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INVOLUTIONS ON GEL'FAND-TSETLIN SCHEMES AND MULTIPLICITIES IN SKEW GL_n-MODULES

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An important problem in the theory of finite-dimensional representations of reductive groups and in numerous physical applications is that of calculating multiplicities under the restriction of irreducible representations of a connected reductive complex group G_1 to a connected reductive subgroup G_2 regularly embedded in G_1. Two special types of multiplicities are particularly important:

a) multiplicities of weights in irreducible G-modules (here $G_1 = G$ and $G_2 = H$ is a Cartan subgroup of G); and

b) multiplicities in a decomposition into irreducible constituents of a tensor product of two irreducible G-modules (in this case $G_1 = G \times G$ and $G_2 \simeq G$ is the diagonal subgroup of G_1).

It is desirable to have a combinatorial expression for a multiplicity, i.e. to represent it as a number of certain combinatorial objects. At present such an expression is known only in some special cases. In this note we suggest a general method for obtaining combinatorial expressions for multiplicities, and illustrate it by the example of the group GL_n.

Weyl's character formula entails a general formula for multiplicities under reduction from G_1 onto G_2 that expresses them as an alternating sum of certain partition functions (see [1]). We call this the *Kostant-Heckman formula* (for multiplicities of weights it becomes Kostant's classical formula). The individual summands in the Kostant-Heckman formula have a combinatorial meaning, so that the only obstacle to a combinatorial interpretation of multiplicities is the alternation. Our method consists of constructing involutions making it possible to cancel all negative summands in the Kostant-Heckman formula against some of the positive ones. This method of involutions has been widely used recently to prove various combinatorial identities (for example, [2]); in a situation close to ours it was used in [3] for an elementary construction of the representation theory of symmetric groups. Using the method of involutions, we obtain combinatorial expressions for multiplicities in a tensor product of so-called skew modules of the group GL_n (the class of skew modules defined below includes all irreducible modules).

The multiplicities under consideration admit a completely elementary formulation in terms of symmetric functions (see [4]): this involves the coefficients $K_{\theta\gamma}$ in the expression of the skew Schur function $s_\theta(x)$ as a linear combination of monomials $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ and also the coefficients $c_{\theta\nu}^{\mu}$ in the expression of a product $s_\theta(x)s_\nu(x)$ (where $s_\nu(x)$ is an ordinary Schur function) as a linear combination of Schur functions $s_\mu(x)$. Combinatorial expressions for the coefficients $K_{\theta\gamma}$ have been known for a long time (see [4], Chapter I, §5) and expressions for the $c_{\theta\nu}^{\mu}$ were obtained in [5]; the method of involutions enables us to derive them very simply, straight from the classical definition of the Schur functions given by Jacobi. We obtain, in particular, a strikingly simple and short proof of the classical Littlewood-Richardson rule for multiplying Schur functions.

We describe the multiplicities $K_{\theta\gamma}$ and $c_{\theta\nu}^{\mu}$ in terms of Gel'fand-Tsetlin schemes. Their advantage over the traditional Young tables [4] or the pictures used in [5] is that the

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answers can be interpreted geometrically as the numbers of lattice points in certain convex polyhedra (see [6] and [7]). A formulation of the Littlewood-Richardson rule in terms of Gel'fand-Tsetlin schemes was given in [6] and [7]; we generalize it to skew modules by using the useful concept of a truncated scheme.

§1. Combinatorial expressions for multiplicities. Let $G_n = \mathrm{GL}_n(\mathbf{C})$. The lattice of weights P_n of the group G_n is identified in the standard way with $\mathbf{Z}^n$; the set P_n^+ of highest weights of finite-dimensional G_n-modules is equal to $\{\lambda = (\lambda_1, \dots, \lambda_n) \in P_n \colon \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$. For each $\lambda \in P_n^+$ let V_λ be an irreducible G_n-module with highest weight λ. For all k, $0 \leq k \leq n$, we embed the group $G_k \times G_{n-k}$ in G_n so that the subgroup G_k acts on the first k vectors of the standard basis of $\mathbf{C}^n$, and G_{n-k} on the last $n-k$ vectors; accordingly we will write the weights $\gamma \in P_{n-k}$ in the form $\gamma = (\gamma_{k+1}, \dots, \gamma_n)$. For all $\lambda \in P_n^+$ and $\rho \in P_k^+$ we define the skew G_{n-k}-module $V_{\lambda/\rho}$ by putting $V_{\lambda/\rho} = \mathrm{Hom}_{G_k}(V_\rho, V_\lambda|_{G_k})$ (when $k = 0$ it is convenient to assume that P_k^+ consists of one element $\rho = \varnothing$ and that $V_{\lambda/\varnothing}$ is the irreducible G_n-module V_λ).

Suppose $\lambda = (\lambda_1, \dots, \lambda_n) \in P_n$ and $\rho = (\rho_1, \dots, \rho_k) \in P_k$. By a (truncated Gel'fand-Tsetlin) *semischeme* of type λ/ρ we will mean a set of weights $\Lambda = \{\lambda^{(j)} = (\lambda_1^{(j)}, \dots, \lambda_j^{(j)}) \in P_j,\ k \leq j \leq n\}$ such that $\lambda^{(k)} = \rho$, $\lambda^{(n)} = \lambda$, and $\lambda_i^{(j+1)} \geq \lambda_i^{(j)}$ whenever $1 \leq i \leq j < n$. By the *weight of the semischeme* Λ we mean the weight $\gamma = (\gamma_{k+1}, \dots, \gamma_n) \in P_{n-k}$ defined by the equalities $\rho_1 + \dots + \rho_k + \gamma_{k+1} + \dots + \gamma_j = \lambda_1^{(j)} + \dots + \lambda_j^{(j)}$ for all $k < j \leq n$. The semischeme Λ is called a (truncated Gel'fand-Tsetlin) *scheme* if $\lambda_i^{(j)} \geq \lambda_{i+1}^{(j+1)}$ for all $1 \leq i \leq j < n$. It is clear that all components $\lambda^{(j)}$ of the scheme Λ are highest weights and, in particular, $\rho \in P_k^+$ and $\lambda \in P_n^+$.

THEOREM 1. *The multiplicity $K_{\lambda/\rho,\gamma}$ of the weight γ in the skew G_{n-k}-module $V_{\lambda/\rho}$ is equal to the number of all truncated Gel'fand-Tsetlin schemes of type λ/ρ and of weight γ.*

The proof will be given in §2.

Suppose $\Lambda = \{\lambda^{(k)}, \dots, \lambda^{(n)}\}$ is a truncated scheme. For all j and i with $k < j < n$ and $1 \leq i \leq j$ we put

$$d_i^{(j)}(\Lambda) = \sum_{1 \leq h < i} (\lambda_h^{(j+1)} - 2\lambda_h^{(j)} + \lambda_h^{(j-1)}) + \lambda_i^{(j+1)} - \lambda_i^{(j)};$$

the numbers $d_i^{(j)}(\Lambda)$ will be called the *exponents* of the scheme Λ (see [6] and [7]). Suppose $\nu \in P_{n-k}^+$; the scheme Λ will be called *ν-bounded* if $d_i^{(j)}(\Lambda) \leq \nu_j - \nu_{j+1}$ for all i and j.

THEOREM 2. *Suppose $\lambda \in P_n^+$, $\rho \in P_k^+$, and $\mu, \nu \in P_{n-k}^+$. Then the multiplicity $c_{\lambda/\rho,\nu}^{\mu}$ of an irreducible G_{n-k}-module V_μ in the tensor product $V_{\lambda/\rho} \otimes V_\nu$ is equal to the number of all ν-bounded truncated schemes of type λ/ρ and of weight $\mu - \nu$.*

The proof will be given in §3.

REMARK. For $k = 0$ Theorem 2 can be found in [7], where it was proved that in this case it is equivalent to the Littlewood-Richardson rule. It is easy to show that in the general case Theorem 2 is equivalent to Theorem 1 of [5].

2. PROOF OF THEOREM 1. Fix numbers $0 \leq k \leq n$ and weights $\lambda \in P_n^+$, $\rho \in P_k^+$, and $\gamma \in P_{n-k}$. Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the standard basis of the lattice $P_n = \mathbf{Z}^n$. We define a partition function p_k on P_n by putting $p_k(\omega)$ equal to the number of representations of the weight $\omega \in P_n$ in the form $\omega = \sum m_{ij}(\varepsilon_i - \varepsilon_j)$, where the sum extends over the set $\{(i,j) \colon 1 \leq i < j \text{ and } k < j \leq n\}$ and all the m_{ij} are nonnegative integers. Put $\delta_n = (n-1, n-2, \dots, 0) \in P_n$ and $\rho\gamma = (\rho_1, \dots, \rho_k, \gamma_{k+1}, \dots, \gamma_n) \in P_n$. The symmetric group S_n acts on the lattice P_n, rearranging the weights ε_i; for $w \in S_n$ we denote by

$\varepsilon(m) = \pm 1$ the parity of w. We define a shifted action of S_n on P_n by putting $w \cdot \lambda = w(\lambda + \delta_n) - \delta_n$; in particular, $(i, i+1) \cdot \lambda = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_i + 1, \lambda_{i+2}, \ldots, \lambda_n)$.

PROPOSITION 1. $K_{\lambda/\rho,\gamma} = \sum_{w \in S_n} \varepsilon(w) p_k(w \cdot \lambda - \rho\gamma)$.

PROOF. By definition of a skew module, the multiplicity $K_{\lambda/\rho,\gamma}$ is equal to that of the irreducible $G_k \times (G_1)^{n-k}$-module $V_\rho \otimes V_{\gamma_{k+1}} \otimes \cdots \otimes V_{\gamma_n}$ in the restriction of the irreducible G_n-module V_λ to the subgroup $G_k \times (G_1)^{n-k}$ in G_n. Therefore Proposition 1 is a special case of the Kostant-Heckman formula ([1], Formula 3.5); note that when $k = 0$ this is the usual formula of Kostant for the multiplicity of a weight in a reducible G_n-module.

PROPOSITION 2. *The number $p_k(w \cdot \lambda - \rho\gamma)$ is equal to the number of all truncated semischemes of type $(w \cdot \lambda)/\rho$ and of weight γ.*

PROOF. A bijective correspondence between the partitions $w \cdot \lambda - \rho\gamma = \sum m_{ij}(\varepsilon_i - \varepsilon_j)$, where $1 \le i < j$ and $k < j \le n$, and the truncated semischemes $(\lambda_i^{(j)})$ of type $(w \cdot \lambda)/\rho$ and of weight γ can be established by the formula $m_{ij} = \lambda_i^{(j)} - \lambda_i^{(j-1)}$.

For all $w \in S_n$ we denote by A_w the set of all truncated semischemes of type $(w \cdot \lambda)/\rho$ and of weight γ. It is easy to see that the weights $w \cdot \lambda$ are distinct, and hence the sets A_w are pairwise disjoint; let $A = \bigcup_{w \in S_n} A_w$. It is also clear that $w \cdot \lambda \in P_n^+$ only when $w = e$, so that a scheme $\Lambda \in A$ can lie only in the subset A_e. In view of Propositions 1 and 2, Theorem 1 is a direct consequence of the following combinatorial result.

THEOREM 3. *There exists an involution σ on A with the following properties:*
(1) *$\sigma\Lambda = \Lambda$ for all schemes $\Lambda \in A$.*
(2) *If $\Lambda \in A_w$ is a semischeme that is not a scheme, then $\sigma\Lambda \in A_{(i,i+1)w}$ for some i.*

PROOF. For all schemes $\Lambda \in A$ we put $\sigma\Lambda = \Lambda$. Now suppose $\Lambda = (\lambda_i^{(j)}) \in A_w$ is a semischeme, but not a scheme. By a *violation* for the semischeme Λ we mean any pair (i, j) such that $\lambda_i^{(j)} < \lambda_{i+1}^{(j+1)}$; the violation (i, j) with the smallest possible j and with the largest i for that j will be called *extreme* for Λ. Let (i_0, j_0) be the extreme violation for the semischeme Λ. We define the semischeme $\sigma\Lambda = \{\tilde{\lambda}^{(k)}, \tilde{\lambda}^{(k+1)}, \ldots, \tilde{\lambda}^{(n)}\}$ by putting $\tilde{\lambda}^{(j)} = \lambda^{(j)}$ if $k \le j \le j_0$, and $\tilde{\lambda}^{(j)} = (i_0, i_0 + 1) \cdot \lambda^{(j)}$ if $j_0 < j \le n$. It is easy to verify that $\sigma\Lambda \in A_{(i_0,i_0+1)w}$ and that the extreme violation for $\sigma\Lambda$ is again (i_0, j_0). It follows that σ is the desired involution.

3. PROOF OF THEOREM 2. We fix numbers $0 \le k \le n$ and highest weights $\lambda \in P_n^+$, $\rho \in P_k^+$, and $\mu, \nu \in P_{n-k}^+$.

PROPOSITION 3. $c^{\mu}_{\lambda/\rho,\nu} = \sum_{w \in S_{n-k}} \varepsilon(w) K_{\lambda/\rho, w \cdot \mu - \nu}$.

This proposition is well known; like Proposition 1, it can be deduced from the Kostant-Heckman formula.

For all $w \in S_{n-k}$ we denote by B_w the set of all truncated schemes of type λ/ρ and of weight $(w \cdot \mu - \nu)$. It is clear that the sets B_w are pairwise disjoint; let $B = \bigcup_{w \in S_{n-k}} B_w$. It is easy to see that the ν-bounded schemes $\Lambda \in B$ can lie only in the subset B_e. Therefore Theorem 2 is a consequence of the following combinatorial result.

THEOREM 4. *There exists an involution τ on B with the following properties:*
(1) *$\tau\Lambda = \Lambda$ for all ν-bounded schemes $\Lambda \in B$.*
(2) *If $\Lambda \in B_w$ is not ν-bounded, then $\tau\Lambda \in B_{(j,j+1)w}$ for some j.*

PROOF. For all ν-bounded schemes $\Lambda \in B$ we put $\tau\Lambda = \Lambda$. Now suppose $\Lambda = (\lambda_i^{(j)}) \in B_w$ is a scheme, but is not ν-bounded. By a *violation* (of ν-boundedness) of the scheme Λ we mean any pair (i, j) such that $d_i^{(j)}(\Lambda) > \nu_j - \nu_{j+1}$; the violation (i, j) with the

smallest possible i and with the largest j for that i will be called *extreme* for Λ. Let (i_0, j_0) be the extreme violation for the scheme Λ.

Let $a_i = \max(\lambda_{i+1}^{(j_0+1)}, \lambda_i^{(j_0-1)})$ and $b_i = \min(\lambda_i^{(j_0+1)}, \lambda_{i-1}^{(j_0-1)})$; thus if we fix all components $\lambda^{(j)}$ of Λ with $j \neq j_0$, the element $\lambda_i^{(j_0)}$ can range over the interval $[a_i, b_i]$. We define the scheme $\tau\Lambda = \{\tilde{\lambda}^{(k)}, \dots, \tilde{\lambda}^{(n)}\}$ as follows.

(1) If $j \neq j_0$, or if $j = j_0$ and $i < i_0$, then $\tilde{\lambda}_i^{(j)} = \lambda_j^{(j)}$.

(2) If $i > i_0$, then $\tilde{\lambda}_i^{(j_0)} = a_i + b_i - \lambda_i^{(j_0)}$ is obtained from $\lambda_i^{(j_0)}$ by reflection with respect to the midpoint of $[a_i, b_i]$.

(3) $\tilde{\lambda}_{i_0}^{(j_0)} = a_{i_0} + d_{i_0}^{(j_0)}(\Lambda) - (\nu_{j_0} - \nu_{j_0+1} + 1)$.

It can be verified directly that $\tau\Lambda \in B_{(j_0, j_0+1)w}$. It is also clear that $d_i^{(j)}(\tau\Lambda) = d_i^{(j)}(\Lambda)$ when $i < i_0$ and when $i = i_0$ and $j > j_0$, and that

$$\begin{aligned} d_{i_0}^{(j_0)}(\tau\Lambda) &= d_{i_0}^{(j_0)}(\Lambda) + \lambda_{i_0}^{(j_0)} - \tilde{\lambda}_{i_0}^{(j_0)} \\ &= (\lambda_{i_0}^{(j_0)} - a_{i_0}) + (\nu_{j_0} - \nu_{j_0+1} + 1) > \nu_{j_0} - \nu_{j_0+1}. \end{aligned}$$

It follows that (i_0, j_0) is the extreme violation of the scheme $\tau\Lambda$ and that $\tau(\tau\Lambda) = \Lambda$.

4. REMARK. The involutions we have constructed enable us to give a geometric interpretation of the continuous analogue of the multiplicities $K_{\lambda/\rho,\gamma}$ and $c^{\mu}_{\lambda/\rho,\nu}$ (the definition of the continuous analogue of the multiplicities is given in [1], where the term "asymptotic multiplicity function" is used). Namely, if we regard truncated semischemes and Gel'fand-Tsetlin schemes as points of the real vector space with coordinates $(\lambda_i^{(j)})$, where $1 \leq i \leq j$ and $k \leq j \leq n$, then the expressions for $K_{\lambda/\rho,\gamma}$ and $c^{\mu}_{\lambda/\rho,\nu}$ given by Theorems 1 and 2 can be interpreted naturally as the numbers of lattice points in certain convex polyhedra in this space; to obtain the continuous analogue we need only replace the number of lattice points of the polyhedron by its volume.

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