INVOLUTIONS ON GEL'FAND-TSETLIN SCHEMES
AND MULTIPlicITIES IN SKEW GL\(_n\)-MODULES

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An important problem in the theory of finite-dimensional representations of reductive groups and in numerous physical applications is that of calculating multiplicities under the restriction of irreducible representations of a connected reductive complex group \(G_1\) to a connected reductive subgroup \(G_2\) regularly embedded in \(G_1\). Two special types of multiplicities are particularly important:

a) multiplicities of weights in irreducible \(G\)-modules (here \(G_1 = G\) and \(G_2 = H\) is a Cartan subgroup of \(G\)); and

b) multiplicities in a decomposition into irreducible constituents of a tensor product of two irreducible \(G\)-modules (in this case \(G_1 = G \times G\) and \(G_2 \simeq G\) is the diagonal subgroup of \(G_1\)).

It is desirable to have a combinatorial expression for a multiplicity, i.e. to represent it as a number of certain combinatorial objects. At present such an expression is known only in some special cases. In this note we suggest a general method for obtaining combinatorial expressions for multiplicities, and illustrate it by the example of the group \(GL_n\).

Weyl's character formula entails a general formula for multiplicities under reduction from \(G_1\) onto \(G_2\) that expresses them as an alternating sum of certain partition functions (see [1]). We call this the Kostant-Heckman formula (for multiplicities of weights it becomes Kostant's classical formula). The individual summands in the Kostant-Heckman formula have a combinatorial meaning, so that the only obstacle to a combinatorial interpretation of multiplicities is the alternation. Our method consists of constructing involutions making it possible to cancel all negative summands in the Kostant-Heckman formula against some of the positive ones. This method of involutions has been widely used recently to prove various combinatorial identities (for example, [2]); in a situation close to ours it was used in [3] for an elementary construction of the representation theory of symmetric groups. Using the method of involutions, we obtain combinatorial expressions for multiplicities in a tensor product of so-called skew modules of the group \(GL_n\) (the class of skew modules defined below includes all irreducible modules).

The multiplicities under consideration admit a completely elementary formulation in terms of symmetric functions (see [4]): this involves the coefficients \(K_{\gamma \nu}\) in the expression of the skew Schur function \(s_\gamma(x)\) as a linear combination of monomials \(x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}\) and also the coefficients \(c^\mu\nu_{\gamma\nu}\) in the expression of a product \(s_\theta(x)s_\nu(x)\) (where \(s_\nu(x)\) is an ordinary Schur function) as a linear combination of Schur functions \(s_\mu(x)\). Combinatorial expressions for the coefficients \(K_{\gamma \nu}\) have been known for a long time (see [4], Chapter I, §5) and expressions for the \(c^\mu\nu_{\gamma\nu}\) were obtained in [5]; the method of involutions enables us to derive them very simply, straight from the classical definition of the Schur functions given by Jacobi. We obtain, in particular, a strikingly simple and short proof of the classical Littlewood-Richardson rule for multiplying Schur functions.

We describe the multiplicities \(K_{\gamma \nu}\) and \(c^\mu\nu_{\gamma\nu}\) in terms of Gel'fand-Tsetlin schemes. Their advantage over the traditional Young tables [4] or the pictures used in [5] is that the
§1. Combinatorial expressions for multiplicities. Let $G_n = GL_n(\mathbb{C})$. The lattice of weights $P_n$ of the group $G_n$ is identified in the standard way with $\mathbb{Z}^n$; the set $P_n^+$ of highest weights of finite-dimensional $G_n$-modules is equal to $\{ (\lambda_1, \ldots, \lambda_n) \in P_n : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}$. For each $\lambda \in P_n^+$ let $V_\lambda$ be an irreducible $G_n$-module with highest weight $\lambda$. For all $k$, $0 \leq k \leq n$, we embed the group $G_k \times G_{n-k}$ in $G_n$ so that the subgroup $G_k$ acts on the first $k$ vectors of the standard basis of $\mathbb{C}^n$, and $G_{n-k}$ on the last $n-k$ vectors; accordingly we will write the weights $\gamma \in P_{n-k}$ in the form $\gamma = (\gamma_{k+1}, \ldots, \gamma_n)$. For all $\lambda \in P_n^+$ and $\rho \in P_k^+$ we define the skew $G_{n-k}$-module $V_\lambda/\rho$ by putting $V_\lambda/\rho = \text{Hom}_{G_k}(V_\gamma, V_\lambda|_{G_k})$ (when $k = 0$ it is convenient to assume that $P_k^+$ consists of one element $\rho = \emptyset$ and that $V_\lambda/\rho$ is the irreducible $G_n$-module $V_\lambda$).

Suppose $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_n$ and $\rho = (\rho_1, \ldots, \rho_k) \in P_k$. By a (truncated Gel'fand-Tsetlin) semischeme of type $\lambda/\rho$ we will mean a set of weights $\Lambda = \{ (\lambda^{(j)}_1, \ldots, \lambda^{(j)}_j) \in P_j, k \leq j \leq n \}$ such that $\lambda^{(k)} = \rho$, $\lambda^{(n)} = \lambda$, and $\lambda^{(j+1)}_i \geq \lambda^{(j)}_i$ whenever $1 \leq i \leq j < n$. By the weight of the semischeme $\Lambda$ we mean the weight $\gamma = (\gamma_{k+1}, \ldots, \gamma_n) \in P_{n-k}$ defined by the equalities $\rho_1 + \cdots + \rho_k + \gamma_{k+1} + \cdots + \gamma_j = \lambda^{(j)}_1 + \cdots + \lambda^{(j)}_j$ for all $k < j \leq n$. The semischeme $\Lambda$ is called a (truncated Gel'fand-Tsetlin) scheme if $\lambda^{(j)}_i \geq \lambda^{(j+1)}_{i+1}$ for all $1 \leq i \leq j < n$. It is clear that all components $\lambda^{(j)}_i$ of the scheme $\Lambda$ are highest weights and, in particular, $\rho \in P_k^+$ and $\lambda \in P_n^+$.

**Theorem 1.** The multiplicity $K_{\lambda/\rho, \gamma}$ of the weight $\gamma$ in the skew $G_{n-k}$-module $V_\lambda/\rho$ is equal to the number of all truncated Gel'fand-Tsetlin schemes of type $\lambda/\rho$ and of weight $\gamma$.

The proof will be given in §2. Suppose $\Lambda = \{ \lambda^{(k)}_1, \ldots, \lambda^{(n)}_n \}$ is a truncated scheme. For all $j$ and $i$ with $k < j < n$ and $1 \leq i \leq j$ we put
\[
d_i^{(j)}(\Lambda) = \sum_{1 \leq h < i} (\lambda^{(j+1)}_h - 2\lambda^{(j)}_h + \lambda^{(j-1)}_h) + \lambda^{(j+1)}_i - \lambda^{(j)}_i;
\]
the numbers $d_i^{(j)}(\Lambda)$ will be called the exponents of the scheme $\Lambda$ (see [6] and [7]). Suppose $\nu \in P_{n-k}^+$; the scheme $\Lambda$ will be called $\nu$-bounded if $d_i^{(j)}(\Lambda) \leq \nu_j - \nu_{j+1}$ for all $i$ and $j$.

**Theorem 2.** Suppose $\lambda \in P_n^+$, $\rho \in P_k^+$, and $\mu, \nu \in P_{n-k}^+$. Then the multiplicity $e_{\lambda/\rho, \nu}^{\mu}$ of an irreducible $G_{n-k}$-module $V_\mu$ in the tensor product $V_\lambda/\rho \otimes V_\nu$ is equal to the number of all $\nu$-bounded truncated schemes of type $\lambda/\rho$ and of weight $\mu - \nu$.

The proof will be given in §3.

Remark. For $k = 0$ Theorem 2 can be found in [7], where it was proved that in this case it is equivalent to the Littlewood-Richardson rule. It is easy to show that in the general case Theorem 2 is equivalent to Theorem 1 of [5].

2. Proof of Theorem 1. Fix numbers $0 \leq k \leq n$ and weights $\lambda \in P_n^+$, $\rho \in P_k^+$, and $\gamma \in P_{n-k}$. Let $\{ \varepsilon_1, \ldots, \varepsilon_n \}$ be the standard basis of the lattice $P_n = \mathbb{Z}^n$. We define a partition function $p_k$ on $P_n$ by putting $p_k(\omega)$ equal to the number of representations of the weight $\omega \in P_n$ in the form $\omega = \sum m_{ij}(\varepsilon_i - \varepsilon_j)$, where the sum extends over the set $\{ (i, j) : 1 \leq i < j \text{ and } k < j \leq n \}$ and all the $m_{ij}$ are nonnegative integers. Put $\delta_n = (n-1, n-2, \ldots, 0) \in P_n$ and $\rho \gamma = (\rho_1, \ldots, \rho_k, \gamma_{k+1}, \ldots, \gamma_n) \in P_n$. The symmetric group $S_n$ acts on the lattice $P_n$, rearranging the weights $\varepsilon_i$; for $w \in S_n$ we denote by
\[ \varepsilon(m) = \pm 1 \] the parity of \( w \). We define a shifted action of \( S_n \) on \( P_n \) by putting \( w \cdot \lambda = w(\lambda + \delta_n) - \delta_n \); in particular, \((i,i+1) \cdot \lambda = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_i + 1, \lambda_{i+2}, \ldots, \lambda_n)\).

**PROPOSITION 1.** \( K_{\lambda/\rho, \gamma} = \sum_{w \in S_n} \varepsilon(w) p_k(w \cdot \lambda - \rho \gamma) \).

**PROOF.** By definition of a skew module, the multiplicity \( K_{\lambda/\rho, \gamma} \) is equal to that of the irreducible \( G_k \times (G_1)^{n-k} \)-module \( V^{} = V_\gamma \otimes V_{\gamma_{n+1}} \otimes \cdots \otimes V_{\gamma_n} \) in the restriction of the irreducible \( G_n \)-module \( V_\lambda \) to the subgroup \( G_k \times (G_1)^{n-k} \) in \( G_n \). Therefore Proposition 1 is a special case of the Kostant-Heckman formula ([1], Formula 3.5); note that when \( k = 0 \) this is the usual formula of Kostant for the multiplicity of a weight in a reducible \( G_n \)-module.

**PROPOSITION 2.** The number \( p_k(w \cdot \lambda - \rho \gamma) \) is equal to the number of all truncated semischemes of type \((w \cdot \lambda) / \rho\) and of weight \( \gamma \).

**PROOF.** A bijective correspondence between the partitions \( w \cdot \lambda - \rho \gamma = \sum m_{ij} (\varepsilon_i - \varepsilon_j) \), where \( 1 \leq i < j \) and \( k < j \leq n \), and the truncated semischemes \((A_i) \) of type \((w \cdot \lambda) / \rho\) and of weight \( \gamma \) can be established by the formula \( m_{ij} = \lambda_i^{(j)} - \lambda_i^{(j-1)} \).

For all \( w \in S_n \) we denote by \( A_w \) the set of all truncated semischemes of type \((w \cdot \lambda) / \rho\) and of weight \( \gamma \). It is easy to see that the weights \( w \cdot \lambda \) are distinct, and hence the sets \( A_w \) are pairwise disjoint; let \( A = \bigcup_{w \in S_n} A_w \). It is also clear that \( w \cdot \lambda \in P_n^+ \) only when \( w = e \), so that a scheme \( \Lambda \in A \) can lie only in the subset \( A_e \). In view of Propositions 1 and 2, Theorem 1 is a direct consequence of the following combinatorial result.

**THEOREM 3.** There exists an involution \( \sigma \) on \( A \) with the following properties:

1. \( \sigma \Lambda = \Lambda \) for all schemes \( \Lambda \in A \).
2. If \( \Lambda \in A_w \) is a semischeme that is not a scheme, then \( \sigma \Lambda \in A_{(i,i+1)w} \) for some \( i \).

**PROOF.** For all schemes \( \Lambda \in A \) we put \( \sigma \Lambda = \Lambda \). Now suppose \( \Lambda = (\lambda_i^{(j)}) \in A_w \) is a semischeme, but not a scheme. By a violation for the semischeme \( \Lambda \), we mean any pair \((i,j) \) such that \( \lambda_i^{(j)} < \lambda_i^{(j+1)} \); the violation \((i,j) \) with the smallest possible \( j \) and with the largest \( i \) for that \( j \) will be called extreme for \( \Lambda \). Let \((i_0,j_0)\) be the extreme violation for the semischeme \( \Lambda \). We define the semischeme \( \sigma \Lambda = (\lambda^{(k)}, \lambda^{(k+1)}, \ldots, \lambda^{(n)}) \) by putting \( \lambda^{(j)} = \lambda^{(j)} \) if \( k \leq j \leq j_0 \), and \( \lambda^{(j)} = (i_0, i_0 + 1) \cdot \lambda^{(j)} \) if \( j_0 < j \leq n \). It is easy to verify that \( \sigma \Lambda \in A_{(i_0, i_0+1)w} \) and that the extreme violation for \( \sigma \Lambda \) is again \((i_0, j_0)\). It follows that \( \sigma \) is the desired involution.

**3. PROOF OF THEOREM 2.** We fix numbers \( 0 \leq k \leq n \) and highest weights \( \lambda \in P_{n}^+, \rho \in P_{k}^+, \mu, \nu \in P_{n-k}^+ \).

**PROPOSITION 3.** \( c_{\lambda/\rho, \mu, \nu} = \sum_{w \in S_{n-k}} \varepsilon(w) K_{\lambda/\rho, w \cdot \mu - \nu} \).

This proposition is well known; like Proposition 1, it can be deduced from the Kostant-Heckman formula.

For all \( w \in S_{n-k} \) we denote by \( B_w \) the set of all truncated schemes of type \( \lambda/\rho \) and of weight \( (w \cdot \mu - \nu) \). It is clear that the sets \( B_w \) are pairwise disjoint; let \( B = \bigcup_{w \in S_{n-k}} B_w \). It is easy to see that the \( \nu \)-bounded schemes \( \Lambda \in B \) can lie only in the subset \( B_e \). Therefore Theorem 2 is a consequence of the following combinatorial result.

**THEOREM 4.** There exists an involution \( \tau \) on \( B \) with the following properties:

1. \( \tau \Lambda = \Lambda \) for all \( \nu \)-bounded schemes \( \Lambda \in B \).
2. If \( \Lambda \in B_w \) is not \( \nu \)-bounded, then \( \tau \Lambda \in B_{(j,j+1)w} \) for some \( j \).

**PROOF.** For all \( \nu \)-bounded schemes \( \Lambda \in B \) we put \( \tau \Lambda = \Lambda \). Now suppose \( \Lambda = (\lambda_i^{(j)}) \in B_w \) is a scheme, but is not \( \nu \)-bounded. By a violation (of \( \nu \)-boundedness) of the scheme \( \Lambda \) we mean any pair \((i,j) \) such that \( d_i^{(j)}(\Lambda) > \nu_j - \nu_{j+1} \); the violation \((i,j) \) with the
smallest possible \( i \) and with the largest \( j \) for that \( i \) will be called extreme for \( \Lambda \). Let \((i_0, j_0)\) be the extreme violation for the scheme \( \Lambda \).

Let \( a_i = \max(\lambda_i^{(j_0+1)}, \lambda_i^{(j_0-1)}) \) and \( b_i = \min(\lambda_i^{(j_0+1)}, \lambda_{i-1}^{(j_0-1)}) \); thus if we fix all components \( \lambda^{(j)} \) of \( \Lambda \) with \( j \neq j_0 \), the element \( \lambda_i^{(j_0)} \) can range over the interval \([a_i, b_i]\). We define the scheme \( \tau \Lambda = \{x^{(k)}, \ldots, x^{(n)}\} \) as follows.

1. If \( j \neq j_0 \), or if \( i < i_0 \), then \( \tilde{\lambda}_i^{(j_0)} = \lambda_i^{(j_0)} \).
2. If \( i > j_0 \), then \( \tilde{\lambda}_i^{(j_0)} = a_i + b_i - \lambda_i^{(j_0)} \) is obtained from \( \lambda_i^{(j_0)} \) by reflection with respect to the midpoint of \([a_i, b_i]\).
3. \( \tilde{\lambda}_{i_0}^{(j_0)} = a_{i_0} + \nu_{j_0} - \nu_{j_0+1} + 1 \).

It can be verified directly that \( \tau \Lambda \in B_{(j_0, j_0+1)_w} \). It is also clear that \( d_i^{(j)}(\tau \Lambda) = d_i^{(j)}(\Lambda) \) when \( i < i_0 \) and when \( i = i_0 \) and \( j > j_0 \), and that

\[
d_{i_0}^{(j_0)}(\tau \Lambda) = d_{i_0}^{(j_0)}(\Lambda) + \lambda_{i_0}^{(j_0)} - \tilde{\lambda}_{i_0}^{(j_0)} = (\lambda_{i_0}^{(j_0)} - a_{i_0}) + (\nu_{j_0} - \nu_{j_0+1} + 1) > \nu_{j_0} - \nu_{j_0+1}.
\]

It follows that \((i_0, j_0)\) is the extreme violation of the scheme \( \tau \Lambda \) and that \( \tau(\tau \Lambda) = \Lambda \).

4. REMARK. The involutions we have constructed enable us to give a geometric interpretation of the continuous analogue of the multiplicities \( K_{\lambda/\rho, \gamma} \) and \( c_{\lambda/\rho, \nu}^{(k)} \) (the definition of the continuous analogue of the multiplicities is given in [1], where the term “asymptotic multiplicity function” is used). Namely, if we regard truncated semischemes and Gel’fand-Tsetlin schemes as points of the real vector space with coordinates \((\lambda_i^{(j)} \), where \( 1 \leq i \leq j \) and \( k \leq j \leq n \), then the expressions for \( K_{\lambda/\rho, \gamma} \) and \( c_{\lambda/\rho, \nu}^{(k)} \) given by Theorems 1 and 2 can be interpreted naturally as the numbers of lattice points in certain convex polyhedra in this space; to obtain the continuous analogue we need only replace the number of lattice points of the polyhedron by its volume.

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