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Triangular Bases in Quantum Cluster Algebras

Arkady Berenstein[1] and Andrei Zelevinsky[2]

[1]Department of Mathematics, University of Oregon, Eugene, OR 97403, USA and [2]Department of Mathematics, Northeastern University, Boston, MA 02115, USA

Correspondence to be sent to: e-mail: andrei@neu.edu

A lot of recent activity has been directed toward various constructions of "natural" bases in cluster algebras. We develop a new approach to this problem which is close in spirit to Lusztig's construction of a canonical basis, and the pioneering construction of the Kazhdan–Lusztig basis in a Hecke algebra. The key ingredient of our approach is a new version of Lusztig's Lemma that we apply to all acyclic quantum cluster algebras. As a result, we construct the "canonical" basis in every such algebra that we call the canonical triangular basis.

1 Introduction and Main Results

One of the main motivations behind the theory of cluster algebras was to create an algebraic formalism for a better understanding of canonical bases in quantum groups. To this end a lot of recent activity has been directed toward various constructions of "natural" bases in cluster algebras. An overview of these approaches with relevant references can be found in [13].

In this paper, we develop a new approach. It is in fact much closer to Lusztig's original way of constructing a canonical basis [11] (and the pioneering construction of the Kazhdan–Lusztig basis in a Hecke algebra). The key ingredient of our approach is

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a version of *Lusztig's Lemma* generalizing [5, Theorem 1.2] (see also [12]). Here is this version (a proof will be given in Section 7).

Theorem 1.1. Let $\mathcal{A}$ be a free $\mathbb{Z}[v, v^{-1}]$-module with a basis $\{E_a : a \in L\}$ indexed by a partially ordered set $(L, \prec)$ such that, for any $a \in L$, the lengths of chains in L with the top element a are bounded from above. Let $x \mapsto \bar{x}$ be a $\mathbb{Z}$-linear involution on $\mathcal{A}$ such that, for all $f \in \mathbb{Z}[v, v^{-1}]$ and $x \in \mathcal{A}$, we have

$$\overline{fx} = \bar{f}\bar{x}, \quad \text{where } \bar{f}(v) = f(v^{-1}). \tag{1.1}$$

Suppose that

$$\bar{E}_a - E_a \in \bigoplus_{a' \prec a} \mathbb{Z}[v, v^{-1}]E_{a'} \quad (a \in L). \tag{1.2}$$

Then, for every $a \in L$, there exists a unique element $C_a \in \mathcal{A}$ such that:

$$\bar{C}_a = C_a; \tag{1.3}$$

$$C_a - E_a \in \bigoplus_{a' \in L} v\mathbb{Z}[v]E_{a'}. \tag{1.4}$$

Moreover, the element C_a satisfies

$$C_a - E_a \in \bigoplus_{a' \prec a} v\mathbb{Z}[v]E_{a'}, \tag{1.5}$$

hence the elements C_a, for $a \in L$, form a $\mathbb{Z}[v, v^{-1}]$-basis in $\mathcal{A}$. □

We will apply Theorem 1.1 in the situation where $\mathcal{A}$ is a *quantum cluster algebra* (in the sense of [1]) with an acyclic quantum seed. To state the main results we need to recall some terminology and notation from [1].

Recall that a (labeled) *quantum seed* is specified by the following data:

(1) Two positive integers $m \geq n$.
(2) An $m \times n$ integer matrix $\tilde{B}$. We represent $\tilde{B}$ as the family of its columns $b_j \in \mathbb{Z}^m$ for $j \in [1, n] = \{1, \ldots, n\}$.
(3) A family of positive integers $d_1, \ldots, d_n$.
(4) A skew-symmetric bilinear form $\Lambda : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ satisfying the *compatibility condition* with $\tilde{B}$:

$$\Lambda(b_j, e_i) = \delta_{ij} d_j, \tag{1.6}$$

for $i \in [1, m]$, $j \in [1, n]$, where $e_1, \ldots, e_m$ are the standard basis vectors in $\mathbb{Z}^m$. We identify Λ with the skew-symmetric $m \times m$ matrix $\Lambda = (\lambda_{ij})$, where $\lambda_{ij} = \Lambda(e_i, e_j)$.

(5) The *based quantum torus* $\mathcal{T} = \mathcal{T}(\Lambda)$, that is, the $\mathbb{Z}[v, v^{-1}]$-algebra with a distinguished $\mathbb{Z}[v, v^{-1}]$-basis $\{X^e : e \in \mathbb{Z}^m\}$ and the multiplication given by

$$X^e X^f = v^{\Lambda(e,f)} X^{e+f} \quad (e, f \in \mathbb{Z}^m). \tag{1.7}$$

We abbreviate $X^{e_i} = X_i$ for $i \in [1, m]$, and call the set $\tilde{\mathbf{X}} = \{X_1, \ldots, X_m\}$ the *extended cluster* of a quantum seed. With some abuse of notation, we will denote the quantum seed simply as $(\tilde{\mathbf{X}}, \tilde{B})$. We call the $n \times n$ submatrix $B = (b_{ij})_{i,j \in [1,n]}$ of $\tilde{B}$ the *exchange matrix* of a quantum seed. Note that the condition that the form Λ is skew-symmetric, together with (1.6), implies that, for every $j, k \in [1, n]$, we have

$$d_j b_{jk} = \Lambda(b_j, b_k) = -\Lambda(b_k, b_j) = -d_k b_{kj}, \tag{1.8}$$

that is, the matrix B is *skew-symmetrizable*.

The elements of $\tilde{\mathbf{X}}$ and their inverses generate $\mathcal{T}$ as a $\mathbb{Z}[v, v^{-1}]$-algebra, subject to the *quasi-commutation relations*

$$X_i X_j = v^{2\lambda_{ij}} X_j X_i \quad (i, j \in [1, m]). \tag{1.9}$$

We call the set $\mathbf{X} = \{X_j : j \in [1, n]\}$ the *cluster* of a quantum seed, and the elements $X_i \in \tilde{\mathbf{X}} - \mathbf{X}$ the *frozen variables*. Let $\mathbb{P}$ denote the multiplicative subgroup in $\mathcal{T}$ generated by v and all frozen variables, and let $\mathbb{ZP}$ be the integer group ring of $\mathbb{P}$, that is, the ring of (noncommutative) Laurent polynomials in the frozen variables with coefficients in $\mathbb{Z}[v, v^{-1}]$. Note that $\mathcal{T}$ satisfies the *Ore condition* (see, e.g., [1, Appendix]), so can be viewed as a $\mathbb{ZP}$-subalgebra of the ambient *skew-field of fractions* $\mathcal{F}$.

The following example shows that every skew-symmetrizable $n \times n$ matrix B is an exchange matrix of a special quantum seed that we call *principal*. This construction will play an important role later in the paper.

Example 1.2 (Principal quantization)**.** Let B be a skew-symmetrizable integer $n \times n$ matrix, that is, B satisfies (1.8) for some positive integers $d_1, \ldots, d_n$; in other words, DB is a skew-symmetric matrix, where D is the diagonal matrix with diagonal entries

$d_1, \ldots, d_n$. We set $m = 2n$, and define a $m \times n$ integer matrix $\tilde{B}$ as

$$\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix}, \tag{1.10}$$

where I_n is the identity $n \times n$ matrix.

A direct inspection shows that the bilinear form Λ on $\mathbb{Z}^{2n}$ with the matrix

$$\Lambda = \begin{pmatrix} 0 & -D \\ D & -DB \end{pmatrix} \tag{1.11}$$

is skew-symmetric, and satisfies (1.6). The corresponding extended cluster $\tilde{\mathbf{X}}$ consists of cluster variables $X_1, \ldots, X_n$ and frozen variables $X_{n+1}, \ldots, X_{2n}$. The quasi-commutation relations (1.9) are as follows: all cluster variables $X_1, \ldots, X_n$ commute with each other, and we have

$$X_i X_k = v^{2d_i \delta_{i,k+n}} X_k X_i, \quad X_i X_j = v^{2d_j b_{ji}} X_j X_i \quad (k \in [1, n],\ i, j \in [n+1, 2n]). \tag{1.12}$$

□

Returning to the general case, for $k \in [1, n]$, the *quantum seed mutation* μ_k transforms $(\tilde{\mathbf{X}}, \tilde{B})$ into a quantum seed $(\tilde{\mathbf{X}}', \tilde{B}')$ defined as follows:

(1) The matrix entries b'_{ij} of $\tilde{B}'$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ [b_{kj}]_+ - [-b_{ik}]_+ [-b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{1.13}$$

(2) The extended cluster $\tilde{\mathbf{X}}'$ is obtained from $\tilde{\mathbf{X}} = \{X_1, \ldots, X_m\}$ by replacing X_k with X'_k given by

$$X'_k = X^{-e_k + [b_k]_+} + X^{-e_k + [-b_k]_+}. \tag{1.14}$$

Here, as before, $b_k \in \mathbb{Z}^m$ is the kth column of $\tilde{B}$, and, for an integer vector b, we use the notation $[b]_+$ for the vector obtained by applying the operation $c \mapsto \max(c, 0)$ to each component of b (in particular, $[b]_+ = \max(b, 0)$ for $b \in \mathbb{Z}$).

Note that $\tilde{\mathbf{X}}'$ generates a based quantum torus $\mathcal{T}'$ defined in the same way as the quantum torus $\mathcal{T}$ above, but with the quasi-commutation relations governed by the new skew-symmetric form Λ' on $\mathbb{Z}^m$ given as follows: $\Lambda'(e_i, e_j) = \Lambda(e_i, e_j)$ for $i, j \neq k$, and

$\Lambda'(e_k, e_j) = \Lambda(e'_k, e_j)$, where we abbreviate

$$e'_k = -e_k + [b_k]_+ \in \mathbb{Z}^m. \tag{1.15}$$

Note also that the skew-field of fractions of $\mathcal{T}' = \mathcal{T}(\Lambda')$ is naturally identified with $\mathcal{F}$.

Now the *quantum cluster algebra* $\mathcal{A} = \mathcal{A}(\tilde{\mathbf{X}}, \tilde{B})$ is defined as the $\mathbb{ZP}$-subalgebra of the ambient skew-field $\mathcal{F}$ generated by all *cluster variables*, that is, by the union of clusters of all quantum seeds obtained from the initial quantum seed by a finite sequence of mutations. According to the *quantum Laurent phenomenon* proved in [1], $\mathcal{A}$ is a subalgebra in the based quantum torus associated with any of its seeds.

As shown in [1, Section 7], the structure of $\mathcal{A}(\tilde{\mathbf{X}}, \tilde{B})$ simplifies dramatically if the initial quantum seed is *acyclic*. (Recall that this means the following: the directed graph $\Gamma(B)$ with the set of vertices $[1, n]$ and oriented edges $j \to i$ for all pairs (i, j) such that $b_{ij} > 0$, has no oriented cycles.) Namely, we have

For an acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$, the cluster algebra $\mathcal{A}$ is

a $\mathbb{ZP}$-subalgebra of $\mathcal{T}$ generated by the elements X_k, X'_k for $k \in [1, n]$. (1.16)

In fact, the results in [1, Section 7] provide the following sharper statement. Let $i \lhd j$ be any linear order on $[1, n]$. For any $a \in \mathbb{Z}^m$, we denote by E°_a the element of $\mathcal{A}$ given by

$$E^\circ_a = X^{\sum_{i\in[1,m]-[1,n]} a_i e_i + \sum_{j\in[1,n]} [a_j]_+ e_j} \cdot \prod_{k\in[1,n]}^{\lhd} (X'_k)^{[-a_k]_+}, \tag{1.17}$$

where the notation $\prod^{\lhd}_{k\in[1,n]}$ means that the product is taken in increasing order with respect to $\lhd$. Then we have (see [1, Theorem 7.3 and (7.4)])

For an acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$, the elements E°_a

for $a \in \mathbb{Z}^m$ form a basis in $\mathcal{A}$ as a $\mathbb{Z}[v, v^{-1}]$-module. (1.18)

Let $X \to \bar{X}$ be the involution of $\mathcal{T}$ ("bar-involution") satisfying (1.1), and such that $\bar{X}^a = X^a$ for all $a \in \mathbb{Z}^m$. An easy check shows that

$$\overline{XY} = \bar{Y}\bar{X} \quad (X, Y \in \mathcal{T}), \qquad \bar{X}'_k = X'_k \quad (k \in [1, n]). \tag{1.19}$$

It follows that the subalgebra $\mathcal{A} \subset \mathcal{T}$ is invariant under the bar-involution. Furthermore, as shown in [1], the bar-involution on $\mathcal{A}$ thus obtained is independent of the choice of an initial quantum seed.

For $a \in \mathbb{Z}^m$, we define

$$r(a) = \sum_{k \in [1,n]} [-a_k]_+. \tag{1.20}$$

We make $\mathbb{Z}^m$ into a partially ordered set by setting

$$a' \prec a \Longleftrightarrow r(a') < r(a); \tag{1.21}$$

note that $L = \mathbb{Z}^m$ satisfies the boundedness condition in Theorem 1.1.

We define the *leading term* $LT(E_a^\circ)$ as an element of $\mathcal{T}$ obtained from the expression (1.17) for E_a° by replacing each X'_k with $X^{e'_k}$. Then we set

$$E_a = v^{\nu(a)} E_a^\circ, \tag{1.22}$$

where the exponent $\nu(a) \in \mathbb{Z}$ is determined from the condition that

$$\text{the element } v^{\nu(a)} LT(E_a^\circ) \in \mathcal{T} \text{ is invariant under the bar-involution.} \tag{1.23}$$

Remark 1.3. It is easy to give an explicit formula for $\nu(a)$ but we will not need it. Note also that $\nu(a)$ and hence the element E_a will not change if we modify the definition of $LT(E_a^\circ)$ by replacing each X'_k with $X^{e'_k - b_k} = X^{-e_k + [-b_k]_+}$ instead of $X^{e'_k}$. □

Finally, everything is ready for stating our application of Theorem 1.1.

Theorem 1.4. Let $\mathcal{A} = \mathcal{A}(\tilde{\mathbf{X}}, \tilde{B})$ be the cluster algebra associated with an acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$, and $\lhd$ be an arbitrary linear order on $[1, n]$. Then the elements E_a, for $a \in \mathbb{Z}^m$, form a basis in $\mathcal{A}$ as a $\mathbb{Z}[v, v^{-1}]$-module, which satisfies the condition (1.2) with respect to the bar-involution on $\mathcal{A}$, and the partial order on the index set $L = \mathbb{Z}^m$ given by (1.21). Therefore, this basis gives rise to a $\mathbb{Z}[v, v^{-1}]$-basis $\mathbf{B} = \mathbf{B}(\tilde{\mathbf{X}}, \tilde{B}; \lhd) = \{C_a : a \in \mathbb{Z}^m\}$ in $\mathcal{A}$ uniquely determined by (1.3) and (1.4). □

We refer to the basis $\mathbf{B} = \mathbf{B}(\tilde{\mathbf{X}}, \tilde{B}; \lhd)$ in Theorem 1.4 as the *triangular basis* associated with an acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$, and a linear order $\lhd$ on $[1, n]$. Theorem 1.4 will be proved in Section 2.

Remark 1.5. Theorem 1.4 allows the following generalization to the case of an arbitrary quantum cluster algebra $\mathcal{A}$ (not necessarily possessing an acyclic quantum seed) and its arbitrary quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$. Namely, we can still consider a *lower bound* $\underline{\mathcal{A}}(\tilde{\mathbf{X}}, \tilde{B})$, that is, a $\mathbb{ZP}$-subalgebra of $\mathcal{A}$ generated by the elements X_k, X'_k for $k \in [1, n]$ (cf. (1.16)). Clearly, $\underline{\mathcal{A}}(\tilde{\mathbf{X}}, \tilde{B})$ is invariant under the bar-involution. As shown in [1], this subalgebra is equal to $\mathcal{A}$ if and only if the quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$ is acyclic. Let $\underline{L} \subseteq L = \mathbb{Z}^m$ be the set of vectors $a \in \mathbb{Z}^m$ such that the full directed subgraph of $\Gamma(B)$ with the set of vertices $\{j \in [1, n] : a_j < 0\}$ is acyclic. It is not hard to deduce from the results in [1] that the elements E_a, for $a \in \underline{L}$, form a basis in $\underline{\mathcal{A}}(\tilde{\mathbf{X}}, \tilde{B})$ as a $\mathbb{Z}[v, v^{-1}]$-module. Then Theorem 1.4 as well as its proof in Section 2 remain valid if one replaces $\mathcal{A}$ with $\underline{\mathcal{A}}(\tilde{\mathbf{X}}, \tilde{B})$, and $L = \mathbb{Z}^m$ with $\underline{L}$ (with the restricted partial order). □

Varying an acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$ and a linear order $\lhd$, we obtain many different triangular bases in $\mathcal{A}$. However, it turns out that many of these bases actually coincide with each other, and so provide us with a more "canonical" choice. To describe it, we recall the following characterization of acyclic seeds:

$$\text{A quantum seed } (\tilde{\mathbf{X}}, \tilde{B}) \text{ is acyclic if and only if there exists a linear order } \lhd \text{ on } [1, n] \text{ such that } b_{ij} \leq 0 \text{ for all } i, j \in [1, n] \text{ with } i \lhd j \tag{1.24}$$

(this follows at once from the well-known fact that every partial order on a finite set extends to a linear order).

Now we can state the main result of this paper.

Theorem 1.6. A triangular basis $\mathbf{B} = \mathbf{B}(\tilde{\mathbf{X}}, \tilde{B}; \lhd)$ in a quantum cluster algebra $\mathcal{A}$ does not depend on the choice of an acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$ and a linear order $\lhd$ on $[1, n]$ provided this linear order satisfies the condition in (1.24). □

We refer to the basis $\mathbf{B}$ in Theorem 1.6 as the *canonical triangular basis*. Theorem 1.6 will be proved in Sections 3–5.

By the construction, $\mathbf{B}$ contains all elements X^a for $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ such that $a_k \geq 0$ for $k \in [1, n]$. We refer to these elements as (normalized) *cluster monomials* associated with a quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$. As an immediate corollary of Theorem 1.6, we obtain the following important property of $\mathbf{B}$.

Corollary 1.7. The canonical triangular basis in $\mathcal{A}$ contains all cluster monomials associated to acyclic quantum seeds of $\mathcal{A}$. □

Remark 1.8. F. Qin (private communication) has informed us that the results of [9] imply the following: if $\mathcal{A}$ has an acyclic quantum seed such that the exchange matrix B is skew-symmetric, and the directed graph $\Gamma(B)$ is *bipartite*, then $\mathbf{B}$ contains *all* cluster monomials associated to quantum seeds of $\mathcal{A}$ (not only acyclic ones). Furthermore, the results in [7] imply that each acyclic quantum cluster algebra $\mathcal{A}$ (with an appropriate choice of frozen variables) can be realized as the localization with respect to frozen variables of the coordinate ring of a quantum unipotent cell (in an appropriate Kac–Moody group) associated with the square of a Coxeter element. Then the technique developed in [7, 8] implies that, under the same assumptions as above (B skew-symmetric, and $\Gamma(B)$ bipartite), the canonical triangular basis becomes identified with the dual canonical basis for the quantum unipotent cell (with the appropriate relation between the formal variable q in [7, 8], and the formal variable v used in this paper). □

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.4. Sections 3–5 are devoted to the proof of Theorem 1.6.

In Section 3, we reduce the proof of Theorem 1.6 to a special case of *principal quantization* described in Example 1.2. This is done by a two-step construction that we find of independent interest. First, we embed any quantum cluster algebra $\mathcal{A}(B)$ with an acyclic exchange matrix B into a "double" algebra $\mathcal{A}^{(2)}(B)$ obtained by adjoining some frozen variables in an appropriate way; and then we show that the principal quantization $\mathcal{A}_\bullet(B)$ can also be embedded in $\mathcal{A}^{(2)}(B)$. Note that the principal quantization appears (in some disguise) in [6] in the context of the "quantum symplectic double" (we are grateful to A. Goncharov for bringing this connection to our attention). It remains to be seen whether the above reduction procedure makes sense in the context of [6].

The case of the principal quantization is treated in Section 5. For the convenience of the reader we treat a special case $n=2$ separately in Section 4, to illustrate the general idea in a somewhat less technical situation.

In Section 6, we illustrate our results in the special case when $m=n=2$ and $\tilde{B}=B=\left(\begin{smallmatrix}0 & -2\\ 2 & 0\end{smallmatrix}\right)$. This cluster algebra and various bases in it were studied in detail in [4, 10]. We make use of the calculations in [4], and we show (in Proposition 6.1) that in this special case our canonical triangular basis coincides with the dual canonical

basis studied in [10], and also with one of the bases constructed in [4] with the help of the quantum Caldero–Chapoton characters associated with indecomposable representations of the Kronecker quiver (see [14]).

The concluding Section 7 contains the proof of Lusztig's Lemma (Theorem 1.1).

2 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. We freely use the terminology and notation from Section 1. Recall that we work with the quantum cluster algebra $\mathcal{A} = \mathcal{A}(\tilde{\mathbf{X}}, \tilde{B})$ associated with an acyclic quantum seed. According to (1.16), $\mathcal{A}$ is a $\mathbb{ZP}$-algebra generated by the elements X_k, X'_k for $k \in [1, n]$. We start by collecting the quasi-commutation relations among these generators. The relations between the generators X_k are given by (1.9). The rest are as follows (recall that the vectors e'_k were introduced in (1.15)).

Lemma 2.1.

(1) For $i \in [1, m]$ and $k \in [1, n]$ with $i \neq k$ we have

$$X_i X'_k = v^{2\Lambda(e_i, e'_k)} X'_k X_i. \tag{2.1}$$

(2) For $k \in [1, n]$ we have

$$v^{-\Lambda(e'_k, e_k)} X'_k X_k - v^{\Lambda(e'_k, e_k)} X_k X'_k = (v^{-d_k} - v^{d_k}) X^{[-b_k]_+}. \tag{2.2}$$

(3) For $j, k \in [1, n]$ with $j \neq k$ we have

$$v^{-\Lambda(e'_j, e'_k)} X'_j X'_k - v^{\Lambda(e'_j, e'_k)} X'_k X'_j = (v^{-d_j b_{jk}} - v^{d_j b_{jk}}) X^{-e_j - e_k + [-\varepsilon_{jk} b_j]_+ + [\varepsilon_{jk} b_k]_+}, \tag{2.3}$$

where ε_{jk} is the sign of the matrix entry b_{jk}. □

Proof. Remembering the notation (1.15), we can rewrite (1.14) as

$$X'_k = X^{e'_k} + X^{e'_k - b_k}. \tag{2.4}$$

Now the equalities (2.1)–(2.3) follow easily from (1.7) and (1.6) (note that (2.3) was essentially proved in [1, Proposition 7.1]). ■

Consider an increasing filtration $\{0\} = F_{-1} \subset F_0 \subset F_1 \subset \cdots$ on $\mathcal{A}$ defined as follows:

$$\begin{aligned} &F_r \text{ is the } \mathbb{ZP}\text{-linear span of (noncommutative) monomials } M \\ &\quad \text{in the elements } X_k \text{ and } X'_k \text{ for } k \in [1, n] \text{ such that the degree} \\ &\quad \text{of } M \text{ with respect to the elements } X'_k \text{ does not exceed } r. \end{aligned} \tag{2.5}$$

By the definition we have $F_r \cdot F_s \subseteq F_{r+s}$ for $r, s \geq 0$. Thus, this filtration gives rise to an *associated graded* $\mathbb{ZP}$*-algebra* $\hat{\mathcal{A}}$ given by

$$\hat{\mathcal{A}} = \bigoplus_{r \geq 0} \hat{\mathcal{A}}_r, \quad \text{where } \hat{\mathcal{A}}_r = F_r / F_{r-1}. \tag{2.6}$$

For an element $E \in F_r - F_{r-1}$ we denote by $\hat{E}$ the (nonzero) image of E in $\hat{\mathcal{A}}_r$.

The relations in Lemma 2.1 imply that (1.18) can be sharpened as follows:

$$\begin{aligned} &\text{For every } r \geq 0, \text{ the elements } E_a^\circ \text{ with } r(a) \leq r \\ &\quad \text{form a basis in } F_r \text{ as a } \mathbb{Z}[v, v^{-1}]\text{-module} \end{aligned} \tag{2.7}$$

(see (1.20)). It follows that

$$\begin{aligned} &\text{For every } r \geq 0, \text{ the elements } \hat{E}_a^\circ \text{ with } r(a) = r \\ &\quad \text{form a basis in } \hat{\mathcal{A}}_r \text{ as a } \mathbb{Z}[v, v^{-1}]\text{-module.} \end{aligned} \tag{2.8}$$

Clearly, (2.7) and (2.8) remain true if we replace E_a° with $E_a = v^{\nu(a)} E_a^\circ$.

Now we turn our attention to the bar-involution on $\mathcal{A}$. In view of (2.5) and (1.19), we have $\bar{F}_r = F_r$ for all r. Therefore, this involution gives rise to an involution on each $\hat{\mathcal{A}}_r$, which we will denote in the same way. In this notation, the desired condition (1.2) for the partial order on the index set $L = \mathbb{Z}^m$ given by (1.21), simply means that every element $\hat{E}_a$ is invariant under the bar-involution.

Fix some $a \in \mathbb{Z}^m$. To prove the desired equality $\bar{\hat{E}}_a = \hat{E}_a$, we can view $\hat{E}_a$ as an element of a new quantum torus $\mathcal{T}(a)$ generated by the following elements:

(1) the frozen variables X_i for $i \in [n+1, m]$;
(2) the elements X_j for $j \in [1, n]$ such that $a_j > 0$; and
(3) the elements $\hat{X}'_k$ for $k \in [1, n]$ such that $a_k < 0$.

The commutation relations among the generators of the first two kinds are given by (1.9), and according to (2.1) and (2.3), those involving one or both generators $\hat{X}'_k$ are of the form

$$X_i\hat{X}'_k = v^{2\Lambda(e_i,e'_k)}\hat{X}'_k X_i, \quad \hat{X}'_j\hat{X}'_k = v^{2\Lambda(e'_j,e'_k)}\hat{X}'_k\hat{X}'_j; \tag{2.9}$$

thus, they are of the same form (1.9), with the following modification: the argument of the form Λ (that appears in the exponent of v), corresponding to each $\hat{X}'_k$ is equal to e'_k. We see that the condition that $\hat{E}_a$ is invariant under the bar-involution, is in fact equivalent to the definition of E_a via (1.22) and (1.23). This completes the proof of Theorem 1.4.

The tools developed in the course of the above proof allow us to show the following.

Proposition 2.2. For a given acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$, the elements E_a associated with a linear order $\lhd$ on $[1, n]$, do not change under the following modification of $\lhd$: interchanging two adjacent indices j and k such that $b_{jk} = 0$. Therefore, the triangular basis $\mathbf{B}(\tilde{\mathbf{X}}, \tilde{B}; \lhd)$ also does not change under such a modification. □

Proof. If $b_{jk} = 0$, then the right-hand side of (2.3) becomes equal to 0, hence the elements X'_j and X'_k quasi-commute (i.e., commute, up to a multiple which is a power of v). Remembering (1.17), we see that interchanging j and k results in multiplying each E°_a by a power of v. But the element E_a can be characterized as an element of the form $v^s E^\circ_a$ such that $\hat{E}_a$ is invariant under the bar-involution. Thus, E_a remains unchanged under the modification in question. ■

3 Proof of Theorem 1.6-I. Reduction to Principal Case

In this and the next two sections, we prove Theorem 1.6, that is, that the triangular basis $\mathbf{B} = \mathbf{B}(\tilde{\mathbf{X}}, \tilde{B}; \lhd)$ in a quantum cluster algebra $\mathcal{A}$ does not depend on the choice of an acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$ and a linear order $\lhd$ on $[1, n]$ as long as $\lhd$ satisfies the condition in (1.24).

First, we fix an acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$ and show that $\mathbf{B}$ does not depend on the choice of a linear order $\lhd$ satisfying (1.24). Indeed, since the directed graph $\Gamma(B)$ has no oriented cycles, it gives rise to the partial order on $[1, n]$ defined as follows: i precedes j if there is an oriented path from i to j in $\Gamma(B)$. Now the condition (1.24) means that $\lhd$ is a linear extension of this partial order. It is well known (and easy to prove) that every two linear extensions of a given finite partial order can be obtained from each other by a sequence of operations of the following kind: interchanging two adjacent elements

provided they are incomparable with respect to the partial order in question. Thus, the desired independence of $\mathbf{B}$ on $\lhd$ follows from Proposition 2.2.

It remains to show that $\mathbf{B}$ is independent of the choice of an acyclic quantum seed $(\tilde{\mathbf{X}}, \tilde{B})$. As shown in [2, Corollary 4], every two acyclic quantum seeds can be obtained from each other by a finite number of *shape-preserving* mutations (recall that a mutation μ_k is shape-preserving if $k \in [1, n]$ is a source or a sink of the directed graph $\Gamma(\tilde{B})$). (In [2], the exchange matrix B was assumed to be skew-symmetric; the general case can be deduced from this one by the standard argument involving *folding*.) Therefore, to complete the proof of Theorem 1.6, it is enough to show that the basis $\mathbf{B}(\tilde{\mathbf{X}}, \tilde{B}; \lhd)$ with the order $\lhd$ satisfying (1.24), does not change under a shape-preserving mutation. Without loss of generality, we assume for the rest of this section that the order $\lhd$ is the usual one: $1 \lhd 2 \lhd \cdots \lhd n$. Thus, we have

$$b_{ij} \le 0 \quad \text{for } 1 \le i < j \le n. \tag{3.1}$$

For a vector $a = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we abbreviate

$$a^{>n} = \sum_{n<i\le m} a_i e_i, \quad a^{\le n} = \sum_{1\le i\le n} a_i e_i. \tag{3.2}$$

Note that the truncation operations $a \mapsto a^{>n}$ and $a \mapsto a^{\le n}$ commute with the operation $a \mapsto [a]_+$ (replacing each component a_i with $[a_i]_+ = \max(a_i, 0)$), hence we have well-defined operations $a \mapsto [a]_+^{>n}$ and $a \mapsto [a]_+^{\le n}$. In this notation, the definition (1.17) takes the form

$$E_a^\circ = X^{a^{>n}+[a]_+^{\le n}} \cdot (X_1')^{[-a_1]_+} (X_2')^{[-a_2]_+} \cdots (X_n')^{[-a_n]_+}. \tag{3.3}$$

Again without loss of generality, as a shape-preserving mutation, we take μ_n, the mutation at the last index n. The mutated quantum seed $\mu_n(\tilde{\mathbf{X}}, \tilde{B}) = (\tilde{\mathbf{X}}', \tilde{B}')$ is defined as follows. The new extended cluster $\tilde{\mathbf{X}}'$ is obtained from $\tilde{\mathbf{X}} = \{X_1, \dots, X_m\}$ by replacing X_n with X_n' given by (1.14) or (2.4) (for $k = n$), where the vector e_n' is given by (1.15). In view of (3.1), we have

$$e_n' = -e_n + b_n^{>n}. \tag{3.4}$$

The set $\tilde{\mathbf{X}}'$ generates a based quantum torus $\mathcal{T}' \subset \mathcal{F}$ defined in the same way as the quantum torus $\mathcal{T}$ above, but with the quasi-commutation relations governed by the new skew-symmetric form Λ' on $\mathbb{Z}^m$ given as follows: $\Lambda'(e_i, e_j) = \Lambda(e_i, e_j)$ for $i, j \ne n$, and $\Lambda'(e_n, e_j) = \Lambda(e_n', e_j)$.

Recall that the new extended exchange matrix $\tilde{B}'$ is obtained from $\tilde{B}$ by the *matrix mutation* in (1.13) with $k=n$. Note that the sign condition $b_{ij} \le 0$ for $1 \le i < j \le n$ implies that the top $n \times n$ part B' of $\tilde{B}'$ is obtained from the top part B of $\tilde{B}$ by simply changing the sign of all entries in the nth row and column. It follows that the condition (1.24) for the new seed will be obeyed by choosing the linear order $\lhd$ on $[1, n]$ as follows:

$$n \lhd 1 \lhd 2 \lhd \cdots \lhd (n-1). \tag{3.5}$$

For every $a \in \mathbb{Z}^m$, we define the element $(E'_a)^\circ \in \mathcal{A}$ is the same way as in (1.17) but with respect to the acyclic quantum seed $(\tilde{\mathbf{X}}', \tilde{B}')$ and the linear order $\lhd$ given by (3.5). To describe $(E'_a)^\circ$ explicitly, we need some more notation. Define the vectors $e''_1, \ldots, e''_n$ by setting

$$e''_k = -e_k + [b'_k]_+ \in \mathbb{Z}^m, \tag{3.6}$$

where b'_k is the kth column of $\tilde{B}'$. Then the mutated cluster variables from the quantum seed $(\tilde{\mathbf{X}}', \tilde{B}')$ are given by the following counterpart of (2.4):

$$X''_k = (X')^{e''_k} + (X')^{e''_k - b'_k}. \tag{3.7}$$

Note that we have $X''_n = X_n$ since the mutation μ_n is an involution; however, it is sometimes convenient to keep the notation X''_n.

Using this notation, we can obtain $(E'_a)^\circ$ from the element E°_a given by (3.3) by the following modifications:

(1) the factor $X^{a^{>n}+[a]_+^{\le n}}$ is replaced with its counterpart

$$(X')^{a^{>n}+[a]_+^{\le n}}$$

in the quantum torus $\mathcal{T}'$ (i.e., in constructing this element, X_n is replaced by X'_n);

(2) the remaining product $(X'_1)^{[-a_1]_+}(X'_2)^{[-a_2]_+} \cdots (X'_n)^{[-a_n]_+}$ is replaced by

$$(X''_n)^{[-a_n]_+}(X''_1)^{[-a_1]_+}(X''_2)^{[-a_2]_+} \cdots (X''_{n-1})^{[-a_{n-1}]_+}.$$

As for the normalized basis element E'_a, it is obtained from $(E'_a)^\circ$ by a suitable modification of (1.22) and (1.23). Namely, we have $E'_a = v^{\nu'(a)}(E'_a)^\circ$, with the exponent $\nu'(a) \in \mathbb{Z}$ determined from the condition that $v^{\nu'(a)} LT((E'_a)^\circ)$ is invariant under the bar-involution, where the leading term $LT((E'_a)^\circ)$ is obtained from $(E_a)^\circ$ by replacing each X''_k with $(X')^{e''_k}$.

In particular, we have

$$\begin{aligned} E'_{\pm e_i} &= E_{\pm e_i} = X_i^{\pm 1} \quad (n < i \le m), \quad E'_{e_k} = E_{e_k} = X_k \quad (1 \le k < n), \\ E'_{e_n} &= E_{-e_n} = X'_n, \quad E'_{-e_n} = E_{e_n} = X_n. \end{aligned} \tag{3.8}$$

Also for $1 \le k < n$, we have $E'_{-e_k} = X''_k$. Let us compute the expansion of this element in the basis $\{E_a\}$.

Recall that, for $r \ge s \ge 0$, the Gaussian binomial coefficient $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]_t$ is an integer polynomial in a variable t given by

$$\begin{bmatrix} r \\ s \end{bmatrix}_t = \frac{(t^r - 1)(t^{r-1} - 1) \cdots (t^{r-s+1} - 1)}{(t^s - 1)(t^{s-1} - 1) \cdots (t - 1)}. \tag{3.9}$$

The corresponding binomial formula is

$$\prod_{p=1}^{r} (1 + u^{2p-1} X) = \sum_{s=0}^{r} u^{s^2} \begin{bmatrix} r \\ s \end{bmatrix}_{u^2} X^s. \tag{3.10}$$

Now, for $1 \le k < n$, we define a vector $\varphi(-e_k) \in \mathbb{Z}^m$ as follows (the notation will be explained in a moment):

$$\varphi(-e_k) = -e_k - b_{nk} e_n + [-b'_k]_+^{>n} - [-b_k]_+^{>n}. \tag{3.11}$$

Then we have

$$E'_{-e_k} = X''_k = E_{\varphi(-e_k)} - \sum_{s=1}^{b_{nk}} v^{s^2 d_n} \begin{bmatrix} b_{nk} \\ s \end{bmatrix}_{v^{2d_n}} E_{e''_k - s b_n}. \tag{3.12}$$

This calculation was essentially done in [1, Proof of Lemma 5.8]. Note that if $b_{nk} = b_{kn} = 0$, then the last sum in (3.12) disappears, and $\varphi(-e_k) = -e_k$, and so, in this case, $X''_k = E'_{-e_k} = E_{-e_k} = X'_k$. And if $b_{kn} < 0$ (hence $b_{nk} > 0$), then, for every $j \in [1, n]$ and $s \in [1, b_{nk}]$, the jth component of the vector $e''_k - s b_n$ is nonnegative, therefore we have $E_{e''_k - s b_n} = X^{e''_k - s b_n}$.

Let $\mathcal{A}_+ = \mathcal{A}_+(\tilde{\mathbf{X}}, \tilde{B})$ denote the $\mathbb{Z}[v]$-linear span of the basis $\{E_a\}$. We refer to $\mathcal{A}_+$ as the *crystal lattice* associated with a given acyclic quantum seed. Our proof of Theorem 1.6 is based on the following fact.

Theorem 3.1. All elements E'_a belong to $\mathcal{A}_+ - v\mathcal{A}_+$. More precisely, there is a bijection $\varphi : \mathbb{Z}^m \to \mathbb{Z}^m$ such that, for every $a \in \mathbb{Z}^m$, we have

$$E'_a - E_{\varphi(a)} \in v\mathcal{A}_+. \tag{3.13}$$

□

Before proving Theorem 3.1, we deduce the following corollary.

Corollary 3.2. The triangular basis $\mathbf{B}' = \mathbf{B}(\tilde{\mathbf{X}}', \tilde{B}')$ coincides with $\mathbf{B} = \mathbf{B}(\tilde{\mathbf{X}}, \tilde{B})$. More precisely, for any $a \in \mathbb{Z}^m$, an element C'_a of $\mathbf{B}'$ is equal to $C_{\varphi(a)}$. □

Proof. Theorem 3.1 implies that the crystal lattice $\mathcal{A}'_+ = \mathcal{A}_+(\tilde{\mathbf{X}}', \tilde{B}')$ is contained in $\mathcal{A}_+$. Thus, the condition (1.4) for the basis $\mathbf{B}'$ implies that

$$C'_a \in E'_a + v\mathcal{A}'_+ \subseteq E_{\varphi(a)} + v\mathcal{A}_+.$$

In view of Theorem 1.4, we have $C'_a = C_{\varphi(a)}$, as claimed. ■

In view of (3.12), the condition (3.13) holds for $a = -e_k$ with $k < n$, and $\varphi(-e_k)$ given by (3.11). And in view of (3.8), it also holds for the rest of the vectors $\pm e_i$, with $\varphi(\pm e_i)$ given as follows:

$$\begin{aligned} &\varphi(\pm e_i) = \pm e_i \quad (n < i \le m), \quad \varphi(e_k) = e_k \quad (k < n), \\ &\varphi(e_n) = -e_n, \quad \varphi(-e_n) = e_n. \end{aligned} \tag{3.14}$$

For every $a \in \mathbb{Z}^m$, we write the element E'_a in terms of the basis $\{E_{a'}\}$ as follows:

$$E'_a = \sum_{a' \in \mathbb{Z}^m} c_a^{a'} E_{a'}, \tag{3.15}$$

where all coefficients $c_a^{a'}$ belong to $\mathbb{Z}[v^{\pm 1}]$, and, for a given a, all but finitely many of them are equal to 0. It is easy to see that to prove Theorem 3.1, it suffices to show the following:

For every $a \in \mathbb{Z}^m$, exactly one of the coefficients

$c_a^{a'}$ is equal to 1, and the rest belong to $v\mathbb{Z}[v]$. (3.16)

The proof of (3.16) will occupy the rest of this section as well as the next two sections. We start with the following lemma.

Lemma 3.3. We have $c_{a+a_\circ}^{a'+a_\circ} = c_a^{a'}$ for all $a, a', a_\circ \in \mathbb{Z}^m$ such that $a_\circ^{\le n} = 0$. □

Proof. The key observation is that the element $(X')^{a_\circ}$ is equal to $X^{a_\circ}$, and that it quasi-commutes (i.e., commutes up to an integer power of v as a multiple) with all elements E'_a and $E_{a'}$; furthermore, $X^{a_\circ}$ has the same quasi-commutation multiple with E'_a and $LT((E'_a)^\circ)$, as well as with $E_{a'}$ and $LT(E_{a'}^\circ)$. Then the definitions readily imply that

$$X^{a_\circ} E'_a = v^{\nu'(a_\circ;a)} E'_{a+a_\circ}, \quad X^{a_\circ} E_{a'} = v^{\nu(a_\circ;a')} E_{a'+a_\circ}, \tag{3.17}$$

where the integer exponents $\nu'(a_\circ; a)$ and $\nu(a_\circ; a')$ are determined from the quasi-commutation relations

$$X^{a_\circ} E'_a = v^{2\nu'(a_\circ;a)} E'_a X^{a_\circ}, \quad X^{a_\circ} E_{a'} = v^{2\nu(a_\circ;a')} E_{a'} X^{a_\circ}. \tag{3.18}$$

Now the fact that $X^{a_\circ}$ quasi-commutes with all the terms in the equality (3.15) implies that the corresponding quasi-commutation multiples are all the same, that is, $\nu'(a_\circ; a) = \nu(a_\circ; a')$ whenever $c_a^{a'} \neq 0$. Thus, the desired equality $c_{a+a_\circ}^{a'+a_\circ} = c_a^{a'}$ follows from (3.17). ■

In view of Lemma 3.3, it is enough to prove (3.16) for $a \in \mathbb{Z}^n$. This will be done in two steps.

The first step (occupying the rest of this section) is a reduction to a special choice of a quantum seed $(\tilde{\mathbf{X}}^\bullet, \tilde{B}^\bullet)$ (for a given exchange matrix B), namely to the *principal quantum seed* introduced in Example 1.2. To do this, we embed the quantum cluster algebra $\mathcal{A}(\tilde{\mathbf{X}}, \tilde{B})$ into a bigger one associated with the "double" quantum seed $(\tilde{\mathbf{X}}^{(2)}, \tilde{B}^{(2)})$ defined by the following data:

(1) $\tilde{B}^{(2)}$ is an integer $2m \times n$ matrix with the top $m \times n$ block $\tilde{B}$, and the bottom $m \times n$ block a zero matrix.

(2) A skew-symmetric bilinear form $\Lambda^{(2)} : \mathbb{Z}^{2m} \times \mathbb{Z}^{2m} \to \mathbb{Z}$ is given by

$$\Lambda^{(2)}((e, e'), (f, f')) = \Lambda(e, f) - \Lambda(e', f') \quad (e, e', f, f' \in \mathbb{Z}^m). \tag{3.19}$$

Note that $\tilde{B}^{(2)}$ and $\Lambda^{(2)}$ satisfy the compatibility condition (1.6) with the same choice of positive integers $d_1, \ldots, d_n$.

The corresponding *double based quantum torus* $\mathcal{T}^{(2)} = \mathcal{T}(\Lambda^{(2)})$ is the $\mathbb{Z}[v, v^{-1}]$-algebra with a distinguished $\mathbb{Z}[v, v^{-1}]$-basis $\{X^{(e,e')} : e, e' \in \mathbb{Z}^m\}$ and the multiplication given by

$$X^{(e,e')} X^{(f,f')} = v^{\Lambda^{(2)}((e,e'),(f,f'))} X^{e+e',\, f+f'} \quad (e, e',\, f,\, f' \in \mathbb{Z}^m). \tag{3.20}$$

We identify the based quantum torus $\mathcal{T} = \mathcal{T}(\Lambda)$ with a subalgebra in $\mathcal{T}^{(2)}$ by setting $X^e = X^{(e,0)}$ for $e \in \mathbb{Z}^m$. This way the quantum cluster algebra $\mathcal{A}(\tilde{\mathbf{X}}, \tilde{B})$ is identified with a subalgebra of $\mathcal{A}(\tilde{\mathbf{X}}^{(2)}, \tilde{B}^{(2)})$.

For $a^{(2)} \in \mathbb{Z}^{2m}$, we denote by $E^{(2)}_{a^{(2)}}$ the element of $\mathcal{A}(\tilde{\mathbf{X}}^{(2)}, \tilde{B}^{(2)})$ constructed as in (1.17) and (1.22). In particular, we have

$$E_a = E^{(2)}_{(a,0)} \tag{3.21}$$

for each $a \in \mathbb{Z}^m$.

It turns out that the principal quantum cluster algebra $\mathcal{A}(\tilde{\mathbf{X}}^\bullet, \tilde{B}^\bullet)$ can also be realized as a subalgebra of $\mathcal{A}(\tilde{\mathbf{X}}^{(2)}, \tilde{B}^{(2)})$. To avoid notational confusion, we will use the symbol $\bullet$ for the objects related to this subalgebra. First, we introduce the based quantum torus $\mathcal{T}^\bullet$ as the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $\mathcal{T}^{(2)}$ generated by the elements $X^\bullet_1, \dots, X^\bullet_{2n}$ given by

$$X^\bullet_j = X^{(e_j, e_j)}, \quad X^\bullet_{n+j} = X^{(b_j^{>n}, -b_j^{\leq n})} \quad (j \in [1, n]), \tag{3.22}$$

where we use the notation (3.2) for truncated vectors, and as before, $b_j \in \mathbb{Z}^m$ is the jth column of $\tilde{B}$.

Lemma 3.4. The based quantum torus $\mathcal{T}^\bullet$ is equal to $\mathcal{T}(\Lambda^\bullet)$, where $\Lambda^\bullet$ is the skew-symmetric bilinear form on $\mathbb{Z}^{2n}$ with the matrix (1.11). □

Proof. First, we show that the $2n$ vectors

$$(e_1, e_1), \dots, (e_n, e_n), (b_1^{>n}, -b_1^{\leq n}), \dots, (b_n^{>n}, -b_n^{\leq n})$$

in $\mathbb{Z}^{2m}$ are linearly independent, implying that $X^\bullet_1, \dots, X^\bullet_{2n}$ are algebraically independent. Indeed, for every $j \in [1, n]$, we have

$$(b_j^{>n}, -b_j^{\leq n}) + \sum_{i=1}^{n} b_{ij}(e_i, e_i) = (b_j, 0).$$

Thus, it is enough to show that the vectors $(e_1, e_1), \dots, (e_n, e_n), (b_1, 0), \dots, (b_n, 0)$ are linearly independent. This follows by noticing that the first n of these vectors have linearly independent second components, and then that the last n vectors are linearly independent in view of (1.6).

It remains to show that the elements $X_1^\bullet, \dots, X_{2n}^\bullet$ obey the commutation relations in (1.12), and also that the first n of them commute with each other. The latter statement follows from (3.20) and (3.19). To show that, for $i, j \in [1, n]$, the elements $X_{n+i}^\bullet$ and $X_{n+j}^\bullet$ satisfy the second equality in (1.12), it is enough to observe that

$$\begin{aligned}\Lambda^{(2)}((b_i^{>n}, -b_i^{\leq n}), (b_j^{>n}, -b_j^{\leq n})) &= \Lambda(b_i^{>n}, b_j^{>n}) - \Lambda(b_i - b_i^{>n}, b_j - b_j^{>n}) \\ &= -\Lambda(b_i, b_j) = d_j b_{ji}\,,\end{aligned}$$

where we used (1.6) and (1.8). The first equality in (1.12) is checked in a similar way. ■

In view of Lemma 3.4, the assignment (3.22) allows us to identify the principal quantum cluster algebra $\mathcal{A}(\tilde{\mathbf{X}}^\bullet, \tilde{B}^\bullet)$ with a subalgebra of $\mathcal{A}(\tilde{\mathbf{X}}^{(2)}, \tilde{B}^{(2)})$. In accordance with the above notational convention, we denote the analogs of elements E_a and E'_a in $\mathcal{A}(\tilde{\mathbf{X}}^\bullet, \tilde{B}^\bullet)$ as $E_{a^\bullet}^\bullet$ and $E_{a^\bullet}^{\bullet\,\prime}$, where $a^\bullet$ runs over $\mathbb{Z}^{2n}$.

Lemma 3.5. There exist injective maps $\psi : \mathbb{Z}^{2n} \to \mathbb{Z}^{2m}$ and $\psi' : \mathbb{Z}^{2n} \to \mathbb{Z}^{2m}$ such that, for every $a \in \mathbb{Z}^{2n}$, we have $E_a^\bullet = E_{\psi(a)}^{(2)}$, $E'^\bullet_a = E'^{(2)}_{\psi'(a)}$. Specifically these maps are uniquely determined by the following properties:

(1) Both ψ and ψ' restrict as linear maps to the sublattice $\{0\} \times \mathbb{Z}^n \subset \mathbb{Z}^{2n}$ generated by $e_{n+1}, \dots, e_{2n}$ and to each of the 2^n coordinate orthants in the sublattice $\mathbb{Z}^n \times \{0\}$ generated by $e_1, \dots, e_n$. Moreover, if $a, a_\circ \in \mathbb{Z}^{2n}$ and $a_\circ^{\leq n} = 0$, then

$$\psi(a + a_\circ) = \psi(a) + \psi(a_\circ), \quad \psi'(a + a_\circ) = \psi'(a) + \psi'(a_\circ).$$

(2) We have

$$\psi(e_{n+k}) = \psi'(e_{n+k}) = (b_k^{>n}, -b_k^{\leq n}) \quad (k \in [1, n]),$$

$$\psi(e_k) = (e_k, e_k), \quad \psi(-e_k) = (-e_k - [-b_k]_+^{>n}, -e_k + [-b_k]_+^{\leq n}) \quad (k \in [1, n]),$$

$$\psi'(e_k) = (e_k, e_k) \quad (k \in [1, n-1]), \quad \psi'(e_n) = (e_n - [-b_n]_+^{>n}, -e_n - b_n^{\leq n}),$$

$$\psi'(-e_k) = (-e_k - [-b'_k]_+^{>n} - b_{nk} \cdot [-b_n]_+^{>n}, -e_k + [-b_k]_+^{\leq n} - b_{nk} \cdot b_n^{\leq n} - b_{nk} e_n)$$

$$(k \in [1, n-1]), \quad \psi'(-e_n) = (-e_n, e_n).$$

In particular, for every $a \in \mathbb{Z}^{2n}$, we have $\psi(a)^{\leq n} = \psi'(a)^{\leq n} = a^{\leq n}$. □

Proof. This is proved by a direct calculation that compares the expressions (1.14) and (3.12) evaluated in the principal algebra $\mathcal{A}(\tilde{\mathbf{X}}^\bullet, \tilde{B}^\bullet)$ with their counterparts in the double algebra $\mathcal{A}(\tilde{\mathbf{X}}^{(2)}, \tilde{B}^{(2)})$. ■

We are finally in a position to show the desired reduction to the principal case, namely that the validity of (3.16) in $\mathcal{A}(\tilde{\mathbf{X}}^\bullet, \tilde{B}^\bullet)$ implies that in $\mathcal{A}(\tilde{\mathbf{X}}, \tilde{B})$. Let $a \in \mathbb{Z}^n$. In view of Lemma 3.3, the coefficients $c_a^{a'}$ in the expansion of E'_a in the basis $\{E_{a'}\}$ of $\mathcal{A}(\tilde{\mathbf{X}}, \tilde{B})$ coincide with the corresponding coefficients in the expansion of $E'^{(2)}_{\psi'(a)}$ in $\mathcal{A}(\tilde{\mathbf{X}}^{(2)}, \tilde{B}^{(2)})$. Since $E'^{(2)}_{\psi'(a)} = E'^{\bullet}_a$, its expansion in $\mathcal{A}(\tilde{\mathbf{X}}^{(2)}, \tilde{B}^{(2)})$ coincides with the corresponding expansion in $\mathcal{A}(\tilde{\mathbf{X}}^\bullet, \tilde{B}^\bullet)$. Therefore, if one of the coefficients in the latter expansion is equal to 1, while the rest of the coefficients belong to $v\mathbb{Z}[v]$, then the same is true for the expansion of E'_a in $\mathcal{A}(\tilde{\mathbf{X}}, \tilde{B})$, as desired.

4 Proof of Theorem 1.6-II. Principal Quantization in Rank 2

In this and the next section, we complete the proof of Theorem 1.6 by showing that Theorem 3.1 and the condition (3.16) hold in the quantum cluster algebra $\mathcal{A}(\tilde{\mathbf{X}}^\bullet, \tilde{B}^\bullet)$ with principal coefficients. To make the argument more clear, we first explain it in the case $n = 2$, avoiding heavy notation.

We fix positive integers b and c, and let the initial extended exchange matrix $\tilde{B}$ be given by

$$\tilde{B} = \begin{pmatrix} 0 & -b \\ c & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.1}$$

(the signs of entries in the first two rows are chosen in accordance with (3.1)). Then we can take

$$d_1 = c, \quad d_2 = b, \tag{4.2}$$

and the matrix Λ given by (1.11) specializes to

$$\Lambda = \begin{pmatrix} 0 & 0 & -c & 0 \\ 0 & 0 & 0 & -b \\ c & 0 & 0 & bc \\ 0 & b & -bc & 0 \end{pmatrix}. \tag{4.3}$$

We denote the corresponding quantum cluster algebra by $\mathcal{A}^{\bullet}(b, c)$.

Relations (1.14) specialize to

$$X_1' = X^{(-1,c,1,0)} + X^{(-1,0,0,0)}, \quad X_2' = X^{(0,-1,0,1)} + X^{(b,-1,0,0)}. \tag{4.4}$$

Remembering the definition of E_a (cf. (1.17), (1.22), and (1.23)), it is easy to see that in our situation it is given by

$$E_a = v^{-ca_1a_3 - ba_2a_4 - bc[-a_2]_+a_3} X^{(0,0,a_3,a_4)} (X_1')^{[-a_1]_+} X_2^{[a_2]_+} X_1^{[a_1]_+} (X_2')^{[-a_2]_+} \tag{4.5}$$

for each $a = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$.

Turning to the elements E_a', we first note that the mutation $\mu_n = \mu_2$ transforms $\tilde{B}$ and Λ into the matrices

$$\tilde{B}' = \begin{pmatrix} 0 & b \\ -c & 0 \\ 1 & 0 \\ c & -1 \end{pmatrix}, \quad \Lambda' = \begin{pmatrix} 0 & 0 & -c & 0 \\ 0 & 0 & -bc & b \\ c & bc & 0 & bc \\ 0 & -b & -bc & 0 \end{pmatrix}. \tag{4.6}$$

Then (3.12) specializes to

$$X_1'' = X_1'(X_2')^c - \sum_{s=1}^{c} v^{bs^2} \begin{bmatrix} c \\ s \end{bmatrix}_{v^{2b}} X^{(bs-1,0,1,c-s)}. \tag{4.7}$$

A direct check shows that, for each $a = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$, we have

$$\begin{aligned} E_a' = {} & v^{bc([-a_1]_+[-a_2]_+ + [-a_1]_+a_4 - c[-a_1]_+a_3 - [a_2]_+a_3) - ca_1a_3 + ba_2a_4} \\ & \cdot X^{(0,0,a_3,a_4)} X_2^{[-a_2]_+} X_1^{[a_1]_+} (X_2')^{[a_2]_+} (X_1'')^{[-a_1]_+}. \end{aligned} \tag{4.8}$$

Proposition 4.1. Theorem 3.1 holds for $\mathcal{A}^\bullet(b,c)$, with the bijection $\varphi:\mathbb{Z}^4\to\mathbb{Z}^4$ given as follows:

$$\varphi(a_1,a_2,a_3,a_4)=(a_1,-c[-a_1]_+-a_2,a_3,a_4+\min(c[-a_1]_+,[-a_2]_+)). \tag{4.9}$$

□

The main idea of our proof of Proposition 4.1 is to include both bases $\{E_a\}$ and $\{E'_a\}$ into a larger family of "crystal monomials". Namely, let I denote the set of 7-tuples of integers $\mathbf{m}=(m_3,m_4,m'_1,m_2,m_1,m'_2,m''_1)$ such that the last five components are nonnegative, and let $I_0=\{\mathbf{m}\in I: m'_1m_1m''_1=0\}$. For $\mathbf{m}=(m_3,m_4,m'_1,m_2,m_1,m'_2,m''_1)\in I$, we define

$$M^\circ_{\mathbf{m}}=X^{(0,0,m_3,m_4)}(X'_1)^{m'_1}X_2^{m_2}X_1^{m_1}(X'_2)^{m'_2}(X''_1)^{m''_1}. \tag{4.10}$$

Then we define the *leading term* $LT(M^\circ_{\mathbf{m}})$ as an element of the ambient quantum torus $\mathcal{T}$ obtained from $M^\circ_{\mathbf{m}}$ by replacing the variable X'_1 with $X^{(-1,0,0,0)}$, the variable X'_2 with $X^{(0,-1,0,1)}$, and the variable X''_1 with $X^{(-1,0,1,c)}$. We set

$$M_{\mathbf{m}}=v^{\nu(\mathbf{m})}M^\circ_{\mathbf{m}}, \tag{4.11}$$

where the exponent $\nu(\mathbf{m})$ is determined from the following condition:

$$\text{the element } v^{\nu(\mathbf{m})-cm'_1m''_1}LT(M^\circ_{\mathbf{m}})\text{is invariant under the bar-involution.} \tag{4.12}$$

A direct calculation provides the following explicit expression for $\nu(\mathbf{m})$:

$$\begin{aligned}\nu(\mathbf{m})=c(m'_1-m_1-(bc-1)m''_1-bm'_2)m_3+b(cm''_1+m'_2-m_2)m_4\\ +cm_1m''_1+bm_2m'_2+bcm_2m''_1.\end{aligned} \tag{4.13}$$

Comparing this definition with the definitions of the elements E_a and E'_a, it is easy to check the following: for every $a=(a_1,a_2,a_3,a_4)\in\mathbb{Z}^4$, we have

$$E_a=M_{(a_3,a_4,[-a_1]_+,[a_2]_+,[a_1]_+,[-a_2]_+,0)},\qquad E'_a=M_{(a_3,a_4,0,[-a_2]_+,[a_1]_+,[a_2]_+,[-a_1]_+)}. \tag{4.14}$$

It follows that the elements E_a (resp. E'_a) are exactly the elements $M_{\mathbf{m}}$ with $m'_1m_1=m_2m'_2=m''_1=0$ (resp. $m'_1=m_2m'_2=m_1m''_1=0$); more precisely, we have

$$\begin{aligned}M_{(m_3,m_4,m'_1,m_2,m_1,m'_2,0)}&=E_{(m_1-m'_1,m_2-m'_2,m_3,m_4)}\quad(m'_1m_1=m_2m'_2=0),\\ M_{(m_3,m_4,0,m_2,m_1,m'_2,m''_1)}&=E'_{(m_1-m''_1,m'_2-m_2,m_3,m_4)}\quad(m_1m''_1=m_2m'_2=0).\end{aligned} \tag{4.15}$$

Let $\pi : I_0 \to \mathbb{Z}^4$ be the mapping given by

$$\pi(m_3, m_4, m_1', m_2, m_1, m_2', m_1'') = (m_1' - m_1 + m_1'', m_2 - m_2' - c(m_1'' - \underline{m}),$$
$$m_3 + \underline{m}, m_4 + \min(m_2 + c\underline{m}, m_2' + cm_1'')), \qquad (4.16)$$

where we abbreviate $\min(m_1, m_1'') = \underline{m}$. We deduce Proposition 4.1 from the following lemma.

Lemma 4.2.

(1) All elements $M_{\mathbf{m}}$ for $\mathbf{m} \in I$ belong to $\mathcal{A}_+$. More precisely, if $\mathbf{m} \in I_0$, then $M_{\mathbf{m}} \in \mathcal{A}_+ - v\mathcal{A}_+$, and if $\mathbf{m} \in I - I_0$, then $M_{\mathbf{m}} \in v\mathcal{A}_+$.

(2) For every $\mathbf{m} \in I_0$, we have $M_{\mathbf{m}} - E_{\pi(\mathbf{m})} \in v\mathcal{A}_+$. □

In view of the second equality in (4.14), to show that Part (2) of Lemma 4.2 implies Proposition 4.1 it is enough to prove that

$$\pi(a_3, a_4, 0, [-a_2]_+, [a_1]_+, [a_2]_+, [-a_1]_+) = \varphi(a)$$

for every $a = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$; but this is immediate from the definitions.

Proof of Lemma 4.2. First of all, note that (4.17) implies

$$X_1' X_1 = 1 + v^c X^{(0,0,1,0)} X_2^c, \quad X_2 X_2' = v^{-b} X^{(0,0,0,1)} + X_1^b. \qquad (4.17)$$

In a similar fashion, we obtain

$$X_1 X_1'' = v^{-c} X^{(0,0,1,c)} + (X_2')^c. \qquad (4.18)$$

In the calculation below, we will use the following easily verified property of the ordering $X_1', X_2, X_1, X_2', X_1''$ of cluster variables that appear in the definition (4.11): any two elements adjacent in this ordering commute with each other. Also all these variables quasi-commute with the elements of the form $X^{(0,0,m_2,m_4)}$, with the multiples of quasi-commutation governed by the matrix Λ in (4.3). Using these properties, it is a matter of a routine check to deduce from the equalities (4.17), (4.18), and (4.7) the following relations between the elements $M_{\mathbf{m}}$.

Let $\mathbf{m} = (m_3, m_4, m_1', m_2, m_1, m_2', m_1'') \in I$. Then we have

- if $m_1' m_1 > 0$, then

$$M_{\mathbf{m}} = v^{cm_1''} M_{(m_3, m_4, m_1'-1, m_2, m_1-1, m_2', m_1'')} + v^{c(m_1'+m_1-1)} M_{(m_3+1, m_4, m_1'-1, m_2+c, m_1-1, m_2', m_1'')}; \tag{4.19}$$

- if $m_2 m_2' > 0$, then

$$M_{\mathbf{m}} = M_{(m_3, m_4+1, m_1', m_2-1, m_1, m_2'-1, m_1'')} + v^{b(m_2+m_2'-1+cm_1'')} M_{(m_3, m_4, m_1', m_2-1, m_1+b, m_2'-1, m_1'')}; \tag{4.20}$$

- if $m_1 m_1'' > 0$, then

$$M_{\mathbf{m}} = v^{cm_1'} M_{(m_3+1, m_4+c, m_1', m_2, m_1-1, m_2', m_1''-1)} + v^{c(m_1+m_1''-1)} M_{(m_3, m_4, m_1', m_2, m_1-1, m_2'+c, m_1''-1)}; \tag{4.21}$$

- if $m_1 = 0$, $m_1'' > 0$, then

$$M_{\mathbf{m}} = M_{(m_3, m_4, m_1'+1, m_2, 0, m_2'+c, m_1''-1)} - \sum_{s=1}^{c} v^{cm_1'+bs(m_2+m_2'+s)} \begin{bmatrix} c \\ s \end{bmatrix}_{v^{2b}} M_{(m_3+1, m_4+c-s, m_1', m_2, bs-1, m_2', m_1''-1)}. \tag{4.22}$$

Now we are ready to prove Lemma 4.2. We start with Part (1).

Consider first the special case when $m_1'' = 0$. Then $\mathbf{m} \in I_0$, so we need to prove that $M_{\mathbf{m}} \in \mathcal{A}_+ - v\mathcal{A}_+$. If $m_1' m_1 = m_2 m_2' = 0$, then by the first equality in (4.15) $M_{\mathbf{m}}$ is one of the elements E_a, and there is nothing to prove. Thus, it suffices to treat the case when either $m_1' m_1 > 0$, or $m_2 m_2' > 0$. Then $M_{\mathbf{m}}$ satisfies (at least) one of the identities (4.19) and (4.20). In both identities the first term on the right appears with coefficient 1 (for (4.19) this is our assumption $m_1'' = 0$), while the coefficient of the second one is a positive power of v. Also both terms on the right have the value of $m_1' + m_2'$ smaller than that of $\mathbf{m}$. Thus, the desired inclusion $M_{\mathbf{m}} \in \mathcal{A}_+ - v\mathcal{A}_+$ follows by induction on $m_1' + m_2'$.

It remains to treat the case $m_1'' > 0$. Then $M_{\mathbf{m}}$ satisfies one of the identities (4.21) and (4.22). If $\mathbf{m} \in I - I_0$, that is if $m_1' m_1 > 0$, then $M_{\mathbf{m}}$ satisfies (4.21), where both coefficients on the right are positive powers of v. On the other hand, if $\mathbf{m} \in I_0$, then it can satisfy either of (4.21) and (4.22), and in each case the first term on the right appears

with coefficient 1, while all the coefficients in the second one are in $v\mathbb{Z}[v]$. Note also that all terms on the right in each of the identities (4.21) and (4.22) have the value of m''_1 smaller than that of $\mathbf{m}$. Thus, the claims in Part (1) follow by induction on m''_1.

To prove Part (2) suppose $\mathbf{m} \in I_0$. If $m'_1 m_1 = m_2 m'_2 = m''_1 = 0$, then $M_{\mathbf{m}}$ is given by the first equality in (4.15), and an easy inspection shows that $M_{\mathbf{m}} = E_{\pi(\mathbf{m})}$, proving the desired claim. Thus, we can assume that at least one of $m'_1 m_1$, $m_2 m'_2$, and m''_1 is nonzero. Then $M_{\mathbf{m}}$ satisfies at least one of the identities (4.19)–(4.22). In all of them the first term on the right appears with coefficient 1, and we refer to this term as $M_{\mathbf{m}^-}$. Furthermore, the rest of the terms on the right appear with coefficients in $v\mathbb{Z}[v]$. Thus, Part (1) implies that in each of the cases we have $M_{\mathbf{m}} - M_{\mathbf{m}^-} \in v\mathcal{A}_+$. To make the correspondence $\mathbf{m} \mapsto \mathbf{m}^-$ well defined, we take the first term in the first of the identities (4.21), (4.22), (4.20), and (4.19) (in this order!), that is applicable to $M_{\mathbf{m}}$. An easy inspection shows that $\pi(\mathbf{m}^-) = \pi(\mathbf{m})$, and that the iteration of the mapping $\mathbf{m} \mapsto \mathbf{m}^-$ terminates (i.e., ends up in the set $\{\mathbf{m} \in I_0 : m'_1 m_1 = m_2 m'_2 = m''_1 = 0\}$) after a finite number of steps for any initial $\mathbf{m} \in I_0$. This completes the proofs of Lemma 4.2 and Proposition 4.1. ■

5 Proof of Theorem 1.6-III. Principal Quantization in any Rank

In this section, we complete the proof of Theorem 1.6 by dealing with the principal acyclic quantum cluster algebra of an arbitrary rank n. Thus, the matrices $\tilde{B}$ and Λ are given by (1.10) and (1.11), and we assume that the matrix entries of the exchange matrix B satisfy (3.1).

The arguments below follow those in Section 4 with some necessary modifications. As before, we include both bases $\{E_a\}$ and $\{E'_a\}$ (where now a runs over $\mathbb{Z}^{2n}$) into a larger family of "crystal monomials". Namely, let I denote the set of $(4n-1)$-tuples of integers

$$\mathbf{m} = (m_{n+1}, \dots, m_{2n}, m'_1, \dots, m'_{n-1}, m_1, \dots, m_n, m'_n, m''_1, \dots, m''_{n-1})$$

such that the last $3n-1$ components are nonnegative, and

$$m'_i m''_j = 0 \quad \text{for } 1 \le j < i \le n-1. \tag{5.1}$$

For $\mathbf{m} \in I$, we define

$$\begin{aligned} M^\circ_{\mathbf{m}} = {} & X^{(0,\dots,0,m_{n+1},\dots,m_{2n})} (X'_1)^{m'_1} \cdots (X'_{n-1})^{m'_{n-1}} \\ & \times X^{(m_1,\dots,m_n,0,\dots,0)} (X'_n)^{m'_n} (X''_1)^{m''_1} \cdots (X''_{n-1})^{m''_{n-1}}. \end{aligned} \tag{5.2}$$

Note that the factors in $M^{\circ}_{\mathbf{m}}$ satisfy the following easily checked commutation properties:

(1) The elements $X_1, \dots, X_n$ commute with each other.
(2) For $i \in [1, n]$, the element X'_i commutes with X_j for $j \in [1, n]$, $j \neq i$.
(3) For $i \in [1, n-1]$, the element X''_i commutes with X'_n and with X_j for $j \in [1, n-1]$, $j \neq i$.

We will use these properties without further notice.

We define the *leading term* $LT(M^{\circ}_{\mathbf{m}})$ as an element of the ambient quantum torus $\mathcal{T}$ obtained from $M^{\circ}_{\mathbf{m}}$ by the following replacement (for $j \in [1, n-1]$):

$$X'_j \mapsto X^{-e_j - b_j^{<j}}, \quad X'_n \mapsto X^{-e_n + e_{2n}}, \quad X''_j \mapsto X^{-e_j + b_j^{>j} + b_{nj}(e_{2n} - e_n)}, \tag{5.3}$$

where as before, b_j stands for the jth column of $\tilde{B}$, and $b_j^{<j}$ and $b_j^{>j}$ are the corresponding truncated vectors. We abbreviate

$$\nu'(\mathbf{m}) = \sum_{j=1}^{n-1} d_j m'_j m''_j, \tag{5.4}$$

and we set

$$M_{\mathbf{m}} = v^{\nu(\mathbf{m})} M^{\circ}_{\mathbf{m}}, \tag{5.5}$$

where the exponent $\nu(\mathbf{m})$ is determined from the following condition:

$$\text{the element } v^{\nu(\mathbf{m}) - \nu'(\mathbf{m})} LT(M^{\circ}_{\mathbf{m}}) \text{ is invariant under the bar-involution.} \tag{5.6}$$

As in the case $n = 2$, it is easy to check that both triangular bases $\{E_a\}$ and $\{E'_a\}$ are contained in the family of elements $M_{\mathbf{m}}$. More precisely, the elements E_a are exactly the elements $M_{\mathbf{m}}$ with

$$m'_1 m_1 = m'_2 m_2 = \cdots = m'_n m_n = m''_1 = m''_2 = \cdots = m''_{n-1} = 0. \tag{5.7}$$

Namely, for every $\mathbf{m} \in I$ satisfying (5.7), we have $M_{\mathbf{m}} = E_a$, where the vector $a \in \mathbb{Z}^{2n}$ is related to $\mathbf{m}$ as follows:

$$\begin{aligned} &a_i = m_i \quad (i \in [n+1, 2n]), \quad a_j = m_j - m'_j \quad (j \in [1, n]), \\ &m_i = a_i \quad (i \in [n+1, 2n]), \quad m_j = [a_j]_+, \quad m'_j = [-a_j]_+ \quad (j \in [1, n]). \end{aligned} \tag{5.8}$$

Similarly, the elements E'_a are exactly the elements $M_{\mathbf{m}}$ with

$$m_1 m''_1 = m_2 m''_2 = \cdots = m_{n-1} m''_{n-1} = m'_n m_n = m'_1 = m'_2 = \cdots = m'_{n-1} = 0, \tag{5.9}$$

and for every $\mathbf{m} \in I$ satisfying (5.9), we have $M_{\mathbf{m}} = E'_a$, where the vector $a \in \mathbb{Z}^{2n}$ is related to $\mathbf{m}$ as follows:

$$\begin{aligned} &a_i = m_i \quad (i \in [n+1, 2n]), \quad a_j = m_j - m'_j \quad (j \in [1, n-1]), \quad a_n = m'_n - m_n, \\ &m_i = a_i \quad (i \in [n+1, 2n]), \quad m_j = [a_j]_+, \quad m'_j = [-a_j]_+ \quad (j \in [1, n-1]), \\ &m_n = [-a_n]_+, \quad m'_n = [a_n]_+. \end{aligned} \tag{5.10}$$

As in the previous section, to complete the proof of Theorem 1.6 it suffices to prove the following generalization of Lemma 4.2.

Proposition 5.1. Let $I_0 = \{\mathbf{m} \in I : m'_j, m_j, m''_j = 0 \text{ for } j \in [1, n-1]\}$.

(1) All elements $M_{\mathbf{m}}$ for $\mathbf{m} \in I$ belong to $\mathcal{A}_+$. More precisely, if $\mathbf{m} \in I_0$, then $M_{\mathbf{m}} \in \mathcal{A}_+ - v\mathcal{A}_+$, and if $\mathbf{m} \in I - I_0$, then $M_{\mathbf{m}} \in v\mathcal{A}_+$.

(2) There exists a mapping $\pi : I_0 \to \mathbb{Z}^{2n}$ such that, for every $\mathbf{m} \in I_0$, we have $M_{\mathbf{m}} - E_{\pi(\mathbf{m})} \in v\mathcal{A}_+$. □

The proof basically follows that of Lemma 4.2 but with some modifications. First, we introduce a subset $I_{00} \subset I_0$ by setting

$$I_{00} = \{\mathbf{m} \in I : m''_j = 0 \text{ for } j \in [1, n-1]\}.$$

Let $\tilde{\mathcal{A}}_+$ denote the $\mathbb{Z}[v]$-submodule of $\mathcal{A}$ generated by the $M_{\mathbf{m}}$ for $\mathbf{m} \in I_{00}$. Clearly, Proposition 5.1 is a consequence of the following two lemmas.

Lemma 5.2.

(1) All elements $M_{\mathbf{m}}$ for $\mathbf{m} \in I - I_{00}$ belong to $\tilde{\mathcal{A}}_+$.

(2) If $\mathbf{m} \in I - I_0$, then $M_{\mathbf{m}} \in v\tilde{\mathcal{A}}_+$.

(3) There exists a mapping $\pi_0 : I_0 \to I_{00}$ such that, for every $\mathbf{m} \in I_0$, we have $M_{\mathbf{m}} - M_{\pi_0(\mathbf{m})} \in v\tilde{\mathcal{A}}_+$. □

Lemma 5.3. There exists a mapping $\pi_{00} : I_{00} \to \mathbb{Z}^{2n}$ such that, for every $\mathbf{m} \in I_{00}$, we have $M_{\mathbf{m}} - E_{\pi_{00}(\mathbf{m})} \in v\mathcal{A}_+$. In particular, we have $\tilde{\mathcal{A}}_+ = \mathcal{A}_+$. □

Proceeding by induction on $\sum_{j=1}^{n-1} m''_j$, we see that Lemma 5.2 is a consequence of the following two identities.

Lemma 5.4. Suppose $\mathbf{m} \in I$ is such that $\sum_{j=1}^{n-1} m''_j > 0$, and let $j \in [1, n-1]$ be the smallest index such that $m''_j > 0$.

(1) If $m_j > 0$, then we have

$$M_{\mathbf{m}} = v^{d_j m'_j} M_{\mathbf{m}^+} + v^{d_j(m_j + m''_j - 1)} M_{\mathbf{m}^-}, \tag{5.11}$$

where $\mathbf{m}^+$ is obtained from $\mathbf{m}$ by the replacement

$$m_j \mapsto m_j - 1, \quad m''_j \mapsto m''_j - 1, \quad m_k \mapsto m_k + b_{kj} \quad (j < k \le n-1),$$
$$m_{n+j} \mapsto m_{n+j} + 1, \quad m_{2n} \mapsto m_{2n} + b_{nj},$$

while $\mathbf{m}^-$ is obtained from $\mathbf{m}$ by the replacement

$$m_j \mapsto m_j - 1, \quad m''_j \mapsto m''_j - 1, \quad m_i \mapsto m_i - b_{ij} \quad (1 \le i < j), \quad m'_n \mapsto m'_n + b_{nj}.$$

(2) If $m_j = 0$, then we have

$$M_{\mathbf{m}} = M_{\mathbf{m}^+} - \sum_{s=1}^{b_{nj}} v^{d_j m'_j + d_n s(m_n + m'_n + s)} \begin{bmatrix} b_{nj} \\ s \end{bmatrix}_{v^{2d_n}} M_{\mathbf{m}^-(s)}, \tag{5.12}$$

where $\mathbf{m}^+$ is obtained from $\mathbf{m}$ by the replacement

$$m''_j \mapsto m''_j - 1, \quad m'_j \mapsto m'_j + 1, \quad m'_n \mapsto m'_n + b_{nj},$$

and $\mathbf{m}^-(s)$ is obtained from $\mathbf{m}$ by the replacement

$$m''_j \mapsto m''_j - 1, \quad m_j \mapsto m_j - sb_{jn} - 1, \quad m_i \mapsto m_i - sb_{in} \quad (1 \le i < j),$$
$$m_k \mapsto m_k + b_{kj} - sb_{kn} \quad (j < k \le n-1), \quad m_{n+j} \mapsto m_{n+j} + 1,$$
$$m_{2n} \mapsto m_{2n} + b_{nj} - s.$$

□

Proof of Lemma 5.4. To prove (5.11), we use the identity

$$X_j X''_j = v^{-d_j} X^{b_j^{>j}+b_{nj}(e_{2n}-e_n)} + X^{-b_j^{<j}}(X'_n)^{b_{nj}} \quad (1 \le j < n), \tag{5.13}$$

which is an easy consequence of (3.7). Recalling the definition of $M^\circ_{\mathbf{m}}$ in (5.2), and the commutation properties collected after it, we can express the factor $X^{(m_1,\dots,m_n,0,\dots,0)}$ in $M^\circ_{\mathbf{m}}$ as $X^{(m_1,\dots,m_n,0,\dots,0)-m_j e_j} X_j^{m_j}$, and then interchange the term $(X''_j)^{m''_j}$ with the commuting term $(X'_n)^{m'_n}$, so that it will stand right after $X_j^{m_j}$. Using (5.13), we replace $X_j^{m_j}(X''_j)^{m''_j}$ with the sum of two terms: $v^{-d_j} X_j^{m_j-1} X^{b_j^{>j}+b_{nj}e_{2n}}(X''_j)^{m''_j-1}$, and $X_j^{m_j-1} X^{-b_j^{<j}}(X'_n)^{b_{nj}}(X''_j)^{m''_j-1}$. Again using the commutation relations we conclude that

$$M^\circ_{\mathbf{m}} = v^{-d_j} M^\circ_{\mathbf{m}^+} + M^\circ_{\mathbf{m}^-}.$$

In view of (5.5), this implies the following:

$$M_{\mathbf{m}} = v^{\nu(\mathbf{m})-\nu(\mathbf{m}^+)-d_j} M_{\mathbf{m}^+} + v^{\nu(\mathbf{m})-\nu(\mathbf{m}^-)} M_{\mathbf{m}^-}. \tag{5.14}$$

Here the exponents of v are determined from the conditions that each of the elements $v^{\nu(\mathbf{m})-\nu'(\mathbf{m})} LT(M^\circ_{\mathbf{m}})$, $v^{\nu(\mathbf{m}^+)-\nu'(\mathbf{m}^+)} LT(M^\circ_{\mathbf{m}^+})$, and $v^{\nu(\mathbf{m}^-)-\nu'(\mathbf{m}^-)} LT(M^\circ_{\mathbf{m}^-})$ is invariant under the bar-involution (see (5.6)); recall that the function $\nu'(\mathbf{m})$ is defined by (5.4).

Remembering the definition (5.3) of the leading term, we see that $LT(M^\circ_{\mathbf{m}^+})$ is obtained from $LT(M^\circ_{\mathbf{m}})$ by replacing $LT(X_j X''_j) = X_j X^{-e_j+b_j^{>j}+b_{nj}(e_{2n}-e_n)}$ with $X^{b_j^{>j}+b_{nj}(e_{2n}-e_n)}$ in the appropriate place in the product expansion of $LT(M^\circ_{\mathbf{m}})$. Since

$$X_j X^{-e_j+b_j^{>j}+b_{nj}(e_{2n}-e_n)} = v^{-d_j} X^{b_j^{>j}+b_{nj}(e_{2n}-e_n)}$$

(see (1.7) and (1.11)), we see that $LT(M^\circ_{\mathbf{m}^+}) = v^{d_j} LT(M^\circ_{\mathbf{m}})$. This implies that $\nu(\mathbf{m}) - \nu'(\mathbf{m}) = d_j + \nu(\mathbf{m}^+) - \nu'(\mathbf{m}^+)$. Therefore, we have

$$\nu(\mathbf{m}) - \nu(\mathbf{m}^+) - d_j = \nu'(\mathbf{m}) - \nu'(\mathbf{m}^+) = d_j m'_j m''_j - d_j m'_j(m''_j - 1) = d_j m'_j$$

showing that $M_{\mathbf{m}^+}$ appears in the right-hand side of (5.14) with the same coefficient $v^{d_j m'_j}$ as in the right-hand side of (5.11).

To prove the similar statement for $M_{\mathbf{m}^-}$, we note that the elements $LT(M^\circ_{\mathbf{m}})$ and $LT(M^\circ_{\mathbf{m}^-})$ can be factored as follows:

$$LT(M^\circ_{\mathbf{m}}) = v^{\gamma-d_j} X^f X^{b_j^{>j}+b_{nj}(e_{2n}-e_n)} X^g, \quad LT(M^\circ_{\mathbf{m}^-}) = v^{\gamma} X^f X^{-b_j^{<j}+b_{nj}(e_{2n}-e_n)} X^g,$$

where $\gamma \in \mathbb{Z}$, and $f, g \in \mathbb{Z}^{2n}$ are integer vectors such that e_j appears in f with the coefficient $m_j - m'_j - 1$ and appears in g with the coefficient $-(m''_j - 1)$ (the statement about the coefficient of e_j in f uses the fact that, according to (5.1), we have $m'_i = 0$ for $1 \le j < i \le n-1$). Using (1.7) and the compatibility condition (1.6), we conclude that $\nu(\mathbf{m}) - \nu'(\mathbf{m}) = d_j + \nu(\mathbf{m}^-) - \nu'(\mathbf{m}^-) + d_j(m_j - m'_j + m''_j - 2)$. It follows that

$$\nu(\mathbf{m}) - \nu(\mathbf{m}^-) = \nu'(\mathbf{m}) - \nu'(\mathbf{m}^-) + d_j(m_j - m'_j + m''_j - 1) = d_j(m_j + m''_j - 1).$$

Thus, the desired identity (5.11) becomes a consequence of (5.14), completing the proof of Part 1 of Lemma 5.4.

The identity (5.12) is proved in a similar way, and we leave the details to the reader (our starting point is the identity

$$X''_j = X'_j (X'_n)^{b_{nj}} - \sum_{s=1}^{b_{nj}} v^{s^2 d_n} \begin{bmatrix} b_{nj} \\ s \end{bmatrix}_{v^{2d_n}} X^{-e_j + b_j^{>j} + b_{nj}(e_{2n} - e_n) - s b_n}, \tag{5.15}$$

which is a specialization of (3.12)). This completes the proof of Lemma 5.4. ■

Now we turn to the proof of Lemma 5.3. We view the index set I_{00} as the set of $3n$-tuples of integers

$$\mathbf{m} = (m_{n+1}, \dots, m_{2n}, m'_1, \dots, m'_n, m_1, \dots, m_n)$$

such that the last $2n$ components are nonnegative. Once again we include the family of monomials $\{M_{\mathbf{m}} : \mathbf{m} \in I_{00}\}$ into a larger family. Namely, we define

$$\hat{I}_{00} = I_{00} \times [1, n-1],$$

and for each $(\mathbf{m}, j) \in \hat{I}_{00}$, define an element $M^{\circ}_{\mathbf{m};j}$ by setting

$$M^{\circ}_{\mathbf{m};j} = X^{(0,\dots,0,m_{n+1},\dots,m_{2n})} (X'_1)^{m'_1} \cdots (X'_j)^{m'_j} \times X^{(m_1,\dots,m_n,0,\dots,0)} (X'_{j+1})^{m'_{j+1}} \cdots (X'_n)^{m'_n}. \tag{5.16}$$

For $(\mathbf{m}, j) \in \hat{I}_{00}$, we define the *leading term* $LT_j(M^{\circ}_{\mathbf{m};j})$ as an element of the quantum torus $\mathcal{T}$ obtained from $M^{\circ}_{\mathbf{m};j}$ by the following replacement:

$$X'_i \mapsto X^{-e_i - b_i^{<i}} \quad (1 \le i \le j), \qquad X'_k \mapsto X^{-e_k + b_k^{>k}} \quad (j+1 \le k \le n). \tag{5.17}$$

Finally, we set

$$M_{\mathbf{m};j} = v^{\nu(\mathbf{m};j)} M^{\circ}_{\mathbf{m};j}, \tag{5.18}$$

where the exponent $\nu(\mathbf{m}; j)$ is determined from the following condition:

$$\text{the element } v^{\nu(\mathbf{m};j)} LT_j(M^{\circ}_{\mathbf{m};j}) \text{ is invariant under the bar-involution.} \tag{5.19}$$

Comparing this definition with (5.2) and (5.6), we see that $M^{\circ}_{\mathbf{m}} = M^{\circ}_{\mathbf{m};n-1}$ and $M_{\mathbf{m}} = M_{\mathbf{m};n-1}$ for every $\mathbf{m} \in I_{00}$. Thus, to prove Lemma 5.3, it suffices to show the following:

$$\text{For every } (\mathbf{m}, j) \in \hat{I}_{00} \text{ there exists } a \in \mathbb{Z}^{2n} \text{ such that } M_{\mathbf{m};j} - E_a \in v\mathcal{A}_+. \tag{5.20}$$

We deduce (5.20) from the following identities.

Lemma 5.5. Suppose $\mathbf{m} \in I_{00}$ is such that $m'_j m_j > 0$ for some $j \in [1, n]$. For $\mathbf{m} \in I_{00}$, let $\mathbf{m}[< j] \in I_{00}$ denote an element obtained from $\mathbf{m}$ by the replacement

$$m_j \mapsto m_j - 1, \quad m'_j \mapsto m'_j - 1, \quad m_i \mapsto m_i - b_{ij} \quad (1 \le i < j),$$

and $\mathbf{m}[> j] \in I_{00}$ denote an element obtained from $\mathbf{m}$ by the replacement

$$m_j \mapsto m_j - 1, \quad m'_j \mapsto m'_j - 1, \quad m_k \mapsto m_k + b_{kj} \quad (j < k \le n), \quad m_{n+j} \mapsto m_{n+j} + 1.$$

(1) If $j < n$, then we have

$$M_{\mathbf{m};j} = M_{\mathbf{m}[<j];j} + v^{d_j(m_j + m'_j - 1)} M_{\mathbf{m}[>j];j}. \tag{5.21}$$

(2) If $1 < j$, then we have

$$M_{\mathbf{m};j-1} = M_{\mathbf{m}[>j];j-1} + v^{d_j(m_j + m'_j - 1)} M_{\mathbf{m}[<j];j-1}. \tag{5.22}$$

□

The identities (5.21) and (5.22) are proved in the same way as (5.11), with the role of (5.13) played by the identities

$$X'_j X_j = X^{-b_j^{<j}} + v^{d_j} X^{b_j^{>j}} \quad (1 \le j < n) \tag{5.23}$$

and

$$X_j X'_j = v^{-d_j} X^{b_j^{>j}} + X^{-b_j^{<j}} \quad (1 < j \le n). \tag{5.24}$$

We leave the details to the reader.

To deduce (5.20) from Lemma 5.5, we first make an easy observation:

$$\text{If } (\mathbf{m}, j) \in \hat{I}_{00} \text{ is such that } 1 < j, \text{ and } m'_j m_j = 0, \text{ then } M_{\mathbf{m};j} = M_{\mathbf{m};j-1}. \tag{5.25}$$

Now let $(\mathbf{m}, k) \in \hat{I}_{00}$ be such that $m'_i m_i > 0$ for some $i \in [1, n]$. Using (5.25) if necessary, we can find $j \in [1, n]$ such that $m'_j m_j > 0$, and $M_{\mathbf{m};k}$ is equal either to $M_{\mathbf{m};j}$ or to $M_{\mathbf{m};j-1}$, so that it satisfies one of the identities (5.21) and (5.22). The fact that $M_{\mathbf{m};k}$ satisfies the desired property (5.20), then follows by induction on $r(\mathbf{m}) = \sum_{i=1}^n m'_i$ (since we have $r(\mathbf{m}[< j]) = r(\mathbf{m}[> j]) = r(\mathbf{m}) - 1$).

This concludes the proof of (5.20) and hence that of Lemma 5.3, Proposition 5.1, and Theorem 1.6.

6 Example: Coefficient-Free Type $A_1^{(1)}$

In this section, we illustrate the above results by computing the canonical triangular basis in the quantum cluster algebra $\mathcal{A}$ with the initial quantum seed given as follows:

(1) $m = n = 2$, $d_1 = d_2 = 2$.
(2) $\tilde{B} = B = \left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$, $\Lambda = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.
(3) $\tilde{\mathbf{X}} = \mathbf{X} = (X_1, X_2)$.

This cluster algebra and various bases in it were studied in detail in [4, 10]. In what follows, we use the results in [4]. Since our choice of $\tilde{B}$ and Λ differs from that of [4] by the sign, to reconcile our setups their formal variable q and our formal variable v are related by $q = v^{-2}$.

As in [4], the set of cluster variables in $\mathcal{A}$ is denoted by $\{X_m : m \in \mathbb{Z}\}$, with the clusters $\{X_m, X_{m+1}\}$ for $m \in \mathbb{Z}$. The commutation relations in $\mathcal{A}$ are

$$X_{m+1} X_m = v^2 X_m X_{m+1} \quad (m \in \mathbb{Z}), \tag{6.1}$$

the exchange relations are

$$X_{m+1} X_{m-1} = v^2 X_m^2 + 1 \quad (m \in \mathbb{Z}), \tag{6.2}$$

and the (bar-involution invariant) cluster monomials are the elements

$$v^{a_1 a_2} X_m^{a_1} X_{m+1}^{a_2} \quad (a_1, a_2 \in \mathbb{Z}_{\geq 0},\ m \in \mathbb{Z}). \tag{6.3}$$

Following [4], we denote by X_δ the element of $\mathcal{A}$ given by

$$X_\delta = v X_3 X_0 - v^3 X_2 X_1. \tag{6.4}$$

Let $S_{-1}(z), S_0(z), S_1(z), \ldots$ be the sequence of (normalized) *Chebyshev polynomials of second kind* given by the initial conditions

$$S_{-1}(z) = 0, \quad S_0(z) = 1, \tag{6.5}$$

and the recurrence relation

$$S_r(z) = z S_{r-1}(z) - S_{r-2}(z) \quad (r \geq 1). \tag{6.6}$$

Thus, we have $S_1(z) = z$, $S_2(z) = z^2 - 1$, etc.

Proposition 6.1. The canonical triangular basis in $\mathcal{A}$ consists of all cluster monomials given by (6.3), and the elements $S_r(X_\delta)$ for all $r \geq 1$. □

Remark 6.2. As shown in [4], the cluster variables and the elements $S_r(X_\delta)$, for all $r \geq 1$, are precisely the quantum Caldero–Chapoton characters associated with indecomposable representations of the Kronecker quiver (see [14]). In this case the canonical triangular basis coincides with the natural quantum version of the dual semicanonical basis introduced for the commutative setting in [3]; this version was also discovered and studied in [10]. □

Proof of Proposition 6.1. The fact that all cluster monomials belong to the canonical triangular basis $\mathbf{B}$ in $\mathcal{A}$, follows from Corollary 1.7 (in our situation, *all* the seeds are acyclic). They are labeled by the lattice $\mathbb{Z}^2$, and our first task is to describe this labeling explicitly.

An easy calculation shows that the elements of the "standard" basis $\{E_a : a \in \mathbb{Z}^2\}$ can be expressed as follows:

$$E_a = v^{a_1 a_2} X_3^{[-a_1]_+} X_1^{[a_1]_+} X_2^{[a_2]_+} X_0^{[-a_2]_+}. \tag{6.7}$$

Because of the translational symmetry of the relations (6.1) and (6.2), there are well-defined mutually inverse algebra automorphisms η_+ and η_- of $\mathcal{A}$ acting on the generators (i.e., the cluster variables) by

$$\eta_+(X_m) = X_{m+1}, \quad \eta_-(X_m) = X_{m-1} \quad (m \in \mathbb{Z}). \tag{6.8}$$

Comparing (6.7) with (4.8) (and ignoring frozen variables), we see that $\eta_-(E_{(a_1,a_2)}) = E'_{(a_2,a_1)}$. Thus, Proposition 4.1 implies that

$$\eta_-(E_{(a_1,a_2)}) - E_{(a_2,-2[-a_2]_+-a_1)} \in v\mathcal{A}_+. \tag{6.9}$$

Since $\eta_+ = \eta_-^{-1}$, we also have

$$\eta_+(E_{(a_1,a_2)}) - E_{(-2[-a_1]_+-a_2,a_1)} \in v\mathcal{A}_+. \tag{6.10}$$

As a consequence, the elements C_a of $\mathbf{B}$ satisfy

$$\eta_-(C_{(a_1,a_2)}) = C_{(a_2,-2[-a_2]_+-a_1)}, \quad \eta_+(C_{(a_1,a_2)}) = C_{(-2[-a_1]_+-a_2,a_1)}. \tag{6.11}$$

Iterating (6.11), we conclude that the cluster monomials in (6.3) are labeled as follows:

$$v^{a_1a_2} X_m^{a_1} X_{m+1}^{a_2} = C_{a_1\alpha(m)+a_2\alpha(m+1)} \quad (a_1, a_2 \in \mathbb{Z}_{\geq 0},\ m \in \mathbb{Z}), \tag{6.12}$$

where the vectors $\alpha(m) \in \mathbb{Z}^2$ are given by

$$\alpha(1-r) = (1-r, -r), \quad \alpha(2+r) = (-r, 1-r) \quad (r \geq 0). \tag{6.13}$$

As an easy consequence of (6.12) and (6.13), we conclude that the cluster monomials are all the elements C_a for $a \in \mathbb{Z}^2 - \{(-r,-r) : r \geq 1\}$. To complete the proof of Proposition 6.1, it remains to show that

$$C_{(-r,-r)} = S_r(X_\delta) \quad (r \geq 1). \tag{6.14}$$

We use the following properties of the elements $S_r(X_\delta)$ (all of them are established in [4]).

Lemma 6.3.

(1) The elements $S_r(X_\delta)$ are invariant under the bar-involution, and under each of the automorphisms η_+ and η_-.

(2) For each $r \geq -1$ we have

$$S_r(X_\delta) = v^r X_{r+2} X_0 - v^{r+2} X_{r+1} X_1. \tag{6.15}$$

□

Recall the notation $\mathcal{A}_+$ for the $\mathbb{Z}[v]$-linear span of the basis $\{E_a : a \in \mathbb{Z}^2\}$. In this notation, the condition (1.4) takes the form $C_a - E_a \in v\mathcal{A}_+$. Recall that we also have a stronger condition (1.5). We have defined the partial order $\prec$ in (1.21). However, in our current situation it is possible to replace this partial order by the following sharper one:

$$a' = (a'_1, a'_2) \prec a = (a_1, a_2) \Longleftrightarrow [-a'_1]_+ < [-a_1]_+, \quad [-a'_2]_+ < [-a_2]_+; \tag{6.16}$$

indeed a direct check shows that the condition (1.2) holds for this sharper partial order.

The last ingredient we need to prove (6.14) is the following lemma.

Lemma 6.4. For every $a = (a_1, a_2) \in \mathbb{Z}^2$, we have

$$v^{-a_1} E_a X_0 - E_{(a_1, a_2 - 1)} \in v\mathcal{A}_+. \tag{6.17}$$

□

Proof. We prove (6.17) by a direct calculation of the left-hand side using (6.7), (6.1), and (6.2). There are four cases to consider. In each case, we just give the result of a calculation leaving the details to the reader.

Case 1: $a_2 \leq 0$. Then we have $v^{-a_1} E_a X_0 - E_{(a_1, a_2 - 1)} = 0$.

Case 2: $a_2 > 0$, $a_1 \geq 0$. Then $v^{-a_1} E_a X_0 - E_{(a_1, a_2 - 1)} = v^{2a_2} E_{(a_1 + 2, a_2 - 1)}$.

Case 3: $a_2 > 0$, $a_1 = -1$. Then

$$v^{-a_1} E_a X_0 - E_{(a_1, a_2 - 1)} = v^{2a_2} E_{(1, a_2 - 1)} + v^{2(a_2 + 2)} E_{(1, a_2 + 1)}.$$

Case 4: $a_2 > 0$, $a_1 \leq -2$. Then

$$\begin{aligned} v^{-a_1} E_a X_0 - E_{(a_1, a_2 - 1)} = {} & v^{2a_2} E_{(a_1 + 2, a_2 - 1)} + (v^{2(a_2 - a_1 - 1)} + v^{2(a_2 - a_1 + 1)}) E_{(a_1 + 2, a_2 + 1)} \\ & + v^{2(a_2 - 2a_1)} E_{(a_1 + 2, a_2 + 3)}. \end{aligned}$$

Since in all the cases the right-hand side belongs to $v\mathcal{A}_+$, we are done. ■

Now everything is ready for the proof of (6.14). For $r = 1$, we have

$$S_r(X_\delta) = X_\delta = vX_3X_0 - v^3X_2X_1 = E_{(-1,-1)} - v^4E_{(1,1)} \in E_{(-1,-1)} + v\mathcal{A}_+;$$

since X_δ is also invariant under the bar-involution (see Lemma 6.3), it follows that $X_\delta = C_{-1,-1}$. Thus, we assume that $r \geq 2$.

As a special case of (6.12), we have $X_{r+2} = C_{(-r,1-r)}$. Applying (1.5) with the partial order $\prec$ given by (6.16), we have

$$X_{r+2} = E_{(-r,1-r)} + \sum_{[-a_1]_+<r,[-a_2]_+<r-1} c_{a_1,a_2}E_{(a_1,a_2)},$$

where $c_{a_1,a_2} \in vZZ[v]$ for all (a_1, a_2). Multiplying both sides on the left with v^r and on the right with X_0, and using (6.17), we obtain

$$v^rX_{r+2}X_0 \in E_{(-r,-r)} + \sum_{[-a_1]_+<r,[-a_2]_+<r-1} v^{r+a_1}c_{a_1,a_2}E_{(a_1,a_2-1)} + v\mathcal{A}_+ \subseteq E_{(-r,-r)} + v\mathcal{A}_+$$

(the last inclusion follows since $r + a_1 > 0$ for $[-a_1]_+ < r$).

By the same token, we have $v^{r-2}X_rX_0 \in E_{(2-r,2-r)} + v\mathcal{A}_+$, implying that

$$v^{r+2}X_{r+1}X_1 = v^4\eta_+(v^{r-2}X_rX_0) \in v^4E_{(2-r,2-r)} + v\mathcal{A}_+ = v\mathcal{A}_+.$$

In view of (6.15), we conclude that

$$S_r(X_\delta) = v^rX_{r+2}X_0 - v^{r+2}X_{r+1}X_1 \in E_{(-r,-r)} + v\mathcal{A}_+.$$

Since $S_r(X_\delta)$ is also invariant under the bar-involution (see Lemma 6.3), it follows that $S_r(X_\delta) = C_{-r,-r}$, completing the proofs of (6.14) and Proposition 6.1. ■

Remark 6.5. A big part of the above proof carries over without difficulty to the general rank 2 case, where $\tilde{B} = B = \left(\begin{smallmatrix} 0 & -b \\ c & 0 \end{smallmatrix}\right)$, with arbitrary positive integers b and c. However, the problem of finding explicit expressions for all the elements C_a is still open in this generality. It definitely deserves a further study. □

7 Proof of Lusztig's Lemma

In this section, we prove Theorem 1.1.

Proof. According to (1.2), we have

$$\bar{E}_a = E_a + \sum_{a' \in L} r_{a,a'} E_{a'}, \tag{7.1}$$

where $r_{a,a'} \in \mathbb{Z}[v, v^{-1}]$, and $r_{a,a'} = 0$ unless $a' \prec a$. Expanding the equality $\bar{\bar{E}}_a = E_a$, we see that the condition that $x \mapsto \bar{x}$ is an involution is equivalent to the following: for all $a, a' \in L$, we have

$$r_{a,a'} + \bar{r}_{a,a'} + \sum_{a''} \bar{r}_{a,a''} r_{a'',a'} = 0. \tag{7.2}$$

Applying the bar-involution on both sides of (7.2), we also obtain

$$r_{a,a'} + \bar{r}_{a,a'} + \sum_{a''} r_{a,a''} \bar{r}_{a'',a'} = 0. \tag{7.3}$$

According to (1.4), the desired element C_a must have the form

$$C_a = E_a + \sum_{a' \in L} p_{a,a'} E_{a'}, \tag{7.4}$$

where only finitely many of the coefficients $p_{a,a'}$ are nonzero, and $p_{a,a'} \in v\mathbb{Z}[v]$. Expanding both sides of the equality (1.3) in the basis $\{E_{a'}\}$, we rewrite it as the system of equations

$$p_{a,a'} - \bar{p}_{a,a'} = r_{a,a'} + \sum_{a''} \bar{p}_{a,a''} r_{a'',a'}. \tag{7.5}$$

Let us first show that equation (7.5) implies that $p_{a,a'} = 0$ unless $a' \prec a$, hence the desired element C_a must satisfy (1.5). Assume for the sake of contradiction that $p_{a,a'} \neq 0$ for some $a' \not\prec a$, and choose a' as some maximal element with this property (this is possible since the sum in (7.4) is finite). For this choice of a', the right-hand side of (7.5) becomes 0. But since $p_{a,a'} \in v\mathbb{Z}[v]$, the condition $p_{a,a'} - \bar{p}_{a,a'} = 0$ implies that $p_{a,a'} = 0$, the desired contradiction.

To show the uniqueness of $p_{a,a'}$ for $a' \prec a$, we proceed by induction on the maximal length of a chain between a' and a. Thus, we can assume that the statement is already known for all $p_{a,a''}$ appearing in the right-hand side of (7.5) since we must have $a' \prec a'' \prec a$ for the corresponding term to be nonzero. Now the uniqueness of $p_{a,a'}$ is obvious for the same reason as above: any $p \in v\mathbb{Z}[v]$ is uniquely determined by $p - \bar{p}$.

To show the existence, we use the following obvious property: a Laurent polynomial $f \in \mathbb{Z}[v, v^{-1}]$ can be written (uniquely) as $p - \bar{p}$ for $p \in v\mathbb{Z}[v]$ if and only if $f + \bar{f} = 0$.

Indeed, we have $p = [f]_+$, where $[f]_+$ stands for the part of the Laurent expansion of f that contains the positive powers of v. Thus, to check that (7.5) has a (unique) solution for $p_{a,a'}$, it suffices to show that the right-hand side f of (7.5) satisfies $f + \bar{f} = 0$. This is done by the following calculation using (7.3) and our inductive assumption:

$$\begin{aligned} f + \bar{f} &= r_{a,a'} + \bar{r}_{a,a'} + \sum_{a''} (\bar{p}_{a,a''} r_{a'',a'} + p_{a,a''} \bar{r}_{a'',a'}) \\ &= \sum_{a''} \left(-r_{a,a''} \bar{r}_{a'',a'} + \bar{p}_{a,a''} r_{a'',a'} + \left(\bar{p}_{a,a''} + r_{a,a''} + \sum_{a'''} \bar{p}_{a,a'''} r_{a''',a''} \right) \bar{r}_{a'',a'} \right) \\ &= \sum_{a''} \bar{p}_{a,a''} \left(r_{a'',a'} + \bar{r}_{a'',a'} + \sum_{a'''} r_{a'',a'''} \bar{r}_{a''',a'} \right) = 0, \end{aligned}$$

as desired.

To complete the proof, it remains to show that our finiteness assumption on the indexing poset L implies that, for a given $a \in L$, only finitely many of the coefficients $p_{a,a'}$ are nonzero. For $a' \preceq a$, let $c(a')$ denote the maximal length m of a chain $a' = a_0 \prec a_1 \prec \cdots \prec a_m = a$ in L. Let $P_m(a)$ denote the set of elements a' such that $c(a') = m$ and $p_{a,a'} \neq 0$. Since by our assumption, the function $c(a')$ is bounded from above, it is enough to show that each $P_m(a)$ is finite. Let $R(a)$ denote the (finite!) set consisting of the elements $a' \in L$ such that $r_{a,a'} \neq 0$. In view of (7.5), we have

$$P_m(a) \subseteq R(a) \cup \bigcup_{k=1}^{m-1} \bigcup_{a'' \in P_k(a)} R(a''),$$

hence the desired finiteness of $P_m(a)$ follows by induction on m. ■

Remark 7.1. Using the notation $[f]_+$ introduced in the course of the proof of Theorem 1.1, we can express the coefficients $p(a, a')$ recursively by

$$p_{a,a'} = \left[r_{a,a'} + \sum_{a''} \bar{p}_{a,a''} r_{a'',a'} \right]_+ . \tag{7.6}$$

□

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