$$\chi^{st}(\iota(u)^{-1}) + \chi^{st}(\iota(u')^{-1}) = -\chi^{st}(u) - \chi^{st}(u') = -f_P(ugu') \ .$$

This finishes the proof of the lemma. □

Furthermore, in order to prove (2.26), it suffices to deal with the standard parabolic crystals of type w_0: $(\mathbf{X}, \mathbf{p}, f) = (\mathbf{Y}, \mathbf{p}', f') = (\mathbf{X}_B, \mathrm{id}_{X_B}, f_B)$. Let $g = u_0 t\overline{w_0}u$, $g' = u't'\overline{w_0}$ with $u, u' \in U$, $t, t' \in T$.

Thus (using the notation $\chi = \chi^{st}$ and the fact that $\Delta = \Delta_{X_B * X_B}$ is $U \times U$-invariant), we obtain

$$\Delta(g * g') = f_B(g) + f_B(g') - f_B(gg') = \chi(u) + \chi(u') - f_T(t\overline{w_0}uu't'\overline{w_0})$$
$$= \chi(uu') - f_B(t\overline{w_0}uu't'\overline{w_0}) = \chi(u'') - f_T(t\overline{w_0}u''t'\overline{w_0}) \ ,$$

where $u'' = uu' = \pi(g * g')$.

Furthermore, using Lemma 2.40 with $L_P = T$ and the fact that $\iota(\overline{w})^{-1} = \overline{w^{-1}}^{-1}$ for any $w \in W$, we obtain

$$\Delta(g * g') = \chi(u'') + f_B(\iota(t\overline{w_0}u''t'\overline{w_0})^{-1}) = \chi(u'') + f_B(t \cdot \overline{w_0}^{-1}\iota(u''t')^{-1} \cdot \overline{w_0}^{-1})$$
$$= \chi(u'') + f_B(t \cdot \overline{w_0}^{-2}\overline{w_0} \cdot \iota(u''t')^{-1} \cdot \overline{w_0}^{-1})$$
$$= \chi(u'') + f_B(\overline{w_0}^{-2}\mathbf{m}(u'', t, t')) = \chi(u'') + f_B(\mathbf{m}(u'', t, t'))$$

because $\overline{w_0}^{-2} \in Z(G) \subset T$.

This proves Proposition 2.39. □

Remark 2.41. The formula (2.26) implies that Δ_{X*Y} is a rational (but not regular) function on $X * Y$ even when Δ_X and Δ_Y are regular because the domain of f_B is Bw_0B while the range of the morphism $g * g' \to \mathbf{m}(\pi(g * g'), \pi^{w_0}(g), \pi^{w_0}(g'))$ is the entire B^- (rather than the intersection $B^- \cap Bw_0B$).

2.6. **The $(U \times U, \chi^{st})$-linear bicrystal Z_w.** Now we explicitly describe the unipotent bicrystal structure on $X_B * X_B = (Bw_0B) * (Bw_0B)$. According to Claim 1.10, $X_B * X_B$ is a regular $U \times U$-variety.

For each $w \in W$, denote by $(X_B * X_B)_w$ the set of all $g * g' \in X_B * X_B$ such that $gg' \in BwB$. Clearly, $(X_B * X_B)_w$ is a locally closed subset in $X_B * X_B$ and $X_B * X_B = \bigsqcup_{w \in W}(X_B * X_B)_w$ and it is a $X_B * X_B$ is a regular $U \times U$-variety.

Following [2], define

$$U^w := U \cap B^- wB^- \tag{2.28}$$

for each $w \in W$ (note a change of the notation). It is easy to see that dim $U^w =$ dim $U^{w_0 w w_0} = l(w)$. For each $w \in W$, let

$$Z_w := (U^{w_0 w w_0} \times T \times T) \times_T BwB \ , \tag{2.29}$$

where the fibered product is taken with respect to $\pi^w : BwB \to T$ (given by (1.20)) and the following morphism $\mu_w : U^{w_0 w w_0} \times T \times T \to T$:

$$\mu_w(u, t, t') = \overline{w}_0^2 \cdot (\overline{w}\,\overline{w^{-1}}) \cdot \pi^w(\mathbf{m}(u, t, t')) \ , \tag{2.30}$$

(where the morphism $\mathbf{m} : U \times T \times T \to B^-$ is defined by (2.25)). By definition, Z_w is a $U \times U$-variety, where the $U \times U$ acts on the second factor.

Now define a morphism $F_w : (X_B * X_B)_w \to U \times T \times T \times BwB$ by the formula

$$F_w(g * g') = (\pi(g * g'), \pi^{w_0}(g), \pi^{w_0}(g'), gg')$$

for all $g * g' \in (X_B * X_B)_w$.

Clearly, each F_w commutes with the $U \times U$-action on $U \times T \times T \times BwB$ (which is the trivial extension of that on BwB).

Proposition 2.42. *For each $w \in W$, the image of F_w is equal to Z_w and the quotient morphism*

$$U\backslash F_w/U : U\backslash(X_B * X_B)_w/U \to U\backslash Z_w/U \tag{2.31}$$

is an isomorphism. In particular, F_{w_0} is an isomorphism of free $U \times U$-varieties $(X_B * X_B)_{w_0} \xrightarrow{\sim} Z_{w_0}$.

Proof. First, note that all $U \times U$-orbits in $X_B * X_B$ are free. And each of these orbits intersects the subset $\overline{w}_0 T * B\overline{w}_0 \subset X_B * X_B$ at exactly one point.

Let us study the subset $(\overline{w}_0 T * B\overline{w}_0)_w$ of $(X_B * X_B)_w$. By definition, this is the set of all $\overline{w}_0 t * t' u \overline{w}_0$, $u \in U, t, t' \in T$ such that $\overline{w}_0 t t' u \overline{w}_0 \in BwB$ or, equivalently, $tt'u \in \overline{w}_0^{-1} BwB \overline{w}_0^{-1} = B^- w_0 w w_0 B^-$, that is, $u \in U^{w_0 w w_0}$. Therefore,

$$(\overline{w}_0 T * B\overline{w}_0)_w = \overline{w}_0 T * T U^{w_0 w w_0} \overline{w}_0 = T\overline{w}_0 * U^{w_0 w w_0} T \overline{w}_0 \ .$$

This characterization of $U \times U$-orbits in $(X_B * X_B)_w$ implies that each element of $(X_B * X_B)_w$ can be uniquely expressed as $g * g'$, where $g = u_1 t \overline{w}_0$, $g' = u t' \overline{w}_0 u_2$ for $u_1, u_2 \in U$, $t, t' \in T$, $u \in U^{w_0 w w_0}$.

Furthermore, by definitions (2.24) and (1.20), we have

$$\pi(g * g') = u, \ \pi^{w_0}(g) = t, \pi^{w_0}(g') = t', gg' = u_1 t \overline{w}_0 u t' \overline{w}_0 u_2 \ .$$

That is,

$$F_w(g * g') = (u, t, t', u_1 t \overline{w}_0 u t' \overline{w}_0 u_2) \in U^{w_0 w w_0} \times T \times T \times BwB \tag{2.32}$$

for $g = u_1 t \overline{w}_0$, $g' = u t' \overline{w}_0 u_2$ for $u_1, u_2 \in U$, $t, t' \in T$, $u \in U^{w_0 w w_0}$. Next, we will prove that, in fact, $F_w(g * g') \in Z_w$. First, note that

$$\pi^w(gg') = \pi^w(u_1 \overline{w}_0 t u \overline{w}_0 t' u_2) = \pi^w(t \overline{w}_0 u t' \overline{w}_0) \ .$$

Furthermore, using Lemma 1.26, we obtain for $u \in U^{w_0 w w_0}$, $t, t' \in T$:

$$\pi^w(\mathbf{m}(u, t, t')) = \pi^w(t\sigma(u \cdot t')) = \pi^w(\iota\left(t\overline{w_0}^{-1} u t' \overline{w_0}\right)^{-1}) = t_w \pi^w(t\overline{w_0}^{-1} u t' \overline{w_0})$$

$$= t_w \pi^w(t\overline{w_0}^{-2} \overline{w_0} u t' \overline{w_0}) = t_w \overline{w_0}^{-2} \pi^w(t \overline{w_0} u t' \overline{w_0}) = t_w \overline{w_0}^{-2} \pi^w(gg') \ .$$

Here we used the fact that $\overline{w_0}^{-2} \in Z(G)$. Putting it together, we obtain:

$$\pi^w(gg') = \overline{w_0}^2 t_w^{-1} \pi^w(\mathbf{m}(u, t, t')) = \mu_w(u, t, t') \ . \tag{2.33}$$

This proves that the image of F_w belongs to Z_w.

Let us prove that the morphism (2.31) is an isomorphism. Consider the diagram:

$$\begin{array}{ccc} (X_B * X_B)_w & \xrightarrow{F_w} & Z_w \\ \uparrow & & \uparrow \\ T\overline{w}_0 U^{w_0 w w_0} * T\overline{w}_0 & \longrightarrow & U^{w_0 w w_0} \times T \times T \end{array}$$

where

1. The left vertical arrow is the natural inclusion $T\overline{w}_0 U^{w_0 w w_0} * T\overline{w}_0 \subset (X_B * X_B)_w$.
2. The right vertical arrow is the injective morphism $U^{w_0 w w_0} \times T \times T \xrightarrow{\sim} Z_w$ defined by $(u,t,t') \mapsto (u,t,t',\mu_w(u,t,t')\cdot\overline{w})$.
3. The bottom horizontal arrow the isomorphism $T\overline{w}_0 U^{w_0 w w_0} * T\overline{w}_0 \to U^{w_0 w w_0} \times T \times T$ defined by $t\overline{w}_0 u * t'\overline{w}_0 \mapsto (u,t,t')$.

It follows from (2.32) and (2.33) that this diagram commutes up to the $U\times U$-action on Z_w. Therefore, passing to the quotient, we obtain the following commutative diagram:

$$\begin{array}{ccc} U\backslash(X_B * X_B)_w/U & \xrightarrow{U\backslash F_w/U} & U\backslash Z_w/U \\ \uparrow & & \uparrow \\ T\overline{w}_0 U^{w_0 w w_0} * T\overline{w}_0 & \longrightarrow & U^{w_0 w w_0} \times T \times T \end{array}$$

Clearly, the vertical arrows in these diagram (as well as the bottom horizontal one) are isomorphisms. This implies the top horizontal arrow $U\backslash F_w/U$ is also an isomorphism.

The proposition is proved. □

Clearly, $(X_B * X_B)_{w_0}$ is a dense $U \times U$-invariant subset in $X_B * X_B$. According to Proposition 2.42, F_{w_0} is an isomorphism of free $U \times U$-varieties $(\mathbf{X}_B * \mathbf{X}_B)_{w_0} \xrightarrow{\sim} \mathbf{Z}_{w_0}$.

Note that as a $U \times U$-variety, $Z_{w_0} = (U^{w_0} \times T \times T) \times_T X_B$, where the fiber product is taken with respect to $\mu_{w_0} : U^{w_0} \times T \times T \to T$ and $\pi^{w_0} : X_B \to T$, where μ_{w_0} is given by (2.30), In our case it simplifies to

$$\mu_{w_0}(u,t,t') = \overline{w_0}\,{}^4\pi^{w_0}(\mathbf{m}(u,t,t')) = \pi^{w_0}(\mathbf{m}(u,t,t'))\ .$$

Using this characterization, define a morphism

$$\mathbf{p}_Z : Z_{w_0} = (U^{w_0} \times T \times T) \times_T X_B \to X_B$$

to be the projection to the second fibered factor and a function $f_Z : Z_{w_0} \to \mathbb{A}^1$ by

$$f_Z((u,t,t'),g) = f_B(\mathbf{m}(u,t,t')) + \chi(u) + f_B(g)$$

for $u \in U^{w_0}$, $t,t' \in T$, and $g \in X_B = Bw_0B$.

Claim 2.43. *The morphism $\mathbf{p}_Z$ commutes with the $U \times U$-action and the function f_Z is $(U \times U, \chi^{st})$-linear, that is, the triple*

$$(\mathbf{Z}_{w_0}, \mathbf{p}_Z, f_Z) \tag{2.34}$$

is a $(U \times U, \chi^{st})$-linear bicrystal.

Note that $(X_B * X_B)_{w_0}$ is open and dense in $X_B * X_B$. Then Proposition 2.39, Proposition 2.42, and Claim 2.43 imply the following corollary.

Corollary 2.44. *The inverse of the isomorphism $F_{w_0} : (X_B * X_B)_{w_0} \xrightarrow{\sim} Z_{w_0}$ from Proposition 2.42 defines an open embedding of $(U \times U, \chi^{st})$-linear bicrystals*

$$(\mathbf{Z}_{w_0}, \mathbf{p}_Z, f_Z) \hookrightarrow (\mathbf{X}_B, \mathrm{id}, f_B) * (\mathbf{X}_B, \mathrm{id}, f_B) \ . \tag{2.35}$$

3. Positive geometric crystals and unipotent bicrystals

3.1. **Toric charts and positive varieties.** Let S be a split algebraic torus defined over $\mathbb{Q}$. Denote by $X_\star(S) = \mathrm{Hom}(\mathbb{G}_m, S)$ the lattice of *co-characters* of S and by $X^\star(S) = \mathrm{Hom}(S, \mathbb{G}_m)$ the lattice of *characters* of S. These lattices are dual: the integer pairing $\langle \mu, \lambda \rangle$ of $\lambda \in X_\star(S)$ and $\mu \in X^\star(S)$ is defined via

$$\mu(\lambda(c)) = c^{\langle \mu, \lambda \rangle}$$

for $c \in \mathbb{G}_m$.

The coordinate ring of S is the group algebra $\mathbb{Q}[X^\star(S)]$.

Definition 3.1. For any split algebraic torus, a *positive* regular function on S is an element of $\mathbb{Q}[X^\star(S)]$ of the form $f = \sum_\mu c_\mu \cdot \mu$, where all c_μ are non-negative integers. A rational function on S is said to be *positive* if it can be expressed as a ratio $\frac{f}{g}$ where both f and g are positive regular functions on S and $g \neq 0$.

Definition 3.2. For any split algebraic tori S and S', we say that a rational morphism f is *positive* if for any character $\mu' : S' \to \mathbb{G}_m$ the composition $\mu' \circ f$ is a positive rational function on S.

Remark 3.3. Equivalently (see e.g., [11] or [26, Section 6]), a rational function f on $(\mathbb{G}_m)^\ell$ is positive if and only if the restriction of f to the positive octant $(\mathbb{Q}_{>0})^l$ is a well-defined function $(\mathbb{Q}_{>0})^\ell \to \mathbb{Q}_{>0}$. Therefore, a rational morphism $f : (\mathbb{G}_m)^\ell \to (\mathbb{G}_m)^k$ is positive if and only if the restriction of f to the positive octant $(\mathbb{Q}_{>0})^l$ is a well-defined map $(\mathbb{Q}_{>0})^\ell \to (\mathbb{Q}_{>0})^k$. Taking into account that each split algebraic torus S is isomorphic to $(\mathbb{G}_m)^\ell$ for some ℓ and the positive octant $S(\mathbb{Q}_{>0}) \cong (\mathbb{Q}_{>0})^l$ in $S(\mathbb{Q})$ does not depend on the choice of the isomorphism $S \cong (\mathbb{G}_m)^\ell$, the above arguments guarantee that a rational morphism $f : S \to S'$ is positive if and only if the restriction of f to $S(\mathbb{Q}_{>0})$ is a well-defined map $S(\mathbb{Q}_{>0}) \to S'(\mathbb{Q}_{>0})$.

Remark 3.4. Obviously, for any positive rational functions $f, g : S \to \mathbb{A}^1$, the functions $f + g$, fg, and $\frac{f}{g}$ are also positive.

Remark 3.5. Not every positive birational isomorphism $S \to S$ has positive inverse. For instance, $f : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ given by $f(x, y) = (x, x + y)$ is a positive isomorphism, but $f^{-1}(x, y) = (x, y - x)$ is not positive.

Definition 3.6. A *toric chart* on a variety X is a birational isomorphism $\theta : S \xrightarrow{\sim} X$, where S is a split algebraic torus.

For any sequence $\mathbf{i} = (i_1, \ldots, i_\ell) \in I^\ell$, define a morphism $\theta^-_{\mathbf{i}} : (\mathbb{G}_m)^\ell \to B^-$ by

$$\theta^-_{\mathbf{i}}(c_1, \ldots, c_\ell) := x_{-i_1}(c_1) \cdot x_{-i_2}(c_2) \cdots x_{-i_\ell}(c_\ell) \tag{3.1}$$

for any $c_1, \ldots, c_\ell \in \mathbb{G}_m$, where $x_{-i} : \mathbb{G}_m \to B^-$ is given by the formula

$$x_{-i}(c) := \phi_i \begin{pmatrix} c^{-1} & 0 \\ 1 & c \end{pmatrix} . \tag{3.2}$$

In a similar manner, following [20], for any sequence $\mathbf{i} = (i_1, \dots, i_\ell) \in I^\ell$, we define a morphism $\theta^+_{\mathbf{i}} : (\mathbb{G}_m)^\ell \to U$ by

$$\theta^+_{\mathbf{i}}(c_1, \dots, c_\ell) := x_{i_1}(c_1) \cdot x_{i_2}(c_2) \cdots x_{i_\ell}(c_\ell) \tag{3.3}$$

for any $c_1, \dots, c_\ell \in \mathbb{G}_m$, where each $x_i : \mathbb{G}_m \to U$ is a generator of U (defined in Section 1.1).

Claim 3.7. [5, Proposition 4.5] *For any reduced decomposition* $\mathbf{i} = (i_1, \dots, i_\ell)$ *of an element* $w \in W$*, one has:*

(a) *The morphism* $\theta^-_{\mathbf{i}}$ *is an open embedding (hence a toric chart)*

$$(\mathbb{G}_m)^\ell \hookrightarrow B^-_w = B^- \cap U\overline{w}U \ .$$

(b) *The morphism* $\theta^+_{\mathbf{i}}$ *is an open embedding (hence a toric chart)*

$$(\mathbb{G}_m)^{l(w)} \hookrightarrow U^w = U \cap B^- w B^- \ .$$

Example 3.8. For $G = GL_3$ and $\mathbf{i} = (1, 2, 1)$, we have

$$\theta^-_{\mathbf{i}}(c_1, c_2, c_3) = \begin{pmatrix} c_1^{-1} & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2^{-1} & 0 \\ 0 & 1 & c_2 \end{pmatrix} \begin{pmatrix} c_3^{-1} & 0 & 0 \\ 1 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{c_1 c_3} & 0 & 0 \\ \frac{c_1}{c_2} + \frac{1}{c_3} & \frac{c_1 c_3}{c_2} & 0 \\ 1 & c_3 & c_2 \end{pmatrix},$$

$$\theta^+_{\mathbf{i}}(c_1, c_2, c_3) = \begin{pmatrix} 1 & c_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_1 + c_3 & c_1 c_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We say that two toric charts $\theta : S \xrightarrow{\sim} X$ and $\theta' : S' \xrightarrow{\sim} X$ are *positively equivalent* if both birational isomorphisms $\theta'^{-1} \circ \theta : S \to S'$ and $\theta^{-1} \circ \theta' : S' \to S$ are positive (see also Definition 4.16).

Claim 3.9. [5, 20] *For any reduced decompositions* $\mathbf{i}$ *and* $\mathbf{i}'$ *of* $w \in W$*, one has:*

(a) *The toric charts* $\theta^-_{\mathbf{i}}$ *and* $\theta^-_{\mathbf{i}'}$ *on* B^-_w *are positively equivalent.*
(b) *The toric charts* $\theta^+_{\mathbf{i}}$ *and* $\theta^+_{\mathbf{i}'}$ *on* U^w *are positively equivalent.*

Definition 3.10. A *positive structure* Θ_X on a variety X is a positive equivalence class of its toric charts (if X is a split algebraic torus S, then we will always denote by Θ_S that positive structure on S which contains the identity isomorphism $\mathrm{id} : S \to S$); we will call a pair (X, Θ_X) a *positive variety.*

Define *supp*(X, Θ) of the positive variety (X, Θ) by

$$\mathrm{Supp}(X, \Theta) = \bigcup_{\theta \in \Theta} \mathrm{dom}(\theta^{-1}) \tag{3.4}$$

(see Definition 2.11). That is, $\mathrm{Supp}(X, \Theta)$ is a dense open subset of X.

For each rational morphism $\theta : S \to X$, where S is a split algebraic torus, we denote by $\theta(\mathbb{Q}_{>0}) \subset X(\mathbb{Q})$ the image of the restriction of θ to $\mathrm{dom}(\theta)(\mathbb{Q}) \cap S(\mathbb{Q}_{>0})$ (see Remark 3.3).

Claim 3.11. *Let (X, Θ) be a positive variety. Then for each $\theta, \theta' \in \Theta$, one has $\theta(\mathbb{Q}_{>0}) = \theta'(\mathbb{Q}_{>0})$.*

Based on this, for each positive variety (X, Θ) define the set of *positive points* $X(\mathbb{Q}_{>0}) \subset X(\mathbb{Q})$ by $X(\mathbb{Q}_{>0}) := \theta(\mathbb{Q}_{>0})$ for any $\theta \in \Theta$.

Definition 3.12. A *morphism of positive varieties* $(X, \Theta_X) \to (Y, \Theta_Y)$ is any rational morphism $f : X \to Y$ such that one has (in the notation of Definition 2.11):

(1) The intersection $\mathrm{ran}(f) \cap \mathrm{Supp}(Y, \Theta_Y)$ is non-empty (hence it is open in $\mathrm{ran}(f)$).
(2) For any toric chart $\theta : S \xrightarrow{\sim} X$ in Θ_X and any toric chart $\theta' : S' \xrightarrow{\sim} Y$ in Θ_Y such that $\mathrm{dom}(\theta'^{-1}) \cap \mathrm{ran}(f) \neq \emptyset$, the composition $\theta'^{-1} \circ f \circ \theta$ is a positive morphism $S \to S'$.

We will sometimes refer to a morphism of positive varieties $(X, \Theta_X) \to (Y, \Theta_Y)$ as a (Θ_X, Θ_Y)*-positive morphism* $X \to Y$.

Claim 3.13. *Let (X, Θ_X) and (Y, Θ_Y) be positive varieties. Then a rational morphism $f : X \to Y$ is a morphism of positive varieties $(X, \Theta_X) \to (Y, \Theta_Y)$, if and only if the restriction of f to $X(\mathbb{Q}_{>0})$ is a well-defined map $X(\mathbb{Q}_{>0}) \to Y(\mathbb{Q}_{>0})$. In the latter case, $\mathrm{dom}(\theta_Y^{-1}) \cap \mathrm{ran}(f) \neq \emptyset$ for any $\theta_Y \in \Theta_Y$.*

Claim 3.14. *Positive varieties and their morphisms form a category $\mathcal{V}_+$. This category is monoidal with respect to the operation*

$$(X, \Theta_X) \times (Y, \Theta_Y) = (X \times Y, \Theta_{X \times Y}) \ ,$$

where $\Theta_{X \times Y}$ is the positive structure on $X \times Y$ such that $\theta \times \theta' \in \Theta_{X \times Y}$ for any $\theta \in \Theta_X$ and $\theta' \in \Theta_Y$.

Following [2], denote by $\mathcal{T}_+$ the category whose objects are split algebraic tori (defined over $\mathbb{Q}$), and arrows are positive rational morphisms.

One has a natural functor $\mathcal{T}_+ \hookrightarrow \mathcal{V}_+$ given by $S \mapsto (S, \Theta_S)$, where Θ_S is the natural positive structure on S.

Lemma 3.15. *The natural functor $\mathcal{T}_+ \hookrightarrow \mathcal{V}_+$ is an equivalence of monoidal categories $\mathcal{T}_+$ and $\mathcal{V}_+$.*

Proof. Clearly, $\mathcal{T}_+ \hookrightarrow \mathcal{V}_+$ is both a full and faithful functor. This functor preserves a monoidal structure.

Claim 3.16. *Given a positive variety (X, Θ), each toric chart $\theta : S \xrightarrow{\sim} X$ from Θ is an isomorphism of positive varieties $(S, \Theta_S) \xrightarrow{\sim} (X, \Theta)$ (where Θ_S is the natural positive structure on the algebraic torus S).*

Claim 3.16 guarantees that each object (X, Θ) is isomorphic to (S, Θ_S). The lemma is proved. □

Define the monoidal category $\mathcal{V}_{++}$ whose objects are triples (X, Θ, θ), where (X, Θ) is a positive variety and $\theta \in \Theta$ is a chosen toric chart $S \xrightarrow{\sim} X$ and each morphism $f : (X, \Theta, \theta) \to (Y, \Theta', \theta')$ in $\mathcal{V}_{++}$ is simply a morphism $f : (X, \Theta) \to (Y, \Theta')$ of positive varieties. By definition, one has a full forgetful monoidal functor $\mathcal{V}_{++} \to \mathcal{V}_+$ via $(X, \Theta, \theta) \mapsto (X, \Theta)$. We will refer to each object of $\mathcal{V}_{++}$ as a *decorated positive variety*.

Claim 3.17.

(a) *The forgetful functor $\mathcal{V}_{++} \to \mathcal{V}_+$ is an equivalence of monoidal categories.*
(b) *A simultaneous choice of the toric chart $\theta : S \to X$ for each object (X, Θ_X) defines a functor $\mathcal{G}^* : \mathcal{V}_+ \to \mathcal{V}_{++}$ adjoint to $\mathcal{G}$, and each adjoint to $\mathcal{G}$ functor is of this form.*

Let us also define a functor $\tau : \mathcal{V}_{++} \to \mathcal{T}_+$ as follows:

(i) $\tau(X, \Theta_X, \theta_X) = S$, where $\theta_X : S \xrightarrow{\sim} X$;
(ii) for each morphism $f : (X, \Theta_X, \theta_X) \to (Y, \Theta_Y, \theta_Y)$ in $\mathcal{V}_{++}$, we define $\tau(f) : S \to S'$ to be the positive morphism of the form

$$\tau(f) = \theta_Y^{-1} \circ f \circ \theta_X \ .$$

According to Claim 3.13, $\tau(f)$ is well-defined.

Claim 3.18.

(a) *The functor $\tau : \mathcal{V}_{++} \to \mathcal{T}_+$ is an equivalence of monoidal categories.*
(b) *All functors $\mathcal{G}^* : \mathcal{V}_+ \to \mathcal{V}_{++}$ from Claim 3.17 are isomorphic to each other.*
(c) *For each functor $\mathcal{G}^* : \mathcal{V}_+ \to \mathcal{V}_{++}$, the composition $\tau \circ \mathcal{G}^*$ is a functor adjoint to the natural inclusion $\mathcal{T}_+ \hookrightarrow \mathcal{V}_+$ (see Lemma 3.15).*

Denote by $\Theta_{\mathbb{A}^1}$ the positive structure on $\mathbb{A}^1$ containing the natural inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1 = \mathbb{G}_m \sqcup \{0\}$; therefore, $(\mathbb{A}^1, \Theta_{\mathbb{A}^1})^n = (\mathbb{A}^n, \Theta_{\mathbb{A}^n})$, where $\Theta_{\mathbb{A}^n}$ is the positive structure containing the natural inclusion $(\mathbb{G}_m)^n \hookrightarrow \mathbb{A}^n$.

Lemma 3.19. *For each $n > 0$, we have* $\mathrm{Supp}(\mathbb{A}^n, \Theta_{\mathbb{A}^n}) = (\mathbb{G}_m)^n$.

Proof. We need the following result.

Claim 3.20. *Let $\theta : \mathbb{A}^n \to \mathbb{A}^n$ be a positive birational isomorphism such that its inverse θ^{-1} is also positive. Let $\theta_{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^n$ be the rational map obtained by the specialization of θ to the real points. Then:*

(a) *The restriction of $\theta_{\mathbb{R}}$ to the positive octant $(\mathbb{R}_{>0})^n$ is a well-defined homeomorphism $(\mathbb{R}_{>0})^n \xrightarrow{\sim} (\mathbb{R}_{>0})^n$.*
(b) *If a boundary point $x \in (\mathbb{R}_{\geq 0})^n \setminus (\mathbb{R}_{>0})^n$ belongs to the domain of $\theta_{\mathbb{R}}$, then $\theta_{\mathbb{R}}(x) \in (\mathbb{R}_{\geq 0})^n \setminus (\mathbb{R}_{>0})^n$.*

To prove the lemma, we show that the assumption that $\mathrm{Supp}(\mathbb{A}^n, \Theta_{\mathbb{A}^n}) \setminus (\mathbb{G}_m)^n$ is non-empty leads to a contradiction. Indeed, this assumption implies that there exists a positive birational isomorphism $\theta : \mathbb{A}^n \to (\mathbb{G}_m)^n$ such that θ^{-1} is also positive and $\mathrm{dom}(\theta) \cap (\mathbb{A}^n \setminus (\mathbb{G}_m)^n) \neq \emptyset$. Passing to the real points, and taking into account that $(\mathbb{A}^n \setminus (\mathbb{G}_m)^n)(\mathbb{R}) = \mathbb{R}^n \setminus (\mathbb{R}_{\neq 0})^n$, we see that the restriction of $\theta_{\mathbb{R}} : \mathbb{R}^n \to (\mathbb{R}_{\neq 0})^n$ to $\mathbb{R}^n \setminus (\mathbb{R}_{\neq 0})^n$ is a real rational map $\mathbb{R}^n \setminus (\mathbb{R}_{\neq 0})^n \to (\mathbb{R}_{\neq 0})^n$. Since $(\mathbb{R}_{\geq 0})^n \setminus (\mathbb{R}_{>0})^n$ contains a non-empty open subset of $\mathbb{R}^n \setminus (\mathbb{R}_{\neq 0})^n$, the restriction of $\theta_{\mathbb{R}}$ to $(\mathbb{R}_{\geq 0})^n \setminus (\mathbb{R}_{>0})^n$ is a rational map $(\mathbb{R}_{\geq 0})^n \setminus (\mathbb{R}_{>0})^n \to (\mathbb{R}_{\neq 0})^n$. But the latter fact contradicts Claim 3.20(b). This contradiction proves the lemma. □

Given a positive variety (X, Θ), we simply say that a non-zero rational or regular function $f : X \to \mathbb{A}^1$ is Θ-positive if f is positive $(\Theta, \Theta_{\mathbb{A}^1})$-positive.

Based on Claim 3.9, for any $w \in W$ we denote by Θ^w (resp. by Θ_w^-) the positive structure on $U^w = U \cap B^- w B^-$ (resp. on $B_w^- = B^- \cap U \overline{w} B$) containing each toric

chart $\theta^+_{\mathbf{i}}$ (resp. $\theta^-_{\mathbf{i}}$) for $\mathbf{i} \in R(w)$. Therefore, (B^-_w, Θ^-_w) and (U^w, Θ^w) are positive varieties. The following result complements Lemma 3.19.

Claim 3.21. [1, Lemma 2.13] *The support of each positive variety (B^-_w, Θ^-_w) and (U^w, Θ^w) has co-dimension at least 2 in B^-_w and U^w, respectively.*

Remark 3.22. It is easy to show (see Example 3.8) that for $G = GL_3$, the complement $U^{w_0} \setminus \mathrm{Supp}(U^{w_0}, \Theta^{w_0})$ is precisely $\begin{pmatrix} 1 & 0 & \mathbb{G}_m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In particular, the morphism $f : \mathbb{G}_m \to U^{w_0}$ given by $c \mapsto \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ satisfies $\mathrm{ran}(f) \cap \mathrm{Supp}(U^{w_0}, \Theta^{w_0}) = \emptyset$. Therefore, f is not a morphism of positive varieties (because it fails the requirement (1) of Definition 3.12).

Claim 3.23. *Let $w, w' \in W$ be such that $l(ww') = l(w) + l(w')$. Then the natural factorizations $U^{ww'} \xrightarrow{\sim} U^w \times U^{w'}$, $B^-_{ww'} = B^-_w \times B^-_{w'}$ are, respectively, the isomorphisms of positive varieties*

$$(U^{ww'}, \Theta^{ww'}) \xrightarrow{\sim} (U^w, \Theta^w) \times (U^{w'}, \Theta^{w'}),\ (B^-_{ww'}, \Theta^-_{ww'}) \xrightarrow{\sim} (B^-_w, \Theta^-_w) \times (B^-_{w'} \times \Theta^-_{w'})\ .$$

Note that for any sub-torus $T' \subset T$ and $w \in W$ we have canonical isomorphisms $T' \cdot U^w \cong T' \times U^w$ and $T' \cdot B^-_w \cong T' \times B^-_w$. We will denote the positive structures on $T' \cdot U^w$ and $T' \cdot B^-_w$ compatible with these isomorphisms by $\Theta_{T'} \cdot \Theta^w$ and $\Theta_{T'} \cdot \Theta^-_w$, respectively. Finally, for each standard parabolic subgroup P, we abbreviate

$$\Theta^P := \Theta_{Z(L_P)} \cdot \Theta^{w_P}, \Theta^-_P := \Theta_{Z(L_P)} \cdot \Theta^-_{w_P}\ ,$$

where L_P is the Levi factor of P.

Note that for $P = B$, the intersection $B^- \cap X_B$ is open and dense in B^-. Therefore, we regard each Θ^-_B as a positive structure on the entire B^-.

Remark 3.24. We expect that $\mathrm{Supp}(B^-, \Theta^-_B) = \mathrm{Supp}(B^-_{w_0}, \Theta^-_{w_0})$.

Recall that π^+ and π^- are the regular projections $B^- \cdot U \to U$ and $B^- \cdot U \to B^-$, respectively (as defined in (1.17)). Since $B^- wB^- \cdot w^{-1} = B^- wU^- \cdot w^{-1} \subset B^- \cdot U$, we can define a morphism $\eta^w : U^w \to B^-$ by

$$\eta^w(u) = \iota(\pi^-(u\overline{w^{-1}})) \tag{3.5}$$

for $u \in U^w$, where $\iota : B^- \to B^-$ is the positive inverse defined in (1.1)). The following fact was proved in [2, Section 4.1].

Claim 3.25. *The morphism η^w is a biregular isomorphism $U^w \xrightarrow{\sim} B^-_w$. The inverse isomorphism $\eta_w = (\eta^w)^{-1} : B^-_w \xrightarrow{\sim} U^w$ is given by*

$$\eta_w(b) = \iota(\pi^+(\overline{w}^{-1}b))\ . \tag{3.6}$$

Claim 3.26. [5, Theorem 4.7] *For each $w \in W$, the morphism $\eta^w : U^w \to B^-_w$ is an isomorphism of positive varieties $(B^-_w, \Theta^-_w) \xrightarrow{\sim} (U^w, \Theta^w)$.*

The following fact was proved (in a slightly different form) in [2, Proposition 4.7].

Claim 3.27. *The multiplication $B^- \times B^- \to B^-$ is a morphism of positive varieties $(B^-, \Theta_B^-) \times (B^-, \Theta_B^-) \to (B^-, \Theta_B^-)$.*

Remark 3.28. Generalizing Claim 3.27 and a part of Claim 3.23, one can easily show (see Claim 1.12 above and [2, Proposition 4.7]) that for any $w, w' \in W$, the multiplication morphisms $U^w \times U^{w'} \to U^{w\star w'}$ and $B_w^- \times B_{w'}^- \to T \cdot B_{w\star w'}^-$ are, respectively, the morphisms of positive varieties $(U^w, \Theta^w) \times (U^{w'}, \Theta^{w'}) \to (U^{w\star w'}, \Theta^{w'\star w''})$ and $(B_w^-, \Theta^w) \times (B_{w'}^-, \Theta_{w'}^-) \to (T \cdot B_{w\star w'}^-, \Theta_T \cdot \Theta_{w'\star w''}^-)$.

3.2. **Positive geometric crystals and positive unipotent bicrystals.** Let $\mathcal{X} = (X, \gamma, \varphi_i, \varepsilon_i, e_i^{\cdot} | i \in I)$ be a geometric pre-crystal and let Θ be a positive structure on the variety X. We say that $(\mathcal{X}, \Theta)$ is a *positive geometric pre-crystal* if:
- $\gamma : X \to T$ is a morphism of positive varieties $(X, \Theta_X) \to (T, \Theta_T)$.
- The functions $\varphi_i, \varepsilon_i : S \to \mathbb{A}^1$ are Θ-positive.
- For each $i \in I$, the action $e_i^{\cdot} : \mathbb{G}_m \times X \to X$ is $(\Theta_{\mathbb{G}_m} \times \Theta, \Theta)$-positive.

Recall that decorated geometric (pre)crystals are defined in Section 2.4. We say that a triple $(\mathcal{X}, f, \Theta)$ is a *positive* decorated geometric pre-crystal if $(\mathcal{X}, f)$ is a decorated pre-crystal, $(\mathcal{X}, \Theta)$ is a positive pre-crystal, and $f : X \to \mathbb{A}^1$ is Θ-positive.

Based on Definitions 2.15, 2.33 and Claim 3.14, we can define product of positive geometric pre-crystals and the product of decorated positive geometric pre-crystals. Both products are associative.

Definition 3.29. A *positive U-bicrystal* of type w is a triple $(\mathbf{X}, \mathbf{p}, \Theta)$, where $(\mathbf{X}, \mathbf{p})$ is a U-bicrystal of type w and Θ is a positive structure on the variety $X^- = \mathbf{p}^{-1}(B^-)$ such that:
(P1) The rational morphism $\mathbf{p}|_{X^-} : X^- \to TB_w^-$ is $(\Theta, \Theta_T \cdot \Theta_w^-)$-positive.
(P2) The pair $(\mathcal{F}(\mathbf{X}, \mathbf{p}), \Theta)$ (see (2.12)) is a positive geometric crystal.

If f is a $(U \times U, \chi^{st})$-linear function on X satisfying:
(P3) The function $f|_{X^-} : X^- \to \mathbb{A}^1$ is Θ-positive,
then we say that the quadruple $(\mathbf{X}, \mathbf{p}, f, \Theta)$ is a positive $(U \times U, \chi^{st})$-linear bicrystal.

In the notation of (2.20), for any positive unipotent bicrystal $(\mathbf{X}, \mathbf{p}, \Theta)$, we denote

$$\mathcal{F}(\mathbf{X}, \mathbf{p}, \Theta) := (\mathcal{F}(\mathbf{X}, \mathbf{p}), \Theta);$$

and for any positive $(U \times U, \chi^{st})$-linear bicrystal $(\mathbf{X}, \mathbf{p}, \Theta)$ we denote

$$\mathcal{F}(\mathbf{X}, \mathbf{p}, f, \Theta) := (\mathcal{F}(\mathbf{X}, \mathbf{p}, f), \Theta) \ .$$

Lemma 3.30.
(a) *$\mathcal{F}(\mathbf{X}, \mathbf{p}, \Theta)$ is a positive geometric crystal for any positive unipotent bicrystal $(\mathbf{X}, \mathbf{p}, \Theta)$.*
(b) *$\mathcal{F}(\mathbf{X}, \mathbf{p}, f, \Theta)$ is a positive decorated geometric crystal for any positive $(U \times U, \chi^{st})$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f, \Theta)$.*

Proof. The condition (P1) implies both the (Θ, Θ_T)-positivity of the morphism γ and the Θ-positivity of the functions φ_i, ε_i involved in the geometric crystal $\mathcal{F}(\mathbf{X}, \mathbf{p})$. This proves (a). Part (b) also follows. The lemma is proved. □

Claim 3.31. *Let $(\mathbf{X}, \mathbf{p}, \Theta)$ be a positive unipotent bicrystal. Then both rational morphisms $hw_X, lw_X : X^- \to T$ (defined in (2.5)) are (Θ, Θ_T)-positive.*

Furthermore, we will investigate the behavior of positive bicrystals under convolution products.

Definition 3.32. Let $(\mathbf{X}, \mathbf{p}, \Theta)$ and $(\mathbf{Y}, \mathbf{p}', \Theta')$ be positive unipotent bicrystals. Denote

$$(\mathbf{X}, \mathbf{p}, \Theta) * (\mathbf{Y}, \mathbf{p}', \Theta') := ((\mathbf{X}, \mathbf{p}) * (\mathbf{Y}, \mathbf{p}'), \Theta * \Theta') \ , \tag{3.7}$$

where $\Theta * \Theta'$ is that positive structure on the variety $(X * Y)^-$ which is obtained from $\Theta \times \Theta'$ via the birational isomorphism $j_{X,Y} : X^- \times Y^- \to (X * Y)^-$ defined in (2.3) (see also Lemma 2.4).

Let $(\mathbf{X}, \mathbf{p}, f, \Theta)$ and $(\mathbf{Y}, \mathbf{p}', f', \Theta')$ be positive $(U \times U, \chi^{st})$-linear bicrystals. Denote

$$(\mathbf{X}, \mathbf{p}, f, \Theta) * (\mathbf{Y}, \mathbf{p}', f', \Theta') := ((\mathbf{X}, \mathbf{p}, f) * (\mathbf{Y}, \mathbf{p}', f'), \Theta * \Theta') \ . \tag{3.8}$$

We can consider the category whose objects are positive unipotent (resp. positive $(U \times U, \chi^{st})$-linear)) bicrystals and arrows are structure-preserving morphisms of positive varieties (see Claim 3.14).

Claim 3.33. *The formula* (3.7) *(resp.* (3.8)*) defines a monoidal structure on the category of positive unipotent (resp. positive* $(U \times U, \chi^{st})$*-linear) bicrystals.*

Recall from Example 2.7 that the pair $(\mathbf{X}_w, \mathrm{id})$ is a unipotent bicrystal.

Claim 3.34. [2, Section 4.4] *For each* $w \in W$*, the triple* $(\mathbf{X}_w, \mathrm{id}, \Theta_w^-)$ *is a positive unipotent bicrystal.*

Definition 3.35. Let $(\mathbf{X}, \mathbf{p}, f, \Theta)$ be a positive parabolic $(U \times U, \chi^{st})$-linear bicrystal of type w_P. We say that $(\mathbf{X}, \mathbf{p}, f, \Theta)$ is *strongly positive* if the rational function $\Delta_X|_{X^-} : X^- \to \mathbb{A}^1$ is Θ-positive (where $\Delta_X : X \to \mathbb{A}^1$ is defined in (2.21)).

Next, we describe an important class of strongly positive parabolic $(U \times U, \chi^{st})$-linear bicrystals.

Lemma 3.36. *For any standard parabolic subgroup* P *of* G*, the quadruple* $(\mathbf{X}_P, \mathrm{id}, f_P, \Theta_P^-)$ *is a strongly positive parabolic* $(U \times U, \chi^{st})$*-linear bicrystal.*

Proof. One can easily show (based on results of [5] and [2]) that the restriction of the function $f_P : X_P \to \mathbb{A}^1$ to X_P^- is a Θ_P^--positive function $X_P^- \to \mathbb{A}^1$. Therefore, $(\mathbf{X}_P, \mathrm{id}, f_P, \Theta_P^-)$ is a positive parabolic $(U \times U, \chi^{st})$-linear bicrystal. At the same time, it is strongly positive since Δ_{X_P} is zero. This proves the lemma. □

Our main result on strongly positive unipotent bicrystals is the following.

Main Theorem 3.37. *For any strongly positive parabolic* $(U \times U, \chi^{st})$*-linear bicrystals* $(\mathbf{X}, \mathbf{p}, f, \Theta)$ *and* $(\mathbf{Y}, \mathbf{p}', f', \Theta')$ *of type* w_0*, their convolution product is also a strongly positive* $(U \times U, \chi^{st})$*-linear bicrystal of type* w_0*.*

Proof. Denote $(\mathbf{Z}, \mathbf{p}'', f'', \Theta'') = (\mathbf{X}, \mathbf{p}, f, \Theta) * (\mathbf{Y}, \mathbf{p}', f', \Theta')$. That is, $Z = X * Y$, $Z^- = (X * Y)^- = (\mathbf{p}'')^{-1}(B^-)$, and $\Theta'' = \Theta * \Theta'$. All we have to show is that the restriction of $\mathbf{p}''$ to Z^- is a positive morphism $Z^- \to B^-$ and the restriction of Δ_Z to Z^- is a positive rational function $Z^- \to \mathbb{A}^1$.

First, the morphism $\mathbf{p}''|_{Z^-} : Z^- \to B^-$ factors as a composition of positive morphisms:

$$Z^- \xrightarrow{j_{X,Y}^{-1}} X^- \times Y^- \xrightarrow{\mathbf{p}\times\mathbf{p}'} B^- \times B^- \to B^- \ .$$

Therefore, $\mathbf{p}''|_{Z^-}$ is a also positive.

Furthermore, (3.6) and Claim 3.26 imply that the morphism $B^- \cap Bw_0B \to U$ defined by $b \mapsto \pi^+(\overline{w}_0^{-1}b)$ (where $\pi^+ : B^- \cdot U \to U$ is the projection to the second factor defined above in (1.17)) are $(\Theta_B^-, \Theta^{w_0})$-positive. It is also easy to see that the morphism and $B^- \cap Bw_0B \to T$ defined by $b \mapsto \pi^{w_0}(b)$ is (Θ_B^-, T) positive. The multiplication morphisms $B^- \times B^- \to B^-$, $U \times U \to U$, and the involution σ are also positive. Therefore, both π and $\mathbf{m}$ (defined in (2.24) and (2.25)) are positive. Finally, the functions $\chi^{st} : U \to \mathbb{A}^1$ and the restriction of f_B to $B^- \cap Bw_0B$ are positive functions. These arguments and (2.26) imply that the restriction of Δ_Z to Z^- is a Θ''-positive function.

This proves Theorem 3.37. □

Remark 3.38. Theorem 3.37 implies that the category of all strongly positive parabolic $(U\times U, \chi^{st})$-linear bicrystals of type w_0 is closed under the convolution product. However, this category is not monoidal because it has no unit object.

Remark 3.39. We expect that the result of Theorem 3.37 holds for any parabolic bicrystals. The difficulty is to prove that Δ_{X*Y} is a positive function.

We finish the section with the construction of a positive structure Θ_Z on the unipotent bicrystal $(\mathbf{Z}_{w_0}, \mathbf{p}_Z, f_Z)$ given by (2.34).

Note that, by definition (2.29)

$$Z_{w_0}^- = (U^{w_0} \times T \times T) \times_T (Bw_0B)^- \cong U^{w_0} \times T \times T \times B_{w_0}^- \ .$$

Using this isomorphism, we define a positive structure Θ_Z on $Z_{w_0}^-$ in such a way that

$$(Z_{w_0}^-, \Theta_Z) \cong (U^{w_0}, \Theta^{w_0}) \times (T, \Theta_T) \times (T, \Theta_T) \times (B_{w_0}^-, \Theta_{w_0}^-) \ .$$

Claim 3.40. *The quadruple* $(\mathbf{Z}_{w_0}, \mathbf{p}_Z, f_Z, \Theta_Z)$ *is a strongly positive* $(U\times U, \chi^{st})$*-linear bicrystal (see* (2.34)*).*

Denote $\mathcal{X}_B = \mathcal{F}(\mathbf{X}_B, \mathrm{id}, f_B)$ and $\mathcal{Z}_{w_0} = \mathcal{F}(\mathbf{Z}_{w_0}, \mathbf{p}_Z, f_Z)$. Then the inverse of the open embedding (2.35) defines a birational isomorphism of decorated geometric crystals

$$\mathcal{X}_B \times \mathcal{X}_B \xrightarrow{\sim} \mathcal{Z}_{w_0} \ . \tag{3.9}$$

Furthermore, Lemma 3.36 and Theorem 3.37 imply that $(\mathcal{X}_B, \Theta_B) \times (\mathcal{X}_B, \Theta_B)$ is a positive decorated geometric crystal. Also Claim 3.40 implies that $(\mathcal{Z}_{w_0}, \Theta_Z)$ is a positive geometric crystal.

Claim 3.41. *The isomorphism* (3.9) *defines a morphism of positive decorated geometric crystals*

$$(\mathcal{X}_B, \Theta_B) \times (\mathcal{X}_B, \Theta_B) \xrightarrow{\sim} (\mathcal{Z}_{w_0}, \Theta_Z) \ . \tag{3.10}$$

We expect that the inverse of (3.10) is also positive (see Conjecture 7.13).

4. Semi-field of polytopes and tropicalization

4.1. **Semi-field of polytopes.** Let E be a finite-dimensional vector space over $\mathbb{R}$. Denote by $\mathcal{P}_E$ the set of all convex polytopes in E.

The *Minkovski sum* $P+Q=\{p+q|p\in P, q\in Q\}$ defines the structure of an abelian monoid on $\mathcal{P}_E$ (with the unit element - the single point 0).

For any $P\in\mathcal{P}_E$, define the *support function* $\chi_P : E^* \to \mathbb{R}$ by

$$\chi_P(\xi)=\min_{p\in P}\xi(p)$$

for any $\xi\in E^*$. Note that

$$\chi_P(\xi)=\min_{v\in \mathrm{Vert}(P)}\xi(v)$$

where $\mathrm{Vert}(P)$ is the set of vertices of P.

Denote by $\mathrm{Fun}(E^*,\mathbb{R})$ the set of all functions $\tilde{f}: E^*\to\mathbb{R}$. Clearly, $\mathrm{Fun}(E^*,\mathbb{R})$ is an abelian monoid under the pointwise addition of functions $(\tilde{f},\tilde{g})\mapsto \tilde{f}+\tilde{g}$.

Claim 4.1. *The correspondence $P\mapsto\chi_P$ is an injective homomorphism of monoids $\chi:\mathcal{P}_E\to Fun(E^*,\mathbb{R})$, i.e., for any $P,Q\in\mathcal{P}_E$, one has:*

(a) $\chi_{P+Q}=\chi_P+\chi_Q$.
(b) $\chi_P=\chi_Q$ *if and only if* $P=Q$.

This immediately implies the following well-known result.

Corollary 4.2. *The monoid $\mathcal{P}_E$ has a cancellation property:*

$$P+R=Q+R <=> P=Q\ . \tag{4.1}$$

For any two functions $\tilde{f},\tilde{g}\in Fun(E^*,\mathbb{R})$, we write $\tilde{f}\le\tilde{g}$ if $\tilde{f}(\xi)\le\tilde{g}(\xi)$ for all $\xi\in E^*$. Clearly, this defines a partial order on $\mathrm{Fun}(E^*,\mathbb{R})$.

Claim 4.3. *We have for any convex polytopes P,Q:*

$$P\supseteq Q <=> \chi_P\le\chi_Q\ ,$$

that is, the two partial orders on $\mathcal{P}_E$ are equal.

For any convex polytopes $P,Q\in\mathcal{P}_E$, let $P\vee Q$ be the convex hull of $P\cup Q$. Clearly,

$$\chi_{P\vee Q}=\min(\chi_P,\chi_Q)$$

for any convex polytopes $P,Q\in\mathcal{P}_E$. This implies the following well-known fact (see e.g., [10]).

Corollary 4.4. *The operation $\vee$ satisfies*

$$(P\vee Q)+R=(P+R)\vee(Q+R)$$

for any $P,Q,R\in\mathcal{P}_E$ (i.e., $\mathcal{P}_E$ is a semi-ring with the "multiplication" $+$ and the "addition" $\vee$).

Therefore, if we make the monoid $\mathrm{Fun}(E^*,\mathbb{R})$ into a semi-ring with the operation of "addition" $(\tilde{f},\tilde{g})\mapsto\min(\tilde{f},\tilde{g})$ and the "multiplication" $(\tilde{f},\tilde{g})\mapsto\tilde{f}+\tilde{g}$, the correspondence $P\mapsto\chi_P$ is an order-preserving homomorphism of monoids.

Let us define the *Grothendieck group* $\mathcal{P}_E^+$ of the monoid $\mathcal{P}_E$ by the standard construction: $\mathcal{P}_E^+$ is the quotient of the free abelian group generated by elements of the

form $[P]$, $P \in \mathcal{P}_E$ by the relations of the form $[P+Q]-[P]-[Q]$, $P, Q \in \mathcal{P}_E$. The group $\mathcal{P}_E^+$ is universal in the sense that one has the canonical homomorphism of monoids $\mathbf{j} : \mathcal{P}_E \to \mathcal{P}_E^+$ $(P \mapsto [P])$ and $\mathbf{j}(\mathcal{P}_E)$ generates the group $\mathcal{P}_E^+$. The homomorphism $\mathbf{j}$ is injective due to Claim 4.1. The elements $[P]-[Q]$ of the group are referred to as *virtual polytopes* (see e.g., [22], [24], [25]).

The above operations and homomorphisms extend to the group $\mathcal{P}_E^+$ as follows.

1. The operation $\vee$ uniquely extends to the group $\mathcal{P}_E^+$ via the "quotient" rule:
$$([P]-[Q]) \vee ([P']-[Q']) = [(P+Q') \vee (P'+Q)] - [Q+Q']$$
and, therefore, turns $\mathcal{P}_E^+$ into a semi-field.
2. The homomorphism $\chi : \mathcal{P}_E \to Fun(E^*, \mathbb{R})$ defines an injective semi-field homomorphism $\mathcal{P}_E^+ \to Fun(E^*, \mathbb{R})$ via
$$[P]-[Q] \mapsto \chi_{[P]-[Q]} = \chi_P - \chi_Q \ .$$
3. The containment relation $\supset$ extends to a relation $\prec$ on $\mathcal{P}_E^+$ via $[P]-[Q] \preceq [P']-[Q']$ if and only if $P+Q' \supseteq P'+Q$. This relation satisfies
$$[P]-[Q] \preceq [P']-[Q'] <=> \chi_{[P]-[Q]} \leq \chi_{[P']-[Q']} \tag{4.2}$$
for any $[P]-[Q], [P']-[Q'] \in \mathcal{P}_E^+$.

4.2. **Newton polytopes and tropicalization.** Let S be a split algebraic torus defined over $\mathbb{Q}$. For a regular function f on S of the form $f = \sum_{\mu \in X^\star(S)} c_\mu \cdot \mu$ define the *Newton polytope* of f in $E = \mathbb{R} \otimes X^\star(S)$ to be the convex hull of all those $\mu \in X^\star(S)$ such that $c_\mu \neq 0$.

Claim 4.5. *For any non-zero regular functions $f, g : S \to \mathbb{A}^1$, one has*
$$N(fg) = N(f) + N(g), \ N(f+g) \subseteq N(f) \vee N(g),$$
i.e, the association $f \mapsto N(f)$ is a homomorphism of monoids $\mathbb{Q}[X^\star(S)]^\times \to \mathcal{P}_E$.

Definition 4.6. For any non-zero rational function $h : S \to \mathbb{A}^1$, define the *virtual Newton polytope* $[N](h)$ by
$$[N](h) = [N(f)] - [N(g)] \ ,$$
where $f, g : S \to \mathbb{A}^1$ are non-zero regular functions such that $h = \frac{f}{g}$.

Clearly, $[N](h)$ is well-defined.

Claim 4.7. *The correspondence $h \mapsto [N](h)$ is a homomorphism of abelian groups $[N] : Frac(S)^\times \to \mathcal{P}_E^+$ (where $Frac(S)$ is the field of rational functions on S), i.e.,*
$$[N](fg) = [N](f) + [N](g)$$
for any non-zero rational functions $f, g : S \to \mathbb{A}^1$. This homomorphism also satisfies
$$[N](f) \vee [N](g) \preceq [N](f+g)$$
for any non-zero rational functions $f, g : S \to \mathbb{A}^1$ such that $f+g \neq 0$.

For each rational function $h : S \to \mathbb{A}^1$, define a function $\mathrm{Trop}(h) : X_\star(S) \to \mathbb{Z}$ by

$$\mathrm{Trop}(h) := \chi_{[N](h)}$$

and refer to $\mathrm{Trop}(h)$ as the *tropicalization* of the rational function h.

In particular, for any non-zero regular function $f : S \to \mathbb{A}^1$, we have

$$\mathrm{Trop}(f)(\lambda) := \chi_{N(f)}(\lambda) = \min_{\mu \in N(f)} \langle \mu, \lambda \rangle$$

(here we used the identification of the lattice $X_\star(S)$ with $(X^\star(S))^*$ via $\lambda \mapsto \langle \cdot, \lambda \rangle$ for each co-character $\lambda \in X_\star(S)$).

Lemma 4.8. *For any non-zero rational functions: $f, g : S \to \mathbb{A}^1$, we have:*

(a) $\mathrm{Trop}(fg) = \mathrm{Trop}(f) + \mathrm{Trop}(g)$, $\mathrm{Trop}(\frac{f}{g}) = \mathrm{Trop}(f) - \mathrm{Trop}(g)$.

(b) *If $f + g \neq 0$ then $\mathrm{Trop}(f + g) \geq \min(\mathrm{Trop}(f), \mathrm{Trop}(g))$; the equality*

$$\mathrm{Trop}(f + g) = \min(\mathrm{Trop}(f), \mathrm{Trop}(g)) \tag{4.3}$$

is achieved if and only if $[N](f + g) = [N](f) \vee [N](g)$.

Proof. Follows from (4.2) and Claim 4.7. □

Below we will list some sufficient conditions for (4.3) to hold.

Definition 4.9. A rational function $f : S \to \mathbb{A}^1$ on a split algebraic torus S is called *positive* if it can be written as a ratio $f = \frac{f'}{f''}$, where f' and f'' are linear combinations of characters with positive integer coefficients.

Corollary 4.10. *Let $f_1, f_2, \ldots, f_k$ be non-zero rational functions on S. Then:*

(a) *Assume additionally that the functions $f_1, f_2, \ldots, f_k$ are regular and the vertex sets $\mathrm{Vert}(N(f_1))$, …, $\mathrm{Vert}(N(f_k))$ of their Newton polytopes are pairwise disjoint, i.e., $\mathrm{Vert}(N(f_i)) \cap \mathrm{Vert}(N(f_j)) = \emptyset$ for all $i \neq j$. Then*

$$N(f_1 + f_2 + \cdots + f_k) = N(f_1) \vee N(f_2) \vee \cdots \vee N(f_k)$$

and $\mathrm{Trop}(f_1 + f_2 + \cdots + f_k) = \min(\mathrm{Trop}(f_1), \mathrm{Trop}(f_2), \ldots, \mathrm{Trop}(f_k))$.

(b) *Let and $n_1, n_2, \ldots, n_k$ be pairwise distinct integers. Let f be the rational function $\mathbb{G}_m \times S \to \mathbb{A}^1$ of the form*

$$f(c, s) = c^{n_1} f_1(s) + c^{n_2} f_2(s) + \cdots + c^{n_k} f_k(s) \ .$$

Then

$$[N](f) = (n_1, [N](f_1)) \vee (n_2, [N](f_2)) \vee \cdots \vee (n_k, [N](f_k))$$

and $\mathrm{Trop}(f) = \min(n_1 \cdot + \mathrm{Trop}(f_1), n_2 \cdot + \mathrm{Trop}(f_2), \ldots, n_k \cdot + \mathrm{Trop}(f_k))$.

(c) *Let $\mu_1, \mu_2, \ldots, \mu_k$ be pairwise distinct characters of an algebraic torus T'. Let f be the rational function $T' \times S \to \mathbb{A}^1$ of the form*

$$f(t, s) = \mu_1(t) f_1(s) + \mu_2(t) f_2(s) + \cdots \mu_n(t) f_k(s) \ .$$

Then

$$[N](f) = (\mu_1, [N](f_1)) \vee (\mu_2, [N](f_2)) \vee \cdots \vee (\mu_k, [N](f_k))$$

and

$$\mathrm{Trop}(f) = \min(\langle \mu_1, \bullet \rangle + \mathrm{Trop}(f_1), \langle \mu_2, \bullet \rangle + \mathrm{Trop}(f_2), \ldots, \langle \mu_k, \bullet \rangle + \mathrm{Trop}(f_k)) \ .$$

(d) *Assume additionally that* $f_1, f_2, \dots, f_k$ *are positive (see Section 3.1). Then*

$$[N](f_1 + f_2 + \cdots + f_k) = [N](f_1) \vee [N](f_2) \vee \cdots \vee [N](f_k)$$

$$\textit{and } \mathrm{Trop}(f_1 + f_2 + \cdots + f_k) = \min(\mathrm{Trop}(f_1), \mathrm{Trop}(f_2), \dots, \mathrm{Trop}(f_k)).$$

Next, we extend the correspondence $f \mapsto \mathrm{Trop}(f)$ to the case when f is a rational morphism of algebraic tori.

Let now S and S' be split algebraic tori defined over $\mathbb{Q}$, and let $f : S \to S'$ be a rational morphism (note that all regular morphisms $S \to S'$ are group homomorphisms up to translations). Define the map $\mathrm{Trop}(f) : X_\star(S) \to X_\star(S')$ by the formula

$$\langle \mu', \mathrm{Trop}(f)(\lambda) \rangle = \mathrm{Trop}(\mu' \circ f)(\lambda)$$

for any $\lambda \in X_\star(S)$, $\mu' \in X^\star(S')$, where the rational function $\mu' \circ f : S \to \mathbb{G}_m$ is considered as a rational function $S \to \mathbb{A}^1$.

Next, we will consider an appropriate category of split algebraic tori for which the correspondence $f \mapsto \mathrm{Trop}(f)$ is a functor.

Definition 4.11. A *marked set* is a pair $(A, \mathbf{0})$ where A is a set, $\mathbf{0} \in A$ is a *marked point*. Denote by $\mathbf{Set_0}$ the category whose objects are marked sets and morphisms are structure-preserving maps.

Recall from Section 3.1 that $\mathcal{T}_+$ is the category whose objects are split algebraic tori (defined over $\mathbb{Q}$), and arrows are positive rational morphisms.

The following is a slightly modified form of the result of [2, Section 2.4].

Theorem 4.12 ([2])**.** *The correspondence* $S \mapsto X_\star(S)$, $f \mapsto \mathrm{Trop}(f)$ *is a functor*

$$\mathrm{Trop} : \mathcal{T}_+ \to \mathbf{Set_0} \ .$$

Remark 4.13. The theorem implies that for any positive birational isomorphism $f : S \xrightarrow{\sim} S'$ such that f^{-1} is also positive the tropicalization $\mathrm{Trop}(f)$ is a bijection $X_\star(S) \xrightarrow{\sim} X_\star(S')$. The converse is not true: if $f : \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{G}_m$ is as in Remark 3.5, i.e., of f^{-1} is not positive, then $\tilde{f} = \mathrm{Trop}(f) : \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by $\tilde{f}(\tilde{x}, \tilde{y}) = (\tilde{x}, \min(\tilde{x}, \tilde{y}))$. Clearly, $\tilde{f}$ is not a bijection.

We conclude this section with a discussion of the tropicalization of those (non-positive) functions which are compositions of positive morphisms with certain rational functions.

Definition 4.14. We say that a rational function $f : S \to \mathbb{A}^1$ is *half-positive* if it can be presented as a difference of two positive functions.

Clearly, positive and each regular functions on S are half-positive. It is also clear that half-positive functions on S are closed under addition, subtraction, and multiplication – they form a sub-ring of the field $Frac(S)$. Also, positive functions act on half-positive ones by multiplication and composition of each half-positive function with any positive morphism is also a half-positive function. The restriction of each half-positive function on $S = (\mathbb{G}_m)^\ell$ to $(\mathbb{R}_{>0})^\ell$ is a well-defined map $(\mathbb{R}_{>0})^\ell \to \mathbb{R}$. One can expect that this property is characteristic of half-positive functions.

Lemma 4.15. *Let $f : S' \to \mathbb{A}^1$ be a non-zero half-positive function. Then for any positive rational morphism $h : S \to S'$, we have*

$$\mathrm{Trop}(f \circ h) \geq \mathrm{Trop}(f) \circ \mathrm{Trop}(h) \ . \tag{4.4}$$

Proof. Let us write f as $f = f_+ - f_-$ where both f_+ and f_- are positive rational functions on S'. If $f_+ = 0$ or $f_- = 0$ then we have nothing to prove because Theorem 4.12 guarantees the equality in (4.4).

Note that, by Theorem 4.12, we have $\mathrm{Trop}(f_+ \circ h) = \mathrm{Trop}(f_+) \circ \mathrm{Trop}(h)$.

Then by Lemma 4.8(b),

$$\begin{aligned}\mathrm{Trop}(f \circ h) &= \mathrm{Trop}(f_+ \circ h - f_- \circ h) \geq \min(\mathrm{Trop}(f_+ \circ h), \mathrm{Trop}(f_- \circ h)) \\ &= \min(\mathrm{Trop}(f_+) \circ \mathrm{Trop}(h), \mathrm{Trop}(f_-) \circ \mathrm{Trop}(h)) \\ &= \min(\mathrm{Trop}(f_+), \mathrm{Trop}(f_-)) \circ \mathrm{Trop}(h) = \mathrm{Trop}(f) \circ \mathrm{Trop}(h)\end{aligned}$$

because $\mathrm{Trop}(f) = \min(Trop(f_+), \mathrm{Trop}(f_-))$.

The lemma is proved. □

Definition 4.16. We say that a birational isomorphism $h : S \to S'$ is a *positive equivalence* if both h and h^{-1} are positive.

Proposition 4.17. *Let $f : S \to \mathbb{A}^1$ be a non-zero rational function on a split algebraic torus S. Then for any positive equivalence $h : S' \to S$ of split algebraic tori, one has*

$$\mathrm{Trop}(f \circ h) = \mathrm{Trop}(f) \circ \mathrm{Trop}(h) \ . \tag{4.5}$$

Proof. First, let us prove the assertion for any half-positive function. Since in this case $f \circ h$ is also half-positive, we obtain by Lemma 4.15. Therefore,

$$\mathrm{Trop}(f) = \mathrm{Trop}(f \circ h \circ h^{-1}) \geq \mathrm{Trop}(f \circ h) \circ \mathrm{Trop}(h^{-1})$$

$$= \mathrm{Trop}(f \circ h) \circ \mathrm{Trop}(h)^{-1} \geq \mathrm{Trop}(f) \circ \mathrm{Trop}(h) \circ \mathrm{Trop}(h)^{-1} = \mathrm{Trop}(f) \ ,$$

that is, this chain of inequalities becomes a chain of equalities, i.e., $\mathrm{Trop}(f) = \mathrm{Trop}(f \circ h) \circ \mathrm{Trop}(h)^{-1}$ which implies (4.5). Finally, note that each non-zero rational function $f : S' \to \mathbb{A}^1$ can be expressed as a ratio $f = \frac{f'}{f''}$ where f' and f'' are regular hence half-positive and, therefore, f' and f'' already satisfy (4.5). Then (4.5) follows for f as well by Lemma 4.8(a).

The lemma is proved. □

We say that two non-zero rational functions $f : S \to \mathbb{A}^1$ and $f' : S' \to \mathbb{A}^1$ are *positively equivalent* if there exists a positive equivalence $h : S' \to S$ such that $f' = f \circ h$.

Corollary 4.18. *Let $f : S \to \mathbb{A}^1$ be a non-zero rational function. Then the isomorphism class of the function $\mathrm{Trop}(f) : X_\star(S) \to \mathbb{Z}$ in $\mathbf{Set_0}$ depends only the positive equivalence class of f.*

5. Kashiwara crystals, perfect bases, and associated crystals

5.1. **Kashiwara crystals and normal crystals.** First, we recall some definitions from [2, Appendix].

Definition 5.1. A *partial bijection* of sets $\tilde{f} : \tilde{A} \to \tilde{B}$ is a bijection $\tilde{A}' \to \tilde{B}'$ of subsets $\tilde{A}' \subset \tilde{A}$, $\tilde{B}' \subset \tilde{B}$. We denote the subset $\tilde{A}'$ by $\mathrm{dom}(f)$ and the subset $\tilde{B}'$ by $\mathrm{ran}(\tilde{f})$.

The inverse $\tilde{f}^{-1}$ of a partial bijection $\tilde{f} : \tilde{A} \to \tilde{B}$ is the inverse bijection $\mathrm{ran}(\tilde{f}) \to \mathrm{dom}(\tilde{f})$. The composition $\tilde{g} \circ \tilde{f}$ of partial bijections $\tilde{f} : \tilde{A} \to \tilde{B}$, $\tilde{g} : \tilde{B} \to \tilde{C}$ is a partial bijection with $\mathrm{dom}(\tilde{g} \circ \tilde{f}) = \mathrm{dom}(\tilde{f}) \cap \tilde{f}^{-1}(\mathrm{ran}(\tilde{f}) \cap \mathrm{dom}(\tilde{g}))$ and $\mathrm{ran}(\tilde{g} \circ f) = \tilde{g}(\mathrm{ran}(\tilde{f}) \cap \ dom(\tilde{g}))$. In particular, for any partial bijection $f : \tilde{B} \to \tilde{B}$ and $n \in \mathbb{Z}$ the n-th power $\tilde{f}^n$ is a partial bijection $\tilde{B} \to \tilde{B}$.

For a partial bijection $\tilde{f} : \tilde{A} \to \tilde{B}$, we will sometimes use a notation: $\tilde{f}(\tilde{a}) \in \tilde{B}$ if and only if $\tilde{a} \in \mathrm{dom}(\tilde{f})$.

Definition 5.2. Following [14] and [2, Appendix], we say that a *Kashiwara crystal* is a 5-tuple $\mathcal{B} = (\tilde{B}, \tilde{\gamma}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$, where $\tilde{B}$ is a set, $\tilde{\gamma} : \tilde{B} \to X_\star(T)$ is a map, $\tilde{\varphi}_i, \tilde{\varepsilon}_i : \tilde{B} \to \mathbb{Z}$ are functions, and each $\tilde{e}_i : \tilde{B} \to \tilde{B}$, $i \in I$, is a partial bijection such that either $\tilde{e}_i = \mathrm{id}_{\tilde{B}}$, $\tilde{\varepsilon}_i = \tilde{\varphi}_i = -\infty$ or:

$$\tilde{\varphi}_i(\tilde{b}) - \tilde{\varepsilon}_i(\tilde{b}) = \left\langle \alpha_i, \tilde{\gamma}(\tilde{b}) \right\rangle$$

for all $\tilde{b} \in \tilde{B}$ and

$$\tilde{\gamma}(\tilde{e}_i^n(\tilde{b})) = \tilde{\gamma}(\tilde{b}) + n\alpha_i^\vee$$

whenever $\tilde{e}_i^n(\tilde{b}) \in \tilde{B}$.

We define $\mathrm{Supp}\,\mathcal{B} = \{i \in I : \tilde{e}_i \neq \mathrm{id}\}$ and call it the *support* of $\mathcal{B}$.

Example 5.3. Along the lines of Example 2.13, $X_\star(T)$ is a *trivial* Kashiwara crystal with $\tilde{\gamma} = \mathrm{id}$, $\tilde{\varepsilon}_i = \tilde{\varphi}_i = -\infty$, $\tilde{e}_i = \mathrm{id}$ for all $i \in I$. The support of this trivial crystal is the empty set $\emptyset$. Another example of a trivial Kashiwara crystal is any subset of $X_\star(T)$, in particular, the single point $0 \in X_\star(T)$. Yet another example of a Kashiwara crystal on $X_\star(T)$ is a 5-tuple $\mathcal{B}_T = (X_\star(T), id_{X_\star(T)}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$, where $\tilde{e}_i^n(\mu) = \mu + n\alpha_i^\vee$ for all $n \in \mathbb{Z}_{\geq 0}$, $\mu \in X_\star(T)$, $i \in I$ and $\widetilde{\varepsilon}_i, \widetilde{\varphi}_i \in X^\star(T)$ are such that $\langle \tilde{\varepsilon}_i, \alpha_i^\vee \rangle = -1$, $\tilde{\varphi}_i = \tilde{\varepsilon}_i + \alpha_i^\vee$ for all $i \in I$.

Definition 5.4. A *homomorphism of Kashiwara crystals* $\tilde{h} : \mathcal{B} \to \mathcal{B}'$ is a pair (h, J), where $h : \tilde{B} \to \tilde{B}'$ is a map and $J \subset \mathrm{Supp}\,\mathcal{B} \cap \mathrm{Supp}\,\mathcal{B}'$ such that $h \circ \tilde{e}_i^n = \tilde{e}_i^n \circ h$ for $i \in \mathrm{Supp}\,\mathcal{B}$, $n \in \mathbb{Z}$ and

$$\tilde{\varphi}'_j \circ h = \varphi_j, \ \tilde{\varepsilon}'_j \circ h = \varepsilon_j$$

for all $j \in J$ (we will refer to this J as the *support* of $\tilde{h}$ and denote by $\mathrm{Supp}\,\tilde{h}$).

The composition of homomorphisms $(\tilde{h}, J) : \mathcal{B} \to \mathcal{B}'$ and $(\tilde{h}', J') : \mathcal{B}' \to \mathcal{B}''$ is defined by $(\tilde{h}', J') \circ (\tilde{h}, J) := (\tilde{h}' \circ \tilde{h}, J \cap J')$ (i.e., $\mathrm{Supp}\,\tilde{h}' \circ \tilde{h} := \mathrm{Supp}\,\tilde{h} \cap \mathrm{Supp}\,\tilde{h}'$).

Remark 5.5. In the case when $\mathrm{Supp}\,\mathcal{B} = \mathrm{Supp}\,\mathcal{B}' = \mathrm{Supp}\,\tilde{h}$, our definition of a homomorphism of Kashiwara crystals $\tilde{h} : \mathcal{B} \to \mathcal{B}'$ coincides with the definition of *strict* homomorphism of crystals from the original work [14].

In what follows, we consider the category whose objects are Kashiwara crystals and morphisms are homomorphisms of Kashiwara crystals.

Definition 5.6. Given Kashiwara crystals $\mathcal{B} = (\tilde{B}, \tilde{\gamma}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$ and $\mathcal{B}' = (\tilde{B}, \tilde{\gamma}', \tilde{e}_i, \tilde{\varphi}'_i, \tilde{\varepsilon}'_i | i \in I)$, the *product* $\mathcal{B} \times \mathcal{B}'$ of Kashiwara crystals is a 5-tuple

$$(\tilde{B} \times \tilde{B}', \tilde{\gamma}'', \tilde{\varphi}''_i, \tilde{\varepsilon}''_i, \tilde{e}_i | i \in I) \ ,$$

where $\operatorname{Supp} \mathcal{B} \times \mathcal{B}' := \operatorname{Supp} \mathcal{B} \cup \operatorname{Supp} \mathcal{B}'$, $\tilde{\gamma}''(\tilde{b}, \tilde{b}') = \tilde{\gamma}(\tilde{b}) + \tilde{\gamma}(\tilde{b}')$ and for each $i \in \operatorname{Supp} \mathcal{B} \times \mathcal{B}'$, one has

$$\tilde{\varphi}_i(\tilde{b}, \tilde{b}') = \max(\tilde{\varphi}_i(\tilde{b}), \tilde{\varphi}_i(\tilde{b}') + \langle \alpha_i, \tilde{\gamma}(\tilde{b}) \rangle), \ \tilde{\varepsilon}_i(\tilde{b}, \tilde{b}') = \max(\tilde{\varepsilon}'_i(\tilde{b}'), \tilde{\varepsilon}_i(\tilde{b}) - \langle \alpha_i, \tilde{\gamma}'(\tilde{b}') \rangle) \ ,$$

and each partial bijection $e_i^n : \tilde{B} \times \tilde{B}' \to \tilde{B} \times \tilde{B}'$, $n \in \mathbb{Z}$, $i \in I$ is given by

$$\tilde{e}_i^n(\tilde{b}, \tilde{b}') = (\tilde{e}_i^{n_1}(\tilde{b}), \tilde{e}_i^{n_2}(\tilde{b}')) \ ,$$

for $(\tilde{b}, \tilde{b}') \in \tilde{B} \times \tilde{B}'$, where

$$n_1 = \max(\tilde{\varepsilon}_i(\tilde{b}), \tilde{\varphi}'_i(\tilde{b}')) - \max(\tilde{\varepsilon}_i(\tilde{b}) - n, \tilde{\varphi}'_i(\tilde{b}')), \tag{5.1}$$

$$n_2 = \max(\tilde{\varepsilon}_i(\tilde{b}), \tilde{\varphi}'_i(\tilde{b}') + n) - \max(\tilde{\varepsilon}_i(\tilde{b}), \tilde{\varphi}'_i(\tilde{b}')) \ . \tag{5.2}$$

It is easy to see that and the product is associative.

Remark 5.7. The above definition agrees with Kashiwara's original definition up to the permutation of factors, i.e., the product $\mathcal{B} \times \mathcal{B}'$ equals the *tensor product* $\mathcal{B}' \otimes \mathcal{B}$ in the notation of [14]. The reason for this is functoriality of the transition from the geometric crystals to Kashiwara crystals (see (6.1) below).

Example 5.8. For a Kashiwara crystal $\mathcal{B}$, the product $X_\star(T) \times \mathcal{B}$ (where $X_\star(T)$ is considered the trivial Kashiwara crystal as in Example 5.3) is a Kashiwara crystal with $\operatorname{Supp}(X_\star(T) \times \mathcal{B}) = \operatorname{Supp} \mathcal{B}$. The projection to the first factor is a homomorphism of Kashiwara crystals $X_\star(T) \times \mathcal{B} \to X_\star(T)$. The support of the homomorphism is $\emptyset$.

Claim 5.9. *Let $\mathcal{B}$ and $\mathcal{B}'$ be Kashiwara crystals and let $i \in \operatorname{Supp} \mathcal{B} \cup \operatorname{Supp} \mathcal{B}'$, and $n \in \mathbb{Z} \setminus \{0\}$. Then for any $\tilde{b} \in \mathcal{B}$ such that $\tilde{e}_i^n(\tilde{b}) \in \mathcal{B}$ and any $\tilde{b}' \in \mathcal{B}'$, one has*

(a) $\tilde{e}_i^n(\tilde{b}', \tilde{b}) = (\tilde{b}', \tilde{e}_i^n(\tilde{b}))$ *if and only if* $\tilde{\varphi}_i(\tilde{b}) \geq \tilde{\varepsilon}'_i(\tilde{b}')$, $n \geq \tilde{\varepsilon}'_i(\tilde{b}') - \tilde{\varphi}_i(\tilde{b})$.
(b) $\tilde{e}_i^n(\tilde{b}, \tilde{b}') = (\tilde{e}_i^n(\tilde{b}), \tilde{b}')$ *if and only if* $\tilde{\varepsilon}_i(\tilde{b}) \geq \tilde{\varphi}'_i(\tilde{b}')$, $n \leq \tilde{\varepsilon}_i(\tilde{b}) - \tilde{\varphi}'_i(\tilde{b}')$.

Following [14], for each Kashiwara crystal $\mathcal{B} = (\tilde{B}, \tilde{\gamma}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$, define the *opposite* Kashiwara crystals $\mathcal{B}^{op} = (\tilde{B}, -\tilde{\gamma}, \tilde{\varepsilon}_i, \tilde{\varphi}_i, \tilde{e}_i^{-1} | i \in I)$.

The following is a combinatorial analogue of Claim 2.19.

Claim 5.10. [14] *The correspondence $\mathcal{B} \mapsto \mathcal{B}^{op}$ defines an involutive covariant functor from the category of Kashiwara crystals into itself. This functor reverses the product, i.e., $(\mathcal{B} \times \mathcal{B}')^{op}$ is naturally isomorphic to $\mathcal{B}'^{op} \times \mathcal{B}^{op}$. On the underlying sets this isomorphism is the permutation of factors $\tilde{B} \times \tilde{B}' \xrightarrow{\sim} \tilde{B}' \times \tilde{B}$.*

Definition 5.11. For each Kashiwara crystal $\mathcal{B} = (\tilde{B}, \tilde{\gamma}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$ and a subset $\tilde{B}' \subset \tilde{B}$, we define the *sub-crystal* $\mathcal{B}|_{\tilde{B}'}$ of $\mathcal{B}$ as follows:

$$\mathcal{B}|_{\tilde{B}'} = (\tilde{B}', \tilde{\gamma}|_{\tilde{B}'}, \tilde{\varphi}_i|_{\tilde{B}'}, \tilde{\varepsilon}_i|_{\tilde{B}'}, \tilde{e}_i|_{\tilde{B}'} \ | i \in I) \ ,$$

where $\tilde{e}_i|_{\tilde{B}'}$ is the partial bijection $\tilde{B}' \to \tilde{B}'$ obtained by the restriction of the partial bijection $\tilde{e}_i : \tilde{B} \to \tilde{B}$ to $\tilde{B}'$ via $\mathrm{dom}(\tilde{e}_i|_{\tilde{B}'}) = \{\tilde{b}' \in \tilde{B}' \cap \mathrm{dom}(\tilde{e}_i) : \tilde{e}_i(\tilde{b}) \in \tilde{B}'\}$.

Clearly, the natural embedding $\tilde{B}' \subset \tilde{B}$ defines an injective homomorphism of Kashiwara crystals $\mathcal{B}|_{\tilde{B}'} \hookrightarrow \mathcal{B}$. If $\tilde{B}'$ is not empty, then $\mathrm{Supp}\,\mathcal{B}' = \mathrm{Supp}\,\mathcal{B}$ and the support of the homomorphism also equals $\mathrm{Supp}\,\mathcal{B}$ (see Definition 5.4).

Definition 5.12. We say that a Kashiwara crystal $\mathcal{B} = (\tilde{B}, \tilde{\gamma}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$ is *connected* if for any $\tilde{b}, \tilde{b}' \in \tilde{B}$ there exist a sequence $(i_1, \dots, i_\ell) \in I^\ell$ and a sequence $(n_1, \dots, n_\ell) \in \mathbb{Z}^\ell$ such that

$$\tilde{b}' = \tilde{e}_{i_1}^{n_1} \cdots \tilde{e}_{i_\ell}^{n_\ell}(\tilde{b}) \ .$$

Claim 5.13. *Each Kashiwara crystal $\mathcal{B}$ equals to the disjoint union of its connected sub-crystal.*

For each Kashiwara crystal $\mathcal{B}$, we define functions $\ell_i, \ell_{-i} : \tilde{B} \to \mathbb{Z}_{\geq 0} \sqcup \{+\infty\}$ by

$$\ell_i(\tilde{b}) = \max\{n \geq 0 : \tilde{e}_i^n(\tilde{b}) \in \tilde{B}\}, \ \ell_{-i}(\tilde{b}) = \max\{n \geq 0 : \tilde{e}_i^{-n}(\tilde{b}) \in \tilde{B}\} \tag{5.3}$$

for $\tilde{b} \in \tilde{B}$.

Definition 5.14. Given a Kashiwara crystal $\mathcal{B}$, we say that an element $\tilde{b} \in \tilde{B}$ is a *highest* (resp. *lowest*) *weight element* if $\ell_i(\tilde{b}) = 0$ (resp. $\ell_{-i}(\tilde{b}) = 0$) for all $i \in \mathrm{Supp}\,\mathcal{B}$. Denote by $\mathcal{B}^+$ (resp. by $\mathcal{B}^-$) the set of highest (resp. lowest) weight elements of $\mathcal{B}$.

Claim 5.15. *$(\mathcal{B}^{op})^+ = \mathcal{B}^-$ and $(\mathcal{B}^{op})^- = \mathcal{B}^+$ for each Kashiwara crystal $\mathcal{B}$ and $(\mathcal{B} \times \mathcal{B}')^+ \subset \mathcal{B}^+ \times \mathcal{B}'^+$, $(\mathcal{B} \times \mathcal{B}')^- \subset \mathcal{B}^- \times \mathcal{B}'^-$ for any Kashiwara crystals $\mathcal{B}$ and $\mathcal{B}'$.*

Definition 5.16. We say that $\mathcal{B} = (\tilde{B}, \tilde{\gamma}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$ is *upper subnormal* (resp. *lower subnormal*) if $\varepsilon_i(\tilde{b}) \geq 0$ (resp. $\varphi_i(\tilde{b}) \geq 0$) for all $i \in \mathrm{Supp}\,\mathcal{B}$, $\tilde{b} \in \tilde{B}$. If $\mathcal{B}$ is both upper and lower subnormal, we will refer to it simply as *subnormal*.

Claim 5.17. *A Kashiwara crystal $\mathcal{B}$ is upper subnormal (resp. lower subnormal) if and only if $\ell_i \leq \tilde{\varepsilon}_i$ (resp. $\ell_{-i} \leq \tilde{\varphi}_i$) for each $i \in \mathrm{Supp}\,\mathcal{B}$.*

Claim 5.18. *Let $\mathcal{B}$, $\mathcal{B}'$ be Kashiwara crystals such that $\mathrm{Supp}\,\mathcal{B} \supseteq \mathrm{Supp}\,\mathcal{B}'$. Then*

(a) *If $\mathcal{B}$ is lower sub-normal, then $\mathcal{B} \times \mathcal{B}'$ is also lower subnormal.*
(b) *If $\mathcal{B}$ is upper sub-normal, then $\mathcal{B}' \times \mathcal{B}$ is also upper subnormal.*

In particular, subnormal Kashiwara crystals of a given support $J \subset I$ from a monoidal subcategory in the category of all Kashiwara crystals.

For each Kashiwara crystal $\mathcal{B} = (\tilde{B}, \tilde{\gamma}, \tilde{e}_i, \tilde{\varphi}_i, \tilde{\varepsilon}_i | i \in I)$, define the subset

$$\overline{\tilde{B}} = \{\tilde{b} \in \tilde{B} : \tilde{\varphi}_i(\tilde{b}) \geq 0, \tilde{\varepsilon}_i(\tilde{b}) \geq 0 \ \forall \ i \in \mathrm{Supp}\,\mathcal{B}, \ \tilde{b} \in \tilde{B}\} \ .$$

We denote by $\overline{\mathcal{B}}$ the restriction of $\mathcal{B}$ to the subset $\overline{\tilde{B}}$ and call $\overline{\mathcal{B}}$ the *sub-normalization* of $\mathcal{B}$. Clearly, $\overline{\mathcal{B}}$ is always subnormal. Also, a Kashiwara crystal $\mathcal{B}$ is subnormal if and only if $\overline{\mathcal{B}} = \mathcal{B}$.

Lemma 5.19. *Let $\mathcal{B}$ and $\mathcal{B}'$ be Kashiwara crystals and let $\mathbf{0} \in \mathcal{B}'$ be such a point that $\tilde{\varepsilon}'_i(\mathbf{0}) = \tilde{\varphi}'_i(\mathbf{0}) = 0$ for $i \in \mathrm{Supp}\,\mathcal{B}'$. Then:*

(a) *The correspondence* $\tilde{b} \mapsto (\mathbf{0}, \tilde{b})$ *(resp.* $\tilde{b} \mapsto (\tilde{b}, \mathbf{0})$*) defines an injective homomorphism of subnormal Kashiwara crystals* $\overline{\mathcal{B}} \hookrightarrow \overline{\mathcal{B}'} \times \overline{\mathcal{B}}$ *(resp.* $\overline{\mathcal{B}} \hookrightarrow \overline{\mathcal{B}} \times \overline{\mathcal{B}'}$*). The support of each of these homomorphisms is* $\operatorname{Supp} \mathcal{B}$.

(b) *If* $\mathcal{B}$ *is upper subnormal, then the correspondence* $\tilde{b} \mapsto (\tilde{b}, \mathbf{0})$ *defines an injective homomorphism of Kashiwara crystals* $\mathcal{B} \hookrightarrow \mathcal{B} \times \mathcal{B}'$*. The support of this homomorphism is* $\operatorname{Supp} \mathcal{B}$.

(c) *If* $\mathcal{B}$ *is lower subnormal, the correspondence* $\tilde{b} \mapsto (\mathbf{0}, \tilde{b})$ *defines an injective homomorphism of Kashiwara crystals* $\mathcal{B} \hookrightarrow \mathcal{B}' \times \mathcal{B}$*. The support of this homomorphism is* $\operatorname{Supp} \mathcal{B}$.

Proof. We need the following special case of Claim 5.9.

Claim 5.20. *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be Kashiwara crystals and let* $\mathbf{0} \in \mathcal{B}'$ *be such a point that* $\tilde{\varepsilon}'_i(\mathbf{0}) = \tilde{\varphi}'_i(\mathbf{0}) = 0$ *for all* $i \in \operatorname{Supp} \mathcal{B}'$*. Then for each* $\tilde{b} \in \mathcal{B}$*,* $i \in \operatorname{Supp} \mathcal{B} \cap \operatorname{Supp} \mathcal{B}'$*, one has*

$$\tilde{\varphi}_i(\mathbf{0}, \tilde{b}) = \max(0, \tilde{\varphi}_i(\tilde{b})), \tilde{\varepsilon}_i(\mathbf{0}, \tilde{b}) = \max(\tilde{\varepsilon}_i(\tilde{b}), \tilde{\varepsilon}_i(\tilde{b}) - \tilde{\varphi}_i(\tilde{b})) \ ,$$

$$\tilde{\varphi}_i(\tilde{b}, \mathbf{0}) = \max(\tilde{\varphi}_i(\tilde{b}), \tilde{\varphi}_i(\tilde{b}) - \tilde{\varepsilon}_i(\tilde{b})), \ \tilde{\varepsilon}_i(\tilde{b}, \mathbf{0}) = \max(0, \tilde{\varepsilon}_i(\tilde{b})) \ ,$$

and for $n \in \mathbb{Z} \setminus \{0\}$*,* $\tilde{b} \in \mathcal{B}$*,* $i \in \operatorname{Supp} \mathcal{B}$ *such that* $\tilde{e}_i^n(\tilde{b}) \in \mathcal{B}$*, one has:*

(i) $\tilde{e}_i^n(\mathbf{0}, \tilde{b}) = (\mathbf{0}, \tilde{e}_i^n(\tilde{b}))$ *if and only if* $\tilde{\varphi}_i(\tilde{b}) \geq \tilde{0}$ *and* $n \geq -\tilde{\varphi}_i(\tilde{b})$.

(ii) $\tilde{e}_i^n(\tilde{b}, \mathbf{0}) = (\tilde{e}_i^n(\tilde{b}), \mathbf{0})$ *if and only if* $\tilde{\varepsilon}_i(\tilde{b}) \geq 0$ *and* $n \leq \tilde{\varepsilon}_i(\tilde{b})$

Prove (a) now. For each $\tilde{b} \in \overline{\mathcal{B}}$, one has

$$\tilde{\varphi}_i(\mathbf{0}, \tilde{b}) = \tilde{\varphi}_i(\tilde{b}), \tilde{\varepsilon}_i(\mathbf{0}, \tilde{b}) = \tilde{\varepsilon}_i(\tilde{b}), \tilde{\varphi}_i(\tilde{b}, \mathbf{0}) = \tilde{\varphi}_i(\tilde{b}), \ \tilde{\varepsilon}_i(\tilde{b}, \mathbf{0}) = \tilde{\varepsilon}_i(\tilde{b}) \ ,$$

Using Claim 5.20(i), we obtain

$$\tilde{e}_i^n(\mathbf{0}, \tilde{b}) = (\mathbf{0}, \tilde{e}_i^n(\tilde{b}))$$

for $\tilde{b} \in \overline{\mathcal{B}}$, $i \in \operatorname{Supp} \mathcal{B}$, $n \in \mathbb{Z}$ if and only if $\tilde{e}_i^n(\tilde{b}) \in \overline{\mathcal{B}}$ and $n \geq -\tilde{\varphi}_i(\tilde{b})$. But the latter inequality holds automatically because $\tilde{\varphi}_i(e_i^n(\tilde{b})) = \tilde{\varphi}_i(\tilde{b}) + n \geq 0$ if $\tilde{e}_i^n(\tilde{b}) \in \overline{\mathcal{B}}$.

Similarly, using Claim 5.20(ii), we obtain

$$\tilde{e}_i^n(\tilde{b}, \mathbf{0}) = (\tilde{e}_i^n(\tilde{b}), \mathbf{0})$$

for $\tilde{b} \in \overline{\mathcal{B}}$, $i \in \operatorname{Supp} \mathcal{B}$, $n \in \mathbb{Z}$ if and only if $\tilde{e}_i^n(\tilde{b}) \in \overline{\mathcal{B}}$. Finally, $\tilde{\gamma}(\tilde{b}, \mathbf{0}) = \gamma(\mathbf{0}, \tilde{b}) = \tilde{\gamma}(\tilde{b})$. This proves (a).

Prove (b). Indeed, if $\mathcal{B}$ is upper normal, then Claim 5.20 guarantees that $\tilde{\varphi}_i(\tilde{b}, \mathbf{0}) = \tilde{\varphi}_i(\tilde{b})$, $\tilde{\varepsilon}_i(\tilde{b}, \mathbf{0}) = \tilde{\varepsilon}_i(\tilde{b}) \geq 0$, and $\tilde{e}_i^n(\tilde{b}, \mathbf{0}) = (\tilde{e}_i^n(\tilde{b}), \mathbf{0})$ for all $n \leq \tilde{\varepsilon}_i(\tilde{b})$. Finally, $\tilde{\gamma}(\tilde{b}, \mathbf{0}) = \gamma(\mathbf{0}, \tilde{b}) = \tilde{\gamma}(\tilde{b})$. This proves (b).

Part (c) follows. The lemma is proved. □

We conclude the section with definitions and results related to normal Kashiwara crystals.

Definition 5.21. Following [14], we say that $\mathcal{B} = (\tilde{B}, \tilde{\gamma}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$ is *upper normal* (resp. *lower normal*) if $\tilde{\varepsilon}_i = \ell_i$ (resp. $\tilde{\varphi}_i = \ell_{-i}$) for all $i \in \operatorname{Supp} \mathcal{B}$ (in the notation (5.3)).

A Kashiwara crystal $\mathcal{B}$ is *normal* if it is both lower normal and upper normal and $\operatorname{Supp} \mathcal{B} = I$.

Lemma 5.22. *If $\mathcal{B}$ is an upper (resp. lower) normal Kashiwara crystal and $\mathcal{B}'$ is normal, then $\mathcal{B}\times B'$ is upper normal (resp. $\mathcal{B}'\times\mathcal{B}$ is lower normal).*

Proof. For the upper normal crystals, the assertion follows from the formula

$$\tilde{e}_i^{\tilde{\varepsilon}_i(\tilde{b},\tilde{b}')+n}(\tilde{b},\tilde{b}')=(\tilde{e}_i^{n_1}(\tilde{b}),\tilde{e}_i^{n_2}(\tilde{b}'))\ ,$$

for $(\tilde{b},\tilde{b}')\in\mathcal{B}\times\mathcal{B}'$, $n\in\mathbb{Z}$, $i\in\operatorname{Supp}\mathcal{B}\cap\operatorname{Supp}\tilde{\mathcal{B}}'$, where

$$n_1=\min(\max(0,\tilde{\varepsilon}_i(\tilde{b})-\tilde{\varphi}_i'(\tilde{b}')),\varepsilon_i'(\tilde{b}')+|\tilde{\varepsilon}_i(\tilde{b})-\tilde{\varphi}_i'(\tilde{b}')|+n),$$

$$n_2=\max(\tilde{\varepsilon}_i'(\tilde{b}')+n,\min(0,\tilde{\varepsilon}_i(\tilde{b})-\tilde{\varphi}_i'(\tilde{b}'))\ .$$

This implies that if $\mathcal{B}$ is upper normal and $\mathcal{B}'$ is normal, then $n_1=\max(0,\tilde{\varepsilon}_i(\tilde{b})-\tilde{\varphi}_i'(\tilde{b}'))\le\varepsilon_i(\tilde{b})$, $n_2=\tilde{\varepsilon}_i'(\tilde{b}')+n$; hence $\tilde{e}_i^{\tilde{\varepsilon}_i(\tilde{b},\tilde{b}')}(\tilde{b},\tilde{b}')\in\mathcal{B}\times\mathcal{B}'$ and $\tilde{e}_i^{\tilde{\varepsilon}_i(\tilde{b},\tilde{b}')+1}(\tilde{b},\tilde{b}')\notin\mathcal{B}\times\mathcal{B}'$. This proves the assertion for the upper normal $\mathcal{B}$. If $\mathcal{B}$ is lower normal, the assertion follows from the above and Claim 5.10 by applying $\mathcal{B}\mapsto\mathcal{B}^{op}$ and $\mathcal{B}'\mapsto\mathcal{B}'^{op}$. □

Thus normal Kashiwara crystals and their homomorphisms form a monoidal category (see also [14]). This category acts from the right (resp. from the left) on the category of upper (resp. lower) normal crystals.

The following result demonstrates that each upper normal (resp. lower) crystal is semisimple.

Claim 5.23. *Let $\mathcal{B}$ be an upper normal crystal, and let $\tilde{B}'$ be a subset of $\tilde{B}$ such that the sub-crystal $\mathcal{B}|_{\tilde{B}'}$ is also upper normal. Then the complement $\mathcal{B}|_{\tilde{B}\setminus\tilde{B}'}$ is upper normal as well.*

Note that if $\{\mathcal{B}_k\}$ is any family of upper normal sub-crystals of an upper normal crystal, then both the intersection $\bigcap_k\mathcal{B}_k$ and the union $\bigcup_k\mathcal{B}_k$ are also upper normal sub-crystals. This prompts the following definition.

Definition 5.24. Given an upper normal crystal $\mathcal{B}$, for any subset $\tilde{C}\subset\mathcal{B}$ denote by $\mathcal{B}[\tilde{C}]$ the intersection of all upper normal sub-crystals of $\mathcal{B}$ containing $\tilde{C}$, i.e., $\mathcal{B}[\tilde{C}]$ is the smallest upper normal sub-crystal of $\mathcal{B}$ containing $\tilde{C}$. We will refer to $\mathcal{B}[\tilde{C}]$ as the upper normal sub-crystal of $\mathcal{B}$ *generated upward* by $\tilde{C}$.

Lemma 5.25. *For any $\tilde{C}\subset\mathcal{B}$, the upper normal sub-crystal $\mathcal{B}[\tilde{C}]$ of $\mathcal{B}$ consists of all elements of the form*

$$\tilde{e}_{i_\ell}^{n_\ell}\tilde{e}_{i_{\ell-1}}^{n_{\ell-1}}\cdots\tilde{e}_{i_1}^{n_1}(\tilde{c}) \tag{5.4}$$

for all $\tilde{c}_0\in\tilde{C}$ and any $i_1,\dots,i_\ell\in\operatorname{Supp}\mathcal{B}$, $n_1,\dots,n_\ell\in\mathbb{Z}_{\ge0}$ are such that

$$n_k\le\tilde{\varepsilon}_{i_k}(\tilde{e}_{i_{k-1}}^{n_{k-1}}\cdots\tilde{e}_{i_1}^{n_1}(\tilde{c}))$$

for $k=1,2,\dots,\ell$.

Proof. Denote by $\mathcal{B}_0$ the sub-crystal of $\mathcal{B}$ which consists of all elements of the form (5.4). Clearly, if $\mathcal{B}'$ is an upper normal sub-crystal of $\mathcal{B}$ containing each $\tilde{c}\in\tilde{C}$, then $\mathcal{B}'$ also contains $\mathcal{B}_0$. It remains to show that $\mathcal{B}_0$ is upper normal. But, by definition of $\mathcal{B}_0$ for each $\tilde{b}_0\in\mathcal{B}_0$, one has

$$\tilde{e}_i^n(\tilde{b}_0)\in\mathcal{B}_0$$

for each $i \in \operatorname{Supp}\mathcal{B}$, $n \in \mathbb{Z}_{\geq 0}$ such that $n \leq \varepsilon_i(\tilde{b}_0)$. This verifies Definition 5.21 and proves the upper normality of $\mathcal{B}_0$. Thus, $\mathcal{B}_0$ is the smallest upper normal crystal containing $\tilde{C}$. The lemma is proved. □

Note that if $\tilde{C} = \{\tilde{b}\}$ is a single element, then $\tilde{c}$ is the *lowest* weight element in $\mathcal{B}[\tilde{c}]$. Moreover, $\mathcal{B}[\tilde{C}] = \bigcup_{\tilde{c}\in\tilde{C}} \mathcal{B}[\{\tilde{c}\}]$.

Claim 5.26. *Let $\mathcal{B}$ be a Kashiwara crystal and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be its normal sub-crystals. Then $\mathcal{B}_1 \cap \mathcal{B}_2$ is also normal. If, in addition, $\mathcal{B}_1$ and $\mathcal{B}_2$ are connected, then either $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ or $\mathcal{B}_1 = \mathcal{B}_2$.*

Claim 5.27. *Any normal crystal is a disjoint union of its normal connected sub-crystals.*

5.2. **Perfect bases, upper normal crystals, and associated crystals.** Let I be a finite set of indices, let Λ be a lattice, and let $\{\alpha_i, i \in I\}$ be a subset of Λ. Let $\{\alpha_j^\vee, j \in I\}$ be a subset of the dual lattice $\Lambda^\vee = \Lambda^*$ (e.g., $\Lambda = X^\star(T)$ is the weight lattice of G, $\Lambda^\vee = X_\star(T)$ is the co-weight lattice of G, and $\alpha_j^\vee$, α_i are respectively co-roots and roots of G). Denote by $\hat{\mathfrak{b}}$ the Lie algebra generated by h_i, e_i, $i \in I$ subject to the relations

$$[h_i, e_j] = \langle \alpha_i, \alpha_j^\vee \rangle e_j$$

for all $i, j \in I$, where $\langle \bullet, \bullet \rangle : \Lambda \times \Lambda^\vee \to \mathbb{Z}$ is the evaluation pairing. By definition, $\hat{\mathfrak{b}}$ is a Borel sub-algebra of a generalized Kac-Moody Lie algebra and $\hat{\mathfrak{b}} = \mathfrak{h}^\vee \ltimes \hat{\mathfrak{n}}$, where $\mathfrak{h}^\vee$ is an Abelian Lie algebra with the basis $\{h_i, i \in I\}$ and $\hat{\mathfrak{n}}$ is the free Lie algebra generated by all e_i.

Denote by $\mathfrak{B}$ the universal enveloping algebra $U(\mathfrak{b})$, that is, $\mathfrak{B}$ is an associative algebra generated over $\mathbb{C}$ by h_i, e_i, $i \in I$ subject to the relations

$$h_i e_j - e_j h_i = \langle \alpha_i, \alpha_j^\vee \rangle e_j$$

for all $i, j \in I$.

We say that a $\mathfrak{B}$-module V is *locally finite* if:

• V has a $\Lambda^\vee$-weight decomposition:

$$V = \bigoplus_{\mu \in \Lambda^\vee} V(\mu) \ . \tag{5.5}$$

• For any $v \in V$, the cyclic sub-module $\mathfrak{B}(v)$ of V is finite-dimensional.

For any non-zero vector $v \in V$ and $i \in I$, denote by $\ell_i(v)$ the smallest positive integer ℓ such that $e_i^{\ell+1}(v) = 0$. For $v = 0$ we will use the convention $\ell_i(0) = -\infty$ for all i.

Given a locally finite $\mathfrak{B}$-module V, for each sequence $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$, $m \geq 1$, we define a binary relation $\preceq_{\mathbf{i}}$ on $V \setminus \{0\}$ as follows.

If $m = 1$, $\mathbf{i} = (i)$, write $v \preceq_{\mathbf{i}} v'$ if and only if $\ell_i(v) \leq \ell_i(v')$. For $\mathbf{i} = (i; \mathbf{i}')$, write $v \preceq_{\mathbf{i}} v'$ if and only if either $\ell_i(v) < \ell_i(v')$ or $\ell_i(v) = \ell_i(v')$, $e_i^{\ell_i(v)}(v) \preceq_{\mathbf{i}'} e_i^{\ell_i(v)}(v')$.

Claim 5.28. *For each $\mathbf{i}$, the relation $\preceq_{\mathbf{i}}$ is a pre-order on $V \setminus \{0\}$.*

Therefore, we define the equivalence relation $\equiv_{\mathbf{i}}$ on $V \setminus \{0\}$ by setting $v \equiv_{\mathbf{i}} v'$ if $v \preceq_{\mathbf{i}} v'$ and $v' \preceq_{\mathbf{i}} v$. Also we will write $v \prec_{\mathbf{i}} v'$ if $v \preceq_{\mathbf{i}} v'$ and $v \not\equiv_{\mathbf{i}} v'$.

Claim 5.29. *For any sequence* $\mathbf{i} \in I^m$*, one has:*

(a) *For any non-zero* $v \in V$*, the set* $\{v' \in V : v' \prec_{\mathbf{i}} v\}$ *is a subspace of* V*.*

(b) *If* $v \not\equiv_{\mathbf{i}} v'$*, then* $v + v' \equiv_{\mathbf{i}} \begin{cases} v & \text{if } v' \prec_{\mathbf{i}} v \\ v' & \text{if } v \prec_{\mathbf{i}} v' \end{cases}$.

For each $i \in I$ and $\ell \geq 0$, define the subspace

$$V_i^{<\ell} := \{v \in V : \ell_i(v) < \ell\} = \{v \in V : e_i^{\ell}(v) = 0\} \ .$$

We say that a basis $\mathbf{B}$ of a locally finite $\mathfrak{B}$-module V is a *weight basis* if $\mathbf{B}$ is compatible with the weight decomposition (5.5), i.e., $\mathbf{B}(\mu) := V(\mu) \cap \mathbf{B}$ is a basis of $V(\mu)$ for any $\mu \in \Lambda^\vee$.

Definition 5.30. We say that a weight basis $\mathbf{B}$ in a locally finite $\mathfrak{B}$-module V is *perfect* if for each $i \in I$ there is a partial bijection $\tilde{e}_i : \mathbf{B} \to \mathbf{B}$ (i.e., a bijection of a subset of $\mathbf{B}$ onto another subset of $\mathbf{B}$, see also Definition 5.1 above) such that $\tilde{e}_i(b) \in \mathbf{B}$ if and only if $e_i(b) \neq 0$, and in the latter case one has

$$e_i(b) \in \mathbb{C}^\times \cdot \tilde{e}_i(b) + V_i^{<\ell_i(b)-1} \ . \tag{5.6}$$

Following [19], we refer to a pair $(V, \mathbf{B})$, where V is a locally finite $\mathfrak{B}$-module and $\mathbf{B}$ is a perfect basis of V, as a *based* $\mathfrak{B}$-module.

Replacing the lattice $X_\star(T)$ by $\Lambda^\vee$, we extend Definition 5.2 of Kashiwara crystals to the setup of this section.

Recall from Definition 5.21 that a Kashiwara crystal $\mathcal{B}$ is upper normal if

$$\varepsilon_i(\tilde{b}) = \ell_i(\tilde{b}) = \max\{n : \tilde{e}_i^n(\tilde{b}) \in B\}$$

for all $\tilde{b} \in \tilde{B}$.

Claim 5.31. *Each based* $\mathfrak{B}$*-module* $(V, \mathbf{B})$ *defines an upper normal Kashiwara crystal* $\mathcal{B}(V, \mathbf{B}) = (\mathbf{B}, \tilde{\gamma}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$*, where*

- $\tilde{\gamma} : \mathbf{B} \to \Lambda^\vee$ *is given by* $\tilde{\gamma}(b) = \mu$ *whenever* $b \in \mathbf{B}(\mu)$*.*
- *the functions* $\tilde{\varepsilon}_i, \tilde{\varphi}_i : \mathbf{B} \to \mathbb{Z}$ *are given by* $\tilde{\varepsilon}_i = \ell_i$*,* $\tilde{\varphi}_i = \tilde{\varepsilon}_i + \langle \alpha_i, \tilde{\gamma}(\bullet) \rangle$ *for all* $i \in I$*.*

In order to establish the functoriality of the association $(V, \mathbf{B}) \mapsto \mathcal{B}(V, \mathbf{B})$, we will define morphisms of based $\mathfrak{B}$-modules. Following [3], for a locally finite $\mathfrak{B}$-module and any sequence $\mathbf{i} \in I^m$, define a map $e_{\mathbf{i}}^{\text{top}} : V \setminus \{0\} \to V \setminus \{0\}$ as follows. First, for $i \in I$ and $v \neq 0$, we set

$$e_i^{\text{top}}(v) := e_i^{\ell_i(v)}(v) \ ,$$

and for $\mathbf{i} = (i_1, \ldots, i_m)$, $m > 1$, define

$$e_{\mathbf{i}}^{\text{top}} = e_{i_m}^{\text{top}} \circ e_{i_{m-1}}^{\text{top}} \circ \cdots \circ e_{i_1}^{\text{top}} \ . \tag{5.7}$$

Denote by V^+ the space of the *highest weight vectors* of V:

$$V^+ = \{v \in V : e_i(v) = 0 \ \forall \ i \in I\} \ . \tag{5.8}$$

Since V is a locally finite module, for each $v \in V$, there exists a sequence $\mathbf{i}$ such that $e_{\mathbf{i}}^{\text{top}}(v) \in V^+$. In particular, $V^+ \neq \{0\}$ if $V \neq \{0\}$.

For each based $\mathfrak{B}$-module $(V, \mathbf{B})$, denote $\mathbf{B}^+ := \mathbf{B} \cap V^+$.

Claim 5.32. *For any perfect basis* $\mathbf{B}$ *for* V*, the subset* $\mathbf{B}^+$ *is a basis for* V^+*.*

Note that $\mathbf{B}^+ = \mathcal{B}(V,\mathbf{B})^+$, the set of highest weight elements in the sense of Definition 5.14.

Definition 5.33. A *homomorphism of based* $\mathfrak{B}$*-modules* $(V,\mathbf{B}) \to (V',\mathbf{B}')$ is any $\mathfrak{B}$-linear map $f : V \to V'$ such that $0 \notin f(\mathbf{B})$, and there exist a map $\tilde{f} : \mathbf{B} \to \mathbf{B}'$ and a function $c : \mathbf{B} \setminus \mathbf{B}^+ \to \mathbb{C}^\times$ $(b \mapsto c_b)$ satisfying $\tilde{f}(b) = f(b)$ for each $b \in \mathbf{B}^+$ and

$$f(b) - c_b\tilde{f}(b) \prec_{\mathbf{i}} f(b) \tag{5.9}$$

for each $b \in \mathbf{B} \setminus \mathbf{B}^+$ and any sequence $\mathbf{i}$ such that $e_{\mathbf{i}}^{\mathrm{top}}(b) \in V^+$.

For example, for each based $\mathfrak{B}$-module $(V,\mathbf{B})$, the map $f : V \oplus V \to V$ given by $f(v,v') = v + v'$ is a morphism of $\mathfrak{B}$-modules $(V \oplus V, \mathbf{B} \sqcup \mathbf{B}) \to (V,\mathbf{B})$.

Lemma 5.34. *The map* $\tilde{f}$ *and the function* $b \mapsto c_b$ *in Definition 5.33 are unique. More precisely, for each* $b \in \mathbf{B} \setminus \mathbf{B}^+$ *and* $\mathbf{i}$ *such that* $e_{\mathbf{i}}^{\mathrm{top}}(b) \in V^+$*, the element* $\tilde{f}(b) \in \mathbf{B}'$ *and the number* c_b *satisfying* (5.9) *are unique.*

Proof. We need some notation. Similarly to (5.7), for a based $\mathfrak{B}$-module $(V,\mathbf{B})$ and any sequence $\mathbf{i} = (i_1,\dots,i_m) \in I^m$, $m \geq 1$ define the maps $\tilde{e}_{\mathbf{i}}^{\mathrm{top}} : \mathbf{B} \to \mathbf{B}$ as follows. First, define $\tilde{e}_i^{\mathrm{top}} : \mathbf{B} \to \mathbf{B}$, $i \in I$, by

$$\tilde{e}_i^{\mathrm{top}}(b) := \tilde{e}_i^{\ell_i(b)}(b) \ ,$$

and for $\mathbf{i} = (i_1,\dots,i_m)$, define

$$\tilde{e}_{\mathbf{i}}^{\mathrm{top}} = \tilde{e}_{i_m}^{\mathrm{top}} \circ \tilde{e}_{i_{m-1}}^{\mathrm{top}} \circ \cdots \circ \tilde{e}_{i_1}^{\mathrm{top}} \ .$$

Claim 5.35. *Let* $(V,\mathbf{B})$ *be a based* $\mathfrak{B}$*-module. Then for any sequence* $\mathbf{i} \in I^m$*,* $m \geq 1$*, we have:*

(a) $e_{\mathbf{i}}^{\mathrm{top}}(b) \in \mathbb{C}^\times \cdot \tilde{e}_{\mathbf{i}}^{\mathrm{top}}(b)$ *for any* $b \in \mathbf{B}$*.*
(b) *If* $e_{\mathbf{i}}^{\mathrm{top}}(b) \in V^+$ *for some* $b \in \mathbf{B}$*, then* $\tilde{e}_{\mathbf{i}}^{\mathrm{top}}(b) \in \mathbf{B}^+$*.*
(c) *If* $b \equiv_{\mathbf{i}} b'$ *and* $\tilde{e}_{\mathbf{i}}^{\mathrm{top}}(b) = \tilde{e}_{\mathbf{i}}^{\mathrm{top}}(b')$*, then* $b = b'$ *(for any* $b, b' \in \mathbf{B}$*).*

In the notation of Lemma 5.34, let $b \in \mathbf{B}$, $b', b'' \in \mathbf{B}' \setminus \mathbf{B}'^+$ be such that $f(b) - c'b' \prec_{\mathbf{i}} f(b)$ and $f(b) - c''b'' \prec_{\mathbf{i}} f(b)$ for some and $c', c'' \in \mathbb{C}^\times$ and some $\mathbf{i}$ such that $e_{\mathbf{i}}^{\mathrm{top}}(b) \in V^+$. We will show that $b' = b''$ and $c' = c''$. Indeed, the above inequalities imply that $c'b' \equiv_{\mathbf{i}} c''b'' \equiv_{\mathbf{i}} f(b)$ and $c'b' - c''b'' \prec_{\mathbf{i}} f(b)$. Therefore, $c'e_{\mathbf{i}}^{\mathrm{top}}(b') = c''e_{\mathbf{i}}^{\mathrm{top}}(b'')$. This and Claim 5.35(a) imply that $\tilde{e}_{\mathbf{i}}^{\mathrm{top}}(b') = \tilde{e}_{\mathbf{i}}^{\mathrm{top}}(b'')$. Finally, Claim 5.35(c) implies that $b' = b''$. In turn, this implies that $c' = c''$.

The lemma is proved. □

Clearly, based $\mathfrak{B}$-modules with such defined morphisms form a category. This category possesses direct sums $(V,\mathbf{B}) \oplus (V',\mathbf{B}') = (V \oplus V', \mathbf{B} \sqcup \mathbf{B}')$. However, the category is not additive, because $\mathrm{Hom}((V,\mathbf{B}),(V',\mathbf{B}'))$ is not an abelian group.

Lemma 5.36. *The association* $(V,\mathbf{B}) \mapsto \mathcal{B}(V,\mathbf{B})$ *is a functor from the category of based* $\mathfrak{B}$*-modules to the category of upper normal crystals. More precisely, each morphism* $f : (V,\mathbf{B}) \to (V',\mathbf{B}')$ *of based* $\mathfrak{B}$*-modules defines the homomorphism* $\tilde{f} = \mathcal{B}(f) : \mathcal{B}(V,\mathbf{B}) \to \mathcal{B}(V',\mathbf{B}')$ *of upper normal crystals.*

Proof. By definition, a morphism of based modules $f : (V, \mathbf{B}) \to (V', \mathbf{B}')$ defines a map $\tilde{f} : \mathbf{B} \to \mathbf{B}'$ satisfying (5.9). In particular, taking $\mathbf{i} = (i_1, i_2, \dots, i_m)$ in (5.9) with $i_1 = i$ proves that $\ell_i(\tilde{f}(b)) = \ell_i(b)$ for all $b \in \mathbf{B}$, $i \in I$. Therefore, in order to prove that $\tilde{f}$ is a homomorphism of Kashiwara crystals, it suffices to show that $\tilde{f}$ is $\tilde{e}_i$-equivariant for each $i \in I$. Let us choose any sequence $\mathbf{i} = (i_1, \dots, i_m) \in I^m$ such that $i_1 = i$ and $e_{\mathbf{i}}^{\mathrm{top}}(b) \in V^+$.

Applying e_i to (5.9) yields (provided that $\ell_i(b) \neq 0$):

$$e_i(\tilde{f}(b)) \equiv_{\mathbf{i}} e_i(f(b)) \ .$$

At the same time, (5.6) implies that

$$e_i(b) \equiv_{\mathbf{i}} \tilde{e}_i(b), \ e_i(\tilde{f}(b)) \equiv_{\mathbf{i}} \tilde{e}_i(\tilde{f}(b)) \ .$$

Therefore, taking into account that $e_i(f(b)) = \varphi(e_i(b)) \equiv_{\mathbf{i}} f(\tilde{e}_i(b))$, we obtain:

$$\tilde{e}_i(\tilde{f}(b)) \equiv_{\mathbf{i}} f(\tilde{e}_i(b)) \ .$$

Therefore, $\tilde{e}_i(\tilde{f}(b)) = \tilde{f}(\tilde{e}_i(b))$ because according to (5.9) and Lemma 5.34, $\tilde{f}(\tilde{e}_i(b))$ is the only element of $\mathbf{B}'$ such that $\tilde{f}(\tilde{e}_i(b)) \equiv_{\mathbf{i}} f(\tilde{e}_i(b))$. This proves the lemma. □

The following is the main result of this section.

Main Theorem 5.37. *Let $\mathbf{B}$ and $\mathbf{B}'$ be perfect bases of a locally finite $\mathfrak{B}$-module such that $\mathbf{B}'^+ = \mathbf{B}^+$. Then the identity map $\mathrm{id} : V \to V$ defines an isomorphism of based $\mathfrak{B}$ modules $(V, \mathbf{B}) \xrightarrow{\sim} (V, \mathbf{B}')$, that is, there is a unique map $\tilde{f} : \mathbf{B} \to \mathbf{B}'$ and a unique function $c : \mathbf{B} \setminus \mathbf{B}^+ \to \mathbb{C}^\times$ satisfying $\tilde{f}(b) = b$ for each $b \in \mathbf{B}$, and*

$$\tilde{f}(b) - c_b b \prec_{\mathbf{i}} b \tag{5.10}$$

for each $b \in \mathbf{B} \setminus \mathbf{B}^+$ and any sequence $\mathbf{i}$ such that $\tilde{e}_{\mathbf{i}}^{\mathrm{top}}(b) \in \mathbf{B}^+$.

In particular, one has an isomorphism of (upper normal) Kashiwara crystals $\mathcal{B}(V, \mathbf{B}) \xrightarrow{\sim} \mathcal{B}(V, \mathbf{B}')$.

Proof. We need the following result.

Lemma 5.38. *Let $(V, \mathbf{B})$ be a based $\mathfrak{B}$-module. Then for any sequence $\mathbf{i} \in I^m$, $m \geq 1$, we have:*

(a) *Each $v \in V \setminus \{0\}$ is in the $\mathbb{C}$-linear span of $\{b \in \mathbf{B} : b \preceq_{\mathbf{i}} v\}$.*
(b) *For each $v \in V \setminus \{0\}$, there exists a unique vector v_0 in the linear span of $\{b \in \mathbf{B} : b \equiv_{\mathbf{i}} v\}$ such that $v - v_0 \prec_{\mathbf{i}} v$.*
(c) *For each $v \in V \setminus \{0\}$ such that $c e_{\mathbf{i}}^{\mathrm{top}}(v) \in \mathbf{B}$ (for some $c \in \mathbb{C}^\times$), there exists a unique element $b \in \mathbf{B}$ such that $v \equiv_{\mathbf{i}} b$. Moreover, $v - cb \prec_{\mathbf{i}} b$ for some $c \in \mathbb{C}^\times$.*

Proof. Prove (a) first. We will proceed by induction in the length m of $\mathbf{i} = (i_1, \dots, i_m)$. Let us write the expansion of $v \in V \setminus \{0\}$:

$$v = \sum_{b \in \mathbf{B}} c_b b \ . \tag{5.11}$$

Denote by $\mathbf{B}_0$ the set of all $b \in \mathbf{B}$ such that $c_b \neq 0$; and denote $n_1 := \max_{b \in \mathbf{B}_0} \ell_{i_1}(b)$. We have to show that $\mathbf{B}_0 \preceq v$. Clearly, $\ell_{i_1}(v) \leq n_1$. Next, we will show that $n_1 = \ell_{i_1}(v)$. Indeed, assume, by contradiction, that $\ell_{i_1}(v) < n_1$. Applying $e_{i_1}^{n_1}$ to (5.11), we obtain

$$0 = \sum_{b \in \mathbf{B}_1} c_b e_{i_1}^{n_1}(b) \ ,$$

where $\mathbf{B}_1 := \{b \in \mathbf{B}_0 : \ell_{i_1}(b) = n_1\}$. In other words, taking into account Claim 5.35(a), we obtain a non-trivial linear dependence

$$0 = \sum_{b' \in \tilde{e}_{i_1}^{n_1}(\mathbf{B}_1)} c'_b b' \ .$$

This is in contradiction to $\tilde{e}_{i_1}^{n_1}(\mathbf{B}_1) \subset \mathbf{B}$ being linearly independent. Therefore, $n_1 = \ell_{i_1}(v)$. This, in particular, proves that $\mathbf{B}_0 \setminus \mathbf{B}_1 \prec_{\mathbf{i}} v$. In order to show that $\mathbf{B}_1 \preceq v$, let us again apply $e_{i_1}^{n_1}$ to (5.11):

$$e_{i_1}^{n_1}(v) = \sum_{b \in \mathbf{B}_1} c_b e_{i_1}^{n_1}(b) \ .$$

Using Claim 5.35(a) again and the inductive assumption with $\mathbf{i}' = (i_2, \ldots, i_m)$, we obtain: $\tilde{e}_{i_1}^{n_1}(\mathbf{B}_1) \preceq_{\mathbf{i}'} e_{i_1}^{n_1}(v)$. Therefore, $\mathbf{B}_1 \preceq_{\mathbf{i}} v$. This proves (a).

Prove (b) now. Denote by $\mathbf{B}'_0$ the set of all $b \in \mathbf{B}$ such that $c_b \neq 0$ and $b \equiv_{\mathbf{i}} v$. Then we set $v_0 := \sum_{b \in \mathbf{B}'_0} c_b b$ in the notation of (5.11). Clearly, $v - v_0$ is in the span of all $b \in \mathbf{B}$ such that $b \prec_{\mathbf{i}} v$. Using Claim 5.29(a), we see that an inequality $v - v'_0 \prec_{\mathbf{i}} v$ implies that $v_0 - v'_0 \prec_{\mathbf{i}} v$, i.e., $v_0 - v'_0$ is in the span of $\mathbf{B} \setminus \mathbf{B}'_0$. Therefore, if v'_0 is in the span of $\mathbf{B}'_0$, this implies that $v'_0 = v_0$, which proves the uniqueness of v_0 in the linear span of $\mathbf{B}'_0$ with the desired property. Part (b) is proved.

To prove (c), note that, under the assumptions of (c), the above set $\mathbf{B}'_0$ consists of a single element b. Therefore, $v_0 = c_b b$ and $v - v_0 \prec_{\mathbf{i}} v \equiv_{\mathbf{i}} v_0 \equiv_{\mathbf{i}} b$. Part (c) is also proved.

Lemma 5.38 is proved. □

Let $b \in \mathbf{B} \setminus \mathbf{B}^+$. We now apply Lemma 5.38(c) with $v = b$ relative to the perfect basis $\mathbf{B}'$. Then for any sequence $\mathbf{i}$ such that $\tilde{e}_{\mathbf{i}}^{\mathrm{top}}(b) \in \mathbf{B}^+ = \mathbf{B}'^+$, there exists a unique element $b' \in \mathbf{B}'$ such that $b \equiv_{\mathbf{i}} b'$ and

$$b = v + cb' \tag{5.12}$$

for some $c \in \mathbb{C}^\times$, where either $v = 0$ or $v \prec_{\mathbf{i}} b$, $v \prec_{\mathbf{i}} b'$. If $v = 0$ then we set $\tilde{f}(b) = b'$ and end the proof here. So we assume that $v \neq 0$. Then, according to Lemma 5.38(a), v is in the span of $\{b_1 \in \mathbf{B} : b_1 \preceq_{\mathbf{i}} v\}$; hence v is the span of $\mathbf{B} \setminus \{b\}$. By the same reasoning, v is in the span of $\mathbf{B}' \setminus \{b'\}$.

Let now $\mathbf{i}'$ be any other sequence such that $\tilde{e}_{\mathbf{i}'}^{\mathrm{top}}(b) \in \mathbf{B}^+$, i.e., $c' e_{\mathbf{i}'}^{\mathrm{top}}(b) \in \mathbf{B}^+$ for some $c' \in \mathbb{C}^\times$. Next, we will show that $v \not\equiv_{\mathbf{i}'} b$. Indeed, if $v \equiv_{\mathbf{i}'} b$, then we would have (again by Lemma 5.38(c)) that $v - c'b$ is in the span of $\mathbf{B} \setminus \{b\}$. But this contradicts the above observation that v itself is in the span of $\mathbf{B} \setminus \{b\}$. Similarly, one shows that $v \not\equiv_{\mathbf{i}'} b'$. Finally, applying Claim 5.29(b) with $\mathbf{i}'$ for $\mathbf{i}$, $v' = cb'$, and $v + v' = b$, we obtain $v \prec_{\mathbf{i}'} b$, $v \prec_{\mathbf{i}'} b'$, and $b \equiv_{\mathbf{i}'} b'$. Thus, we proved that $b \equiv_{\mathbf{i}'} b'$ for any $\mathbf{i}'$ such that $\tilde{e}_{\mathbf{i}'}^{\mathrm{top}}(b) \in \mathbf{B}^+$.

Therefore, the identity map $V \to V$ defines a homomorphism $(V, \mathbf{B}) \to (V, \mathbf{B}')$ of based $\mathfrak{B}$-modules, which is, obviously, an isomorphism. Applying Lemma 5.36, we obtain an isomorphism of the corresponding upper normal crystals.

Theorem 5.37 is proved. □

Corollary 5.39. *Let V be any locally finite $\mathfrak{B}$-module such that* $\dim V^+ = 1$. *Then for any perfect bases $\mathbf{B}$ and $\mathbf{B}'$ for V the upper normal crystals $\mathcal{B}(V, \mathbf{B})$ and $\mathcal{B}(V, \mathbf{B}')$ are isomorphic.*

This allows for the following definition.

Definition 5.40. For each locally finite $\mathfrak{B}$-module V such that $V^+ \cong \mathbb{C}$ and such that V admits a perfect basis, define the *associated crystal* $\mathcal{B}(V)$ to be isomorphic to each of the Kashiwara crystals $\mathcal{B}(V, \mathbf{B})$, where $\mathbf{B}$ is any perfect basis of V. Furthermore, if $V = \oplus_k V_k$, where each V_k is a locally finite $\mathfrak{B}$-module such that $V_k^+ \cong \mathbb{C}$ and V_k admits a perfect basis, then we define the *associated crystal* $\mathcal{B}(V)$ by $\mathcal{B}(V) := \bigsqcup_k \mathcal{B}(V_k)$.

5.3. **Perfect bases of $\mathfrak{b}^\vee$-modules and associated crystals.** Let $G^\vee$ be the Langlands dual group of a reductive algebraic group G, and $\mathfrak{g}^\vee$ be the Lie algebra of $G^\vee$, and let $\mathfrak{g}^\vee = \mathfrak{u}_-^\vee \oplus \mathfrak{h}^\vee \oplus \mathfrak{u}^\vee$ be the Cartan decomposition. Denote by $\mathfrak{b}^\vee := \mathfrak{h}^\vee \oplus \mathfrak{u}^\vee$ the Borel sub-algebra of $\mathfrak{g}^\vee$.

Recall that in the setup of Section 5.2, I is the vertex set of Dynkin diagram of G, $\Lambda = X^\star(T)$, $\Lambda = X_\star(T)$, and $\langle \bullet, \bullet \rangle : \Lambda \times \Lambda^\vee \to \mathbb{Z}$ is the canonical pairing. By definition, in the notation of Section 5.2, we have a natural surjective homomorphism $\hat{\mathfrak{b}} \twoheadrightarrow \mathfrak{b}^\vee$ via $h \mapsto h$, $e_i \mapsto e_i$ for $i \in I$, where the latter e_i are Chevalley generators of $\mathfrak{u}^\vee$. Therefore, each $\mathfrak{b}^\vee$-module is a $\mathfrak{B}$-module, and all definitions and results of Section 5.2 are valid for locally finite $\mathfrak{b}^\vee$-modules.

In this section we will construct some important examples of based $\mathfrak{b}^\vee$-modules and the associated crystals.

Denote by $T^\vee$, $B^\vee$, $U^\vee$ the Lie groups of $\mathfrak{h}^\vee$, $\mathfrak{b}^\vee$, and $\mathfrak{u}^\vee$, respectively. By definition, $B^\vee$ is a Borel subgroup in $G^\vee$ such that $B^\vee = T^\vee \cdot U^\vee = U^\vee \cdot T^\vee$.

Let us define the right $B^\vee$-action on $U^\vee$ via the isomorphism $U^\vee = T^\vee \setminus B^\vee$. This defines a $\mathcal{B}^\vee$-action on U and, therefore, structure of the left $B^\vee$-module on the algebra $\mathbb{C}[U^\vee]$. This locally finite $B^\vee$-module is naturally a $\mathfrak{b}^\vee$-module.

Claim 5.41. *The $\mathfrak{b}^\vee$-module $\mathbb{C}[U^\vee]$ is locally finite.*

For each $\lambda \in X_\star(T)^+$, denote by V^λ the restriction to $\mathfrak{b}^\vee$ of the finite-dimensional $\mathfrak{g}^\vee$-module with the highest weight λ.

It is well-known (see e.g., [19]) that

$$V^\lambda = U(\mathfrak{u}^\vee)/I_\lambda \ , \tag{5.13}$$

where $I_\lambda = \sum U(\mathfrak{u}^\vee) \cdot e_i^{\langle \alpha_i, \lambda \rangle + 1}$.

Therefore, taking the graded dual of the $\mathfrak{b}^\vee$-equivariant quotient map $U(\mathfrak{u}^\vee) \twoheadrightarrow V^\lambda$, we obtain an embedding of $\mathfrak{b}$-modules:

$$\mathbf{j}_\lambda : V^\lambda \hookrightarrow \mathbb{C}[U^\vee] \ . \tag{5.14}$$

for each $\lambda \in X_\star(T)^+$.

Denote by $\iota^* : \mathbb{C}[U^\vee] \to \mathbb{C}[U^\vee]$ the pullback of the positive inverse $\iota : U^\vee \to U^\vee$ (defined in (1.1)) above. By twisting the $\mathfrak{b}^\vee$-action by ι^*, we obtain another $\mathfrak{b}^\vee$-action on $\mathbb{C}[U^\vee]$. Denote

$$e_i^* := \iota^* \circ e_i \circ \iota^* \ . \tag{5.15}$$

Claim 5.42. *Each embedding* $\mathbf{j}_\lambda$ *satisfies*

$$\mathbf{j}_\lambda(V^\lambda) = \{f \in \mathbb{C}[U^\vee] : e_i^{*\langle \alpha_i, \lambda\rangle + 1}(f) = 0 \ \forall \ i \in I\} \ . \tag{5.16}$$

Therefore,

$$\mathbf{j}_\lambda(V^\lambda) \subset \mathbf{j}_{\lambda+\mu}(V^{\lambda+\mu})$$

for any $\lambda, \mu \in X_\star(T)^+$*; and* $\mathbf{j}_\lambda(V^\lambda) = \mathbf{j}_{\lambda'}(V^{\lambda'})$ *whenever* $\lambda - \lambda' \in Z(G)$*.*

Corollary 5.43. *For each* $\lambda \in X_\star(T)^+$*, one has*

$$\mathbb{C}[U^\vee] = \lim_{\lambda \in X_\star(T)^+} V^\lambda = \bigcup_{\lambda \in X_\star(T)^+} \mathbf{j}_\lambda(V^\lambda) \ .$$

This result can be generalized to $\mathbb{C}[U_P^\vee]$, where P is a standard parabolic subgroup of G, $U_P \subset U$ is the unipotent radical, i.e., $U_P = U \cap \overline{w_P} U^- \overline{w_P}^{-1}$ (see e.g., Example 1.7), and $U_P^\vee \subset U$ is the *dual* unipotent radical, i.e., $U_P^\vee = U^\vee \cap \overline{w_P}\,(U^-)^\vee\,\overline{w_P}^{-1}$.

Proposition 5.44. *For each* $\lambda \in X_\star(Z(L_P)) \cap X_\star(T)^+$*, the* $\mathfrak{b}^\vee$*-linear embedding* (5.14) *factors through* $V^\lambda \hookrightarrow \mathbb{C}[U_P^\vee]$*. With respect to these embeddings,*

$$\mathbb{C}[U_P^\vee] = \lim_{\lambda \in X_\star(Z(L_P)) \cap X_\star(T)^+} V^\lambda = \bigcup_{\lambda \in X_\star(Z(L_P)) \cap X_\star(T)^+} \mathbf{j}_\lambda(V^\lambda) \ .$$

Proof. Taking $\lambda \in X_\star(Z(L_P))$, we see that $\langle \alpha_i, \lambda\rangle = 0$ for all $i \in J(P)$ (i.e., such that $e_i \in \mathfrak{l}^\vee$). That is, according to (5.18), $\mathbf{j}_\lambda(V^\lambda) \subset \mathbb{C}[U_P^\vee]$. Finally, taking $\lambda \to \infty$ in $X_\star(Z(L_P)) \cap X_\star(T)^+$, Claim 5.42 implies that $\mathbf{j}_\lambda(V^\lambda) \subset \mathbb{C}[U_P^\vee]$. □

Note that the dual unipotent radical $U_P^\vee$ is being acted on by $B^\vee$. Therefore, the coordinate algebra $\mathbb{C}[U_P^\vee]$ is a locally finite $\mathfrak{b}^\vee$-module.

Denote by $\mathbf{B}^{\mathrm{dual}}$ the *dual canonical basis* for $\mathbb{C}[U^\vee]$ which is the specialization at $q = 1$ of the dual canonical basis of the quantized coordinate algebra $\mathbb{C}_q[U^\vee]$ (see, e.g., [4]).

Claim 5.45. *The basis* $\mathbf{B}^{\mathrm{dual}}$ *is a perfect basis of the locally finite* $\mathfrak{b}^\vee$*-module* $\mathbb{C}[U^\vee]$*. Moreover, for each* $\lambda \in X_\star(T)^+$*, the intersection* $\mathbf{B}^\lambda = \mathbf{B}^{\mathrm{dual}} \cap \mathbf{j}_\lambda(V^\lambda)$ *is a perfect basis in* $\mathbf{j}_\lambda(V^\lambda)$*.*

Therefore, according to Corollary 5.39 and Definition 5.40, each $\mathfrak{b}^\vee$-module of the form V^λ defines the associated crystal $\mathcal{B}(V^\lambda)$ and one has the crystal $\mathcal{B}(\mathbb{C}[U^\vee])$ associated with the $\mathfrak{b}^\vee$-module $\mathbb{C}[U^\vee]$. These crystals are related by a family of injective homomorphisms of upper Kashiwara crystals

$$\tilde{\mathbf{j}}_\lambda : \mathcal{B}(V^\lambda) \hookrightarrow \mathcal{B}(\mathbb{C}[U^\vee]) \ . \tag{5.17}$$

Moreover,

$$\mathcal{B}(\mathbb{C}[U^\vee]) = \lim_{\lambda \in X_\star(T)^+} \mathcal{B}(V^\lambda) = \bigcup_{\lambda \in X_\star(T)^+} \tilde{\mathbf{j}}_\lambda(\mathcal{B}(V^\lambda)) \ .$$

Proposition 5.46. *For each parabolic subgroup $P^\vee \subset G^\vee$, the intersection $\mathbf{B}_P^{\mathrm{dual}} = \mathbf{B}^{\mathrm{dual}} \cap \mathbb{C}[U_P^\vee]$ is a perfect basis in the locally finite $\mathfrak{b}^\vee$-module $\mathbb{C}[U_P^\vee]$.*

Proof. Denote by $U_L^\vee$ the intersection $U^\vee \cap \overline{w_P}\, U^\vee\, \overline{w_P}^{\,-1}$ (see e.g., Example 1.7). By definition, one has a $B^\vee$-equivariant isomorphism $U_P^\vee \cong U_L^\vee \backslash U^\vee$, where the $U^\vee$-action on the quotient $U_L^\vee \backslash U^\vee$ is the right multiplication, and the $T^\vee$-action is the conjugation. Therefore, one has a surjective map $U^\vee \to U_P^\vee$ and hence, a $\mathfrak{b}^\vee$-equivariant embedding of algebras: $\mathbb{C}[U_P^\vee] \hookrightarrow \mathbb{C}[U^\vee]$. In fact, the image of $\mathbb{C}[U_P^\vee]$ is the algebra $\mathbb{C}[U^\vee]^{U_L^\vee}$ of $U_L^\vee$-invariants.

Lemma 5.47. *The intersection of the dual canonical basis $\mathbf{B}^{\mathrm{dual}}$ for $\mathbb{C}[U^\vee]$ with $\mathbb{C}[U^\vee]^{U_{L^\vee}}$ is a basis of $\mathbb{C}[U^\vee]^{U_{L^\vee}}$.*

Proof. Note that

$$\mathbb{C}[U^\vee]^{U_{L^\vee}} = \{f \in \mathbb{C}[U^\vee] : e_j^*(f) = 0\ \forall j \in J\}\ , \tag{5.18}$$

where $J = J(P^{op}) = \{j \in I : U_j^\vee \subset U_{L^\vee}\}$ and e_i^* is defined above in (5.15). It is well-known that $\iota^*(\mathbf{B}^{\mathrm{dual}}) = \mathbf{B}^{\mathrm{dual}}$. This implies that the basis $\mathbf{B}^{\mathrm{dual}}$ is perfect under the twisted $\mathfrak{b}^\vee$-action. Replacing $B^\vee$ with $B_P^\vee = T^\vee \cdot U_{L^\vee}$, we see that under the twisted $B_P^\vee$-action the basis $\mathbf{B}^{\mathrm{dual}}$ is still a perfect basis of the locally finite $B_P^\vee$-module $\mathbb{C}[U^\vee]$ and that $\mathbb{C}[U^\vee]^{U_{L^\vee}}$ is the set of all highest weight vectors in this module. Therefore, the assertion follows from Claim 5.32. The lemma is proved. □

Therefore, the basis $\mathbf{B}_P^{\mathrm{dual}} := \mathbf{B}^{\mathrm{dual}} \cap \mathbb{C}[U^\vee]^{U_{L^\vee}}$ is the desired perfect basis of the $\mathfrak{b}^\vee$-module $\mathbb{C}[U_P^\vee]$. Proposition 5.46 is proved. □

Corollary 5.39 and Definition 5.40 imply existence of the *crystal basis* $\mathcal{B}(\mathbb{C}[U_P^\vee])$ for $\mathbb{C}[U_P^\vee]$. For each $\lambda \in X_\star(Z(L_P)) \cap X_\star(T)^+$, the homomorphism (5.17) factors through $\mathcal{B}(\mathbb{C}[U_P^\vee])$

$$\tilde{\mathbf{j}}_\lambda : \mathcal{B}(V^\lambda) \hookrightarrow \mathcal{B}(\mathbb{C}[U_P^\vee])\ . \tag{5.19}$$

and

$$\mathcal{B}(\mathbb{C}[U_P^\vee]) = \bigcup_{\lambda \in X_\star(Z(L_P)) \cap X_\star(T)^+} \tilde{\mathbf{j}}_\lambda(\mathcal{B}(V^\lambda))\ . \tag{5.20}$$

In fact, the associated crystal $\mathcal{B}(\mathbb{C}[U_P^\vee])$ can be constructed as a limit of a directed family

$$\tilde{\mathbf{f}}_{\lambda,\mu} : \mathcal{B}(V^\lambda) \to \mathcal{B}(V^{\lambda+\mu}) \tag{5.21}$$

the category of upper normal crystals. First, define $\tilde{\mathbf{j}}_{\lambda,\mu} : \mathcal{B}(V^\lambda) \to \mathcal{B}(V^\lambda) \times \mathcal{B}(V^\mu)$ by $\tilde{\mathbf{j}}_{\lambda,\mu} : \tilde{b} \mapsto (\tilde{b}, \tilde{b}_\mu)$. According to Lemma 5.19(b), each $\tilde{\mathbf{j}}_{\lambda,\mu}$ is an injective homomorphism of upper normal crystals $\mathcal{B}(V^\lambda) \hookrightarrow \mathcal{B}(V^\mu) \times \mathcal{B}(V^\lambda)$. Moreover, the range of $\tilde{\mathbf{f}}_{\lambda,\mu}$ belongs to the range of the embedding $\mathcal{B}(V^{\lambda+\mu}) \hookrightarrow \mathcal{B}(V^\lambda) \times \mathcal{B}(V^\mu)$. This defines an injective homomorphism (5.21) and, therefore, a directed family $(\mathcal{B}(V^\lambda), \tilde{\mathbf{f}}_{\lambda,\mu})$, $\lambda, \mu \in X_\star(T)$ of upper normal crystals.

Corollary 5.48. *For each standard parabolic subgroup P of G, the limit of the directed family $(\mathcal{B}(V^\lambda), \tilde{\mathbf{f}}_{\lambda,\mu})$ over $\lambda, \mu \in X_\star(Z(L_P)) \cap X_\star(T)^+$ is isomorphic to the associated crystal $\mathcal{B}(\mathbb{C}[U_P^\vee])$.*

5.4. **Perfect bases of $\mathfrak{g}^\vee$-modules and associated crystals.** Let $\mathfrak{b}^\vee = \mathfrak{h}^\vee \oplus \mathfrak{u}^\vee$ and $\mathfrak{b}^\vee_- = \mathfrak{u}^\vee_- \oplus \mathfrak{h}^\vee$ be the opposite Borel sub-algebras in $\mathfrak{g}^\vee$.

Denote by $U(\mathfrak{g}^\vee)$ the universal enveloping algebra of $\mathfrak{g}^\vee$ and say that $\mathfrak{g}^\vee$-module V is *locally finite* if, for each $v \in V$, the $\mathfrak{g}^\vee$-module $U(\mathfrak{g}^\vee)(v)$ is finite-dimensional and V admits the weight decomposition (5.5). Note that each locally finite $\mathfrak{g}^\vee$-module is also locally finite as both the $\mathfrak{b}^\vee_-$-module and the $\mathfrak{b}^\vee$-module.

Similarly to Section 5.2, for each locally finite $\mathfrak{g}^\vee$-module V, we define:
- the functions $\ell_i : V \setminus \{0\} \to \mathbb{Z}_{\geq 0}$, $i \in I \sqcup -I$,
- the subspaces $V_i^{<\ell} := \{v \in V : \ell_i(v) < \ell\}$ for $i \in I \sqcup -I$, $l \geq 0$,
- the space $V^+ = V^{\mathfrak{b}^\vee}$ of highest weight vectors.

Now we propose the following analogue of Definition 5.30.

Definition 5.49. We say that a weight basis $\mathbf{B}$ in a locally finite $\mathfrak{g}^\vee$-module V is $\mathfrak{g}^\vee$-*perfect* (or simply *perfect*) if for each $i \in I \sqcup -I$, there is a partial bijection $\tilde{e}_i : \mathbf{B} \to \mathbf{B}$ (i.e., a bijection of a subset of $\mathbf{B}$ onto another subset of $\mathbf{B}$, see also Definition 5.1 above) such that $\tilde{e}_i(b) \in \mathbf{B}$ if and only if $e_i(b) \neq 0$. And in the latter case, one has

$$e_i(b) \in \mathbb{C}^\times \cdot \tilde{e}_i(b) + V_i^{<\ell_i(b)-1} \ . \tag{5.22}$$

That is, each $\mathfrak{g}^\vee$-perfect basis is $\mathfrak{b}^\vee_-$-perfect and $\mathfrak{b}^\vee$-perfect at the same time.

Lemma 5.50. *For each locally finite $\mathfrak{g}^\vee$-module V, there exists a perfect basis. Moreover, if V' is a $\mathfrak{g}^\vee$-submodule of V, then there exists a perfect basis $\mathbf{B}$ for V such that $V' \cap \mathbf{B}$ is a (perfect) basis of V'.*

Proof. One can think of V as a specialization at $q = 1$ of a locally finite module V_q over the quantized enveloping algebra $U_q(\mathfrak{g}^\vee)$. G. Lusztig constructed in [17, 18] the *canonical basis* for each simple V_q which defines a basis $\mathbf{B}_q$ for each locally finite V_q. It follows from [18, Theorem 7.5] that for each $b \in \mathbf{B}_q$ and $i \in I \sqcup -I$ with $e_i(b) \neq 0$, there exists a unique element $b' \in \mathbf{B}_q$ such that

$$e_i(b) \in [\ell_i(b)]_i \cdot b' + V_{q,i}^{<\ell_i(b)-1} \ , \tag{5.23}$$

where $[n]_i$ is a q-analogue of the number n (the formula (5.23) also holds for any *global crystal basis* of V_q (see [13, Proposition 5.3.1])).

Substituting $q = 1$ into (5.23) we see that the weight basis $\mathbf{B}_q|_{q=1}$ also satisfies (5.22) and, therefore, is a perfect basis of V (moreover, $\mathbf{B}_q|_{q=1}$ defines a basis in each isotypic component of V).

Now let V' be a $\mathfrak{g}^\vee$-submodule of V. Clearly, V' is also locally finite and there exists a complementary (locally finite) submodule V'' of V such that $V \cong V' \oplus V''$. Then one can choose a perfect basis $\mathbf{B}'$ for V', a perfect basis $\mathbf{B}''$ for V'', and let $\mathbf{B} = \mathbf{B}' \sqcup \mathbf{B}''$ be the desired perfect basis of V.

The lemma is proved. □

According to Claim 5.31, a perfect basis $\mathbf{B}$ for a $\mathfrak{g}^\vee$-module V defines an upper normal Kashiwara crystal $\mathcal{B}(V, \mathbf{B})_+$ - when V is considered $\mathfrak{b}^\vee$-module and a lower normal Kashiwara crystal $\mathcal{B}(V, \mathbf{B})_-$ - when V is considered $\mathfrak{b}^\vee_-$-module (see Section 5.1).

Lemma 5.51. *The Kashiwara crystals $\mathcal{B}(V,\mathbf{B})_+$ and $\mathcal{B}(V,\mathbf{B})_-$ are canonically isomorphic and constitute a normal Kashiwara crystal $\mathcal{B}(V,\mathbf{B})$.*

Proof. The isomorphism between $\mathcal{B}(V,\mathbf{B})_+$ and $\mathcal{B}(V,\mathbf{B})_-$ follows from the fact that the partial bijections $\tilde{e}_{-i}$ and $\tilde{e}_i$ are inverse to each other.

The normality of $\mathcal{B}(V,\mathbf{B})$ follows from the fact that for each vector $v \in V(\mu)$ (e.g., for each $b \in \mathbf{B}(\mu)$) and $i \in I$, one has

$$\ell_i(b) = \ell_{-i}(b) - \langle \alpha_i, \mu \rangle \ , \tag{5.24}$$

where $\langle \bullet, \bullet \rangle$ is the pairing between roots and co-weights of G. □

Remark 5.52. Every $\mathfrak{g}^\vee$-perfect basis in V is *good* in the sense of Gelfand-Zelevinsky ([9]), that is, it is compatible with each subspace $V_i^{<\ell}$ of V.

Remark 5.53. According to [8, Lemma 4.3], every *positive basis* $\mathbf{B}$ for V (i.e., satisfying $e_i(b) \in \mathbb{Q}_{\geq 0} \cdot \mathbf{B}$ for any $b \in \mathbf{B}$, $i \in I \sqcup -I$) is perfect.

Claim 5.54. *For any perfect basis $\mathbf{B}$ for V, the subset $\mathbf{B}^+ = V^+ \cap \mathbf{B}$ is a basis for V^+. In particular,*

$$V = \bigoplus_{b \in \mathbf{B}^+} U(\mathfrak{g}^\vee) \cdot b \ . \tag{5.25}$$

The following is the main result of this section.

Main Theorem 5.55. *For any perfect bases $\mathbf{B}$ and $\mathbf{B}'$ in a locally finite $\mathfrak{g}^\vee$-module V, the normal crystals $\mathcal{B}(V,\mathbf{B})$ and $\mathcal{B}(V,\mathbf{B}')$ are isomorphic.*

Proof. We will construct the isomorphism explicitly. First, we need the following fact.

Claim 5.56. *Let $\mathbf{B}^+$ and $\mathbf{B}'^+$ be any weight bases of V^+. Then*

(a) *There exists a weight-preserving bijection $\varphi^+ : \mathbf{B}^+ \hookrightarrow \mathbf{B}'^+$.*
(b) *Each such weight-preserving bijection $\varphi^+ : \mathbf{B}^+ \xrightarrow{\sim} \mathbf{B}'^+$ uniquely extends to an isomorphism of the $\mathfrak{g}^\vee$-modules $\varphi : V \xrightarrow{\sim} V$.*

Indeed, for any $\mathfrak{g}^\vee$-module automorphism $\varphi : V \xrightarrow{\sim} V$ the set $\varphi(\mathbf{B})$ is a perfect basis of $\varphi(V) = V$. Therefore, we may assume, without loss of generality, that φ is the identity and hence $\mathbf{B}^+ = \mathbf{B}'^+$. This and Theorem 5.37 guarantee an isomorphism of normal crystals $\mathcal{B}(V,\mathbf{B}) \xrightarrow{\sim} \mathcal{B}(V,\mathbf{B}')$.

Theorem 5.55 is proved. □

Remark 5.57. In the case when $\mathbf{B}$ is any perfect basis and $\mathbf{B}' = \mathbf{B}_q|_{q=1}$ is the specialization of a canonical basis ([17, 18]) or a global crystal basis ([13]), the normalized perfect basis $\mathbf{B}_{norm} = \{c_b \cdot b \,|\, b \in \mathbf{B}\}$ satisfies the specialization of (5.23) at $q = 1$, where the function $\mathbf{B} \to \mathbb{C}^\times$: $b \mapsto c_b$ is defined in (5.10) and $c_b := 1$ for $b \in \mathbf{B}^+$.

Remark 5.58. Theorem 5.55 implies that the canonical basis, the dual canonical basis, the (upper and lower) global crystal bases (see e.g., [13]), and the semi-canonical basis ([21]) in each irreducible $\mathfrak{g}$-module V_λ, $\lambda \in X_\star(T)^+$ induce the same associated crystal $\mathcal{B}(V_\lambda)$.

Theorem 5.55 leads to the following definition.

Definition 5.59. For each locally finite $\mathfrak{g}$-module V, define the *associated crystal* $\mathcal{B}(V)$ to be isomorphic to each of the Kashiwara crystals of the form $\mathcal{B}(V, \mathbf{B})$, where $\mathbf{B}$ is any perfect basis of V.

Furthermore, Theorem 5.55 and Lemma 5.50 imply the following result.

Corollary 5.60. *Any injective homomorphism of locally finite $\mathfrak{g}^\vee$-modules $V' \hookrightarrow V$ defines an injective homomorphism $\mathcal{B}(V') \hookrightarrow \mathcal{B}(V)$ of associated crystals.*

6. Tropicalization of geometric crystals and unipotent bicrystals

6.1. **From decorated geometric crystals to normal Kashiwara crystals.** Recall from Claim 3.14 that $\mathcal{V}_+$ is the category of positive varieties and from Claim 3.17 that $\mathcal{V}_{++}$ is the category of decorated positive varieties. Recall also from Section 4.2 that $\mathbf{Set_0}$ is the category of marked sets.

Claim 6.1. *In the notation of Claim 3.18 and Theorem 4.12, the composition of functors* $\mathrm{Trop} \circ \tau \circ \mathcal{G}^*$ *is a monoidal functor* $\mathcal{V}_+ \to \mathbf{Set_0}$ *such that the image of each positive variety* (X, Θ) *under this functor is isomorphic (in* $\mathbf{Set_0}$*) to a lattice of the same dimension.*

The monoidal functor $\mathcal{V}_+ \to \mathbf{Set_0}$ from Claim 6.1 is unique up to isomorphism due to Claim 3.18(b). Throughout the end of the paper, we fix one of these isomorphic functors and, similarly to Theorem 4.12, refer to it as the tropicalization of positive varieties, and denote it by $\mathrm{Trop} : \mathcal{V}_+ \to \mathbf{Set_0}$.

In fact, due to Corollary 4.18 of [2, Appendix], the tropicalization is well-defined for any non-zero function on each positive variety. Namely, given a positive variety (X, Θ) and two non-zero rational functions $f, f' : X \to \mathbb{A}^1$, we say that f and f' are *positively equivalent* if there is an isomorphism of positive varieties $h : (X, \Theta) \xrightarrow{\sim} (X, \Theta)$ such that $f' = f \circ h$. Proposition 4.17 implies that for positively equivalent functions f and f', one has

$$\mathrm{Trop}(f') = \mathrm{Trop}(f) \circ \tilde{h} \ ,$$

where $\tilde{h}$ is that bijection $\mathrm{Trop}(X, \Theta) \to \mathrm{Trop}(X, \Theta)$ which is the tropicalization of the positive equivalence $h : (X, \Theta) \to \mathrm{Trop}(X, \Theta)$. Therefore, the following version of Corollary 4.18 holds.

Claim 6.2. *Let (X, Θ) be a positive variety, and $f : X \to \mathbb{A}^1$ be a non-zero rational function. Then the isomorphism class of the function* $\mathrm{Trop}(f) : \mathrm{Trop}(X, \Theta) \to \mathbb{Z}$ *in* $\mathbf{Set_0}$ *depends only on the positive equivalence class of f.*

Let $(\mathcal{X}, \Theta) = ((X, \gamma, \varphi_i, \varepsilon_i, e_i^\cdot | i \in I), \Theta)$ be a positive a geometric pre-crystal (see Section 3.2). We define the 5-tuple $\mathcal{B}_\Theta = \mathrm{Trop}(\mathcal{X}, \Theta) := (\widetilde{X}, \tilde{\gamma}, \tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{e}_i | i \in I)$, where

(i) $\widetilde{X} = \mathrm{Trop}(X, \Theta)$,
(ii) the map $\tilde{\gamma} : \widetilde{X} \to X_\star(T)$ is given by

$$\tilde{\gamma} = \mathrm{Trop}(\gamma) \ ,$$

(iii) the functions $\tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{\alpha}_i : \widetilde{X} \to \mathbb{Z}$:

$$\tilde{\varphi}_i = -\operatorname{Trop}(\varphi_i), \tilde{\varepsilon}_i = -\operatorname{Trop}(\varepsilon_i), \tilde{\alpha}_i = \operatorname{Trop}(\alpha_i \circ \gamma) = \langle \alpha_i, \tilde{\gamma}(\cdot) \rangle \ ,$$

(iv) the $\mathbb{Z}$-action $\tilde{e}_i^{\cdot} : \mathbb{Z} \times \widetilde{X} \to \widetilde{X}$ is the tropicalization of the $(\Theta_{\mathbb{G}_m} \times \Theta, \Theta)$-positive $\mathbb{G}_m$-action $\tilde{e}_i^{\cdot} : \mathbb{G}_m \times X \to X$. This $\mathbb{Z}$-action defines a bijection $\tilde{e}_i : \widetilde{X} \to \widetilde{X}$ via $\tilde{e}_i = \tilde{e}_i^1$.

Claim 6.3. [2, Theorem 2.11] *For any geometric pre-crystal $\mathcal{X}$, the 5-tuple $\mathcal{B}_\Theta = \operatorname{Trop}(\mathcal{X}, \Theta)$ is a Kashiwara crystal in the category $\mathbf{Set_0}$ of marked sets. This crystal is torsion-free in the sense that each $\tilde{e}_i$ is a bijection.*

Given a positive geometric pre-crystal $(\mathcal{X}, \Theta)$, clearly, the pair $(\mathcal{X}^{op}, \Theta)$ is also positive (see Claim 2.19).

Claim 6.4. *For any positive geometric pre-crystal $(\mathcal{X}, \Theta)$, one has (in the notation of Claim 5.10):*

$$\mathcal{B}_{\Theta^{op}} = (\mathcal{B}_\Theta)^{op} \ ,$$

where $\mathcal{B}_{\Theta^{op}} := \operatorname{Trop}(\mathcal{X}^{op}, \Theta)$.

The following result explains why our Definition 5.6 differs from the original definition of [14] by a permutation of factors.

Lemma 6.5. *Let $(\mathcal{X}, \Theta_X)$ and $(\mathcal{Y}, \Theta_Y)$ be positive geometric pre-crystals. Then one has a canonical isomorphism of torsion-free Kashiwara crystals:*

$$\operatorname{Trop}(\mathcal{X} \times \mathcal{Y}, \Theta_X \times \Theta_Y) = \operatorname{Trop}(\mathcal{X}, \Theta_X) \times \operatorname{Trop}(\mathcal{Y}, \Theta_Y) \ . \tag{6.1}$$

Proof. Indeed, applying Trop to Definition 2.15, we obtain for $\tilde{x} \in \tilde{X}$, $\tilde{y} \in \tilde{Y}$:

$$\tilde{\gamma}''(\tilde{x}, \tilde{y}) = \tilde{\gamma}(\tilde{x}) + \tilde{\gamma}'(\tilde{y}) \ ,$$

and for each $i \in \operatorname{Supp} \mathcal{X} \cup \operatorname{Supp} \mathcal{Y}$, the functions $\tilde{\varphi}_i'', \tilde{\varepsilon}_i'' : \tilde{X} \times \tilde{Y} \to \mathbb{Z}$ given by

$$-\tilde{\varphi}_i''(\tilde{x}, \tilde{y}) = \min(-\tilde{\varphi}_i(\tilde{x}), -\tilde{\varphi}_i'(\tilde{y}) - \langle \alpha_i, \tilde{\gamma}(\tilde{x}) \rangle = -\max(\tilde{\varphi}_i(\tilde{x}), \tilde{\varphi}_i'(\tilde{y}) + \langle \alpha_i, \tilde{\gamma}(\tilde{x}) \rangle,$$

$$-\tilde{\varepsilon}_i''(\tilde{x}, \tilde{y}) = \min(-\tilde{\varepsilon}_i'(\tilde{y}), -\tilde{\varepsilon}_i(\tilde{x}) + \langle \alpha_i, \tilde{\gamma}'(\tilde{y}) \rangle) = -\max(\tilde{\varepsilon}_i'(\tilde{y}), \tilde{\varepsilon}_i(\tilde{x}) - \langle \alpha_i, \tilde{\gamma}'(\tilde{y}) \rangle);$$

the bijection $\tilde{e}_i^n : \tilde{X} \times \tilde{Y} \to \tilde{X} \times \tilde{Y}$, $n \in \mathbb{Z}$ is given by the formula

$$\tilde{e}_i^n(x, y) = (\tilde{e}_i^{n_1}(\tilde{x}), \tilde{e}_i^{n_2}(\tilde{y})) \ ,$$

where

$$\begin{aligned} n_1 &= \min(n - \tilde{\varepsilon}_i(\tilde{x}), -\tilde{\varphi}_i'(\tilde{y})) - \min(-\tilde{\varepsilon}_i(\tilde{x}), -\tilde{\varphi}_i'(\tilde{y})) \\ &= \max(\tilde{\varepsilon}_i(\tilde{x}), \tilde{\varphi}_i'(\tilde{y})) - \max(\tilde{\varepsilon}_i(\tilde{x}) - n, \tilde{\varphi}_i'(\tilde{y})) \ , \\ n_2 &= \min(-\tilde{\varepsilon}_i(\tilde{x}), -\tilde{\varphi}_i'(\tilde{y})) - \min(-\tilde{\varepsilon}_i(\tilde{x}), -n - \tilde{\varphi}_i'(\tilde{y})) \\ &= \max(\tilde{\varepsilon}_i(\tilde{x}), n + \tilde{\varphi}_i'(\tilde{y})) - \max(\tilde{\varepsilon}_i(\tilde{x}), \tilde{\varphi}_i'(\tilde{y})) \ . \end{aligned}$$

The above equations agree with Definition 5.6. The lemma is proved. □

Furthermore, recall from Section 3.2 that a triple $(\mathcal{X}, f, \Theta)$ is a positive decorated geometric pre-crystal if $(\mathcal{X}, f)$ is a decorated geometric pre-crystal, $(\mathcal{X}, \Theta)$ is a positive geometric pre-crystal, and the function f is Θ-positive.

For each positive decorated geometric pre-crystal $(\mathcal{X}, f, \Theta)$, one defines:

(v) the function $\tilde{f} : \widetilde{X} \to \mathbb{Z}$ by $\tilde{f} = \operatorname{Trop}(f)$,

(vi) the set $\tilde{B}_{\tilde{f}} \subset \widetilde{X}$ by $\tilde{B}_{\tilde{f}} := \{\tilde{x} \in \widetilde{X} : \tilde{f}(\tilde{x}) \geq 0\}$.

Denote by $\mathcal{B}_{f,\Theta}$ the Kashiwara crystal obtained by restricting the torsion-free Kashiwara crystal $\mathcal{B}_\Theta = \mathrm{Trop}(\mathcal{X}, \Theta)$ to the subset $\tilde{B}_{\tilde{f}}$, i.e.,

$$\mathcal{B}_{f,\Theta} = (\tilde{B}_{\tilde{f}}, \tilde{\gamma}|_{\tilde{B}_{\tilde{f}}}, \tilde{\varphi}_i|_{\tilde{B}_{\tilde{f}}}, \tilde{\varepsilon}_i|_{\tilde{B}_{\tilde{f}}}, \tilde{e}_i|_{\tilde{B}_{\tilde{f}}} | i \in I) \ . \tag{6.2}$$

Proposition 6.6. *For any positive decorated geometric pre-crystal $(\mathcal{X}, f, \Theta)$, the Kashiwara crystal $\mathcal{B}_{f,\Theta}$ is normal.*

Proof. Let us rewrite (2.18) as follows:

$$f(e_i^c(x)) = f_0(x) + \frac{c}{\varphi_i(x)} + \frac{c^{-1}}{\varepsilon_i(x)} \tag{6.3}$$

for $x \in X$, $c \in \mathbb{G}_m$, $i \in I$, where $f_0(x) = f(x) - \frac{1}{\varphi_i(x)} - \frac{1}{\varepsilon_i(x)}$.

Denote $\widetilde{X} := \mathrm{Trop}(X, \Theta)$ and let $\tilde{\varphi}_i, \tilde{\varepsilon}_i, \tilde{f}, \tilde{f}_0 : \tilde{X} \to \mathbb{Z}$ be the Θ-tropicalizations of the functions $\frac{1}{\varphi_i}$, $\frac{1}{\varepsilon_i}$, f, and f_0, respectively (if $f_0 = 0$, then $\tilde{f}_0 = +\infty$). Note that the function $\mathbb{G}_m \times X \to \mathbb{A}^1$ given by $(c, x) \mapsto f(e_i^c(x))$ is $\Theta_{\mathbb{G}_m} \times \Theta$-positive. It follows from Theorem 4.12 that the tropicalization of this positive function is the function $\mathbb{Z} \times \widetilde{X} \to \mathbb{Z}$ given by $(n, \tilde{b}) \mapsto \tilde{f}(\tilde{e}_i^n(\tilde{b}))$. Therefore, applying Corollary 4.10(b) to the identity (6.3), we obtain:

$$\tilde{f}(\tilde{e}_i^n(\tilde{b})) = \min(\tilde{f}_0(\tilde{b}), n + \tilde{\varphi}_i(\tilde{b}), -n + \tilde{\varepsilon}_i(\tilde{b})) \tag{6.4}$$

for $\tilde{b} \in \widetilde{X}$, $n \in \mathbb{Z}$, $i \in \mathrm{Supp}\, \mathcal{X}$. In particular, taking $n = 0$, we obtain

$$\tilde{f}(\tilde{b}) = \min(\tilde{f}_0(\tilde{b}), \tilde{\varphi}_i(\tilde{b}), \tilde{\varepsilon}_i(\tilde{b})) \ .$$

Since $\tilde{b} \in \tilde{B}_{\tilde{f}}$ if and only if $\tilde{f}(\tilde{b}) \geq 0$, the above identity implies that $\tilde{b} \in \tilde{B}_{\tilde{f}}$ if and only if $\tilde{f}_0(\tilde{b}) \geq 0$, $\tilde{\varphi}_i(\tilde{b}) \geq 0$, $\tilde{\varepsilon}_i(\tilde{b}) \geq 0$. Therefore, according to (6.4), $\tilde{e}_i^n(\tilde{b}) \in \tilde{B}_{\tilde{f}}$ for $\tilde{b} \in \tilde{B}_{\tilde{f}}$ if and only if $-\tilde{\varphi}_i(\tilde{b}) \leq n \leq \varepsilon_i(\tilde{b})$. On the other hand, by definition (5.3), $\tilde{e}_i^n(\tilde{b}) \in \tilde{B}_{\tilde{f}}$ for some $\tilde{b} \in \tilde{B}_{\tilde{f}}$ if and only if $-\ell_{-i}(\tilde{b}) \leq n \leq \ell_i(\tilde{b})$. This implies that

$$\ell_i(\tilde{b}) = \tilde{\varepsilon}_i(\tilde{b}), \ell_{-i}(\tilde{b}) = \tilde{\varphi}_i(\tilde{b})$$

for all $\tilde{b} \in \tilde{B}_{\tilde{f}}$. That is, $\mathcal{B}_{f,\Theta}$ is normal. The proposition is proved. □

As we argued in Section 3.2, based on Definition 2.33, one defines product of positive decorated geometric pre-crystals by

$$(\mathcal{X}, f, \Theta) \times (\mathcal{Y}, f', \Theta') := (\mathcal{X} \times \mathcal{Y}, f * f', \Theta \times \Theta') \ ,$$

where $\Theta \times \Theta'$ is the natural positive structure on the product $X \times Y$ and $f * f' : X \times Y \to \mathbb{A}^1$ is given by $(f * f')(x, y) = f(x) + f'(y)$.

Proposition 6.7. *For any positive decorated geometric pre-crystals $(\mathcal{X}, f, \Theta)$ and $(\mathcal{Y}, f', \Theta')$, one has*

$$\mathcal{B}_{f*f',\Theta\times\Theta'} = \mathcal{B}_{f,\Theta} \times \mathcal{B}_{f',\Theta'} \ . \tag{6.5}$$

Proof. Due to the canonical isomorphism (6.1) of torsion-free Kashiwara crystals it suffices to show that the subset of $\widetilde{X \times Y} = \mathrm{Trop}(X \times Y, \Theta \times \Theta') = \widetilde{X} \times \widetilde{Y}$ involved in the left hand side of (6.5) is equal to the set involved in the right hand side. Indeed, by definition,

$$\tilde{B}_{\widetilde{f*f'}} = \{(\tilde{x}, \tilde{y}) \in \widetilde{X} \times \widetilde{Y} : \widetilde{f * f'}(\tilde{x}, \tilde{y}) \geq 0\} \ ,$$

where $\widetilde{f * f'} : \widetilde{X} \times \widetilde{Y} \to \mathbb{Z}$ is the $\Theta \times \Theta'$-tropicalization of $f * f'$. Since $f : X \to \mathbb{A}^1$, $f' : Y \to \mathbb{A}^1$, and $f * f' : X \times Y \to \mathbb{A}^1$ are positive functions, Corollary 4.10(d) guarantees that their tropicalizations $\tilde{f} : \widetilde{X} \to \mathbb{Z}$, $\tilde{f}' : \widetilde{Y}^{\to}\mathbb{Z}$, and $\widetilde{f * f'} : \widetilde{X} \times \widetilde{Y} \to \mathbb{Z}$, respectively, satisfy

$$\widetilde{f * f'}(\tilde{x}, \tilde{y}) = \min(\tilde{f}(\tilde{x}), \tilde{f}'(\tilde{y}))$$

for all $\tilde{x} \in \widetilde{X}$, $\tilde{y} \in \widetilde{Y^-}$. Hence, $\widetilde{f * f'}(\tilde{x}, \tilde{y}) \geq 0$ if and only if $\tilde{f}(\tilde{x}) \geq 0$ and $\tilde{f}'(\tilde{y}) \geq 0$. Therefore,

$$\tilde{B}_{\widetilde{f*f'}} = \{(\tilde{x}, \tilde{y}) : \tilde{x} \in \tilde{B}_{\tilde{f}}, \tilde{y} \in \tilde{B}_{\tilde{f}'}\} = \tilde{B}_{\tilde{f}} \times \tilde{B}_{\tilde{f}'} \ .$$

The proposition is proved. □

6.2. **From positive unipotent bicrystals to normal Kashiwara crystals.** Let $(\mathbf{X}, \mathbf{p}, \Theta)$ be a positive unipotent bicrystal. As in Section 6.1 above, we denote by $\mathcal{B}_\Theta$ the torsion-free Kashiwara crystals associated with the positive geometric crystal of the form $(\mathcal{X}, \Theta) = \mathcal{F}(\mathbf{X}, \mathbf{p}, \Theta)$ (in the notation of Lemma 3.30(a)).

Clearly, all the definitions and results of the above Section 6.1 are valid for positive decorated geometric crystals obtained this way from positive unipotent bicrystals.

The following result provides the tropicalization of the morphism $\mathbf{f}_w$ from (2.16).

Claim 6.8. *Given a positive unipotent bicrystal* $(\mathbf{X}, \mathbf{p}, \Theta)$ *of type* w*, the tropicalization of the* $(\Theta, \Theta_T \cdot \Theta_w^-)$*-positive morphism* $\mathbf{f} = \mathbf{p}|_{X^-} : X^- \to TB_w^-$ *is a homomorphism of torsion-free Kashiwara crystals*

$$\tilde{\mathbf{f}}_w : \mathcal{B}_\Theta \to \mathcal{B}_{\Theta_T \cdot \Theta_w^-} = X_\star(T) \times \mathcal{B}_{\Theta_w^-} \ , \tag{6.6}$$

where $X_\star(T)$ *is the trivial Kashiwara crystal as in Example 5.3. The support of* $\tilde{\mathbf{f}}_w$ *is* $|w|$*.*

For a positive $(U \times U, \chi^{st})$-linear unipotent bicrystal $(\mathbf{X}, \mathbf{p}, f, \Theta)$, we define the map $\widetilde{hw} : \mathcal{B}_\Theta \to \mathcal{B}_{\Theta_T \cdot \Theta_w^-} = X_\star(T)$ to be the composition of $\tilde{\mathbf{f}}_w$ with the projection to the first factor $X_\star(T) \times \mathcal{B}_{\Theta_w^-} \to X_\star(T)$. This is a homomorphism of Kashiwara crystals (see Example 5.8).

Note that according to Claim 2.28 and Claim 3.31, $\widetilde{hw}$ is the tropicalization of the (Θ, Θ_T)-positive rational morphism $hw_X : \mathcal{F}(\mathbf{X}, \mathbf{p}) \to T$ defined in (2.5).

For each $\lambda \in X_\star(T)$, define $\mathcal{B}_\Theta^\lambda := \widetilde{hw}^{-1}(\lambda)$. In particular, the underlying set $\mathcal{B}_\Theta^\lambda$ is $\{\tilde{b} \in \tilde{B} : \widetilde{hw}(\tilde{b}) = \lambda\}$.

Claim 6.9. *For a positive unipotent bicrystal* $(\mathbf{X}, \mathbf{p}, \Theta)$ *of type* w*, one has:*

(a) *Each* $\mathcal{B}_\Theta^\lambda$ *is invariant under the action of* $\tilde{e}_i^n$*,* $i \in I$*,* $n \in \mathbb{Z}$ *and, therefore, is a (torsion-free or empty) sub-crystal of the torsion-free Kashiwara crystal* $\mathcal{B}_\Theta$*.*

(b) *The direct decomposition* $\mathcal{B}_\Theta = \bigsqcup_\lambda \mathcal{B}_\Theta^\lambda$*.*

(c) $(\mathcal{B}^{\lambda}_{\Theta})^{op} = \mathcal{B}^{-w\lambda}_{\Theta^{op}}$ *for each* $\lambda \in X_\star(T)$*, where* $\mathcal{B}_{\Theta^{op}} = \mathrm{Trop}(\mathcal{F}(\mathbf{X}, \mathbf{p}, \Theta)^{op})$.

Now let $(\mathbf{X}, \mathbf{p}, f, \Theta)$ be a positive $(U \times U, \chi^{st})$-linear bicrystal. Denote by $\mathcal{B}_{f,\Theta}$ the tropicalization of the positive decorated geometric crystal $\mathcal{F}(\mathbf{X}, \mathbf{p}, f, \Theta)$ (see Lemma 3.30 and (6.2)). According to Proposition 6.6, $\mathcal{B}_{f,\Theta}$ is normal. Furthermore, for each $\lambda \in X_\star(T)$, denote

$$\mathcal{B}^{\lambda}_{f,\Theta} := \mathcal{B}_{f,\Theta} \cap \mathcal{B}^{\lambda}_{\Theta} \ . \tag{6.7}$$

Claim 6.10. *For a positive* $(U \times U, \chi^{st})$*-linear bicrystal* $(\mathbf{X}, \mathbf{p}, f, \Theta)$ *of type* w*, one has:*

(a) *Each* $\mathcal{B}^{\lambda}_{f,\Theta}$ *is invariant under the action of* $\tilde{e}_i^n$*,* $i \in I$*,* $n \in \mathbb{Z}$ *and, therefore, is a normal sub-crystal of* $\mathcal{B}_{f,\Theta}$*.*
(b) *The direct decomposition* $\mathcal{B}_{f,\Theta} = \bigsqcup_\lambda \mathcal{B}^{\lambda}_{f,\Theta}$*.*
(c) $(\mathcal{B}^{\lambda}_{f,\Theta})^{op} = \mathcal{B}^{-w\lambda}_{f,\Theta^{op}}$ *for* $\lambda \in X_\star(T)$*, where* $\mathcal{B}_{f,\Theta^{op}} = \mathrm{Trop}(\mathcal{F}(\mathbf{X}, \mathbf{p}, f, \Theta)^{op})$.

Remark 6.11. Each $\mathcal{B}^{\lambda}_{f,\Theta}$ can be thought of as an isotypic component of $\mathcal{B}_{f,\Theta}$. Later on, in Theorem 6.15, we will make this analogy precise.

6.3. **From strongly positive unipotent bicrystals to crystals associated with modules.** Given a strongly positive parabolic $(U \times U, \chi^{st})$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f, \Theta)$ of type w_P, we denote by $\tilde{\Delta}_X : \widetilde{X^-} \to \mathbb{Z}$ the Θ-tropicalization of the positive function $\Delta_X|_{X^-} : X^- \to \mathbb{A}^1$. For each $\lambda \in X_\star(Z(L))$ and $n \in \mathbb{Z}$, we denote

$$\mathcal{B}^{\lambda}_{f,\Theta;n} := \{\tilde{b} \in \mathcal{B}^{\lambda}_{f,\Theta} : \tilde{\Delta}_X(\tilde{b}) = n\} \ .$$

Claim 6.12. *The function* $\tilde{\Delta}_X$ *is invariant under all crystal operators* $\tilde{e}_i^n$*. In particular, each non-empty* $\mathcal{B}^{\lambda}_{f,\Theta;n}$ *is a normal sub-crystal of the normal crystal* $\mathcal{B}^{\lambda}_{f,\Theta}$*.*

Due to its $\tilde{e}_i^n$-invariance, will refer to the function $\tilde{\Delta}_X$ as the *combinatorial central charge.*

Recall that $(X_P, \mathrm{id}, f_P, \Theta^-_P)$ is the strongly positive $(U \times U, \chi^{st})$-linear bicrystal defined in Section 3.1.

Proposition 6.13. *Let* $(\mathbf{X}, \mathbf{p}, f, \Theta)$ *be a strongly positive parabolic* $(U \times U, \chi^{st})$*-linear bicrystal of type* w_P*. Then*

$$\tilde{\mathbf{f}}_{w_P}(\mathcal{B}_{f,\Theta}) \subset \mathcal{B}_{f_P,\Theta^-_P} \ ,$$

where $\tilde{\mathbf{f}}_{w_P} : \mathcal{B}_\Theta \to \mathcal{B}_{\Theta^-_P}$ *is given by* (6.6)*. This defines a homomorphism of normal crystals*

$$\tilde{\mathbf{m}} : \mathcal{B}_{f,\Theta} \to \mathcal{B}_{f_P,\Theta^-_P} \tag{6.8}$$

such that $\tilde{\mathbf{m}}(\mathcal{B}^{\lambda}_{f,\Theta}) \subset \mathcal{B}^{\lambda}_{f,\Theta^-_P}$ *for each* $\lambda \in X_\star(Z(L))$*.*

Proof. We need the following simple fact.

Claim 6.14. *In the notation of Proposition 6.13, let* $\tilde{f} : \widetilde{X^-} \to \mathbb{Z}$*,* $\tilde{f}_P : \widetilde{X^-_P} \to \mathbb{Z}$*, and* $\tilde{\Delta}_X : \widetilde{X^-} \to \mathbb{Z}$ *be the tropicalizations of positive functions* $f|_{X^-} : X^- \to \mathbb{A}^1$*,* $f_P|_{X^-_P} : X^-_P \to \mathbb{A}^1$*, and* $\Delta_X|_{X^-} : X^- \to \mathbb{A}^1$*, respectively. Then*

$$\tilde{f} = \min(\tilde{f}_P \circ \tilde{\mathbf{f}}_{w_P}, \tilde{\Delta}_X) \ .$$

In particular,

$$\tilde{f}(\tilde{x}) \leq \tilde{f}_P(\tilde{\mathbf{f}}_{w_P}(\tilde{x})) \tag{6.9}$$

for any $\tilde{x} \in \widetilde{X^-}$.

The inequality (6.9) guarantees that $\tilde{\mathbf{f}}_{w_P}(\tilde{B}_{\tilde{f}}) \subset \tilde{B}_{\tilde{f}_P}$, i.e., the restriction of $\tilde{\mathbf{f}}_{w_P}$ to $\tilde{B}_{\tilde{f}}$ defines a homomorphism of normal crystals $\mathcal{B}_{f,\Theta} \to \mathcal{B}_{f_P,\Theta_P^-}$.

The proposition is proved. □

Recall that the associated crystal $\mathcal{B}(V)$ of each locally finite $\mathfrak{g}^\vee$-module V is defined in Section 5.4 above. In particular, each $\mathcal{B}(V)$ is the union of $\mathcal{B}(V_\lambda)$, where V_λ is an irreducible finite-dimensional $\mathfrak{g}^\vee$-module with the highest (co-)weight $\lambda \in X_\star(T)^+$, where $X_\star(T)^+$ is the monoid of all dominant co-weights defined in Section 1.1.

Recall also from Claim 6.10 that each Kashiwara crystal of the form $\mathcal{B}_{f,\Theta}$ is the disjoint union of $\mathcal{B}^\lambda_{f,\Theta}$, $\lambda \in X_\star(T)$.

The following is our first main result on the Kashiwara crystals coming from strongly positive unipotent bicrystals.

Main Theorem 6.15. *For each* $\lambda \in X_\star(T)$, *we have:* $\mathcal{B}^\lambda_{f_B,\Theta_B^-}$ *is empty if* $\lambda \notin X_\star(T)^+$ *and* $\mathcal{B}^\lambda_{f_B,\Theta_B^-}$ *is isomorphic to the associated crystal* $\mathcal{B}(V_\lambda)$ *if* $\lambda \in X_\star(T)^+$.

Proof. The proof is based on Joseph's characterization of the associated crystals $\mathcal{B}(V_\lambda)$. Following [12, Section 6.4.21], we say that a family $\mathcal{C}. = \{\mathcal{C}_\lambda | \lambda \in X_\star(T)^+\}$ of normal crystals is *closed* if

• For each $\lambda \in X_\star(T)^+$, there is a unique highest weight element $c_\lambda \in \mathcal{C}_\lambda$ (see Definition 5.14) such that $\tilde{\gamma}(c_\lambda) = \lambda$.

• For each $\lambda, \mu \in X_\star(T)^+$, the correspondence $c_{\lambda+\mu} \mapsto (c_\lambda, c_\mu) \in \mathcal{C}_\lambda \times \mathcal{C}_\mu$ defines an injective homomorphism of normal crystals $\mathcal{C}_{\lambda+\mu} \hookrightarrow \mathcal{C}_\lambda \times \mathcal{C}_\mu$.

Theorem 6.16 ([12])**.** *If* $\mathcal{C}. = \{\mathcal{C}_\lambda | \lambda \in X_\star(T)^+\}$ *is a closed family of crystals, then each* $\mathcal{C}_\lambda$ *is isomorphic to the associated crystal* $\mathcal{B}(V_\lambda)$.

We will show that the crystals $\mathcal{B}^\lambda_{f_B,\Theta_B^-}$, $\lambda \in X_\star(T)^+$ form a closed family. First, we prove that each $\mathcal{B}^\lambda_{\Theta_B^-}$ has a unique highest weight element. This is the most technical part of the entire proof (Theorem 6.17). After this we will use a relatively simple argument based on the strong positivity of Θ_B^- and $\Theta_{B^-} * \Theta_{B^-}$ to construct the embeddings $\mathcal{B}^{\lambda+\mu}_{\Theta_B^-} \hookrightarrow\to \mathcal{B}^\lambda_{\Theta_B^-} \times \mathcal{B}^\mu_{\Theta_B^-}$ and thus finish the proof of that crystals $\mathcal{B}^\lambda_{\Theta_B^-}$ form a closed family.

In fact, we will prove the existence and uniqueness of the highest weight elements in a more general situation. Recall that $X_P = UZ(L_P)\overline{w_P}U$, where L_P is the Levi factor of P and $Z(L_P) \subset T$ is the center of L_P. In order to formulate the following result, we identify $X_P^- = Z(L_P)B^-_{w_P}$ with $Z(L_P) \times B^-_{w_P}$ and, passing to the tropicalization, we identify $\widetilde{Z(L_P)B^-_{w_P}}$ with $X_\star(Z(L_P)) \times \widetilde{B^-_{w_P}}$.

Theorem 6.17. *Let* P *be a standard parabolic subgroup of* G. *Then for each* $\lambda \in X_\star(Z(L_P))$, *we have: if* λ *is not dominant, then* $\mathcal{B}^\lambda_{f_P,\Theta_P^-} = \emptyset$; *and if* λ *is dominant, then the normal crystal* $\mathcal{B}^\lambda_{f_P,\Theta_P^-}$ *has a unique highest weight element* $\tilde{b}_\lambda$ *of the weight*

λ or, more precisely, under the identification $\widetilde{Z(L_P)B^-_{w_P}} = X_\star(Z(L_P)) \times \widetilde{B^-_{w_P}}$, one has

$$\tilde{b}_\lambda = (\lambda, \mathbf{0}) \ ,$$

where $\mathbf{0} \in \widetilde{B^-_{w_P}}$ is the marked point (see Sections 4.2 and 6.1).

Proof. In the notation of Theorem 6.17, denote by $\tilde{f}_P : X_\star(Z(L_P)) \times \widetilde{B^-_{w_P}} \to \mathbb{Z}$ be the Θ^-_P-tropicalization of the positive function $f_P : Z(L_P) \times B^-_{w_P} \to \mathbb{A}^1$, and by $\tilde{\varepsilon}_i : X_\star(Z(L_P)) \times \mathbb{Z}^{l(w_P)} \to \mathbb{Z}$ – the Θ^-_P-tropicalization of the positive function $\varepsilon_i : Z(L_P) \times B^-_{w_P} \to \mathbb{A}^1$ for $i \in I$. Therefore, it suffices to prove the following result.

Proposition 6.18. *Let $(\lambda, \tilde{b}) \in X_\star(Z(L_P)) \times \widetilde{B^-_{w_P}}$ be any point such that*

$$\tilde{f}_P(\lambda, \tilde{b}) \geq 0, \ \tilde{\varepsilon}_i(\lambda, \tilde{b}) \leq 0$$

for all $i \in I$. Then $\tilde{b} = \mathbf{0}$.

Proof. For a character $\chi : U \to \mathbb{A}^1$ define a regular function $f_{w,\chi} : U\overline{w}U \to \mathbb{A}^1$ by

$$f_{w,\chi}(g) = \chi(\pi^+(\overline{w}^{-1}g)) \tag{6.10}$$

for $g \in BwB$.

It follows from Lemma 1.24 that

$$f_P(g) = f_{w,\chi^{st}}(g) + \sum_{j \in J(P)} h_j(g)$$

for any $g \in X_P = UZ(L_P)\overline{w_P}U$, where $h_j(g) = \chi_j(\pi^+(\overline{w_P^{-1}}^{-1}\iota(g)))$. Using the fact that $f_{w,\chi^{st}}(tg) = f_{w,\chi^{st}}(g)$ and $h_j(tg) = \alpha_j(t)f_j(g)$ for $g \in Bw_PB$, $t \in T$, $j \in J(P)$, we obtain

$$f_P(t \cdot b) = f_{w,\chi^{st}}(b) + \sum_{j \in J(P)} \alpha_j(t) f_j(b) \tag{6.11}$$

for $t \in Z(L_P)$, $b \in B^-_{w_P} = U\overline{w_P}U \cap B^-$.

Therefore, applying the Θ^-_P-tropicalization to (6.11) and using Corollary 4.10(c) with $T' = T$, $\mu_j = \alpha_j$, we obtain:

$$\tilde{f}_P(\lambda, \tilde{b}) = \min(\tilde{f}_{w,\chi^{st}}(\tilde{b}), \min_{j \in J(P)}(\langle \alpha_j, \lambda \rangle + \tilde{h}_j(\tilde{b}))) \leq \tilde{f}_{w,\chi^{st}}(\tilde{b}) \tag{6.12}$$

for all $\tilde{b} \in \widetilde{B^-_{w_P}}$, where $\tilde{f}_{w,\chi^{st}} : \widetilde{B^-_{w_P}} \to \mathbb{Z}$ is the $\Theta^-_{w_P}$-tropicalization of $f_{w,\chi^{st}}|_{B^-_{w_P}}$, and $\tilde{f}_j : X_\star(Z(L_P)) \times \widetilde{B^-_{w_P}} \to \mathbb{Z}$ is the Θ^-_P-tropicalization of $f_j|_{X^-_P}$.

In particular,

$$\{\tilde{b} \in \widetilde{B^-_{w_P}} : \tilde{f}_P((\lambda, \tilde{b}) \geq 0\} \subset \{\tilde{b} \in \widetilde{B^-_{w_P}} : \tilde{f}_{w,\chi^{st}}(\tilde{b}) \geq 0\} \ .$$

Therefore, taking into account that $\varepsilon_i(t \cdot b) = \varepsilon_i(b)$ for any $b \in B^-_w$, $t \in T$ and hence $\tilde{\varepsilon}_i(\lambda, \tilde{b}) = \tilde{\varepsilon}_i(\tilde{b})$ for any $\tilde{b} \in \widetilde{B^-_{w_P}}$, $\lambda \in X_\star(T)$, Proposition 6.18 follows from the following result.

Lemma 6.19. *For any $w \in W$, the tropicalizations $\widetilde{\varepsilon}_i : \widetilde{B^-_w} \to \mathbb{Z}$ and $\widetilde{f}_{w,\chi^{st}} : \widetilde{B^-_w} \to \mathbb{Z}$ of Θ^-_w-positive functions $\varepsilon_i : B^-_w \to \mathbb{A}^1$ and $f_{w,\chi^{st}} : B^-_w \to \mathbb{Z}$ satisfy*

$$\{\tilde{b} \in \widetilde{B^-_w} : \tilde{f}_{w,\chi^{st}}(\tilde{b}) \geq 0, \tilde{\varepsilon}_i(\tilde{b}) \leq 0 \ \forall i \in I\} = \{\mathbf{0}\} \ . \tag{6.13}$$

Proof. We need the following recursive formula for $f_{w,\chi}$.

Lemma 6.20. *For any $w_1, w_2 \in W$ such that $l(w_1w_2) = l(w_1) + l(w_2)$, one has*

$$f_{w_1w_2,\chi}(g_1g_2) = f_{w_2,\chi}(u_1g_2)$$

for any $g_1 \in Bw_1B$, $g_2 \in Bw_2B$, where $u_1 = \pi^+(\overline{w_1}^{-1}g_1)$.

Proof. Indeed,

$$\pi^+(\overline{w_1w_2}^{-1}g_1g_2) = \pi^+(\overline{w_2}^{-1}\overline{w_1}^{-1}g_1g_2) = \pi^+(\overline{w_2}^{-1}b_1u_1g_2) = \pi^+(\overline{w_2}^{-1}b_1u_1g_2)\ ,$$

where $u_1 = \pi^+(\overline{w_1}^{-1}g_1)$ and $b_1 = \pi^-(\overline{w_1}^{-1}g_1) \in T \cdot (U^- \cap \overline{w_1}^{-1}U\overline{w_1})$ (we used the fact that $\overline{w_2}^{-1}T \cdot (U^- \cap \overline{w_1}^{-1}U\overline{w_1})\overline{w_2} \subset T \cdot (U^- \cap \overline{w_1w_2}^{-1}U\overline{w_1w_2}) \subset B^-$). Therefore,

$$f_{w_1w_2,\chi}(g_1g_2) = \chi(\pi^+(\overline{w_1w_2}^{-1}g_1g_2)) = \chi(\pi^+(\overline{w_2}^{-1}b_1u_1g_2)) = f_{w_2,\chi}(u_1g_2)\ .$$

The lemma is proved. □

Lemma 6.21. *Let $j \in I$ be such that $w = s_jw'$ and $l(w') = l(w) - 1$. Then*

$$f_{s_jw',\chi^{st}}(x_{-j}(c) \cdot b) = f_{w',\chi^{st}}(x') + \sum_{k\geq 1} c^k f_k(b) \tag{6.14}$$

for any $b \in B^-_{w'}$, $c \in \mathbb{G}_m$, where each $f_k : B^-_{w'} \to \mathbb{A}^1$ is a regular function and $f_k \neq 0$ for at least one $k > 1$.

Proof. We need the following simple fact.

Claim 6.22. *Let Y be a variety equipped with a left U-action (which we denote by $(u, y) \mapsto u \cdot y$) and let $f : Y \to \mathbb{A}^1$ be a regular function. Then*

$$f(u \cdot y) = f(y) + \sum_{k\geq 1} h_k(u)f_k(y) \tag{6.15}$$

for all $u \in U$, $y \in Y$, where the sum is finite, each f_k is a regular function on Y, and each h_k is a regular function on U such that $h_k(e) = 0$.

Now, using Lemma 6.20 with $w_1 = s_j$, $w_2 = w'$, $g_1 = x_{-j}(c)$, $g_2 = b$, and taking into account that $\pi^+(\overline{s_j}^{-1}x_{-j}(c)) = x_j(c)$, the formula (6.15) with $f = f_{w',\chi^{st}}$ becomes

$$f_{w,\chi^{st}}(x_{-j}(c) \cdot b) = f_{w',\chi^{st}}(x_j(c) \cdot b) = f_{w',\chi^{st}}(b) + \sum_{k\geq 1} h_k(x_j(c))f_k(b)\ .$$

Without loss of generality, we can assume that $h_k(x_j(c)) = c^k$ for $k \geq 1$.

Finally, it is clear that the function $(c, b) \mapsto f_{w',\chi^{st}}(x_j(c)\cdot b)$ depends on c, therefore, $f_k \neq 0$ for some k. The lemma is proved. □

We continue the proof of Lemma 6.19 by induction on the length $l(w)$. Indeed, if $l(w) = 0$, i.e., $w = e$, we have nothing to prove. Now let us write $w = s_jw'$ so that $l(w') = l(w) - 1$. Then, according to Claim 3.23, the positive structure Θ^-_w factors as $\Theta^-_w = \Theta^-_{s_j} \times \Theta^-_{w'}$. Therefore, we can identify the tropicalizations $\widetilde{B^-_w} = \mathbb{Z} \times \widetilde{B^-_{w'}}$.

Then, tropicalizing (6.14), we obtain (using Corollary 4.10(b))

$$\tilde{f}_{w,\chi^{st}}(m, \tilde{b}) = \min(\tilde{f}_{w',\chi^{st}}(\tilde{b}), \min_{k>1}(km + \tilde{f}_k(\tilde{b}))) \tag{6.16}$$

for $(m, \tilde{b}) \in \mathbb{Z} \times \widetilde{B^-_{w'}}$, where $\tilde{f}_k$ is the tropicalization of f_k (e.g., $\tilde{f}_k = +\infty$ whenever $f_k = 0$). In particular, the inequality $\tilde{f}_{w,\chi^{st}}(m, \tilde{b}) \geq 0$ implies

$$\tilde{f}_{w',\chi^{st}}(\tilde{b}) \geq 0, km \geq -\tilde{f}_k(\tilde{b}), \tag{6.17}$$

for each $k > 0$ such that $f_k \neq 0$ (such k always exists by Lemma 6.21).

On the other hand, $\widetilde{B^-_w} = \mathbb{Z} \times \widetilde{B^-_{w'}}$ is the product of torsion-free Kashiwara crystals. Then we have by Definition 5.6:

$$\tilde{\varepsilon}_i(m, \tilde{b}) = \max(\tilde{\varepsilon}'_i(\tilde{b}), \tilde{\varepsilon}_i(m) - \langle \alpha_i, \tilde{\gamma}'(\tilde{b})\rangle) = \max(\tilde{\varepsilon}'_i(\tilde{b}), \delta_{ij} \cdot m - \langle \alpha_i, \tilde{\gamma}'(\tilde{b})\rangle)$$

since $\tilde{\varepsilon}'_i(m) = \delta_{ij} \cdot m$.

In particular, the inequalities $\tilde{\varepsilon}_i(m, \tilde{b}) \leq 0$ for $i \in I$ imply $m \leq \langle \alpha_j, \tilde{\gamma}'(\tilde{b})\rangle$ and

$$\tilde{\varepsilon}'_i(\tilde{b}) \leq 0 \tag{6.18}$$

for all $i \in I$.

Summarizing, if $(m, \tilde{b}) \in \widetilde{B^-_w} = \mathbb{Z} \times \widetilde{B^-_{w'}}$ satisfies the inequalities (6.13), then $(m, \tilde{b})$ satisfies (6.16) and (6.18). Therefore, $\tilde{b} \in \widetilde{B^-_{w'}}$ satisfies the inductive hypothesis (6.13) with w' for w. Therefore, by this inductive hypothesis, $\tilde{b} = \mathbf{0}$. This and the above inequalities imply that $m = 0$. This finishes the induction. Lemma 6.19 is proved. □

Therefore, Proposition 6.18 is proved. □

Now we will show that $\mathcal{B}^\lambda_{f_P,\Theta_{w_P}}$ is empty if λ is not dominant and that $\mathcal{B}^\lambda_{f_P,\Theta_{w_P}}$ is non-empty and has a highest weight element if λ is dominant. We need the following result.

Lemma 6.23. *For each $(\lambda, \tilde{b}) \in \mathcal{B}^\lambda_{f_P,\Theta_{w_P}}$, the co-weight $\tilde{\gamma}(\lambda, \tilde{b}) \in X_\star(T)$ is bounded from the above by λ, i.e.,*

$$\langle \mu, \tilde{\gamma}(\lambda, \tilde{b}) - \lambda\rangle \ \leq 0 \tag{6.19}$$

for all $\mu \in X^\star(T)^+$.

Proof. Recall from Section 3.1 that for each $w \in W$ the isomorphism $\eta^w : U^w \xrightarrow{\sim} B^-_w$ is (Θ^w, Θ^-_w)-positive and its inverse $\eta_w : B^-_w \xrightarrow{\sim} U^w$ is (Θ^-_w, Θ^w)-positive. Note that, by definition (3.6) of η_w one has, for each $\mu \in X^\star(T)^+$,

$$\Delta_{\mu,w^{-1}\mu}(u) = \Delta_{\mu,\mu}(u\overline{w}^{-1}) = \Delta_{\mu,\mu}(\iota(b)) = \mu(\gamma(b))^{-1}$$

for all $u \in U^w$, where $b = \eta_w(u) \in B^-_w$, $\gamma : B^- \to T$ is the natural projection, and $\Delta_{\mu,w^{-1}\mu} : G \to \mathbb{A}^1$ is the generalized minor (see Section 1.1). One also has

$$f_{w,\chi^{st}}(b) = \chi^{st}(u)$$

for any $u \in U^w$, where $b = \eta_w(u) \in B^-_w$.

Passing to the tropicalization, we obtain

$$\left\langle \mu, \tilde{\gamma}(\tilde{b})\right\rangle = -\tilde{\Delta}_{\mu,w^{-1}\mu}(\tilde{a}) \ , \tilde{f}_{w,\chi^{st}}(\tilde{b}) = \widetilde{\chi^{st}}(\tilde{u}) \tag{6.20}$$

for any $\tilde{b} \in \widetilde{B^-_w}$, $\mu \in X^\star(T)^+$, where $\tilde{u} = \widetilde{\eta_w}(\tilde{b}) \in \widetilde{U^w} = \mathrm{Trop}(U^w, \Theta^w)$, $\tilde{\Delta}_{\mu,w^{-1}\mu}$ is the Θ^w-tropicalization of the restriction $\Delta_{\mu,w^{-1}\mu}|_{U^w}$, and $\widetilde{\chi^{st}} : \widetilde{U^w} \to \mathbb{Z}$ is the tropicalization of the Θ^w-positive function $\chi^{st}|_{U^w} : U^w \to \mathbb{A}^1$.

Now choose a reduced decomposition $\mathbf{i} = (i_1, \dots, i_\ell)$ of $w \in W$ and recall from Section 3.1 that $\theta^+_{\mathbf{i}}$ is a toric chart of the class Θ_w. Note that the regular function $\Delta_{\mu, w^{-1}\mu}(\theta^+_{\mathbf{i}}(a_1, \dots, a_\ell))$ is a monomial in $a_1, \dots, a_\ell$ (see e.g., [5, (6.3)]) and

$$\chi^{st}(\theta^+_{\mathbf{i}}(a_1, \dots, a_\ell)) = a_1 + \dots + a_\ell \ .$$

Therefore, the tropicalizations of $\chi^{st} \circ \theta^+_{\mathbf{i}}$ and $\Delta_{\mu, w^{-1}\mu} \circ \theta^+_{\mathbf{i}}$ satisfy $\widetilde{\chi^{st}}(\tilde{a}_1, \dots, \tilde{a}_\ell) = \min\limits_{1 \le k \le \ell} \tilde{a}_k$ for all $(\tilde{a}_1, \dots, \tilde{a}_\ell) \in \mathbb{Z}^\ell$; and $\tilde{\Delta}_{\mu, w^{-1}\mu}(\tilde{a}_1, \dots, \tilde{a}_\ell)$ is a linear function in $\tilde{a}_1, \dots, \tilde{a}_\ell$ with non-negative coefficients. Therefore, $\tilde{\Delta}_{\mu, w^{-1}\mu}(\tilde{a}_1, \dots, \tilde{a}_\ell) \ge 0$ whenever $\widetilde{\chi^{st}}(\tilde{a}_1, \dots, \tilde{a}_\ell) \ge 0$.

Therefore, the first equation (6.20) implies that $\left\langle \mu, \tilde{\gamma}(\tilde{b}) \right\rangle \le 0$ for all $\mu \in X^\star(T)^+$ whenever $\tilde{f}_{w, \chi^{st}}(\tilde{b}) \ge 0$. Finally, for $w = w_P$ we have $\tilde{\gamma}(\lambda, \tilde{b}) = \lambda + \tilde{\gamma}(\tilde{b})$ for all $(\lambda, \tilde{b}) \in \mathcal{B}^\lambda_{f_P, \Theta_{w_P}}$ and (6.19) follows. □

Therefore, since weights of all elements of each non-empty normal crystal $\mathcal{B}^\lambda_{f_P, \Theta^-_B}$ are bounded from above by λ, this normal crystal must have at least one highest weight element. This and Proposition 6.18 imply that $(\lambda, \mathbf{0})$ is, indeed, the only highest weight element in $\mathcal{B}^\lambda_{f_P, \Theta_{w_P}}$.

Theorem 6.17 is proved. □

According to Theorem 3.37, $(X_B, \mathrm{id}, f_B, \Theta^-_B) * (X_B, \mathrm{id}, f_B, \Theta^-_B)$ is a strongly positive $(U \times U, \chi^{st})$-linear bicrystal type w_0. Applying Proposition 6.7 and Proposition 6.13, we obtain a homomorphism of Kashiwara crystals

$$* : \mathcal{B}_{f_B, \Theta^-_B} \times \mathcal{B}_{f_B, \Theta^-_B} \to \mathcal{B}_{f_B, \Theta^-_B} \ . \tag{6.21}$$

Furthermore, let us denote by c_λ the unique highest weight element of the normal crystal $\mathcal{C}_\lambda := \mathcal{B}^\lambda_{f_B, \Theta^-_B}$. By definition, the product of the combinatorial crystals $\mathcal{C}_\lambda \times \mathcal{C}_\mu$ contains the element (c_λ, c_μ). Clearly, this is a unique element of the weight $\lambda + \mu$ in $\mathcal{C}_\lambda \times \mathcal{C}_\mu$. And this (c_λ, c_μ) is a highest weight element in $\mathcal{C}_\lambda \times \mathcal{C}_\mu$ since weights of all other elements in $\mathcal{C}_\lambda \times \mathcal{C}_\mu$ are less than $\lambda + \mu$. Clearly, $c_\lambda * c_\mu = c_{\lambda+\mu}$ (in terms of 6.21). Denote by $\mathcal{C}'_{\lambda+\mu} \subset \mathcal{C}_\lambda \times \mathcal{C}_\mu$ the preimage of $\mathcal{C}_{\lambda+\mu}$ under the homomorphism $*$.

Lemma 6.24. *The restriction of $*$ to $\mathcal{C}'_{\lambda+\mu}$ is an isomorphism $\mathcal{C}'_{\lambda+\mu} \xrightarrow{\sim} \mathcal{C}_{\lambda+\mu}$.*

Proof. Clearly, $(c_\lambda, c_\mu) \in \mathcal{C}'_{\lambda+\mu}$. Let us show that $\mathcal{C}'_{\lambda+\mu}$ contains no other highest weight elements. Indeed, if $z \in \mathcal{C}'_{\lambda+\mu}$ is another highest weight element, then z has a weight less than $\lambda + \mu$ and the image $*(z)$ is a highest weight element of $\mathcal{C}_{\lambda+\mu}$ different from $c_{\lambda+\mu}$, which contradicts the uniqueness of the highest weight element in $\mathcal{C}_{\lambda+\mu}$.

Therefore, (c_λ, c_μ) is the only highest weight element of the normal crystal $\mathcal{C}'_{\lambda+\mu}$. This and Claim 5.26 imply that any homomorphism $\mathcal{C}'_{\lambda+\mu} \to \mathcal{C}_{\lambda+\mu}$ is an isomorphism. The lemma is proved. □

Thus, Lemma 6.24 asserts that the assignment $\tilde{c}_{\lambda+\mu} \mapsto \tilde{c}_\lambda \times \tilde{c}_\mu$ defines an injective homomorphism of normal crystals $\mathcal{C}_{\lambda+\mu} \hookrightarrow \mathcal{C}_{\lambda+\mu} \times \mathcal{C}_{\lambda+\mu}$ for all $\lambda, \mu \in X_\star(T)^+$. This and Theorem 6.17 prove that the family of crystals $\mathcal{C}_\cdot = \{\mathcal{B}^\lambda_{f_B, \Theta^-_B} | \lambda \in X_\star(T)^+\}$ is

closed. In turn, Theorem 6.16 guarantees that each $\mathcal{B}^{\lambda}_{f_B,\Theta^-_B}$ is isomorphic to the associated crystal $\mathcal{B}(V_\lambda)$.

Theorem 6.15 is proved. □

Note that Claim 2.36(c) and Claim 6.10(c) imply the following corollary.

Corollary 6.25. *For each $\lambda \in X_\star(T)^+$, one has $(\mathcal{B}^{\lambda}_{f_B,\Theta^-_B})^{op} \cong \mathcal{B}^{-w_0\lambda}_{f_B,\Theta^-_B}$.*

This result agrees with Theorem 6.15 and the fact that the dual of the $\mathfrak{g}^\vee$-module V_λ is isomorphic to $V_{-w_0\lambda}$.

Example 6.26. Let $G = GL_3$, so that $T = \{t = diag(t_1, t_2, t_3)\} \subset GL_3$. We fix a reduced decomposition $\mathbf{i} = (1, 2, 1)$ of $w_0 \in W = S_3$ and choose a toric chart $\theta : T \times \mathbb{G}_m^3 \to TB^-_w$ of the class Θ^-_B as follows:

$$\theta(t; c_1, c_2, c_3) = t \cdot \theta^-_{\mathbf{i}}(c_1, c_2, c_3) = \begin{pmatrix} t_1\frac{1}{c_1c_3} & 0 & 0 \\ t_2(\frac{c_1}{c_2} + \frac{1}{c_3}) & t_2\frac{c_1c_3}{c_2} & 0 \\ t_3 & t_3c_3 & t_3c_2 \end{pmatrix}$$

(see Example 3.8). Therefore, the restriction of f_B to TB^-_w is given by (in the new coordinates $(t; c_1, c_2, c_3)$):

$$f_B(t; c_1, c_2, c_3) = c_1 + \frac{c_2}{c_3} + c_3 + \frac{t_2}{t_3} \cdot \left(\frac{c_1}{c_2} + \frac{1}{c_3}\right) + \frac{t_1}{t_2} \cdot \frac{1}{c_1} \ .$$

And the rest of the decorated geometric crystal structure $\mathcal{X}$ on $TB^-_{w_0}$ is given by the morphism γ, the actions $e^{\cdot}_i$, and the functions φ_i, ε_i, $i = 1, 2$:

$$\gamma(t; c_1, c_2, c_3) = \left(t_1\frac{1}{c_1c_3}, t_2\frac{c_1c_3}{c_2}, t_3c_2\right) \ ,$$

$$e^d_1(t; c_1, c_2, c_3) = \left(t; c_1\frac{c_2 + c_1c_3}{d \cdot c_2 + c_1c_3}, c_2, c_3\frac{c_2 + d^{-1} \cdot c_1c_3}{c_2 + c_1c_3}\right) \ ,$$

$$e^d_2(t; c_1, c_2, c_3) = (t; c_1, d^{-1} \cdot c_2, c_3) \ ,$$

$$\varphi_1(t; c_1, c_2, c_3) = \frac{t_2}{t_1} \cdot \left(\frac{c_1^2c_3}{c_2} + c_1\right), \varphi_2(t; c_1, c_2, c_3) = \frac{t_3}{t_2} \cdot \frac{c_2}{c_1} \ ,$$

$$\varepsilon_1(t; c_1, c_2, c_3) = \frac{1}{c_3} + \frac{c_2}{c_1c_3^2}, \ \varepsilon_2(t; c_1, c_2, c_3) = \frac{c_3}{c_2} \ .$$

The isomorphism $\mathcal{X} \xrightarrow{\sim} \mathcal{X}^{op}$ is given by

$$(t; c_1, c_2, c_3) \mapsto \left(t^{op}; \frac{t_2}{t_3} \cdot c_3^{-1}, \frac{t_1}{t_3} \cdot c_2^{-1}, \frac{t_1}{t_2} \cdot c_1^{-1}\right) \ ,$$

where $t^{op} = diag(t_1, t_2, t_3)^{op} = diag(t_3, t_2, t_1)$.

The central charge $\Delta = \Delta_{X\times X} : \mathcal{X} \times \mathcal{X} \to \mathbb{A}^1$ in these coordinates is given for $x = (t; c_1, c_2, c_3)$, $x' = (t'; c'_1, c'_2, c'_3)$ by the formula

$$\Delta(x, x') = c_1 + \frac{c_2}{c_3} + c_3 + \frac{t'_2}{c'_3t'_3} + \frac{c'_1t'_2}{c'_2t'_3} + \Delta_1 + \Delta_2 \ ,$$

where

$$\Delta_1 = \frac{c_1'(c_1' + \frac{c_2'}{c_3'})t_2't_2 + c_2'(t_1' + c_3c_1't_2')t_3 + (c_1 + \frac{c_2}{c_3})c_1'c_2't_3't_2}{t_3 \cdot (\frac{c_2'}{c_3'}t_1' + c_3c_1'(c_1' + \frac{c_2'}{c_3'})t_2' + c_2c_1'c_2't_3')}$$

$$\Delta_2 = \frac{c_1'(c_1' + \frac{c_2'}{c_3'})t_1't_2't_2 + c_2'(t_1' + c_3c_1't_2')t_3't_1 + (c_1 + \frac{c_2}{c_3})c_1'c_2't_1't_3't_2}{t_2 \cdot (c_1't_1't_2' + (c_1 + \frac{c_2}{c_3})c_2't_1't_3' + c_1c_3c_1'c_2't_2't_3')} \ .$$

The tropicalization of the above structures consists of:

• The set $\tilde{X} = \Lambda^\vee \times \mathbb{Z}^3$ where $\lambda \in \Lambda^\vee = X_\star(T) = \mathbb{Z}^3$.

• The functions $\tilde{f}_B, \tilde{\varphi}_i, \tilde{\varepsilon}_i : \Lambda^\vee \times \mathbb{Z}^3 \to \mathbb{Z}$, $i = 1, 2$:

$$\tilde{f}_B(\lambda; \mathbf{m}) = \min(m_1, m_2 - m_3, m_3, \lambda_2 - \lambda_3 - \max(m_3, m_2 - m_1), \lambda_1 - \lambda_2 - m_1) \ ,$$

$$\tilde{\varphi}_1(\lambda; \mathbf{m}) = \lambda_1 - \lambda_2 - \min(m_1, 2m_1 + m_3 - m_2), \ \tilde{\varphi}_2(\lambda; \mathbf{m}) = \lambda_2 - \lambda_3 + m_1 - m_2 \ ,$$

$$\tilde{\varepsilon}_1(\lambda; \mathbf{m})) = \max(m_3, m_1 + 2m_3 - m_2), \ \tilde{\varepsilon}_2(\lambda; \mathbf{m}) = m_2 - m_3$$

for $(\lambda; \mathbf{m}) = ((\lambda_1, \lambda_2, \lambda_3); (m_1, m_2, m_3)) \in \tilde{X}$.

• The set $\tilde{B}_{\tilde{f}_B}$ which consists of all those $(\lambda; \mathbf{m}) \in \tilde{B}$ that satisfy $\tilde{f}_B(\lambda; \mathbf{m}) \geq 0$, equivalently: $m_1 \geq 0$, $m_2 \geq m_3 \geq 0$, $\lambda_1 - \lambda_2 \geq m_1$, $\lambda_2 - \lambda_3 \geq m_3, \lambda_2 - \lambda_3 \geq m_2 - m_1$. That is, each point of $(\lambda; \mathbf{m}) \in \tilde{B}$ is a Gelfand-Tsetlin pattern:

$$\begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \lambda_2 + m_1 & & \lambda_2 + m_3 & \\ & & \lambda_3 + m_2 & & \end{pmatrix}$$

• The bijection $e_i^n : \tilde{B} \to \tilde{B}$, $i = 1, 2$:

$$\tilde{e}_1^n(\lambda; \mathbf{m}) = (\lambda; m_1 + \max(\delta - n, 0) - \max(\delta, 0), m_2, m_3 + \max(\delta, 0) - \max(\delta, n)) \ ,$$

where $\delta = m_1 + m_3 - m_2$,

$$\tilde{e}_2^n(\lambda; \mathbf{m}) = (\lambda; m_1, m_2 - n, m_3) \ .$$

Therefore, one has the decomposition into the connected components.

$$\mathcal{B}_{f_B, \Theta_{w_0}^-} \cong \bigsqcup_{\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)} \mathcal{B}(V_\lambda) \ .$$

The following is the generalization of Theorem 6.15 to all standard parabolic bicrystals.

Theorem 6.27. *For each $\lambda \in X_\star(Z(L_P))$, we have:*

(a) *If λ is not dominant, then $\mathcal{B}^\lambda_{f_P, \Theta_P^-}$ is empty.*

(b) *If λ is dominant, then $\mathcal{B}^\lambda_{f_P, \Theta_P^-} \cong \mathcal{B}(V_\lambda)$.*

Proof. Recall that $X_w = BwB$ is the Bruhat cell for each $w \in W$. Clearly, X_w has a unique factorization

$$X_w = V(w)T\overline{w}U \ ,$$

where $V(w) = U \cap \overline{w}U^-\overline{w}^{-1}$ is the Schubert cell. Recall that for any w, w' such that $l(ww') = l(w) + l(w')$ the multiplication $B^- \times B^- \to B^-$ defines an open embedding $B_w^- \times B_{w'}^- \hookrightarrow B_{ww'}^-$.

Let now P be a standard parabolic subgroup, L_P be the Levi factor of P, $U_P = U \cap P$ be the unipotent radical of P, and $U_{L_P} = U \cap L_P$ be the maximal unipotent subgroup of L_P. Obviously, $V(w_P) = U_P$. That is, $X_P = U_P Z(L_P)\overline{w_P}U$ (see Example 1.7)

Recall from (1.3) that w_0^P is the longest element in the Weyl group W_P of L_P and $w_P = w_0^P w_0$ so that $w_0^P w_P = w_0$. Then the multiplication in B^- defines an open embedding (and hence a birational isomorphism) of geometric crystals

$$B^-_{w_0^P} \times Z(L_P)B^-_{w_P} \hookrightarrow Z(L_P)B^-_{w_0} \ . \tag{6.22}$$

This isomorphism is $(\Theta^-_{w_0^P} \times \Theta^-_P, \Theta_{Z(L_P)} \cdot \Theta_{w_P})$-positive; therefore, applying the tropicalization to (6.22), one obtains an isomorphism of the corresponding torsion-free Kashiwara crystals (in the notation of Section 6.2):

$$\mathcal{B}_{\Theta^-_{w_0^P}} \times \mathcal{B}_{\Theta^-_P} \xrightarrow{\sim} \mathcal{B}_{\Theta_{Z(L_P)}\cdot B^-_{w_0}} \ .$$

In what follows we will simply identify $\mathcal{B}_{\Theta_{Z(L_P)}\cdot B^-_{w_0}}$ with the product $\mathcal{B}_{\Theta^-_{w_0^P}} \times \mathcal{B}_{\Theta^-_P}$.

For any sub-torus $T' \subset T$, denote temporarily $X_{B;T'} = UT'w_0U$. Clearly, the quadruple $(X_{B;T'}, \mathrm{id}, f_P|_{X_{B;T'}}, \Theta_{T'} \cdot \Theta^-_{w_0})$ is a strongly positive $(U \times U, \chi^{st})$-linear bicrystal; and the natural inclusion $X'_B = UT'w_0U \subset X_B = UTw_0U$ extends to an embedding of $(U \times U, \chi^{st})$-linear bicrystals, and, therefore, to the natural embedding of normal crystals

$$\mathcal{B}_{f_B,\Theta_{T'}\cdot\Theta^-_{w_0}} \subset \mathcal{B}_{f_B,\Theta_T\cdot\Theta^-_{w_0}} = \mathcal{B}_{f_B,\Theta^-_B} \ ,$$

or, more precisely,

$$\mathcal{B}_{f_B,\Theta_{T'}\cdot\Theta^-_{w_0}} = \bigsqcup_{\lambda \in X_\star(T')} \mathcal{B}^\lambda_{f_B,\Theta^-_B} \ .$$

In turn, Theorem 6.15 implies that $\mathcal{B}^\lambda_{f_B,\Theta_{T'}\cdot\Theta^-_{w_0}} \cong \mathcal{B}(V_\lambda)$ if $\lambda \in X_\star(Z(L_P)) \cap X_\star(T)^+$, and $\mathcal{B}^\lambda_{f_B,\Theta_{Z(L_P)}\cdot\Theta^-_{w_0}}$ is empty if $\lambda \in X_\star(Z(L_P)) \setminus X_\star(T)^+$. That is,

$$\mathcal{B}_{f_B,\Theta_{T'}\cdot\Theta^-_{w_0}} \cong \bigsqcup_{\lambda \in X_\star(T') \cap X_\star(T)^+} \mathcal{B}(V_\lambda) \ . \tag{6.23}$$

Furthermore, taking $T' = Z(L_P)$, we have a natural inclusion of Kashiwara crystals:

$$\mathcal{B}_{f_B,\Theta_{Z(L_P)}\cdot\Theta^-_{w_0}} \subset \mathcal{B}_{\Theta_{Z(L_P)}\cdot\Theta^-_{w_0}} \ .$$

According to Theorem 6.17, the normal crystal $\mathcal{B}^\lambda_{f_B,\Theta_{Z(L_P)}\cdot\Theta^-_{w_0}}$ has a unique highest weight element $\tilde{b}_\lambda$. Under the identifications $\mathcal{B}_{\Theta^-_P} = X_\star(Z(L_P)) \times \mathcal{B}_{\Theta^-_{w_P}}$ and $\mathcal{B}_{\Theta_{Z(L_P)}\cdot B^-_{w_0}} = \mathcal{B}_{\Theta^-_{w_0^P}} \times \mathcal{B}_{\Theta^-_P}$, we have $\tilde{b}_\lambda = (\mathbf{0}, \lambda, \mathbf{0}')$, where $\mathbf{0} \in \mathcal{B}_{\Theta^-_{w_0^P}}$ and $\mathbf{0}' \in \mathrm{Trop}(B^-_{w_P}, \Theta^-_{w_P})$ are the marked points (see Sections 4.2 and 6.1).

Theorem 6.17 implies that for each $\lambda \in X_\star(Z(L_P)) \cap X_\star(T)^+$ the normal crystal $\mathcal{B}^\lambda_{f_P,\Theta^-_P}$ has a unique highest weight element $\tilde{b}'_\lambda = (\lambda, \mathbf{0}')$ (under the above identification $\mathcal{B}_{\Theta^-_P} = X_\star(Z(L_P)) \times \mathrm{Trop}(B^-_{w_P}, \Theta^-_{w_P})$). Clearly, $b'_\lambda = (\lambda, \mathbf{0}') \in \overline{\mathcal{B}_{\Theta^-_P}}$. In particular, $\mathcal{B}_{f_P,\Theta^-_P}$ is a normal sub-crystal of $\overline{\mathcal{B}_{\Theta^-_P}}$.

Lemma 5.19(a) guarantees that the correspondence $\tilde{b} \mapsto (\mathbf{0}, \tilde{b})$ is an injective homomorphism of subnormal Kashiwara crystals $\mathbf{j} : \overline{\mathcal{B}_{\Theta_P^-}} \hookrightarrow \overline{\mathcal{B}_{\Theta_{w_0^P}^-}} \times \overline{\mathcal{B}_{\Theta_P^-}}$.

Furthermore, Theorem 6.17 guarantees that $\mathcal{B}^\lambda_{f_B, \Theta_{Z(L_P)} \cdot \Theta^-_{w_0}} = \mathcal{B}^\lambda_{f_P, \Theta_P^-} = \emptyset$ if $\lambda \in X_\star(Z(L_P)) \setminus X_\star(T)^+$.

Therefore, in order to finish the proof of Theorem 6.27, it suffices to show that

$$\mathbf{j}(\mathcal{B}^\lambda_{f_P, \Theta_P^-}) = \mathcal{B}^\lambda_{f_B, \Theta_{Z(L_P)} \cdot \Theta^-_{w_0}} \tag{6.24}$$

for each $\lambda \in X_\star(Z(L_P)) \cap X_\star(T)^+$.

Indeed, $\mathbf{j}(\tilde{b}'_\lambda) = \mathbf{j}(\lambda, \mathbf{0}') = (\mathbf{0}, \lambda, \mathbf{0}') = \tilde{b}_\lambda$, i.e., the normal crystals $\mathbf{j}(\mathcal{B}^\lambda_{f_P, \Theta_P^-})$ and $\mathcal{B}^\lambda_{f_B, \Theta_{Z(L_P)} \cdot \Theta^-_{w_0}}$ share the (unique) highest weight element $\tilde{b}_\lambda = (\mathbf{0}, \lambda, \mathbf{0}')$. Therefore, these normal crystals are equal by Claim 5.26 and we obtain (6.24).

Theorem 6.27 is proved. □

Corollary 6.28. *Let $(\mathbf{X}, \mathbf{p}, f, \Theta)$ be a strongly positive parabolic $(U \times U, \chi^{st})$-linear bicrystal of type w_P. Then, in the notation of Proposition 6.13, for each $\lambda \in X_\star(Z(L))$, one has:*

(a) *If λ is not dominant, then $\mathcal{B}^\lambda_{f,\Theta}$ is empty.*

(b) *If λ is dominant and $\mathcal{B}^\lambda_{f,\Theta}$ is non-empty, then the restriction of $\tilde{\mathbf{m}}$ to $\mathcal{B}^\lambda_{f,\Theta}$ is a surjective homomorphism of normal crystals*

$$\mathcal{B}^\lambda_{f,\Theta} \twoheadrightarrow \mathcal{B}(V_\lambda) \ .$$

6.4. **From right unipotent bicrystals to crystals associated with $\mathfrak{b}^\vee$-modules.** In this section we construct a number of upper normal crystals (see Definition 5.16 above) based on certain unipotent bicrystals which we will refer to as *right linear* unipotent bicrystals.

Definition 6.29. Given $U \times U$-variety $\mathbf{X}$, subgroups $U', U'' \subset U$, and a character $\chi : U \to \mathbb{A}^1$, we say that a regular function $f : X \to \mathbb{A}^1$ is $(U' \times U'', \chi)$-*linear* if

$$f(u'xu'') = \chi(u') + f(x) + \chi(u'')$$

for all $u' \in U'$, $u'' \in U''$, $x \in X$. In particular, we refer to each $(e \times U, \chi)$-linear (resp. $(U \times e, \chi)$-linear) function f as a *right* (U, χ)-*linear* (resp. *left* (U, χ)-*linear*) function.

Similarly to Section 2.3, for any U-bicrystal $(\mathbf{X}, \mathbf{p})$ and any $(U' \times U'', \chi)$-linear function f on X we will refer to the triple $(\mathbf{X}, \mathbf{p}, f)$ as to $(U' \times U'', \chi)$-linear bicrystal. In particular, if f is a right (resp. left) (U, χ)-linear function f, then we will refer to $(\mathbf{X}, \mathbf{p}, f)$ as a *right* (resp. *left*) (U, χ)-linear bicrystal.

Claim 6.30. *In the notation of Claim 2.10, for each right $(U' \times U'', \chi)$-linear bicrystal $(\mathbf{X}, \mathbf{p}, f)$, the triple $(\mathbf{X}, \mathbf{p}, f)^{op} = ((\mathbf{X}, \mathbf{p})^{op}, f)$ is a $(U'' \times U', \chi)$-linear bicrystal. In particular, the correspondence $(\mathbf{X}, \mathbf{p}, f) \mapsto (\mathbf{X}, \mathbf{p}, f)^{op}$ takes right (U, χ)-linear bicrystals to the left ones and vice versa.*

Note that for any right (U, χ)-linear bicrystal $(\mathbf{X}, \mathbf{p}, f)$ and a $(U \times U, \chi)$-linear bicrystal $(\mathbf{Y}, \mathbf{p}', f')$ the convolution product $(\mathbf{X}, \mathbf{p}, f) * (\mathbf{Y}, \mathbf{p}', f')$ given by (2.17) and is a well-defined right (U, χ)-linear bicrystal. In other words, the category of

$(U \times U, \chi)$-linear bicrystals acts (from the right) on the category of right (U, χ)-linear bicrystals. (This observation is parallel to Lemma 5.22, which implies that the category of normal Kashiwara crystals acts from the right on the category of the upper normal ones.)

Similarly to Section 3.2, a positive right (U, χ^{st})-linear bicrystal $(\mathbf{X}, \mathbf{p}, f, \Theta)$ is a positive U-bicrystal $(\mathbf{X}, \mathbf{p}, \Theta)$ such that the restriction of f to $X^- = \mathbf{p}^{-1}(B^-)$ is Θ-positive. To each such positive right (U, χ^{st})-linear bicrystal, we associate a Kashiwara crystal $\mathcal{B}_{f,\Theta}$ by the formula (6.2).

The following result is an "upper" analogue of Proposition 6.6.

Proposition 6.31. *Let $(\mathbf{X}, \mathbf{p}, f, \Theta)$ be a positive right (resp. left) (U, χ^{st})-linear bicrystal. Then the Kashiwara crystal $\mathcal{B}_{f,\Theta}$ is upper (resp. lower) normal.*

Proof. We claim that for any $x \in X_w$, $u \in U$ and $i \in I$, one has

$$f(x_i(a) \cdot x \cdot u) = \chi^{st}(u) + f(x) + \sum_{k>0} a^k f_i^{(k)}(x) \ , \tag{6.25}$$

where each $f_i^{(k)}(x)$ is a regular function on X.

Indeed, by definition, $f(xu) = f(x) + \chi^{st}(u)$ for any $x \in X$, $u \in U$. Since the correspondence $(a, x) \mapsto f(x_i(a) \cdot x)$ is a regular function on $\mathbb{A}^1 \times X$, we obtain

$$f(x_i(a) \cdot x \cdot u) = \chi^{st}(u) + f(x_i(a) \cdot x) = \chi^{st}(u) + f(x) + \sum_{k \geq 0} a^k f_i^{(k)}(x) \ .$$

But $f_i^{(0)}(x) = f(x)$ because $f(x_i(0) \cdot x) = f(x)$. This proves (6.25).

Now let $\mathcal{X} := \mathcal{F}(\mathbf{X}, \mathbf{p}) = (X^-, \gamma, \varphi_i, \varepsilon_i, e_i^{\cdot} | i \in I)$ be the corresponding geometric crystal (as defined in (2.12)).

Then (6.25) taken with $a = \frac{c-1}{\varphi_i(x)}$, $u = x_i(a')$, $a' = \frac{c^{-1}-1}{\varepsilon_i(x)}$, $i \in \operatorname{Supp} \mathcal{X}$, $x \in X^-$ gives in conjunction with (2.14):

$$f(e_i^c(x)) = \frac{c^{-1} - 1}{\varepsilon_i(x)} + f(x) + \sum_{k>0} (c-1)^k \frac{f_i^{(k)}(x)}{\varphi_i(x)^k} \ .$$

Equivalently,

$$f(e_i^c(x)) = \frac{c^{-1}}{\varepsilon_j(x)} + f'(x) + \sum_{k>0} c^k h_i^{(k)}(x) \ ,$$

where f' and each $h_i^{(k)}$ are rational functions on X^-.

The rest of the proof is nearly identical to the proof of Proposition 6.6. □

Similarly to (6.7), for each $\lambda \in X_\star(T)$, denote $\mathcal{B}_{f,\Theta}^\lambda := \mathcal{B}_{f,\Theta} \cap \mathcal{B}_\Theta^\lambda$. We obtain the following corollary from Proposition 6.31.

Corollary 6.32. *Let $(\mathbf{X}, \mathbf{p}, f, \Theta)$ be a positive right (resp. left) (U, χ^{st})-linear bicrystal. Then each non-empty Kashiwara crystal $\mathcal{B}_{f,\Theta}^\lambda$ is upper (resp. lower) normal.*

In what follows, we will construct a number of right (U, χ)-linear bicrystals. The first observation is that one can "truncate" a given upper normal crystal using some right-invariant functions. Indeed, if f is a right (U, χ)-linear function on $(\mathbf{X}, \mathbf{p})$ and $f' : X \to \mathbb{A}^1$ is a right U-invariant function, then $f + f'$ is also right (U, χ)-linear.

Claim 6.33. *For any positive right (U,χ)-linear bicrystal $(\mathbf{X},\mathbf{p},f,\Theta)$ and any Θ-positive right U-invariant function $f' : X \to \mathbb{A}^1$, one has a natural injective homomorphism of upper normal crystals:*

$$\mathcal{B}_{f+f',\Theta} \hookrightarrow \mathcal{B}_{f,\Theta}\ .$$

Moreover, for a given f', the association $(\mathbf{X},\mathbf{p},f) \mapsto (\mathbf{X},\mathbf{p},f+f')$ is a covariant invertible functor from the category of right (U,χ)-linear bicrystals into itself.

Another approach to constructing new right (U,χ)-linear bicrystals consists of twisting the left U-action on X with morphisms $u_0 : X \to U$.

Claim 6.34. *Let $(\mathbf{X},\mathbf{p},f)$ be a right (U,χ)-linear bicrystal and let $u_0 : X \to U$ be any morphism. Then the function f_{u_0} on X given by*

$$f_{u_0}(x) = f(u_0(x)\cdot x) \tag{6.26}$$

for $x \in X$ is (U,χ)-linear. More precisely, for a given morphism u_0 the correspondence $(\mathbf{X},\mathbf{p},f) \mapsto (\mathbf{X},\mathbf{p},f_{u_0})$ is a covariant invertible functor from the category of right (U,χ)-linear bicrystals into itself.

Proposition 6.35. *Let $(\mathbf{X},\mathbf{p},f,\Theta)$ be a positive right (U,χ^{st})-linear bicrystal, let $u_0 : X \to U$ be a morphism such $u_0(X^-) \subset U^{w'}$ for some $w' \in W$, and the restriction $u_0|_{X^-}$ is a $(\Theta,\Theta^{w'})$-positive morphism $X^- \to U^{w'}$. Then:*

(a) *The restriction of f_{u_0} to X^- is a Θ-positive function on X^-, that is, the quadruple $(\mathbf{X},\mathbf{p},f_{u_0},\Theta)$ is a positive right (U,χ^{st})-linear bicrystal.*

(b) *The tropicalized functions $\tilde{f},\tilde{f}_{u_0} : \widetilde{X^-} \to \mathbb{Z}$ satisfy*

$$\tilde{f}_{u_0}(\tilde{b}) \le \tilde{f}(\tilde{b}) \tag{6.27}$$

for all $\tilde{b} \in \widetilde{X^-} = \mathrm{Trop}(X^-,\Theta)$ and, therefore, $\mathcal{B}_{f_{u_0},\Theta}$ is an upper normal sub-crystal of the upper normal crystal $\mathcal{B}_{f_w,\Theta}$.

Proof. We start with the following result.

Lemma 6.36. *Let $(\mathbf{X},\mathbf{p},f,\Theta)$ be a positive right (U,χ^{st})-linear bicrystal. Then:*

(a) *For any sequence $\mathbf{i} = (i_1,\dots,i_\ell) \in I^\ell$, the regular function $(\mathbb{G}_m)^\ell \times X^- \to \mathbb{A}^1$ given by*

$$(c_1,\dots,c_\ell,x^-) \to f(x_{i_1}(c_1)\cdots x_{i_\ell}(c_\ell)\cdot x^-) \tag{6.28}$$

is $\Theta_{(\mathbb{G}_m)^\ell} \times \Theta$-positive.

(b) *For any $w' \in W$, the regular function $U^{w'} \times X^- \to \mathbb{A}^1$ given by*

$$(u,x^-) \to f(u\cdot x^-) \tag{6.29}$$

is $\Theta^{w'} \times \Theta$-positive.

Proof. Prove (a). By the definition from Section 3.2, the rational morphism $(\mathbb{G}_m)^\ell \times X^- \to X^-$ given by

$$(c'_1,\dots,c'_\ell,x^-) \to e_{i_1}^{c'_1}\cdots e_{i_\ell}^{c'_\ell}(x^-)$$

is $(\Theta_{(\mathbb{G}_m)^\ell} \times \Theta, \Theta)$-positive. Denote $x_k^- = e_{i_k}^{c'_k} \cdots e_{i_\ell}^{c'_\ell}(x^-)$ for $k = 1, \ldots, \ell$ and $x_{\ell+1}^- := x^-$. Then substituting $c'_k = c_k \varphi_{i_k}(x_{k+1}^-) + 1$, we see that the rational morphism $F_{\mathbf{i}} : (\mathbb{G}_m)^\ell \times X^- \to X^-$ given by

$$F_{\mathbf{i}}(c_1, \ldots, c_\ell; x^-) = e_{i_1}^{c_k \varphi_{i_1}(x_2^-)+1} \cdots e_{i_\ell}^{c_k \varphi_{i_\ell}(x_{\ell+1}^-)+1}(x^-)$$

is positive.

Taking into account (2.14) and (2.15), we see that the latter positive morphism is given by

$$F_{\mathbf{i}}(c_1, \ldots, c_\ell; x^-) = x_{i_1}(c_1) \cdots x_{i_\ell}(c_\ell) \cdot x^- \cdot x_{i_\ell}(-c''_\ell) \cdots x_{i_1}(-c''_1) \ ,$$

where $c''_k = \frac{c_k}{\alpha_{i_k}(\gamma(x_{k+1}^-))(1+c_k \varphi_{i_k}(x_{k+1}^-))}$ for $k = 1, \ldots, \ell$.

Using the right (U, χ^{st})-linearity of f, we obtain:

$$f(x_{i_1}(c_1) \cdots x_{i_\ell}(c_\ell) \cdot x^-) = f(F_{\mathbf{i}}(c_1, \ldots, c_\ell; x^-)) + \sum_{k=1}^{\ell} c''_k \ .$$

Therefore, the positivity of $F_{\mathbf{i}}$, $f|_{X^-}$, and of each function $c''_k : (\mathbb{G}_m)^\ell \times X^- \to \mathbb{A}^1$ implies the positivity of the function given by (6.28). This proves (a).

Part (b) follows from (a) by taking $\mathbf{i} = (i_1 \ldots, i_\ell)$ to be a reduced decomposition for w'. Lemma 6.36 is proved. □

Since $u_0|_{X^-} : X^- \to U^{w'}$ is a $(\Theta, \Theta^{w'})$-positive morphism and (6.29) defines a $\Theta^{w'} \times \Theta$-positive function by Lemma 6.36(b), it follows that the restriction of f_{u_0} to X^- is a Θ-positive function on X^-. This proves part (a) of Proposition 6.35.

Prove (b) now. Taking again $\mathbf{i} = (i_1, \ldots, i_\ell)$ to be a reduced decomposition of w' and $u = \theta_{\mathbf{i}}^+(c_1, \ldots, c_\ell) = x_{i_1}(c_1) \cdots x_{i_\ell}(c_\ell)$, we obtain for all $x \in X$ (based on Claim 6.22):

$$f(x_{i_1}(c_1) \cdots x_{i_\ell}(c_\ell) \cdot x) = f(x) + \sum_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^\ell \setminus \{\mathbf{0}\}} c^{\mathbf{n}} \cdot f_{\mathbf{n}}(x) \ ,$$

where we abbreviated $c^{\mathbf{n}} = c_1^{n_1} \cdots c_\ell^{n_\ell}$ for $\mathbf{n} = (n_1, \ldots, n_\ell)$, and each $f_{\mathbf{n}}$ is a regular function on X. If we denote by $\tilde{f}, \tilde{f}_{\mathbf{n}}$, the respective Θ-tropicalizations of $f|_{X^-}$ and $f_{\mathbf{n}}|_{X^-}$, and by $\tilde{F}_{\mathbf{i}}$ the tropicalization of the function given by (6.28), then Corollary 4.10(c) guarantees that

$$\tilde{F}_{\mathbf{i}}(\mathbf{m}, \tilde{x}^-) = \min \left(\tilde{f}(\tilde{x}^-), \min_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^\ell \setminus \{\mathbf{0}\}} \left(\mathbf{m} \cdot \mathbf{n} + \tilde{f}_{\mathbf{n}}(\tilde{x}^-) \right) \right)$$

for all $\mathbf{m} \in \mathbb{Z}^\ell$, $\tilde{x}^- \in \widetilde{X^-} = \mathrm{Trop}(X^-, \Theta)$, where we abbreviated $\mathbf{m} \cdot \mathbf{n} = m_1 n_1 + \cdots + m_\ell n_\ell$ for $\mathbf{m} = (m_1, \ldots, m_\ell)$, $\mathbf{n} = (n_1, \ldots, n_\ell)$. This implies that

$$\tilde{F}_{\mathbf{i}}(\mathbf{m}, \tilde{x}^-) \leq \tilde{f}(\tilde{x}^-)$$

for all $\mathbf{m} \in \mathbb{Z}^\ell$, $\tilde{x}^- \in \widetilde{X^-}$. More invariantly, if we denote by $\tilde{F} : \widetilde{U^{w'}} \times \widetilde{X^-} \to \mathbb{Z}$ the $\Theta^{w'} \times \Theta$-tropicalization of the positive function given by (6.29), we obtain

$$\tilde{F}(\tilde{u}, \tilde{x}^-) \leq \tilde{f}(\tilde{x}^-)$$

for all $\tilde{u} \in \widetilde{U^{w'}}$, $\tilde{x}^- \in \widetilde{X^-}$. Taking into account that $\tilde{f}_{u_0}(\tilde{x}^-) = \tilde{F}(\tilde{u}_0(\tilde{x}^-), \tilde{x}^-)$, where $\tilde{u}_0 : \widetilde{X^-} \to \widetilde{U^{w'}}$ is the tropicalization of the positive morphism u_0, we obtain the inequality (6.27). This proves (b).

Therefore, Proposition 6.35 is proved. □

Our main example of right (U, χ)-linear functions is as follows.

Lemma 6.37. *The function $f_{w,\chi}$ on (BwB, id) defined in (6.10) is right (U, χ)-linear.*

Proof. Indeed, for $x \in BwB$, $u \in U$, we have

$$\begin{aligned} f_{w,\chi}(xu) &= \chi(\pi^+(\overline{w}^{-1}xu)) = \chi(\pi^+(\overline{w}^{-1}x)u) \\ &= \chi(\pi^+(\overline{w}^{-1}x)) + \chi(u) = f_{w,\chi}(x) + \chi(u) \ . \end{aligned}$$

This proves the lemma. □

Corollary 6.38. *Let $(\mathbf{X}, \mathbf{p})$ be any U-bicrystal such that $\mathbf{p}(X) \subset BwB$ for some $w \in W$. Then for any morphism $u_0 : X \to U$ and a character $\chi : U \to \mathbb{A}^1$, the function $f_{w,\chi,u_0} : X \to \mathbb{A}^1$ given by $f_{w,\chi,u_0}(x) = f_{w,\chi}(u_0(x) \cdot \mathbf{p}(x))$ is right (U, χ)-linear. That is, the correspondence $(\mathbf{X}, \mathbf{p}) \mapsto (\mathbf{X}, \mathbf{p}, f_{w,\chi,u_0})$ is a functor from a full sub-category of the category of U-bicrystals into the category of right (U, χ)-linear bicrystals.*

Remark 6.39. The requirement $\mathbf{p}(X) \subset BwB$ is not restrictive at all because for each unipotent bicrystal $(\mathbf{X}, \mathbf{p})$ of type w, there exists a dense $U \times U$-invariant subvariety X_0 of X such that $\mathbf{p}(X_0) \subset BwB$.

For each $w \in W$, denote simply by f_w the restriction of the function $f_{w,\chi^{st}}$ to $\overline{X}_w = U\overline{w}U$ (see Example 2.7). Then Claim 3.34 implies the following result.

Claim 6.40. *The quadruple $(\overline{\mathbf{X}}_w, \mathrm{id}, f_w, \Theta_w^-)$ is a positive right (U, χ^{st})-linear bicrystal. Therefore, in the notation of Proposition 6.31, the Kashiwara crystal $\mathcal{B}_{f_w,\Theta_w^-}$ is upper normal.*

Note that $\mathcal{B}^\lambda_{f_w,\Theta_w^-}$ is empty unless $\lambda = 0$, and $\mathcal{B}^0_{f_w,\Theta_w^-} = \mathcal{B}_{f_w,\Theta_w^-}$.

Theorem 6.41. *For any standard parabolic subgroup P of G, the upper normal crystal $\mathcal{B}_{f_{w_P},\Theta_{w_P}^-}$ is isomorphic to the associated crystal $\mathcal{B}(\mathbb{C}[U_P^\vee])$.*

Proof. In view of Corollary 5.48, it will suffice to show that $\mathcal{B}_{f_{w_P},\Theta_{w_P}^-}$ is also a limit of the directed family $(\mathcal{B}(V^\lambda), \tilde{\mathbf{f}}_{\lambda,\mu})$ (where $\mathcal{B}(V^\lambda)$ is the crystal associated to the $\mathfrak{b}^\vee$-module V^λ, which is the restriction of simple $\mathfrak{g}^\vee$-module to $\mathfrak{b}^\vee$).

Define the projection $pr_P : X_P = UZ(L_P)\overline{w_P}U \to \overline{X}_{w_0} = U\overline{w_0}U$ by

$$pr_P(tu\overline{w_P}u') = u\overline{w_P}u' \tag{6.30}$$

for each $u, u' \in U$, $t \in Z(L_P)$. Clearly, pr_P defines a surjective morphism of unipotent bicrystals $(\mathbf{X}_P, \mathrm{id}) \to (\overline{\mathbf{X}}_{w_P}, \mathrm{id})$, which is $(\Theta_P^-, \Theta_{w_P}^-)$-positive and, therefore, it defines a projection $\widetilde{pr_P}$ of marked sets $\widetilde{Z(L_P)B_{w_0}^-} = X_\star(Z(L_P)) \times \widetilde{B_{w_P}^-} \to \widetilde{B_{w_P}^-}$ (see Sections 4.2 and 6.1), where $\widetilde{Z(L_P)B_{w_P}} = \mathrm{Trop}(Z(L_P)B_{w_P}^-, \Theta_P^-)$ and $\widetilde{B_{w_P}} = \mathrm{Trop}(B_{w_P}^-, \Theta_{w_P}^-)$.

According to Theorem 6.27, the normal crystal $\mathcal{B}_{f_P,\Theta_P^-}$ being considered as an upper normal crystal is isomorphic to the union of all associated crystals $\mathcal{B}(V^\lambda)$.

Claim 6.42. *For each standard parabolic subgroup $P \subset G$, one has:*

(a) *The restriction of $\widetilde{pr_P}$ to the normal crystal $\mathcal{B}_{f_P,\Theta_P^-}$ is a surjective homomorphism of upper normal Kashiwara crystals $\mathcal{B}_{f_P,\Theta_P^-} \to \mathcal{B}_{f_{w_P},\Theta_{w_P}^-}$.*

(b) *For each $\lambda \in X_\star(Z(L_P)) \cap X_\star(T)^+$, the restriction of $\widetilde{pr_P}$ to $\mathcal{B}(V_\lambda) \subset \mathcal{B}_{f_P,\Theta_P^-}$ is an injective homomorphism $\tilde{\mathbf{j}}'_\lambda : \mathcal{B}(V^\lambda) \hookrightarrow \mathcal{B}_{f_{w_P},\Theta_{w_P}^-}$ of upper normal Kashiwara crystals.*

(c) *For each $\lambda, \mu \in X_\star(Z(L_P)) \cap X_\star(T)^+$, one has*

$$\tilde{\mathbf{j}}'_\lambda(\mathcal{B}(V^\lambda)) \subset \tilde{\mathbf{j}}'_{\lambda+\mu}(\mathcal{B}(V^{\lambda+\mu})$$

so that the induced homomorphism $\mathcal{B}(V^\lambda) \hookrightarrow \tilde{\mathcal{B}}(V^{\lambda+\mu})$ equals $\tilde{\mathbf{f}}_{\lambda,\mu}$ from Corollary 5.48.

Therefore, the limit of $(\mathcal{B}(V^\lambda), \tilde{\mathbf{f}}_{\lambda,\mu})$, $\lambda, \mu \in X_\star(Z(L_P)) \cap X_\star(T)^+$ is isomorphic to $\mathcal{B}_{f_{w_P},\Theta_{w_P}^-}$. The uniqueness of the limit and Corollary 5.48 finish the proof of Theorem 6.41. □

Remark 6.43. An isomorphism $\mathcal{B}(\mathbb{C}[U_P]) \xrightarrow{\sim} \mathcal{B}_{f_{w_P},\Theta_{w_P}^-}$ was constructed in [5] by, first, choosing a reduced decomposition $\mathbf{i}$ of w_P and, second, using an explicit Kashiwara parametrization of the dual canonical basis of $\mathbb{C}[U_P]$.

7. Conjectures and open questions

7.1. **Tropicalization and associated crystals.** We start the following conjecture which complements results of Section 6.2.

Conjecture 7.1. *Let $(\mathbf{X}, \mathbf{p}, f, \Theta)$ be a positive $(U \times U, \chi^{st})$-linear bicrystal of type w. Then the restriction of the structure map $\tilde{\mathbf{f}}_w : \mathcal{B}_\Theta \to \mathcal{B}_{\Theta_{B^-}}$ (see (6.6)) to each connected component of $\mathcal{B}_{f,\Theta}$ is injective.*

Remark 7.2. Informally speaking, Conjecture 7.1 means that $U \times U$-orbits in X correspond to the components in $\mathcal{B}_\Theta$ or, more precisely, we expect a correspondence between classes of orbits in X^- under the rational U-action (2.2) and connected components of $\mathcal{B}_\Theta$.

The following is an immediate corollary from this conjecture and a refinement of Corollary 6.28.

Conjecture 7.3. *Let $(X, \mathbf{p}, f, \Theta)$ be a strongly positive parabolic $(U \times U, \chi^{st})$-linear bicrystal of type w_P. Then each non-empty the normal crystal $\mathcal{B}^\lambda_{f,\Theta}$ is isomorphic to the union of copies of the associated crystal $\mathcal{B}(V_\lambda)$.*

Proof of the implication Conjecture 7.1 ⇒ Conjecture 7.3. By Theorem 6.27, each non-empty crystal $\mathcal{B}^\lambda_{f_P,\Theta_P^-}$ is isomorphic to the crystal $\mathcal{B}(V_\lambda)$ associated to the finite-dimensional $\mathfrak{g}^\vee$-module V_λ.

Denote by $\mathcal{B}_0$ a connected component of $\mathcal{B}^\lambda_{f,\Theta}$. We have to prove that $\mathcal{B}_0 \cong \mathcal{B}^\lambda_{f_P,\Theta_P^-}$. According to Claim 5.27, $\mathcal{B}_0$ is also a normal crystal. Since the homomorphism $\tilde{\mathbf{f}}$ from Proposition 6.13 commutes with the tropicalization of the highest weight morphism hw_X, we obtain $\tilde{\mathbf{f}}(\mathcal{B}_0) \subset \mathcal{B}^\lambda_{f_P,\Theta_P^-}$. Since $\tilde{\mathbf{f}}(\mathcal{B}_0)$ is normal and $\mathcal{B}^\lambda_{f_P,\Theta_P^-}$ is connected, we

obtain by Claim 5.27 that $\tilde{\mathbf{f}}(\mathcal{B}_0) = \mathcal{B}^{\lambda}_{f_P,\Theta_P^-}$. But according to Conjecture 7.1, the restriction of $\tilde{\mathbf{f}}$ to $\mathcal{B}_0$ is an injective map $\mathcal{B}_0 \hookrightarrow \mathcal{B}^{\lambda}_{f_P,\Theta_P^-}$. Therefore, the restriction of $\tilde{\mathbf{f}}$ to $\mathcal{B}_0$ is an isomorphism $\mathcal{B}_0 \xrightarrow{\sim} \mathcal{B}^{\lambda}_{f_P,\Theta_P^-}$. □

For a complex projective variety $X^\vee$, we denote by $\widehat{X^\vee}$ the affine cone over X. The following is an equivalent reformulation of Conjecture 7.3.

Conjecture 7.4. *Let $(\mathbf{X}, \mathbf{p}, f, \Theta)$ of type w_P be a strongly positive parabolic (U, χ^{st})-linear bicrystal. Then there exist a based $\mathfrak{g}^\vee$-module $(V_\Theta, \mathbf{B}_\Theta)$ and a $\mathfrak{g}^\vee$-linear map $h : V_\Theta \to \mathbb{C}[\widehat{G^\vee/P^\vee}]$ such that the associated normal crystal $\mathcal{B}(V_\Theta, \mathbf{B}_\Theta)$ is isomorphic to $\mathcal{B}_{f,\Theta}$ and $h(\mathbf{B}_\Theta)$ is a perfect basis for $\mathbb{C}[\widehat{G^\vee/P^\vee}]$.*

In fact, we expect that the (infinite-dimensional) module V_Θ is the coordinate algebra $\mathbb{C}[\widehat{X^\vee}]$ of some projective $G^\vee$-variety $X^\vee$. Theorems 6.41 guarantees that the variety $X^\vee = U_P^\vee$ (with the dual Levi factor $L_P^\vee$ for $G^\vee$) is a suitable candidate. However, we do not expect that any $G^\vee$-variety to $X^\vee$ is related to a unipotent bicrystal. In fact, we will show in a separate paper that for $G^\vee = GL_2(\mathbb{C})$, and the 4-dimensional simple $G^\vee$-module $Y^\vee = S^3(\mathbb{C}^2)$ there is no unipotent bicrystal that would parametrize the associated crystal $\mathcal{B}(\mathbb{C}[Y^\vee])$.

Below (Conjecture 7.6), we will lay out sufficient conditions which would allow to construct $(U \times U, \chi^{st})$-linear unipotent bicrystals for certain $G^\vee$-varieties.

Definition 7.5. Given a commutative $\mathbb{C}$-algebra $\mathcal{A}$ without zero divisors and a totally ordered free abelian group Γ, a map $\nu : \mathcal{A} \setminus \{0\} \to \Gamma$ is said to be a *valuation* if

$$\nu(xy) = \nu(x) + \nu(y)$$

for all $x, y \in \mathcal{A} \setminus \{0\}$,

$$\nu(x + y) = \min(\nu(x), \nu(y))$$

for any $x, y \in \mathcal{A} \setminus \{0\}$ such that $\nu(x) \neq \nu(y)$ (we use the convention that $\nu(0) = +\infty$, where $+\infty$ is greater than any element of Γ).

We say that ν is *saturated* if for each $\lambda \in \nu(\mathcal{A} \setminus \{0\})$, the entire "half-line" $\mathbb{Q}_{\geq 0} \cdot \lambda \cap \Gamma$ also belongs to the semi-group $\nu(\mathcal{A} \setminus \{0\})$.)

Conjecture 7.6. *Let $Y^\vee$ be an affine $G^\vee$-variety, S be a split algebraic torus, and let $\prec$ be a total ordering on the co-character lattice $X_\star(S)$. Assume that there exist a saturated valuation $\nu : \mathbb{C}[Y^\vee] \setminus \{0\} \to (X_\star(S), \prec)$ and a perfect basis $\mathbf{B}$ for $\mathbb{C}[Y^\vee]$ (see Section 5.4) such that the restriction of ν to $\mathbf{B}$ is an injective map*

$$\mathbf{B} \hookrightarrow X_\star(S) \ . \tag{7.1}$$

Then there exists a strongly positive (U, χ^{st})-linear bicrystal $(\mathbf{X}, \mathbf{p}, f, \Theta)$ (where $\theta : S \xrightarrow{\sim} X^-$) such that the image of (7.1) is $\mathcal{B}_{f,\Theta} \subset X_\star(S)$, i.e., (7.1) defines an isomorphism of normal crystals $\mathcal{B}(\mathbb{C}[Y^\vee]) \xrightarrow{\sim} \mathcal{B}_{f,\Theta}$.

7.2. **Schubert cells and upper normal crystals.** In the notation of Section 5.3, let $B^\vee\backslash G^\vee$ be the (right) flag variety for $G^\vee$. For $w \in W$ let $X_w^\vee := B^\vee\backslash B^\vee w B^\vee$ be the Schubert cell. Clearly, one has a $B^\vee$-equivariant isomorphism $X_w^\vee \cong B^\vee(w)\backslash B^\vee$, where $B^\vee(w) = B^\vee \cap w^{-1}B^\vee w$ (Note that if $w = {w_P}^{-1}$ is the inverse of the parabolic element $w_P \in W$ defined in (1.3), then $X_{{w_P}^{-1}}^\vee = U_P^\vee$, in particular, $X_{w_0}^\vee = U^\vee$).

Therefore, one has a $B^\vee$-equivariant surjective map $\pi_w : U^\vee \twoheadrightarrow X_w^\vee$. In turn, this defines an embedding of coordinate algebras (and locally finite $\mathfrak{b}^\vee$-modules) $\pi_w^* : \mathbb{C}[X_w^\vee] \hookrightarrow \mathbb{C}[U^\vee]$. In particular, the above embedding $\mathbb{C}[U_P^\vee] \hookrightarrow \mathbb{C}[U^\vee]$ constructed in Section 5.3 is $\pi^*_{{w_P}^{-1}}$.

Conjecture 7.7. *For each $w \in W$, there exists a perfect basis of $\mathbb{C}[X_w^\vee]$.*

It follows from Proposition 5.46 that the conjecture is true for $w = {w_P}^{-1}$.

Conjecture 7.7 would imply (based on Definition 5.30) the existence of the associated crystal $\mathcal{B}(\mathbb{C}[X_w^\vee])$ for each $w \in W$ (where the coordinate algebra $\mathbb{C}[X_w^\vee]$ of the Schubert cell $X_w^\vee$ is regarded as a locally finite $\mathfrak{b}^\vee$-module). The following is a (yet conjectural) refinement of Conjecture 7.7.

Conjecture 7.8. *For any $w \in W$, the crystal basis $\mathcal{B}(\mathbb{C}[X_{w^{-1}}^\vee])$ is isomorphic to the upper normal crystal $\mathcal{B}_{f_w,\Theta_w^-}$ (see Claim 6.40).*

We can provide the following partial justification of Conjecture 7.8.

Note that the complex torus $T^\vee$ (see Section 5.3) is dual to T, e.g., each cocharacter μ of T is a character of $T^\vee$. Indeed, for each $\mu \in X_\star(T)$, denote by $[\mu] : T^\vee \to \mathbb{C}^\times$ the corresponding character.

For each locally finite $B^\vee$-module V, define the character $ch(V) \in \mathbb{Q}[[T^\vee]]$ by the formula

$$ch(V) = \sum_{\mu \in X_\star(T)} (\dim_{\mathbb{C}} V(\mu)) \cdot [\mu] \ ,$$

where $V(\mu)$ is the μ-th weight component of V (see Section 5.3).

For each Kashiwara crystal $\mathcal{B}$ such that all fibers of $\tilde{\gamma} : \tilde{B} \to X_\star(T)$ are finite, define the *character* $ch(\mathcal{B})$ by the formula

$$ch(\mathcal{B}) = \sum_{\tilde{b} \in \mathcal{B}} [\tilde{\gamma}(\tilde{b})] \ .$$

Claim 7.9. *If $\mathcal{B} = \mathcal{B}(V)$ is the crystal associated to V, then $ch(V) = ch(\mathcal{B})$.*

Lemma 7.10. *For each $w \in W$, one has $ch(\mathbb{C}[X_{w^{-1}}^\vee]) = ch(\mathcal{B}_{f_w,\Theta_w^-})$.*

Proof. We need the following well-known facts.

Claim 7.11. *For each $w \in W$, the character $ch(\mathbb{C}[X_{w^{-1}}^\vee])$ is given by the formula*

$$ch(\mathbb{C}[X_{w^{-1}}^\vee]) = \prod_{\alpha^\vee \in R_+^\vee \cap w^{-1}(-R_+^\vee)} \frac{1}{1 - [\alpha^\vee]} \ ,$$

where $R_+^\vee \subset X_\star(T)$ is the set of positive coroots of G.

Claim 7.12. [5, Formula (6.3)] *For each reduced decomposition* $\mathbf{i} = (i_1, \dots, i_\ell)$ *of an element* $w \in W$*, one has*

$$\Delta_{\mu, w^{-1}\mu}(\theta^+_{\mathbf{i}}(c_1, \dots, c_\ell)) = \prod_{k=1}^{\ell} c_k^{\langle \mu, \alpha^\vee_{(k)} \rangle} \ , \tag{7.2}$$

where $\alpha^\vee_{(k)} = s_{i_1} \cdots s_{i_{k-1}} \alpha^\vee_{i_k}$, $k = 1, \dots, \ell$ *is a normal ordering of* $R^\vee_+ \cap w^{-1}(-R^\vee_+)$.

Denote by $\widetilde{\eta^w} : \widetilde{U^w} \to \widetilde{B^-_w}$ the (Θ^w, Θ^-_w)-tropicalization of the positive isomorphism $\eta^w : U^w \to B^-_w$ (See (3.5) and Claim 3.26). By Claim 3.26, $\widetilde{\eta^w}$ is a bijection $\widetilde{U^w} \xrightarrow{\sim} \widetilde{B^-_w}$.

Since $\Delta_{\mu,\mu}(b) = \Delta_{\mu, w\mu}(\eta^{w^{-1}}(b))$ for each $b \in B^-_w$, $\mu \in X^\star(T)^+$, applying the tropicalization to (7.2) with respect to the toric chart $\theta^+_{\mathbf{i}} : (\mathbb{G}_m)^\ell \xrightarrow{\sim} U^w$(and, therefore, identifying $\widetilde{U^w}$ with $\mathbb{Z}^\ell$), we obtain

$$\tilde{\gamma}(\tilde{b}) = \sum_{k=1}^{\ell} \tilde{u}_k \alpha^\vee_{(k)} \ ,$$

where

$$(\tilde{u}_1, \dots, \tilde{u}_\ell) = (\widetilde{\eta^w})^{-1}(\tilde{b})$$

and we identified $\widetilde{U^w}$ with $\mathbb{Z}^\ell$ using the toric chart $\theta^+_{\mathbf{i}}$. Note also that $\tilde{f}_w(\tilde{b}) \geq 0$ if and only if $\tilde{u}_k \geq 0$ for all k. Therefore,

$$\begin{aligned} ch(\mathcal{B}_{f_w, \Theta^-_w}) &= \sum_{\tilde{u}_1, \dots, \tilde{u}_\ell \in (\mathbb{Z}_{\geq 0})^\ell} \left[\sum_{k=1}^{\ell} \tilde{u}_k \alpha^\vee_{(k)} \right] = \sum_{\tilde{u}_1, \dots, \tilde{u}_\ell \in (\mathbb{Z}_{\geq 0})^\ell} \prod_{k=1}^{\ell} \left[\alpha^\vee_{(k)} \right]^{\tilde{u}_k} \\ &= \prod_{k=1}^{\ell} \frac{1}{1 - [\alpha^\vee_{(k)}]} \ . \end{aligned}$$

Since $\alpha^\vee_{(1)}, \dots, \alpha^\vee_{(\ell)}$ is a linear ordering of the set $R^\vee_+ \cap w^{-1}(-R^\vee_+)$, we obtain in conjunction with Claim 7.11 the identity $ch(\mathbb{C}[X^\vee_{w^{-1}}]) = ch(\mathcal{B}_{f_w, \Theta^-_w})$.

The lemma is proved. □

7.3. **Tensor product multiplicities and combinatorial central charge.** We start with a conjecture which is a complement of Claim 3.41.

Conjecture 7.13. *The inverse of* (3.10) *is an isomorphism of positive decorated geometric crystals*

$$(\mathcal{Z}_{w_0}, \Theta_Z) \xrightarrow{\sim} (\mathcal{X}_B, \Theta_B) \times (\mathcal{X}_B, \Theta_B) \ .$$

Denote by $\mathcal{B}_{f_Z, \Theta_Z}$ the normal Kashiwara crystal obtained by the tropicalization of (2.34) with respect to the strongly positive structure Θ_Z from Claim 3.40.

By definition (2.29), one has an invariant projection $\mathcal{Z}_{w_0} \to T \times T$, which is obviously $(\Theta_Z, \Theta_T \times \Theta_T)$-positive; and denote by $\tilde{\pi}' : \mathcal{B}_{\Theta_Z} \to X_\star(T) \times X_\star(T)$ its tropicalization. In fact, the restriction of $\tilde{\pi}$ to the normal sub-crystal $\mathcal{B}_{f_Z, \Theta_Z}$ is an invariant projection $\mathcal{B}_{f_Z, \Theta_Z} \to X_\star(T)^+ \times X_\star(T)^+$. For each $\lambda, \nu \in X_\star(T)^+$, we denote by $\mathcal{B}_{f_Z, \Theta_Z; \lambda, \nu}$ the fiber of the latter projection.

By definition, we have a decomposition into the normal Kashiwara sub-crystals

$$\mathcal{B}_{f_Z,\Theta_Z} = \bigsqcup_{\lambda,\nu\in X_\star(T)^+} \mathcal{B}_{f_Z,\Theta_Z;\lambda,\nu} \ .$$

Lemma 7.14. *For each $\lambda,\nu \in X_\star(T)^+$, the component $\mathcal{B}_{f_Z,\Theta_Z;\lambda,\nu}$ is isomorphic to the associated crystal $\mathcal{B}(V_\lambda \otimes V_\nu)$.*

Proof. One can easily see that each choice of toric chart $\mathbf{j}_{\mathbf{i},\mathbf{i}'} \in \Theta_Z$ defines an isomorphism of normal Kashiwara crystals

$$\mathcal{B}_{f_Z,\Theta_Z;\lambda,\nu} \cong \bigsqcup_{\mu\in X_\star(T)^+} \mathcal{C}^\mu_{\lambda,\nu} \times \mathcal{B}^\mu_{f_B,\Theta_B^-} \ ,$$

where each $\mathcal{C}^\mu_{\lambda,\nu} = \mathcal{C}^\mu_{\lambda,\nu}(\mathbf{i})$ is a finite set considered as a trivial Kashiwara crystal. It is easy to see that $\mathcal{C}^\mu_{\lambda,\nu}$ is precisely the set defined by conditions (1)-(4) in [5, Theorem 2.3]. On the other hand, the latter result asserts that the cardinality of $\mathcal{C}^\mu_{\lambda,\nu}$ is equal to the multiplicity of V_μ in the tensor product $V_\lambda \otimes V_\nu$. This and the isomorphism $\mathcal{B}^\mu_{f_B,\Theta_B^-} \cong \mathcal{B}(V_\mu)$ finish the proof of the lemma. □

Then Conjecture 7.13 implies the following refinement of Lemma 7.14.

Conjecture 7.15. *The tropicalization of* (3.10) *is an isomorphism of torsion-free Kashiwara crystals $\mathcal{B}_{\Theta_B^-} \times \mathcal{B}_{\Theta_B^-} \to \mathcal{B}_{f_Z,\Theta_Z}$ and its restriction to $\mathcal{B}_{f_B,\Theta_B^-} \times \mathcal{B}_{f_B,\Theta_B^-}$ is an isomorphism of normal Kashiwara crystals $\tilde{\mathbf{F}}_{w_0} : \mathcal{B}_{f_B,\Theta_B^-} \times \mathcal{B}_{f_B,\Theta_B^-} \to \mathcal{B}_{f_Z,\Theta_Z}$. The restriction $\tilde{\mathbf{F}}_{w_0}$ to each $\mathcal{B}^\lambda_{f_B,\Theta_B^-} \times \mathcal{B}^\nu_{f_B,\Theta_B^-}$, $\lambda,\nu \in X_\star(T)^+$ is an isomorphism of Kashiwara crystals*

$$\mathcal{B}(V_\lambda) \times \mathcal{B}(V_\nu) \xrightarrow{\sim} \mathcal{B}_{f_Z,\Theta_Z;\lambda,\nu}. \tag{7.3}$$

Remark 7.16. Note that the positivity of the birational isomorphism (3.10) implies only surjectivity of each (7.3). Injectivity of (7.3) follows from [5, Theorem 2.3] or from Conjecture 7.13 above. Conversely, Claim 3.41 and Conjecture 7.13 taken together imply Theorems 2.3 and 2.4 of [5].

Let $(\mathbf{X},\mathbf{p},f,\Theta)$ be a strongly positive parabolic $(U\times U,\chi^{st})$-linear bicrystal of type w_P and let $\kappa : X \to T'$ be a $U\times U$-invariant morphism (where T' is an algebraic torus) such that the restriction of κ to X^- is $(\Theta,\Theta_{T'})$-positive. This data defines a normal Kashiwara crystal $\mathcal{B}_{f,\tilde{\Theta}}$ along with $\tilde{e}_i^n$-invariant maps $\tilde{\kappa} : \mathcal{B}_{f,\Theta} \to X_\star(S)$ and $\tilde{\Delta}_X : \mathcal{B}_{f,\Theta} \to \mathbb{Z}$. For each $\mu \in X_\star(S)$, let $\mathcal{B}_{\lambda'} = \{\tilde{b} \in \mathcal{B} : \tilde{\kappa}(\tilde{b}) = \lambda'\}$, i.e., $\mathcal{B}_{\lambda'}$ is the λ'-th fiber of κ. By definition, $\mathcal{B}_{\lambda'}$ is a normal sub-crystal of $\mathcal{B}$. For each $\mu \in X_\star(Z(L_P)) \cap X_\star(T)^+$, denote by $\mathcal{C}^{(\mu)}_{\lambda'}$ the set of connected components in $\mathcal{B}_{\lambda'}$ of type (i.e., of the highest weight) μ. In particular, if $\mathcal{B}_{\lambda'}$ is an associated crystal of some based $\mathfrak{g}^\vee$-module, then $\mathcal{C}^{(\mu)}_{\lambda'}$ is the multiplicity of $\mathcal{B}(V_\mu)$ in $\mathcal{B}_{\lambda'}$. Note that the restriction of $\tilde{\Delta}_X$ to each $\mathcal{C}^{(\mu)}_{\lambda'}$ is a well-defined function $\mathcal{C}^{(\mu)}_{\lambda'} \to \mathbb{Z}$.

For each $\lambda' \in X_\star(S)$ and $\mu \in X_\star(Z(L_P)) \cap X_\star(T)^+$ such that $|\mathcal{C}^{(\mu)}_{\lambda'}| < \infty$, define the *q-multiplicity* function $[\mathcal{C}^{(\mu)}_{\lambda'}]_q$ by

$$[\mathcal{C}^{(\mu)}_{\lambda'}]_q = \sum_{\tilde{c}\in\mathcal{C}^{(\mu)}_{\lambda'}} q^{\tilde{\Delta}_X(\tilde{c})}$$

We apply this construction in the case when

$$(\mathbf{X}, \mathbf{P}, f, \Theta_X) = (Bw_0B, \mathrm{id}, f, \Theta_B^-) * (Bw_0B, \mathrm{id}, f, \Theta_B^-) * \cdots * (Bw_0B, \mathrm{id}, f, \Theta_B^-)$$

is the k-th power of the standard strongly positive $(U \times U, \chi^{st})$-linear bicrystal $(Bw_0B, \mathrm{id}, f, \Theta_B^-)$, and $\kappa : \mathbf{X} \to T^k$ is given by

$$\kappa(x_1 * x_2 \cdots * x_k) = (hw(x_1), hw(x_2), \ldots, hw(x_k)) \ ,$$

where $hw : Bw_0B \to T$ is the highest weight morphism (see (2.5) above) given by $hw(ut\overline{w_0}u') = t$ for all $u, u' \in U$. Tropicalizing with respect to $\Theta_X := \Theta_B^- * \Theta_B^- * \cdots * \Theta_B^-$, one obtains for each $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in (X_\star(T)^+)^k = X_\star(T^k)^+$:

$$\mathcal{B}_{(\lambda_1,\lambda_2,\ldots,\lambda_k)} = \mathcal{B}(V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_k}) = \mathcal{B}(V_{\lambda_1}) \otimes \mathcal{B}(V_{\lambda_2}) \otimes \cdots \otimes \mathcal{B}(V_{\lambda_k}) \ .$$

Therefore, the polynomial $[\mathcal{C}^{(\mu)}_{(\lambda_1,\lambda_2,\ldots,\lambda_k)}]_q$ is a new q-deformation of the tensor product multiplicity $[V_\mu : V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_k}]$.

We expect that the polynomials $[\mathcal{C}^{(\mu)}_{(\lambda_1,\lambda_2,\ldots,\lambda_k)}]_q$ are related to parabolic Kazhdan-Lusztig polynomials (see [15]).

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Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: `arkadiy@math.uoregon.edu`

Department of Mathematics, Hebrew University, Jerusalem, Israel
E-mail address: `kazhdan@math.huji.ac.il`