DOUBLE CANONICAL BASES

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ABSTRACT. We introduce a new class of bases for quantized universal enveloping algebras $U_q(g)$ and other doubles attached to semisimple and Kac-Moody Lie algebras. These bases contain dual canonical bases of upper and lower halves of $U_q(g)$ and are invariant under many symmetries including all Lusztig's symmetries if $g$ is semisimple. It also turns out that a part of a double canonical basis of $U_q(g)$ spans its center.

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The goal of this paper is to construct a canonical basis $B_g$ of a quantized enveloping algebra $U_q(g)$ where $g$ is a semisimple or a Kac-Moody Lie algebra. For instance, if $g = sl_2$, then $B_g$ is given by

$$B_{sl_2} = \{ q^{n(-m_+)}K^nC^{(m)}F^{-m_+} | n \in \mathbb{Z}, m_0, m_\pm \in \mathbb{Z}_{\geq 0}, \min(m_-, m_+) = 0 \} ,$$

(1.1)

where we used a slightly non-standard presentation of $U_q(sl_2)$ (obtained from the more familiar one by rescaling generators $E \mapsto (q^{-1} - q)E$, $F \mapsto (q - q^{-1})F$)

$$U_q(sl_2) := \langle E, F, K^{\pm 1} : K E K^{-1} = q^2 E, K F K^{-1} = q^2 F, E F - F E = (q^{-1} - q)(K - K^{-1}) \rangle .$$

Here the $C^{(m)}$ are central elements of $U_q(sl_2)$ defined by $C^{(0)} = 1$, $C = C^{(1)} = EF - q^{-1}K - qK^{-1} = F E - qK - q^{-1}K^{-1}$ and $C \cdot C^{(m)} = C^{(m+1)} + C^{(m-1)}$ for $m \geq 1$. We establish properties of $B_{sl_2}$ double canonical because of the following remarkable properties (we will explain later, in §4.1 the reason why we must use Chebyshev polynomials $C^{(m)}$ instead of $C^m$).

1. Each element of $B_{sl_2}$ is homogeneous and is fixed by the bar-involution $u \mapsto \overline{u}$, which is the $Q$-anti-automorphism of $U_q(sl_2)$ given by $\overline{q} = q^{-1}$, $\overline{E} = E$, $\overline{F} = F$, $\overline{K} = K$.

2. $B_{sl_2}$ is invariant, as a set, under the $Q(q)$-linear anti-automorphisms $u \mapsto u^*$ and $u \mapsto u^t$ given respectively by $E^* = E$, $F^* = F$, $K^* = K^{-1}$ and $E^t = F$, $F^t = E$, $K^t = K$; and under the rescaled Lusztig’s symmetry $T$ given by $T(E) = qFK^{-1}, T(F) = q^{-1}KE, T(K) = K^{-1}$.

3. Each monomial in $E, F, K^{\pm 1}$ is in the $\mathbb{Z}_{\geq 0}[q, q^{-1}]$-span of $B_{sl_2}$.

4. $B_{sl_2}$ is compatible with the filtered mock Peter-Weyl components $J_s = \sum_{r=0}^{s}(ad U_q(sl_2))(K^r)$ (see e.g. [13]), where ad denotes an adjoint action of the Hopf algebra $U_q(sl_2)$ on itself.

**Remark 1.1.** It should be noted that this basis is rather different from Lusztig’s canonical basis since the latter is in the *modified* quantized enveloping algebra $\check{U}_q(sl_2)$, as defined in [19, §23.1.1] and we are not aware of any relationship between these bases. It would also be interesting to compare our bases with the ones announced by Fan Qin in [21, 22]. Finally, it should be noted that John Foster constructed in [11] a basis of $U_q(sl_2)$ which differs from (1.1) in that Chebyshev polynomials $C^{(m)}$ are replaced by $C^m$.

We establish properties of $B_{sl_2}$ in §§4.1.4.2 and §5.4.

To construct $B_g$ for any symmetrizable Kac-Moody Lie algebra $g$ we need some notation. Fix a triangular decomposition $g = n_- \oplus h \oplus n_+$ and let $\tilde{g} := g \oplus h$, which we view as the Drinfeld double of the Borel subalgebra $b_+ = n_+ \oplus h$. Let $U_q(\tilde{g})$ be the quantized enveloping algebra of $\tilde{g}$ of adjoint type over $k = \mathbb{Q}(q^2)$, Thus, $U_q(\tilde{g})$ is the $k$-algebra generated by the $E_i, F_i, K_{\pm i}, i \in I$ subject to the relations: $K = \{ K_{+i}, K_{-i} : i \in I \}$ is commutative and

$$[E_i, F_j] = \delta_{ij}(q_i - q_j)(K_{+i} - K_{-i}), \quad K_{+i}E_j = q_i^{a_{ij}}E_jK_{+i}, \quad K_{+i}(F_j = q_i^{-a_{ij}}F_jK_{+i} ,$$

(1.2)

$$\sum_{r,s \geq 0, r+s=1-a_{ij}} (-1)^sE_i^{(r)}E_jE_i^{(s)} = \sum_{r,s \geq 0, r+s=1-a_{ij}} (-1)^sF_i^{(r)}F_jF_i^{(s)} = 0$$

(1.3)
for all $i, j \in I$, where $A = (a_{ij})_{i,j \in I}$ is the Cartan matrix of $\mathfrak{g}$, the $d_i$ are positive integers such that $DA = (d_i a_{ij})_{i,j \in I}$ is symmetric, $q_i = q^{d_i}$, $X_i^{(k)} := (\prod_{s=1}^{k} \langle s \rangle q_i)^{-1} X_i^{s\bar{k}}$ and $\langle s \rangle_v = v^s - v^{-s}$.

**Remark 1.2.** The reason for choosing such a non-standard presentation (1.2)-(1.3) of $U_q(\hat{\mathfrak{g}})$ is that one can now view $U_q(\hat{\mathfrak{g}})$ as a quantized coordinate algebra of $\mathcal{O}_q(\hat{G}^*)$, where $\hat{G}^*$ is the Poisson dual group of the Lie group $\hat{G}$ of $\mathfrak{g}$. This agrees with Drinfeld’s observation that the dual Hopf algebra of the complete Hopf algebra $U_h(\hat{G}^*)$ (where $\hat{G}^*$ is the Lie dual bialgebra of the Lie bialgebra $\mathfrak{g}$) is, on the one hand, $\mathcal{O}_h(\hat{G}^*)$ and, on the other hand, is isomorphic to $U_h(\hat{\mathfrak{g}})$. In particular, our basis $B_0$ will have a “dual canonical” flavor.

Our strategy for constructing $B_0$ is as follows. First, we define quantum Heisenberg algebras $H^\pm_q(\mathfrak{g})$ by $H^\pm_q(\mathfrak{g}) := U_q(\hat{\mathfrak{g}})/\langle K_{\pm i}, i \in I \rangle$. Then we use a variant of Lusztig’s Lemma (Proposition 2.3) to construct the double canonical basis $B_0^\pm$ of $H^\pm_q(\mathfrak{g})$ (see Theorem 1.3 below). Furthermore, using a natural embedding of $k$-vector spaces $t_+ : H^+_q(\mathfrak{g}) \hookrightarrow U_q(\mathfrak{g})$, which splits the canonical projection $\pi_+ : U_q(\hat{\mathfrak{g}}) \twoheadrightarrow H^+_q(\mathfrak{g})$ and the Lusztig’s lemma variant again, we build the double canonical basis $B_0$ of $U_q(\hat{\mathfrak{g}})$ out of $t_+(B_0^\pm)$. Finally, the desired basis $B_0$ is just the image of $B_0^\pm$ under the canonical projection $U_q(\hat{\mathfrak{g}}) \twoheadrightarrow U_q(\mathfrak{g}) = U_q(\mathfrak{g})/\langle K_{\pm i} K_{-i} - 1, i \in I \rangle$.

More precisely, by a slight abuse of notation we denote by $E_i, F_i, K_{+i}$ (respectively $K_{-i}$) the images of $E_i, F_i, K_{+i}$ (respectively $K_{-i}$) under the canonical projection $\pi_+ : U_q(\hat{\mathfrak{g}}) \twoheadrightarrow H^+_q(\mathfrak{g})$ (respectively under $\pi_- : U_q(\hat{\mathfrak{g}}) \twoheadrightarrow H^-_q(\mathfrak{g})$). It is obvious (and well-known) that, applying $\pi_\pm$ to the triangular decomposition $U_q(\mathfrak{g}) = K_- \otimes K_+ \otimes U^- \otimes U^+$, where $U^- = \langle E_i : i \in I \rangle$, $U^+ = \langle E_i : i \in I \rangle$, $K_\pm = \langle K_{\pm i} : i \in I \rangle$, one obtains a triangular decomposition $H^\pm_q(\mathfrak{g}) = K_\pm \otimes U^- \otimes U^+$.

Let $\Gamma = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ be a free abelian monoid, $\overline{\Gamma} = \Gamma \oplus \Gamma$ and set $\alpha_{-i} = (\alpha_i, 0), \alpha_{+i} = (0, \alpha_i) \in \overline{\Gamma}$. Then it is easy to see that $U_q(\hat{\mathfrak{g}})$ and $H^\pm_q(\mathfrak{g})$ are graded by $\overline{\Gamma}$ via $\deg \overline{\Gamma} E_i = \alpha_{+i}, \deg \overline{\Gamma} F_i = \alpha_{-i}$ and $\deg \overline{\Gamma} K_{\pm i} = \alpha_{\pm i}$. Denote by $K_\pm$ (respectively, $K_\mp$) the submonoid of $K$ generated by the $K_{\pm i}$ (respectively, the $K_{-i}$), $i \in I$ and let $K = K_- K_+$. Sometimes it is convenient to regard $U_q^+$ as graded by $\overline{\Gamma}$.

Denote by $B_n^\pm$ the **dual canonical basis** of $U_q^\pm$ (see [19, Chapter 14] and Section 3 for the details) i.e. the upper global crystal basis of $[18]$. By definition, each element of $B_n^\pm$ is homogeneous and is fixed under the involutive $\mathbb{Q}$-linear anti-automorphism $\overline{\gamma}$ of $U_q(\hat{\mathfrak{g}})$ determined by $\overline{q^{\overline{x}}} = q^{\overline{x}}, \overline{E_i} = E_i, \overline{F_i} = F_i, \overline{K_{\pm i}} = K_{\pm i}$. For instance, if $\mathfrak{g} = \mathfrak{sl}_2$ then $B_n^\pm = \{ E^r : r \in \mathbb{Z}_{\geq 0} \}$ and $B_n^\pm = \{ F^r : r \in \mathbb{Z}_{\geq 0} \}$.

We have an action $\diamondsuit$ of the algebra $K$ on $U_q(\mathfrak{g})$ defined by

$$K_{\pm i} \diamondsuit x := q^{\frac{1}{2} \alpha_i^\vee(\deg \overline{\gamma} x)} K_{\pm i} x, \quad (1.4)$$

where $\alpha_i^\vee \in \text{Hom}_{\mathbb{Z}}(\overline{\Gamma}, \mathbb{Z})$ is defined by $\alpha_i^\vee(\alpha_{\pm j}) = \pm a_{ij}$ and $x \in U_q(\hat{\mathfrak{g}})$ is homogeneous. This action is more suitable for our purposes than the left multiplication due to the following easy property

$$\overline{K} \circ K = K \circ \overline{x}, \quad (1.5)$$

Note that this action, as well as the involution $\overline{\gamma}$, factors through to a $K_{\pm}$-action and an anti-involution $\overline{\gamma}$ on $H^\pm_q(\mathfrak{g})$ via the canonical projection $\pi_\pm : U_q(\hat{\mathfrak{g}}) \twoheadrightarrow H^\pm_q(\mathfrak{g})$ and (1.5) holds.

We will show (Propositions 2.7 and 3.13) that for each pair $(b_-, b_+) \in B_n^- \times B_n^+$ there exists a unique monic $d_{b-, b_+} \in \mathbb{Z}[q + q^{-1}]$ of minimal degree such that in $U_q(\hat{\mathfrak{g}})$ one has

$$d_{b-, b_+} (b_+ b_- - b_- b_+) \in \sum_{K \in K \setminus \{1\}, b'_+ \in B_n^+} \mathbb{Z}[q, q^{-1}] d_{b'_-, b'_+} K \diamondsuit (b'_- b'_+) \quad (1.6)$$
It turns out all \( \mathbf{d}_{b_- \cdot b_+} \) are, up to a power of \( q \), products of cyclotomic polynomials in \( q \) (Proposition 3.9) and that for \( \mathfrak{g} \) semisimple \( \mathbf{d}_{b_- \cdot b_+} = 1 \) for all \( b_{\pm} \in B_{n_{\pm}} \) (Theorem 3.11). Some examples are shown in §4.3.

**Main Theorem 1.3.** For any \( (b_-, b_+) \in B_{n_-} \times B_{n_+} \) there is a unique element \( b_- \circ b_+ \in \mathcal{H}^+_q(\mathfrak{g}) \) fixed by \( \tau \) and satisfying

\[
b_- \circ b_+ - \mathbf{d}_{b_- \cdot b_+} b_- b_+ \in \sum q\mathbb{Z}[q] \mathbf{d}_{b_- \cdot b_+} K_+ \circ (b_- b_+)
\]

where the sum is over \( K_+ \in K_+ \setminus \{1\} \), \( b_{\pm} \in B_{n_{\pm}} \) such that \( \deg b_- b_+ + \deg K_+ = \deg b_- b_+ \).

We prove this theorem in Section 3 using a variant of Lusztig’s Lemma (Proposition 2.3) which we refer to as the equivariant Lusztig’s Lemma.

**Corollary 1.4.** The set \( B^+_{\mathfrak{g}} := \{ K_+ \circ (b_- \circ b_+) : (b_-, b_+) \in B_{n_-} \times B_{n_+}, K_+ \in K_+ \} \) is a \( \tau \)-invariant \( \mathbb{Q}(q^{\frac{1}{2}}) \)-linear basis of \( \mathcal{H}^+_q(\mathfrak{g}) \).

We call \( B^+_{\mathfrak{g}} \) the double canonical basis of \( \mathcal{H}^+_q(\mathfrak{g}) \) (the double canonical basis \( B^-_{\mathfrak{g}} \) of \( \mathcal{H}^-_q(\mathfrak{g}) \) is defined verbatim, with \( q \) replaced by \( q^{-1} \).

Furthermore, we have a natural, albeit not \( \tau \)-equivariant, inclusion \( \iota_+ : \mathcal{H}^+_q(\mathfrak{g}) = K_+ \circ U^- \circ U^+ \hookrightarrow K_- \circ (K_+ \circ U^- \circ U^+) = U_q(\tilde{\mathfrak{g}}) \).

**Main Theorem 1.5.** For any \( (b_-, b_+) \in B_{n_-} \times B_{n_+} \) there is a unique element \( b_- \bullet b_+ \in U_q(\tilde{\mathfrak{g}}) \) fixed by \( \tau \) and satisfying

\[
b_- \bullet b_+ - \iota_+(b_- \circ b_+) \in \sum q^{-1} \mathbb{Z}[q^{-1}] K \circ \iota_+(b_- \circ b_+)
\]

where the sum is taken over \( K \in K \setminus K_+ \) and \( b_{\pm} \in B_{n_{\pm}} \) such that \( \deg b_- b_+ + \deg K = \deg b_- b_+ \).

We prove this Theorem in Section 2 using the equivariant Lusztig’s Lemma (Proposition 2.3).

**Corollary 1.6.** The set \( B^\tau_{\mathfrak{g}} := \{ K \circ (b_- \bullet b_+), (b_-, b_+) \in B_{n_-} \times B_{n_+}, K \in K \} \) is a \( \mathbb{Q}(q^{\frac{1}{2}}) \)-basis of \( U_q(\tilde{\mathfrak{g}}) \).

We call \( B^\tau_{\mathfrak{g}} \) the double canonical basis of \( U_q(\tilde{\mathfrak{g}}) \).

**Remark 1.7.** Note that \( B^\tau_{\mathfrak{g}} \) contains both bases \( B_{n_{\pm}} \) as subsets and therefore has a “dual flavor”.

Let \( U_q(\tilde{\mathfrak{g}}, J) \) (respectively, \( U_q(J_-, \tilde{\mathfrak{g}}) \)), \( J \subset I \) be the subalgebra of \( U_q(\tilde{\mathfrak{g}}) \) generated by the \( KU^+_q \) and \( F_j \), \( j \in J \) (respectively, \( KU^-_q \) and \( E_j \), \( j \in J \)) and let \( U_q(J_-, \tilde{\mathfrak{g}}) = U_q(\tilde{\mathfrak{g}}, J_+) \cap U_q(J_-, \tilde{\mathfrak{g}}) \).

**Theorem 1.8.** For any \( J_{\pm} \subset I \), \( B^\tau_{\mathfrak{g}} \cap U_q(J_-, \tilde{\mathfrak{g}}) \) is a basis of \( U_q(J_-, \tilde{\mathfrak{g}}, J_+) \).

**Remark 1.9.** Analogously to the classical \( (q = 1) \) case (cf. e.g. [14]), it is natural to call \( U_q(J_-, \tilde{\mathfrak{g}}, J_+) \) quantum bi-parabolic (or seaweed) algebras.

As one should expect from a canonical basis, \( B^\tau_{\mathfrak{g}} \) is preserved, as a set, by various symmetries of \( U_q(\tilde{\mathfrak{g}}) \). First, let \( x \mapsto x^t \) and \( x \mapsto x^* \) be the \( \mathbb{Q}(q^{\frac{1}{2}}) \)-linear anti-automorphism of \( U_q(\tilde{\mathfrak{g}}) \) defined by

\[
E_i^t = F_i, \quad F_i^t = E_i, \quad (K_{\pm})^t = K_{\mp}, \quad \text{and} \quad E_i^* = E_i, \quad F_i^* = F_i, \quad (K_{\pm})^* = K_{\mp}.
\]

Then \( B^t_{\mathfrak{g}} = B_{\mathfrak{g}}^\tau \) while \( * \) preserves both \( B_{n_{\pm}} \) as sets.

**Theorem 1.10.** \( B^t_{\mathfrak{g}} = B^\tau_{\mathfrak{g}} \). More precisely, for all \( b_{\pm} \in B_{n_{\pm}}, K \in K \) be have \( (K \circ (b_- \bullet b_+))^t = K \circ (b_-)^t \bullet (b_+)^t \).

We prove this Theorem in Section 2.

**Conjecture 1.11.** \( B^*_{\mathfrak{g}} = B^\tau_{\mathfrak{g}} \). More precisely, for all \( b_{\pm} \in B_{n_{\pm}}, K \in K \) be have \( (K \circ (b_- \bullet b_+))^* = K^* \circ (b_-)^* \bullet (b_+)^* \).
Remark 1.12. It is easy to see that this conjecture implies that $B_{\tilde{g}}$ can also be obtained by replacing $\mathcal{H}_q^+(g)$ with $\mathcal{H}_q^-(g)$ and interchanging $q$ and $q^{-1}$ in Theorems 1.3 and 1.5.

It turns out that $B_{\tilde{g}}$ and $B_{\tilde{g}}$ are preserved by appropriately modified Lusztig’s symmetries. First of all, set $\tilde{U}_q(\tilde{g}) = U_q(\tilde{g})[K^{-1}]$. Clearly, $\gamma$, $\delta$ and $\ast$ extend naturally to that algebra.

Theorem 1.13. (a) For each $i \in I$ there exists a unique automorphism $T_i$ of $\tilde{U}_q(\tilde{g})$ which satisfies $T_i(K_{\pm j}) = K_{\pm j}K_{\pm i}^{\mp a_{ij}}$ and

$$T_i(E_j) = \begin{cases} q_i^{-1}K_{\pm 1}^{-1}F_i, & i = j \\ \sum_{r+s=-a_{ij}} (-1)^r q_i^{r+s}b_{ij} E_i^{(r)}E_j^{(s)}, & i \neq j \end{cases}$$

$$T_i(F_j) = \begin{cases} q_i^{-1}K_{\pm 1}^{-1}E_i, & i = j \\ \sum_{r+s=-a_{ij}} (-1)^r q_i^{r+s}b_{ij} F_i^{(r)}F_j^{(s)}, & i \neq j \end{cases}$$

(b) For all $x \in \tilde{U}_q(\tilde{g})$, $T_i(x) = T_i(x)$, $(T_i(x))^\ast = T_i^{-1}(x^\ast)$ and $(T_i(x))^\dagger = T_i^{-1}(x^\dagger)$.

(c) The $T_i$, $i \in I$ satisfy the braid relations on $\tilde{U}_q(\tilde{g})$, that is, they define a representation of the Artin braid group $Br_{\tilde{g}}$ of $g$ on $\tilde{U}_q(\tilde{g})$.

We prove this Theorem in Section 5.

Remark 1.14. Since for each $i \in I$, $T_i$ preserves the ideal $\mathfrak{J} = (K_{\pm j}K_{-j} - 1 : j \in I)$, $T_i$ factors through to an automorphism of $U_q(g) = U_q(\tilde{g})/3$ which, for $x \in U_q(g)$ homogeneous, equals $q_i^{\text{deg}(x)} T_i^{w_{i+1}}(x)$ where $T_i^{w_{i+1}}$ is one of Lusztig’s symmetries defined in [19, §37.1] (see Lemma 5.2).

Clearly, $\diamond$ extends to the group generated by $K$ acting on $\tilde{U}_q(\tilde{g})$. Then the set $\tilde{B}_{\tilde{g}} := K^{-1} \diamond \tilde{B}_{\tilde{g}}$ is a $\tilde{g}$-invariant basis of $\tilde{U}_q(\tilde{g}) = U_q(\tilde{g})[K^{-1}]$.

Conjecture 1.15. Let $g$ be semisimple. Then for all $i \in I$, $T_i(\tilde{B}_{\tilde{g}}) = \tilde{B}_{\tilde{g}}$. In other words, $Br_{\tilde{g}}$ acts on $\tilde{B}_{\tilde{g}}$ by permutations.

We prove supporting evidence for this conjecture in Section 5. In view of Remark 1.14, the conjecture implies that $T_i(B_{\tilde{g}}) = B_{\tilde{g}}$.

If $g$ is infinite dimensional, this does not hold for all elements of $\tilde{B}_{\tilde{g}}$ (see Example 5.6). To amend this conjecture we introduce the following notion. We say that $b \in \tilde{B}_{\tilde{g}}$ is tame if $T_i(b) \in \tilde{B}_{\tilde{g}}$ for all $i \in I$. We prove (Theorem 5.13) that all elements of $B_{n+}$ are tame.

Conjecture 1.16. If $b \in \tilde{B}_{\tilde{g}}$ is tame then $T(b) \in \tilde{B}_{\tilde{g}}$ for all $T \in Br_{\tilde{g}}$.

We provide supporting evidence for this conjecture in Section 5. We show some of it below for which more notation is necessary. Let $W$ be the Weyl group of $g$. Following [19, §39.4.4], for each $w \in W$ define $T_w \in Br_{\tilde{g}}$ recursively as $T_{s_i} = T_i$ and $T_w = T_w'T_w''$ for any non-trivial reduced factorization $w = w'w''$, $w', w'' \in W$ (see [5.1 for the details). Define the quantum Schubert cells $U_q^+(w)$ and $U_q^-(w)$, $w \in W$ by $U_q^+(w) := T_w(KU_q^+ \cap U_q^+)$ and $U_q^-(w) := U_q^\ast \cap T_w^{-1}(KU_q^+)$.

Clearly, these are subalgebras of $U_q^\pm$. For $g$ semisimple we provide an elementary proof (Proposition 5.4) that $U_q^+(w)$ coincides with the subspace $U_q^+(w, 1)$ of $U_q^+$ defined by Lusztig ([19, §40.2]) via a choice of a reduced decomposition of $w$, and conjectured that this is the case for all Kac-Moody $g$ (Conjecture 5.3). Let $B_{n+}(w) = B_{n+} \cap U_q^+(w)$ (since, conjecturally, $U_q^+(w, 1) = U_q^+(w)$, by [16, Theorem 4.22] $B_{n+}^+(w)$ is a basis of $U_q^+(w)$). The following refines Conjecture 1.15.

\footnote{While preparing the final version of the present paper we learned that Conjecture 5.3 was proved by Tanisaki in [25]; shortly after an alternative proof was provided by Kimura ([17]).}
Conjecture 1.17. $T_w^{-1}(B_{n_+}(ww')) \subset K^{-1} \cdot B_{n_-}(w) \cdot B_{n_+}(w')$ for all $w, w' \in W$ such that the factorization $ww'$ is reduced.

Remark 1.18. Note also that this conjecture implies that $K^{-1} \cdot B_{n_-}(w) \cdot B_{n_+}(w')$ is a basis in the double Schubert cell $KU_q^+(w)U_q^+(w') = T_w^{-1}(U_q^+(ww'))$.

Another application of our construction is a double canonical basis in each quantum Weyl algebra $A_q^e(g)$. Given a function $e : I \to \{+,-\}$, $e(i) = e_i$, let $A_q^e(g)$ be a $k$-algebra generated by the $x_i$, $y_i \in I$ subject to the following relations

$$x_i y_i - y_i x_i = e_i (q_i^{-1} - q_i), \quad x_i y_j = q_i^{\delta_{ij} - \epsilon_{ij}} y_j x_i,$$

$$\sum_{r+s=1-a_{ij}} (-1)^r q_i^{r e_{ij} - s a_{ij}} x_i^{(r)} x_j^{(s)} = 0 = \sum_{r+s=1-a_{ij}} (-1)^r q_i^{r e_{ij} - s a_{ij}} y_i^{(r)} y_j^{(s)).} \quad i \neq j. \quad (1.6)$$

We will show (see Proposition 3.17) that each $A_q^e(g)$ is naturally a subalgebra of a Heisenberg algebra $H_q^e(g)$ which “interpolates” between $H_q^+(g)$ and $H_q^-(g)$ (see §3.4 for the details) and obtain the following result.

Theorem 1.19. Each quantum Weyl algebra $A_q^e(g)$ has a double canonical basis $B_{n_+} \circ e \cdot B_{n_+}$.

We prove this Theorem in §3.4.

Remark 1.20. In fact, the $A_q^e(g)$ are closely related to braided Weyl algebras (see e.g. [15]). Note that algebras $A_q^e(g)$ and $A_q^{-e}(g)$ are not (anti)isomorphic if $e \neq -e'$. Thus, the resulting bases $B_{n_+} \circ e \cdot B_{n_+}$ and $B_{n_-} \circ e' \cdot B_{n_+}$ are rather different. To the best of our knowledge, these bases admit an alternative description similar to that in Theorem 1.3 only when $e$ is a constant function, i.e. $e_i = +$ (respectively, $e_i = -$) for all $i \in I$.

Next we discuss the properties of the decomposition of elements of the natural basis of $U_q(g)$ with respect to $B_{\hat{g}}$. Define $C_{b_-,b_+}^{b'_-,b'_+} \in k$ for all $b_-, b'_- \in B_{n_\pm}$ and $K \in K$ by

$$d_{b_-,b_+} b_- b_+ = \sum_{b'_-,b'_+} C_{b_-,b_+}^{b'_-,b'_+} \cdot K \cdot b'_- b'_+. $$

Then Main Theorem 1.5 immediately implies that $C_{b_-,b_+}^{b'_-,b'_+} \in \mathbb{Z}[q, q^{-1}]$. These Laurent polynomials play the role similar to that of Kazhdan-Lusztig polynomials due to the following conjectural result.

Conjecture 1.21. If $g$ is semisimple then $C_{b_-,b_+}^{b'_-,b'_+} \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$ for all $b_-, b'_- \in B_{n_\pm}, K \in K$.

We provide some examples in Section 4.

Remark 1.22. It is well-known (cf. [19]) that if the Cartan matrix of $g$ is symmetric then the structure constants of $B_{n_\pm}$ belong to $\mathbb{Z}_{\geq 0}[q, q^{-1}]$. However, we expect that Conjecture 1.21 holds even for those $g$ (with non-symmetric Cartan matrix) for which such positivity fails.

Next we discuss the relation between the adjoint action of $U_q(\hat{g})$ on itself and the double canonical basis. We expect that the basis $B_{\hat{g}}$ is perfect in the sense of the following extension of Definition 5.30 from [5].

Definition 1.23. Let $\mathcal{Y}$ be a $k$-vector space with linear endomorphisms $e_i$, $i \in I$ and functions $\varepsilon_i : \mathcal{Y} \setminus \{0\} \to \mathbb{Z}$ such that $\varepsilon_i(e_i(v)) = \varepsilon_i(v) - 1$ for all $v \notin \ker e_i$. We say that a basis $B$ of $\mathcal{Y}$ is perfect if for all $i \in I$ and $b \in B$ either $e_i(b) = 0$ or there exists a unique $b' \in B$ with $\varepsilon_i(b') = \varepsilon_i(b) - 1$ such that

$$e_i(b) \in k^x b' + \sum_{b' \in B : \varepsilon_i(b') < \varepsilon_i(b')} k b'',$$
Consider the adjoint action of $\hat{U}_q(\mathfrak{g})$ on itself which factors through an action of $U_q(\mathfrak{g})$ via
\[ F_i(x) := F_i x - K_{-i} x K_{-i}^{-1} F_i, \quad E_i(x) := [E_i, x] K_{i}^{-1}, \quad K_i(x) := K_i x K_i^{-1} \]
for all $i \in I$, $x \in \hat{U}_q(\mathfrak{g})$; here $K_i$ denotes the canonical image of $K_i$ in $U_q(\mathfrak{g})$. It is curious that this action preserves the subalgebra $U_q(\mathfrak{g})[K_i^\pm] \subset \hat{U}_q(\mathfrak{g})$ and its ideal generated by the $K_{-i}, i \in I$, hence descends to $H^+_q(\mathfrak{g})[K_i^\pm]$.

**Conjecture 1.24.** For any symmetrizable Kac-Moody algebra $\mathfrak{g}$, the bases $K_i^\pm \odot B_{n_-} \odot B_{n_+}$ and $\hat{B}_\mathfrak{g}$ are perfect with respect to the action (1.7) of $U_q(\mathfrak{g})$ on $H^+_q(\mathfrak{g})[K_i^\pm]$ and $\hat{U}_q(\mathfrak{g})$, respectively.

We prove this conjecture for $\mathfrak{g} = \mathfrak{sl}_2$ in §4.2.

We now further the behavior of the double basis with respect to the action (1.7) by using an extension of the remarkable $U_q(\mathfrak{g})$-equivariant map $U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined in [10, 23]. In particular, this map yields Joseph’s decomposition of the locally finite part of $U_q(\mathfrak{g})$ and, in the finite dimensional case, the center of $U_q(\mathfrak{g})$ (see [4] and Proposition 1.27 below).

Let $V$ be a lowest weight $U_q(\mathfrak{g})$-module (e.g., a Verma module or its unique simple quotient) of lowest weight $-\mu \in \Lambda$ where $\Lambda$ is an integral weight lattice for $\mathfrak{g}$ (see [3.1]). Let $\hat{\mathfrak{g}} := \mathfrak{g}(1)$, where $\hat{\mathfrak{g}}$ belongs to the double canonical basis of $\hat{U}_q(\mathfrak{g})$.

**Theorem 1.25.** For any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ we have
(a) For any lowest weight module $V$, $\Xi$ is a homomorphism of $U_q(\mathfrak{g})$-modules $V \otimes V \to \hat{U}_q(\mathfrak{g})$ whereby the action of $U_q(\mathfrak{g})$ on $V \otimes V$ (respectively, $\hat{U}_q(\mathfrak{g})$) is defined by $K_i(v \otimes v') = K_i^{-1}(v) \otimes K_i(v')$, $E_i(v \otimes v') = v \otimes E_i(v') - K_{-i}^{-1} F_i(v) \otimes K_{-i}(v')$, $F_i(v \otimes v') = K_{-i}(v) \otimes F_i(v')$, and $E_i K_i(v) \otimes v'$ for all $i \in I$, $v, v' \in V$ while the $U_q(\mathfrak{g})$-action on $\hat{U}_q(\mathfrak{g})$ is defined by (1.7).
(b) If $V$ is simple integrable of lowest weight $-\mu$ then $V \otimes V$ is integrable, $\Xi$ and its dual $\Xi^*$ are injective and $J_V := \Xi^*(V \otimes V)$ is the corresponding Joseph’s component (see §4.4).

Our proof of Theorem 1.25 (see §4.4) relies on results of [4].

It is very tempting to relate some known bases in $V \otimes V$ with our basis $\hat{B}_\mathfrak{g}$. The relation is not immediate. However, as all interesting bases contain the canonical $U_q(\mathfrak{g})$-invariant element $1_V \in V \otimes V$ (cf. §4.4), we suggest the following Conjecture

**Conjecture 1.26.** Let $\mathfrak{g}$ be semisimple and let $V$ be the simple finite dimensional $U_q(\mathfrak{g})$-module of lowest weight $-\mu$. Then $(-1)^{2\rho^\vee(\mu)} C_V$, where $C_V := \Xi(1_V)$ and $2\rho^\vee$ is the sum of all positive coroots of $\mathfrak{g}$ belongs to the double canonical basis of $\hat{U}_q(\mathfrak{g})$.

We prove this conjecture for $\mathfrak{g} = \mathfrak{sl}_2$ and provide other supporting evidence for $\mathfrak{sl}_n$ and $\mathfrak{sp}_4$ in §4.4. Theorem 1.25 implies that the $C_V$ and $\hat{C}_V := \Xi(1_V)$ are central. Their importance for the representation theory of $\hat{U}_q(\mathfrak{g})$ is due to following result (see e.g. [4, Theorem 1.11] which in turn was inspired by Drinfeld’s construction from [10]).
Theorem 1.27. For any semisimple Lie algebra $\mathfrak{g}$ the map assigning to a simple $U_q(\mathfrak{g})$-module $V$ the element $\overline{C}_V := \Xi(1_V)$ defines an isomorphisms between the Grothendieck ring $\mathbb{k} \otimes \mathbb{Z} K_0(\mathfrak{g})$ of the category of finite dimensional $U_q(\mathfrak{g})$-modules and the center of $\hat{U}_q(\mathfrak{g})$.

Thus, the canonical basis of the Grothendieck ring of the category of finite dimensional $\mathfrak{g}$-modules identifies with a subset of the double canonical basis $B_\mathfrak{g}$ and so $B_{\mathfrak{sl}_n}$ contains (the canonical basis of) all Schur polynomials $s_\lambda$. Namely, Conjecture 1.26 and Theorem 1.27 imply that the map assigning to the simple lowest weight module $V(-\mu)$ of lowest map $-\mu$ the element $C_\mu := (-1)^{2\rho(\mu)}C_V(-\mu)$ defines a homomorphism of rings $K_0(\mathfrak{g}) \to \hat{U}_q(\mathfrak{g})$ and that the $C_\mu$ belong to the double canonical basis of $\hat{U}_q(\mathfrak{g})$. Furthermore, it would be interesting to extend these observations to the case when $V$ is simple infinite dimensional. In that case $1_V$ is a well-defined element of a certain completion $V \otimes V$ of $V \otimes V$ and is image $C_V$ under $\Xi$ belongs to a completion of $\hat{U}_q(\mathfrak{g})$. It would be interesting to relate these elements with the quantum Casimir defined in [12]. We believe that these elements $C_V$ should be important for physical applications, for instance when $\mathfrak{g}$ is affine and $V$ is its basic module.

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2. Equivariant Lusztig’s Lemma and bases of Heisenberg and Drinfeld doubles

2.1. An equivariant Lusztig’s Lemma. Let $\Gamma$ be an abelian monoid and let $R$ be a unital $\Gamma$-graded ring $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ where $R_0$ is central in $R$. Suppose that $\cdot$ is an involution of abelian groups $R \to R$ satisfying

$$r \cdot r' = r' \cdot \overline{r}, \quad r, r' \in R$$

and $R_\gamma = R_\gamma$, $\gamma \in \Gamma$. Let $R_+ = \bigoplus_{\gamma \in \Gamma \setminus \{0\}} R_\gamma$, and $\varepsilon : R \to R/R_+ \cong R_0$ be the canonical projection. Note that $\varepsilon$ commutes with $\cdot$.

Let $\hat{E} = \bigoplus_{\gamma \in \Gamma} \hat{E}_\gamma$ be a $\Gamma$-graded left $R$-module where each $\hat{E}_\gamma$ is assumed to be free as an $R_0$-module. Suppose that $\cdot$ is an involution of abelian groups on $\hat{E}$ satisfying $\overline{\alpha} = \overline{\alpha} \cdot \overline{\varepsilon}$ for all $x \in R_0$, $e \in \hat{E}$ and $\overline{E}_\gamma = \hat{E}_\gamma$, $\gamma \in \Gamma$. Assume also that $\overline{R_\gamma E} \subset \bigoplus_{\alpha \in \Gamma} R_{\alpha + \gamma} \hat{E}$ for all $\gamma \in \Gamma \setminus \{0\}$, or, equivalently, $\overline{R_\gamma E} \subset R_+ \hat{E}$. Then $E = \hat{E} / R_+ \hat{E}$ is naturally a $\Gamma$-graded $R_0$-module and $\cdot$ factors through to an involution of abelian groups on $E$ which also satisfies $\overline{\alpha} \cdot \overline{\varepsilon} = \overline{\alpha} \cdot \overline{\varepsilon}$, $x \in R_0$, $e \in E$.

Suppose now that $E$ is also free as an $R_0$-module. Since $\hat{E}$ and $E$ are free as $R_0$-modules and the canonical projection $\pi : \hat{E} \to E$ is a morphism of $\Gamma$-graded $R_0$-modules, it admits a homogeneous splitting $\iota : E \to \hat{E}$.

Define a relation $\prec$ on $\Gamma$ by $\alpha \prec \beta$ if there exists $\gamma \in \Gamma \setminus \{0\}$ such that $\alpha + \gamma = \beta$. Assume that there exists a function $\ell : \Gamma \to \mathbb{Z}_{\geq 0}$ such that for all $\gamma \in \Gamma$, $\gamma_{s} < \gamma_{s-1} < \cdots < \gamma_1 < \gamma$ implies that $s \leq \ell(\gamma)$. For example, this assumption holds for every monoid $\Gamma$ which admits a character $\chi : \Gamma \to \mathbb{Z}_{\geq 0}$ with $\chi(\gamma) > 0$ if $\gamma \neq 0$, which is the case for $\Gamma = \mathbb{Z}_{I \geq 0}$ where $I$ is finite. We will call such a monoid $\Gamma$ bounded. If $\Gamma$ is bounded then, in particular, $\geq$ is a partial order and $0$ is the unique minimal element of $\Gamma$.

Lemma 2.1. Let $\Gamma$ be a bounded monoid. Then $\iota(E)$ generates $\hat{E}$ as an $R$-module.
Proof. We have \( \hat{E} = \iota(E) \oplus R_+ \hat{E} \) as \( \Gamma \)-graded \( R_0 \)-modules, hence \( \hat{E}_\gamma = \iota(E_\gamma) \oplus (R_+ \hat{E})_\gamma \) for all \( \gamma \in \Gamma \). We prove by induction on \((\Gamma, \prec)\) that \( \hat{E}_\gamma \subset \iota(E) \). Since \( 0 \in \Gamma \) is minimal, \( \hat{E}_0 \cap R_+ \hat{E} = 0 \) hence \( \hat{E}_0 \subset \iota(E) \) and the induction begins. For the inductive step, let \( \gamma \in \Gamma \setminus \{0\} \) and assume that \( \bigoplus_{\alpha \prec \gamma} \hat{E}_\alpha \subset \iota(E) \). Then \( \hat{E}_\gamma \cap R_+ \hat{E} \subset \sum_{\alpha + \beta = \gamma : \alpha \in \Gamma \setminus \{0\}} R_\alpha \hat{E}_\beta \subset \iota(E) \) by the induction hypothesis. Thus, \( \hat{E}_\gamma = \iota(E_\gamma) \oplus (R_+ \hat{E})_\gamma \subset \iota(E) \). \( \square \)

From now on we will assume that \( \Gamma \) is bounded.

Let \( \mathcal{E} \) be a homogeneous basis of \( E \) satisfying \( \bar{e} = e \) for all \( e \in \mathcal{E} \). Clearly
\[
\iota(e) - \iota(e) \in R_+ \hat{E}.
\]

The following Lemma is obvious.

**Lemma 2.2.** Let \( \mathcal{R} \subset R, 1 \in \mathcal{R} \). The following are equivalent:

(i) \( \{rt(e) : (r, e) \in \mathcal{R} \times \mathcal{E}\} \) is an \( R_0 \)-basis of \( \hat{E} \)

(ii) As an \( R_0 \)-module, \( R = \text{Ann}_{R} \iota(E) \oplus \bigoplus_{r \in \mathcal{R}} R_0 r \).

Given a homogeneous element \( x \) of \( R, \hat{E} \) or \( E \) we denote its degree by \( |x| \).

**Proposition 2.3.** Suppose that \( R_0 = \mathbb{Z}[\nu, \nu^{-1}] \) and that \( \bar{\nu} : R_0 \to R_0 \) is the unique ring automorphism satisfying \( \iota = \nu^{-1} \). Fix an \( R_0 \)-module splitting \( \iota : E \to \hat{E} \) of the canonical projection \( E \to \hat{E} / R_+ \hat{E} \cong E \). Suppose that there exists a subset \( \mathcal{R} \subset R \) of homogeneous elements containing \( 1 \) such that

(i) As an \( R_0 \)-module, \( R = \text{Ann}_{R} \iota(E) \oplus \bigoplus_{r \in \mathcal{R}} R_0 r \);

(ii) For all \( r \in \mathcal{R}, e \in \mathcal{E} \)
\[
\frac{\iota(r t(e)) - \iota(e)}{\iota(e)} \in \bigoplus_{(r', e') \in \mathcal{R} \times \mathcal{E} : |r'| + |e'| = |r| + |e|, |e'| < |e|} R_0 r' \iota(e').
\]

Then for each \( (r, e) \in \mathcal{R} \times \mathcal{E} \) there exists a unique \( C_{r, e} \in \hat{E} \) such that \( \overline{C_{r, e}} = C_{r, e} \) and
\[
C_{r, e} - \iota(e) \in \bigoplus_{(r', e') \in \mathcal{R} \times \mathcal{E} : |r'| + |e'| = |r| + |e|, |e'| < |e|} \nu \mathbb{Z}[\nu] r' \iota(e').
\]

In particular, the set \( B_{\mathcal{R}, \mathcal{E}} := \{C_{r, e} : (r, e) \in \mathcal{R} \times \mathcal{E}\} \) is an \( R_0 \)-basis of \( \hat{E} \).

**Proof.** Define a relation \( \prec \) on \( \mathcal{R} \times \mathcal{E} \) by \( (r', e') \prec (r, e) \) if \( |e'| < |e| \) and \( |r'| + |e'| = |r| + |e| \). It is easy to see that \( (r', e') \prec (r, e) \) implies that \( 0 < |r'| \) (otherwise \( |e'| = |r| + |e| \) hence \( |e| \leq |e'| < |e| \)). Then \( \prec \) is a partial order and all assumptions of [6, Theorem 1.1] for \( L := \mathcal{R} \times \mathcal{E}, A = \hat{E} \) and \( E_{(r, e)} = \iota(e), (r, e) \in L \) are satisfied. Thus, the assertion follows from the aforementioned result. \( \square \)

We conclude this section with a discussion of some symmetries of the \( B_{\mathcal{R}, \mathcal{E}} \) constructed in Proposition 2.3. Consider the data \( (R, \mathcal{R}, \hat{E}, E, \mathcal{E}, \iota) \) satisfying the assumptions of Proposition 2.3.

**Definition 2.4.** We say that a homogeneous \( \overline{\iota(e)} \)-equivariant \( R_0 \)-module automorphism \( \psi \) of \( \hat{E} \) is **triangular** if there exists a permutation \( \phi \) of \( \mathcal{R} \) with \( \phi(1) = 1 \) and a permutation \( \overline{\psi(e)} \) of \( \mathcal{E} \) such that
\[
\psi(rt(e)) - \phi(r)\psi(e)) \in \bigoplus_{(r', e') \in \mathcal{R} \times \mathcal{E} : |r'| + |e'| = |r| + |e|, |e'| < |e|} \nu \mathbb{Z}[\nu] r' \overline{\psi(e')}.
\]

Using the same argument as in the proof of Proposition 2.3, we conclude that in all non-zero terms in the right-hand side we have \( 0 < |r'| \).

**Lemma 2.5.** Suppose that \( \psi : \hat{E} \to \hat{E} \) is triangular. Then
\[
\psi(C_{r, e}) = C_{\phi(r), \overline{\psi(e)}}, \quad r \in \mathcal{R}, e \in \mathcal{E}.
\]
Proof. Since $\psi$ commutes with $\tilde{\gamma}$, $\tilde{\psi}(\Gamma) = \psi(\Gamma)$. Applying $\psi$ to $(2.2)$ we obtain

$$\psi(C_{r,e}) - \psi(r\iota(e)) \in \sum_{(r',e') \in R \times E \cup |r'|+|e'|=|r|+|e|, |e'|<|e|} \nu Z[\nu](\psi(r'\iota(e')))$$

Applying $(2.3)$ to the left and the right hand side we conclude that

$$\psi(C_{r,e}) - \phi(r)\iota(\psi(e)) \in \sum_{(r',e') \in R \times E \cup |r'|+|e'|=|r|+|e|, |e'|<|e|} \nu Z[\nu](\psi(r'\iota(e')))$$

Proposition $2.3$ then implies that $\psi(C_{r,e}) = C_{\phi(r),\psi(e)}$. \qed

2.2. Double bases of Heisenberg and Drinfeld doubles. In this section we will use the notation and the setup of $\S$A.8,A.9.

Let $\Gamma$ be a bounded abelian monoid as defined in $\S$2.1. Let $k = \mathbb{Q}(\nu), R_0 = \mathbb{Z}[\nu, \nu^{-1}]$. Let $H = k[\hat{\Gamma}]$ be the monoidal algebra of $\hat{\Gamma} = \Gamma \bigoplus \Gamma$ with a basis $\{K_{\alpha_-,\alpha_+} : \alpha_\pm \in \Gamma\}$ and let $R = \bigoplus_{\alpha_\pm \in \Gamma} R_0 K_{\alpha_-,\alpha_+}$.

Let $V^\pm = \bigoplus V^\pm_\alpha$ be $\Gamma$-graded vector spaces. We regard $V^+$ (respectively, $V^-$) as a right (respectively, left) Yetter-Drinfeld module over the localization $\hat{H}$ of $H$ with respect to the monoid $\{K_{\alpha_-,\alpha_+} : \alpha_\pm \in \Gamma\}$ (see $\S$A.9 for the details). Let $\langle \cdot, \cdot \rangle : V^- \otimes V^+ \to k$ be a pairing such that $\langle V^-_\beta, V^+_\alpha \rangle = 0$, $\alpha \neq \beta$ and $\langle \cdot, \cdot \rangle |_{V^-_\alpha \otimes V^+_\alpha}$ is non-degenerate. Set $\Gamma_0 = \{\alpha \in \Gamma : V^+_\alpha \neq 0\}$ and assume that $\Gamma$ is generated by $\Gamma_0$. Let $\chi : \Gamma \times \Gamma \to R_0^* = \pm \nu Z$ be a symmetric bicharacter.

Given $t_+, t_- \in k$, let $U_{t_+,t_-}(V^-, V^+)$ be the algebra $U_{\chi}(V^-, V^+)$ defined in $\S$A.9 with $\langle \cdot, \cdot \rangle_\pm = t_\pm \langle \cdot, \cdot \rangle$. We have in $U_{\chi,t_+,t_-}(V^-, V^+)$

$$K_{\alpha_-,\alpha_+}v^+ = \frac{\chi(\alpha_+, \deg v^+)v^+}{\chi(\alpha_-, \deg v^+)} K_{\alpha_-,\alpha_+}, \quad K_{\alpha_-,\alpha_+}v^- = \frac{\chi(\alpha_-, \deg v^-)v^-}{\chi(\alpha_+, \deg v^-)} K_{\alpha_-,\alpha_+}, \quad (2.4)$$

and

$$[v^+, v^-] = t_- K_{\deg v^-, 0}\langle v^-, v^+ \rangle - t_+ K_{0, \deg v^+}\langle v^-, v^+ \rangle, \quad (2.5)$$

for all $v^\pm \in B(V^\pm)$ homogeneous and $\alpha_\pm \in \Gamma$. We regard $U_{\chi,t_+,t_-}(V^-, V^+)$ as graded by $\hat{\Gamma}$ with $\deg\hat{\Gamma}v^+ = (0, \deg v^+), \deg\hat{\Gamma}v^- = (\deg v^-, 0)$ and $\deg\hat{\Gamma}K_{\alpha_-,\alpha_+} = (\alpha_- + \alpha_+, \alpha_+ + \alpha_+)$, where $v^\pm \in V^\pm$ are homogeneous and $\alpha_\pm \in \Gamma$.

Denote

$$H^0_{\chi}(V^-, V^+) := U_{\chi,0,0}(V^-, V^+), \quad H^\pm_{\chi}(V^-, V^+) = U_{\chi,1,0}(V^-, V^+), \quad H^-_{\chi}(V^-, V^+) := U_{\chi,0,1}(V^-, V^+), \quad U_{\chi}(V^-, V^+) = U_{\chi,1,1}(V^-, V^+).$$

Thus, all these algebras have the same underlying vector space, namely $B(V^-) \otimes H \otimes B(V^+)$ and differ only in the cross relations between $B(V^-)$ and $B(V^+)$. Let $\hat{\gamma} : k \to k$ be the unique field involution defined by $\hat{\nu} = \nu^{-1}$. Fix its extension to $V^\pm$ preserving the grading and assume that $\hat{\langle \cdot, \cdot \rangle} = -\langle \cdot, \cdot \rangle$, $v^\pm \in V^\pm$. Assume also that $\chi$ satisfies $\chi(\alpha, \alpha') = \chi(\alpha, \alpha')^{-1}$ for all $\alpha, \alpha' \in \Gamma$. Then all algebras described above admit an anti-linear $\tilde{\gamma}$-anti-involution extending $\hat{\gamma}$, $\hat{\nu} : V^\pm \to V^\pm$ and satisfying $\hat{K}_{\alpha, \alpha'} = K_{\alpha, \alpha'}$, $\alpha, \alpha' \in \Gamma$.

Assume that $\chi(\alpha, \alpha') \in \nu Z^2$ for all $\alpha, \alpha' \in \Gamma$ and let $\hat{\chi} : \Gamma \times \Gamma \to \pm \nu Z$ be a bicharacter satisfying $(\hat{\chi}^{\pm}_{\alpha} \alpha, \alpha')^2 = \chi(\alpha, \alpha')$, $\alpha, \alpha' \in \Gamma$. Extend $\hat{\chi}^{\pm}_{\alpha}$ to a bicharacter of $\hat{\Gamma}$ via

$$\hat{\chi}^{\pm}_{\alpha}(\alpha_-, \alpha_+)(\beta_-, \beta_+) = \frac{\hat{\chi}^{\pm}_{\alpha}(\alpha_+, \beta_+)(\alpha_-, \beta_-)}{\hat{\chi}^{\pm}_{\alpha}(\alpha_+, \beta_-)(\alpha_-, \beta_+)} , \quad \alpha_\pm, \beta_\pm \in \Gamma.$$
Then we set for all $\alpha_{\pm} \in \Gamma$ and for all $x \in \mathcal{U}_\chi,t_-,t_+(V^-,V^+) \text{ homogeneous with respect to } \widehat{\Gamma}$

$$K_{\alpha_-,\alpha_+} \circ x = (\chi^{\frac{1}{2}}((\alpha_-,\alpha_+), \deg x))^{-1}K_{\alpha_-,\alpha_+} x$$

$$= (\chi^{\frac{1}{2}}((\alpha_-,\alpha_+), (\deg x)\xi))^{-1}x K_{\alpha_-,\alpha_+},$$

where $\xi : \widehat{\Gamma} \to \widehat{\Gamma}$ is defined by $(\alpha_-,\alpha_+)^\xi = (\alpha_+,\alpha_-)$, $\alpha_{\pm} \in \Gamma$. The following Lemma is obvious.

**Lemma 2.6.** For all $t_+ \in \mathcal{k}$, (2.6) defines a structure of a left $H$-module on $\mathcal{U}_\chi,t_-,t_+(V^-,V^+)$ satisfying

$$K_{\alpha_-,\alpha_+} \circ x = K_{\alpha_-,\alpha_+} \circ \text{F}, \quad x \in \mathcal{U}_\chi,t_-,t_+(V^-,V^+), \alpha_{\pm} \in \Gamma.$$
Lemma 2.8. For any finite subset $F \subset \mathbb{Q}(\nu)$ there exists a unique $d(F) \in \mathbb{Z}[\nu + \nu^{-1}]$ such that $d(F)F \subset \mathbb{Z}[\nu, \nu^{-1}]$, the degree of $d(F)$ in $\nu + \nu^{-1}$ is minimal and the highest coefficient of $d(F)$ is positive and minimal. Moreover, if all poles of elements of $F$ are roots of unity then $d(F) = c(\nu + \nu^{-1} - 2)^{m_1}(\nu + \nu^{-1} + 2)^{m_2}\prod_{k \geq 3}(\nu^{\frac{1}{k}}\Phi_k(\nu))^{m_k}$ with $c, m_k \in \mathbb{Z}_{\geq 0}$, $c \neq 0$, where $\Phi_k$ is the $k$th cyclotomic polynomial and $\varphi(k) = \text{deg} \Phi_k$ is the Euler function.

Proof. Let $F = \{f_1, \ldots, f_t\}$, $f_i = a_i/b_i$ where $a_i, b_i \in \mathbb{Z}[\nu]$ and are coprime. Then there exists a unique $f \in \mathbb{Z}[\nu]$ of minimal degree such that $f f_i \in \mathbb{Z}[\nu]$ for all $1 \leq i \leq r$, namely, $f$ is the least common factor of the $b_i$. Write $f = c\prod_{j=1}^t p_j^{m_j}$, where $c \in \mathbb{Z}$, each $p_j \in \mathbb{Z}[\nu]$ is irreducible and $m_j \in \mathbb{Z}_{>0}$. We may assume without generality that $c$ as well as the highest coefficient of each of the $p_j$ is positive. Given an irreducible $p \in \mathbb{Z}[\nu]$ of positive degree, define

$$\tilde{p} = \begin{cases} q^{-\frac{1}{2}\text{deg}p}p, & \text{if } p \in \mathbb{Z}[\nu] \\ p, & \text{otherwise.} \end{cases}$$

Then $\tilde{p} \in \mathbb{Z}[\nu + \nu^{-1}]$ and is irreducible in that ring. It follows that $d(F) := c\prod_{j=1}^t \tilde{p}_j^{m_j}$ has the desired properties. This proves the first assertion.

If the only zeroes of all the $b_i$, $1 \leq i \leq r$ are roots of unity then the only non-constant irreducible factors of $f$ are cyclotomic polynomials. Clearly, $\tilde{\Phi}_1 = \nu + \nu^{-1} - 2$ and $\tilde{\Phi}_2 = \nu + \nu^{-1} + 2$. Since $\varphi(k)$ is even for all $k \geq 3$, it follows that $\tilde{\Phi}_k = \nu^{-\frac{1}{k}\varphi(k)}\Phi_k$. \hfill \Box

Denote $\mathcal{F}_{b_-, b_+} := \{d_{\nu^j\nu^{j'}}b_{\nu^j\nu^{j'}}^\pm b_{\nu^j}\pm b_{\nu^{j'}}^\pm : b_{\nu^j}\pm \in \mathbb{B}_\pm, (\alpha_-, \alpha_+) \in \tilde{\Gamma} \setminus \{(0, 0)\}\}$. Then $\mathcal{F}_{b_-, b_+}$ is finite and we set $d_{b_-, b_+} = d(\mathcal{F}_{b_-, b_+})$. Then by the above computation

$$d_{b_-, b_+}(b_{\nu^j\nu^{j'}}b_{\nu^j}\pm b_{\nu^{j'}}^\pm - b_{\nu^j}\pm b_{\nu^{j'}}^\pm) = \sum_{(\alpha_-, \alpha_+) \in \tilde{\Gamma} \setminus \{(0, 0)\}} R_0 d_{\nu^j\nu^{j'}} K_{\alpha_-, \alpha_+} b_{\nu^j\pm} b_{\nu^{j'}}^\pm.$$

It remains to observe that $R_0 K_{\alpha_-, \alpha_+} b_{\nu^j\pm} b_{\nu^{j'}}^\pm = R_0 K_{\alpha_-, \alpha_+} b_{\nu^j\pm} b_{\nu^{j'}}^\pm$. The uniqueness of $d$ is obvious.

Part (b) is immediate from (2.8). \hfill \Box

Theorem 2.9. Suppose that $\mathbb{B}_\pm$ are $\Gamma$-homogeneous bases of $\mathcal{B}(V^\pm)$ and $\overline{b}_\pm = b_\pm$ for all $b_\pm \in \mathbb{B}_\pm$.

Then for each $(b_-, b_+) \in \mathcal{B}_- \times \mathcal{B}_+$ there exist
(a) a unique element $b_- \circ b_+ \in \mathcal{H}_X^+(V^-, V^+)$ such that $\overline{b_- \circ b_+} = \overline{b_-} \circ \overline{b_+}$ and

$$b_- \circ b_+ - d_{b_-, b_+} b_- b_+ \in \sum_{\alpha_+ \in \tilde{\Gamma} \setminus \{0\}, b_{\nu^j}\pm \in \mathbb{B}_\pm : \deg b_{\nu^j}\pm + \alpha = \deg b_- b_+} \nu\mathbb{Z}[\nu] d_{\nu^j\nu^{j'}} K_{0, \alpha} b_{\nu^j}\pm b_{\nu^{j'}}^\pm.$$

The elements $\{K_{\alpha_-, \alpha_+} \circ (b_- \circ b_+) : \alpha_\pm \in \tilde{\Gamma}, b_\pm \in \mathbb{B}_\pm\}$ form a $\Gamma$-invariant basis of $\mathcal{H}_X^+(V^-, V^+)$. (b) a unique element $b_- \bullet b_+ \in \mathcal{U}_X(V^-, V^+)$ such that $\overline{b_- \bullet b_+} = \overline{b_-} \bullet \overline{b_+}$ and

$$b_- \bullet b_+ - b_- \circ b_+ \in \sum_{\alpha_\pm \in \tilde{\Gamma} \setminus \{0\}, b_{\nu^j}\pm \in \mathbb{B}_\pm : \deg b_{\nu^j}\pm + \alpha = \deg b_- b_+} \nu^{-1}\mathbb{Z}[\nu^{-1}] K_{\alpha_-, \alpha_+} b_{\nu^j}\pm b_{\nu^{j'}}^\pm.$$

The elements $\{K_{\alpha_-, \alpha_+} \circ (b_- \bullet b_+) : \alpha_\pm \in \tilde{\Gamma}, b_\pm \in \mathbb{B}_\pm\}$ form a $\Gamma$-invariant basis of $\mathcal{U}_X(V^-, V^+)$. \hfill \Box

Proof. To prove (a), we apply Proposition 2.3 with the following data. Let $\hat{E}$ be the free $R_0$-module generated by $\{d_{b_-, b_+} K_{\alpha_- \alpha_+} \circ (b_- b_+) : b_\pm \in \mathbb{B}_\pm, \alpha_\pm \in \tilde{\Gamma}\}$, which is clearly a $\tilde{\Gamma}$-graded $R$-module via the $\circ$ action. Then $E$ identifies with the $R_0$-submodule of the algebra $\mathcal{H}_X^0(V^-, V^+)$ generated by $\mathcal{E} := \{d_{b_-, b_+} b_- b_+ : b_\pm \in \mathbb{B}_\pm\}$. Let $\chi$ be the identity map of vector spaces $\mathcal{H}_X^0(V^-, V^+) \to \mathcal{H}_X^+(V^-, V^+)$. Let $R = \{K_{\alpha_- \alpha_+} : \alpha_\pm \in \tilde{\Gamma}\}$. Then in $\mathcal{H}_X^+(V^-, V^+)$ we have by Proposition 2.7

$$d_{b_-, b_+} K_{\alpha_- \alpha_+} \circ (b_- b_+) - d_{b_-, b_+} K_{\alpha_- \alpha_+} \circ (b_- b_+) = d_{b_-, b_+} K_{\alpha_- \alpha_+} \circ (b_+ b_- - b_- b_+).$$
\[ \sum_{\alpha \in \Gamma \setminus \{0\}, \nu_{\pm} \in B_{\pm} : \deg \nu_{\pm} + \alpha = \deg \nu_{\pm}} R_{0} d_{b_{-} b_{+}} K_{\alpha_{-}, \alpha_{+}} \circ (b'_{-} b'_{+}) \]

Thus, all assumptions of Proposition 2.3 are satisfied and hence for each \((\alpha_{-}, \alpha_{+}) \in \Gamma \oplus \Gamma, (b_{-}, b_{+}) \in B_{-} \times B_{+}\) there exists a unique element \(C_{\alpha_{-}, \alpha_{+}, b_{-}, b_{+}} \in \mathcal{H}_{\bar{\chi}}^{+}(V^{-}, V^{+})\) such that \(C_{\alpha_{-}, \alpha_{+}, b_{-}, b_{+}} = C_{C_{\alpha_{-}, \alpha_{+}, b_{-}, b_{+}}}\) and

\[ C_{\alpha_{-}, \alpha_{+}, b_{-}, b_{+}} = \sum_{\nu_{\pm} \in B_{\pm}, \alpha \in \Gamma \setminus \{0\}, \deg \nu_{\pm} = \deg \nu_{\pm} + \alpha} \nu Z[\nu] d_{b_{-} b_{+}} K_{\alpha_{-}, \alpha_{+}} \circ (b'_{-} b'_{+}) \]

Set \(b_{-} \circ b_{+} = C_{0, 0, b_{-}, b_{+}}.\) Then \(K_{\alpha_{-}, \alpha_{+}} \circ b_{-} \circ b_{+}\) has the same properties as \(C_{\alpha_{-}, \alpha_{+}, b_{-}, b_{+}}\) hence they coincide. This completes the proof of part (a).

To prove part (b), we again employ Proposition 2.3. Let \(E\) be the free \(R_{0}\)-submodule of \(\mathcal{U}_{\chi}(V^{-}, V^{+})\) generated by \(K_{\alpha_{-}, \alpha_{+}} \circ b_{-} \circ b_{+} : \alpha_{-}, \alpha_{+} \in \Gamma, b_{-}, b_{+} \in B_{\pm}\), which is clearly a \(\Gamma\)-graded \(R_{0}\)-module. Then \(E\) identifies with the free \(R_{0}\)-submodule of \(\mathcal{H}_{\bar{\chi}}^{+}(V^{-}, V^{+})\) generated by \(K_{0, \alpha} b_{-} \circ b_{+} : \alpha \in \Gamma, b_{-}, b_{+} \in B_{\pm}\). Let \(R = \{K_{0, \alpha} : \alpha \in \Gamma\}\). By part (a) we have

\[ \sum_{b_{-} \circ b_{+} = C_{0, 0, b_{-}, b_{+}}} \nu Z[\nu] K_{0, \alpha} \circ (b'_{-} b'_{+}) = 0. \]

Together with Proposition 2.7 this implies that in \(\mathcal{U}_{\chi}(V^{-}, V^{+})\)

\[ \sum_{b_{-} \circ b_{+} = C_{0, 0, b_{-}, b_{+}}} \nu Z[\nu] K_{0, \alpha} \circ (b'_{-} b'_{+}) = 0. \]

which together with (a) yields

\[ \sum_{b_{-} \circ b_{+} = C_{0, 0, b_{-}, b_{+}}} \nu Z[\nu] K_{0, \alpha} \circ (b'_{-} b'_{+}) = 0. \]

Note that in the last sum only terms with \(\alpha_{-} \neq 0\) may occur with non-zero coefficients, since \(b_{-} \circ b_{+}\) is \(\sim\)-invariant in \(\mathcal{H}_{\bar{\chi}}^{+}(V^{-}, V^{+})\). Thus, all assumptions of Proposition 2.3 are satisfied, and, using it with \(\nu\) replaced by \(\nu^{-1}\) we obtain the desired basis. The rest of the argument is essentially the same as in part (a) and is omitted. \(\square\)

**Remark 2.10.** In view of Remark 1.12, it would be interesting to compare our elements \(b_{-} \bullet b_{+}\) with those obtained by interchanging \(\nu\) and \(\nu^{-1}\) in and/or \(\mathcal{H}_{\bar{\chi}}^{+}(V^{-}, V^{+})\) with \(\mathcal{H}_{\bar{\chi}}^{-}(V^{-}, V^{+})\) in Theorem 2.9.
Choose bases $\B^0_+ = \{E_i\}_{i \in I}$ of $V^+$ and $\B_{0,-} = \{F_i\}_{i \in I}$ of $V^-$ such that $\deg E_i = \deg F_i$, $i \in I$; thus, $\Gamma_0 = \{\deg E_i\}_{i \in I}$. Assume that $E_i = F_i$ and $F_i = E_i$. Let $t$ be the unique anti-involution $\xi$, as defined in Lemma A.37(c), such that $E^t_i = F_i$, $F^t_i = E_i$.

**Proposition 2.11.** Let $\B_\pm$ be a $\Gamma$-homogeneous basis of $\B(V^\mp)$ consisting of $\gamma$-invariant and containing $\B^0_\pm$ and let $\B_- = \B_+ t$. Then for all $b_\pm \in \B_\pm$, $\alpha \in \Gamma$ we have

$$(K_{\alpha,-\alpha} \circ b_\pm \circ b_\pm)^t = K_{\alpha,-\alpha} \circ b_\pm^t \circ b_\pm^-,$$

$$(K_{\alpha,-\alpha} \circ b_- \bullet b_+)^t = K_{\alpha,-\alpha} \circ b_+^t \bullet b_-^-.$$

**Proof.** Since $\deg b_\pm^t = \deg b_\pm$, we have in $\H^+_\chi(V^-,V^+)$

$$(K_{\alpha,-\alpha} \circ b_- \circ b_+)^t = (\chi^{\frac{1}{2}}((\alpha_-,-\alpha_+),(\deg b_-,\deg b_+)))^{-1}b_+^t b_-^- K_{\alpha,-\alpha}.$$

Thus, the anti-automorphism $t$ of $\H^+_\chi(V^-,V^+)$ is triangular in the sense of Definition 2.4, hence $(K_{\alpha,-\alpha} \circ b_- \circ b_+)^t = K_{\alpha,-\alpha} \circ b_+^t \bullet b_-^-$. This implies that the anti-automorphism $t$ of $\U_\chi(V^-,V^+)$ is also triangular in the sense of Definition 2.4 with $\nu$ replaced by $\nu^{-1}$, and the second assertion follows. 

\[\Box\]

3. **Dual canonical bases and proofs of Theorems 1.3, 1.5, 1.10 and 1.19**

We fix some notation which will be used repeatedly throughout the rest of the paper. Define in $\mathbb{Q}(\nu)$

$$[a]_\nu = \frac{\nu^a - 1}{\nu - 1}, \quad [a]_\nu! = \prod_{j=1}^{a} [j]_\nu, \quad \left[\begin{array}{c} a \\ n \end{array}\right]_\nu = \frac{[a]_\nu [a-1]_\nu \cdots [a-n+1]_\nu}{[n]_\nu!} \quad (3.1)$$

$$\langle a \rangle_\nu = \frac{\nu^a - \nu^{-a}}{\nu - \nu^{-1}}, \quad \langle a \rangle_\nu! = \prod_{j=1}^{a} \langle j \rangle_\nu, \quad \left[\begin{array}{c} a \\ n \end{array}\right]_\nu = \frac{\langle a \rangle_\nu [a-1]_\nu \cdots [a-n+1]_\nu}{[n]_\nu!} \quad (3.2)$$

and

$$\langle a \rangle_\nu = \nu^a - \nu^{-a}, \quad \langle a \rangle_\nu! = \prod_{j=1}^{a} \langle j \rangle_\nu. \quad (3.3)$$

We always use the convention that $\left[\begin{array}{c} a \\ n \end{array}\right]_\nu = 0$ if $n < 0$. If $a, n$ are non-negative integers, then all expressions in (3.1) lie in $1 + \nu \mathbb{Z}_{\geq 0}[\nu]$ while all expressions in (3.2) are in $\mathbb{Z}_{\geq 0}[\nu + \nu^{-1}]$. Clearly,

$$[a]_\nu^2 = \nu^a (a)_\nu = \nu^{-a} (\nu - \nu^{-1})^{-1} \langle a \rangle_\nu,$$

hence

$$[a]_\nu!^2 = \nu^{\langle a \rangle_\nu!} (a)_\nu! = \nu^{\langle a \rangle_\nu!} (\nu - \nu^{-1})^{-a} \langle a \rangle_\nu!, \quad \left[\begin{array}{c} a \\ n \end{array}\right]_\nu^2 = \nu^{\langle a \rangle_\nu! [a-n]_\nu} \left[\begin{array}{c} a \\ n \end{array}\right]_\nu \quad (3.4)$$

(thus, there is no need to introduce “angular” $\nu$-binomial coefficients). Finally,

$$\left[\begin{array}{c} a \\ n \end{array}\nu^{-2} = \nu^{n(a-n)} \left[\begin{array}{c} a \\ n \end{array}\right]_\nu = \nu^{2n(a-n)} \left[\begin{array}{c} a \\ n \end{array}\nu^{-2}. \quad (3.4)$$

For every symbol $X_i, i \in I$ such that $X^n_i$ is defined we set $X^{(n)}_i = X^n_i / q_i = (q_i - q_i^{-1})^n X^n_i$. 


3.1. Bicharacters, pairings, lattices and inner products. Let $k = \mathbb{Q}(q^{\frac{1}{2}})$ and let $R_0 = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Let $g$ be a symmetrizable Kac-Moody Lie algebra and let $A = (a_{ij})_{i,j \in I}$ be its Cartan matrix. Fix positive integers $d_i$, $i \in I$ such that $d_ia_{ij} = a_{ij}d_j$, $i,j \in I$. Let $K$ be the monoidal algebra of $\hat{\Gamma}$ with the basis $\{K_{\alpha \cdot \alpha} : \alpha \in \Gamma\}$ and denote $K_{\pm i} := K_{\alpha_{\pm i}}$. The monoid $\Gamma$ (and hence $\hat{\Gamma}$) clearly affords a sign character (cf. §A.8).

Define a symmetric bicharacter $\cdot : \Gamma \times \Gamma \rightarrow \mathbb{Z}$ by $\alpha_i \cdot \alpha_j = d_i$, $i \in I$. We extend $\cdot$ to a bicharacter of $\hat{\Gamma}$ via $\alpha_{\pm i} \cdot \alpha_{\pm j} = -d_ia_{ij}$, $i,j \in I$ and $\eta$ to a character of $\hat{\Gamma}$ via $\eta(\alpha_{\pm i}) = \eta(\alpha_i)$, $i \in I$. Define $\gamma : \Gamma \rightarrow \mathbb{Z}$ by $\gamma(\alpha) = \frac{1}{2}\alpha \cdot \alpha - \eta(\alpha)$, $\alpha \in \Gamma$. Then

$$\gamma(\alpha_i) = 0, \quad \gamma(\alpha + \alpha') = \gamma(\alpha) + \gamma(\alpha') + \alpha \cdot \alpha', \quad i \in I, \alpha, \alpha' \in \Gamma.$$

This implies that $\gamma : \Gamma \rightarrow k^\times$ defined by $\gamma(\alpha) = q^{\gamma(\alpha)}$, $\alpha \in \Gamma$ is the function discussed in §A.8.

Let $V^+ = \bigoplus_{i \in I} kE_i$, $V^- = \bigoplus_{i \in I} kF_i$. We regard $V^\pm$ as $\Gamma$-graded with $\deg E_i = \deg F_i = \alpha_i$. It is well-known (cf. [19, Chapter 1] and §§A.1, A.8) that $U_q^\pm$ is the Nichols algebra $B(V^\pm, \Psi^\pm)$ where the braiding $\Psi^\pm$ is defined via the bicharacter $\chi$ as in §A.8.

Define a pairing $\langle \cdot, \cdot \rangle : V^- \otimes V^+ \rightarrow k$ by $\langle F_i, E_j \rangle = \delta_{ij}(q_i - q_i^{-1})$. Then $\langle \cdot, \cdot \rangle$ extends to a pairing of braided Hopf algebras $U_q^- \otimes U_q^+ \rightarrow k$ (see §§A.3, A.8). The algebra $U_{k,t,\ldots,t}(V^-, V^+)$, $t \in k$, is then $\hat{\Gamma}$-graded as in §2.2.

**Proposition 3.1.** The algebra $U_q(\hat{g})$ is isomorphic to $U_{k,1,1}(V^-, V^+)$ while $\mathcal{H}_q^\pm(\hat{g})$ identify with the subalgebra of $\mathcal{H}_\chi^\pm(V^-, V^+)$ generated by the $K_{\pm i}$ (respectively, $K_{\pm i}$), $E_i$ and $F_i$, $i \in I$, in the notation of §2.2.

**Proof.** After [19, Proposition 1.4.3], (1.3) hold in $B(V^\pm)$, while (A.38) yield (1.2). Thus, $U_{k,1,1}(V^-, V^+)$ is a $\hat{\Gamma}$-graded quotient of $U_q(\hat{g})$, and it remains to observe that their homogeneous subspaces have the same dimensions. The assertion about $\mathcal{H}_q^\pm(\hat{g})$ is proved similarly. \qed

Define $\overline{\cdot} : V^- \rightarrow V^\pm$ as the unique anti-linear map satisfying $\overline{E_i} = E_i$, $\overline{F_i} = F_i$, $i \in I$. Then $\langle v^-, \overline{v}^+ \rangle = -\langle v^-, v^+ \rangle$, hence by Lemma A.37 $U_q(\hat{g})$ admits an anti-linear anti-involution $\overline{\cdot}$ preserving the generators, an anti-involution $^* $ preserving the $E_i$ and the $F_i$, $i \in I$ and satisfying $K_{\pm i}^* = \overline{K_{\pm i}}$, $i \in I$, and an anti-involution $^t$ which restricts to anti-isomorphisms $U_q^\pm \rightarrow U_q^\pm$ such that $E_i^t = F_i$, $F_i^t = E_i$, and preserves the $K_{\pm i}$, $i \in I$. In particular, $^t$ is an involution which restricts to isomorphisms $U_q^\pm \rightarrow U_q^\pm$.

Let $\mathbb{Z}U^+$ (respectively, $\mathbb{Z}U^-$) be the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_q^+$ (respectively, $U_q^-$) generated by the $E_i^{(n)}$ (respectively, $F_i^{(n)}$, $i \in I$, $n \in \mathbb{Z}_{\geq 0}$; thus, $\mathbb{Z}U^\pm$ is the preimage under $\psi_\pm$ of the subalgebra of $U_q^\pm$ generated by the usual divided powers ([19, §1.4.7]). Define

$$U^+_\mathbb{Z} = \{x \in U^+_q : \langle x, \mathbb{Z}U^- \rangle \subset \mathbb{Z}[q,q^{-1}]\}, \quad U^-_{\mathbb{Z}} = \{x \in U^-_q : \langle x, \mathbb{Z}U^+ \rangle \subset \mathbb{Z}[q,q^{-1}]\}$$

**Proposition 3.2.** $U^\pm_{\mathbb{Z}}$ is a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U^\pm_q$ satisfying $\Delta(U^\pm_{\mathbb{Z}}) \subset U^\pm_{\mathbb{Z}} \otimes \mathbb{Z}[q,q^{-1}] U^\pm_{\mathbb{Z}}$.

**Proof.** We prove the statements for $U^+_{\mathbb{Z}}$ only, the argument for $U^-_{\mathbb{Z}}$ being similar. Let $R = \mathbb{Z}[q,q^{-1}]$. The following result is immediate from [19, Lemma 1.4.1]

**Lemma 3.3.** $\mathbb{Z}U^\pm$ is an $R$-subalgebra of $U^\pm_q$ satisfying $\Delta(\mathbb{Z}U^\pm) \subset \mathbb{Z}U^\pm \otimes_R \mathbb{Z}U^\pm$.

Since $U^+_\mathbb{Z}$ is a direct sum of free $R$-modules of finite length, $U^+_{\mathbb{Z}}$ is canonically isomorphic to the graded $\text{Hom}_R(\mathbb{Z}U^-, R)$, which immediately implies the proposition. \qed
3.2. Dual canonical bases. Let $\psi : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the homomorphism defined by $E_i \mapsto (q_i^{-1} - q_i)^{-1} E_i$, $F_i \mapsto (q_i - q_i^{-1})^{-1} F_i$, $K_{\pm i} \mapsto K_{\mp i}$. Denote by $\tilde{\psi}$ its restrictions to $U_q^\pm$. Clearly, the images of generators of $U_q(\mathfrak{g})$ under $\psi$ satisfy the relations of the “standard” presentation of $U_q(\mathfrak{g})$; for example
\[ [\psi(E_i), \psi(F_j)] = \delta_{ij}\frac{K_{i+1} - K_{i-1}}{q_i - q_i^{-1}}. \]

Let $B_{\text{can}}$ be the preimage under $\tilde{\psi}$ of Lusztig’s canonical basis of $U_q^-$ ([19, Chapter 14]). By [19, Theorem 14.4.3], $B_{\text{can}}$ is a $\mathbb{Z}[q,q^{-1}]$-basis of $U_q^-$. If $\mathfrak{g} = \mathfrak{sl}_2$, $B_{\text{can}} = \{ F^{(r)} : r \in \mathbb{Z}_{>0} \}$.

Let $\langle \cdot, \cdot \rangle : U_q^- \otimes U_q^- \to \mathbb{k}$ be the pairing defined in \S A.9 with $\xi$ being the anti-involution $^t$ described above. Since $\langle \cdot, \cdot \rangle$ is non-degenerate and restricts to non-degenerate bilinear forms on finite dimensional graded components of $U_q^-$, for each $b \in B_{\text{can}}$ there exists a unique $\delta_b \in U_q^-$ such that $\langle \delta_b, b' \rangle = \delta_{b,b'}$ for all $b' \in B_{\text{can}}$.

**Definition 3.4.** The dual canonical basis $B_{n-}$ of $U_q^-$ is the set $\{ \delta_b : b \in B_{\text{can}} \}$. The dual canonical basis $B_{n+}$ of $U_q^+$ is defined as $B_{n+} = B_{n-}^\ast$.

This definition is justified by the following Lemma.

**Lemma 3.5.** For all $b_\pm \in B_{n \pm}$, $\tilde{b}_\pm = b_\pm$ and $b'_\pm \in B_{n \pm}$.

**Proof.** Note that for all $b \in B_{\text{can}}$, $\overline{\psi(b)} = \psi(b)$, hence $\tilde{b}_\pm = \text{sgn}(b) b_\pm$. Moreover, $b_\pm \in B_{\text{can}}$ by [19, \S 14.4]. It remains to apply (A.46). \hfill \Box

**Proposition 3.6.** The set $\{ q^{-\frac{1}{2}\gamma(b_\pm)} b_\pm : b_\pm \in B_{n \pm} \}$ is a $\mathbb{Z}[q,q^{-1}]$-basis of $U_q^\pm$.

**Proof.** It suffices to prove that $\{ q^{-\frac{1}{2}\gamma(b)} \delta_b^\ast : b \in B_{\text{can}} \}$ generates $U_q^+$ as a $\mathbb{Z}[q,q^{-1}]$-module. Let $b, b' \in B_{\text{can}}$. Then
\[ q^{-\frac{1}{2}\gamma(b_\pm)} \delta_b^\ast \in U_q^\pm. \]

Therefore, $q^{-\frac{1}{2}\gamma(b_\pm)} \delta_b^\ast \in U_q^+$. Let $x \in U_q^+$ and write $x = \sum_{b \in B_{\text{can}}} q^{-\frac{1}{2}\gamma(b)} c_b \delta_b^\ast$, $c_b \in \mathbb{k}$. Then for all $b \in B_{\text{can}}$,
\[ \langle b, x \rangle = \sum_{b' \in B_{\text{can}}} q^{-\frac{1}{2}\gamma(b')} c_{b'} \langle b, \delta_{b'}^\ast \rangle = c_b. \]

Thus, $c_b \in \mathbb{Z}[q,q^{-1}]$ for all $b \in B_{\text{can}}$. \hfill \Box

Then Propositions 3.2, 3.6 and (3.5) imply the following.

**Corollary 3.7.** The structure constants $\tilde{C}_{b_\pm, b'_\pm}^{b_\pm', b'_\pm'}$, $b_\pm, b'_\pm, b'_\pm' \in B_{n \pm}$ defined by
\[ b_\pm b'_\pm = q^{-\frac{1}{2} \deg b_\pm - \deg b'_\pm} \sum_{b''_\pm' \in B_{n \pm}} \tilde{C}_{b_\pm, b'_\pm}^{b_\pm', b''_\pm'} b''_\pm', \quad \Delta(b_\pm) = \sum_{b'_\pm, b''_\pm' \in B_{n \pm}} q^{\frac{1}{2} \deg b'_\pm - \deg b''_\pm'} \tilde{C}_{b_\pm, b'_\pm}^{b_\pm', b''_\pm'} b'_\pm \otimes b''_\pm' \]
belong to $\mathbb{Z}[q,q^{\pm1}]$.

It follows immediately from the above Corollary that for any $b_\pm \in B_{n \pm}$
\[ \Delta(b_\pm) = \sum_{b''_\pm' \in B_{n \pm}} q^{\frac{1}{2} \deg b'_\pm - \deg b''_\pm'} \tilde{C}_{b_\pm, b''_\pm'}^{b_\pm', b''_\pm'} b'_\pm \otimes b''_\pm', \quad (3.6) \]
where $\tilde{C}_{b_\pm, b''_\pm'}^{b_\pm', b''_\pm'} = \sum_{b_\pm \in B_{n \pm}} C_{\cdot b'_\pm \cdot}^{b'_\pm b''_\pm'} C_{\cdot b''_\pm' \cdot}^{b''_\pm' b''_\pm'} \in \mathbb{Z}[q,q^{\pm1}]$. 

Remark 3.8. It is easy to check that for any \( b, b', b'' \in B^\text{can} \) we have
\[
bb' = \sum_{b'' \in B^\text{can}} C^{\delta b b''}_{\delta b' b''} b'' , \quad \Delta(b'') = \sum_{b,b' \in B^\text{can}} C^{\delta b b''}_{\delta b' b''} b \otimes b'.
\]
After [19, §14.4.14], these structure constants are Laurent polynomials in \( q \).

Proposition 3.9. \((B_{n-}, B_{n+}) \subset \langle U^-_{Z, U^-} \rangle \subset \mathbb{Z}[q, q^{-1}, \Phi^{-1}_k : k > 0] \), where \( \Phi_k \in \mathbb{Z}[q] \) is the \( k \)th cyclotomic polynomial.

Proof. Indeed, it is immediate from properties of \( \langle \cdot, \cdot \rangle \) that \( \langle F^{(b_1)}_{j_1} \ldots F^{(b_s)}_{j_s}, E^{(a_1)}_{i_1} \ldots E^{(a_r)}_{i_r} \rangle \in \mathbb{Z}[q, q^{-1}] \) for any \((i_1, \ldots, i_r) \in I^r, (j_1, \ldots, j_s) \in I^s\) and for any \( a = (a_1, \ldots, a_r) \in \mathbb{Z}_{\geq 0}^r, b = (b_1, \ldots, b_s) \in \mathbb{Z}_{\geq 0}^s\). Therefore,
\[
\langle F^{(b_1)}_{j_1} \ldots F^{(b_s)}_{j_s}, E^{(a_1)}_{i_1} \ldots E^{(a_r)}_{i_r} \rangle \in R' := \mathbb{Z}[q, q^{-1}, \Phi^{-1}_k, k > 0] .
\]
This implies that \( \langle zU^-, zU^+ \rangle \subset R' \). We need the following, apparently well-known result.

Lemma 3.10. Let \( \alpha \in \Gamma \). Let \( B^-_{\alpha} \) be any basis of \( zU^-_{\alpha} \) \( \{ u \in zU^- : \deg u = \alpha \} \) and set \( G_{\alpha} = (\langle b, b' \rangle)_{b, b' \in B^-_{\alpha}} \) be the corresponding Gram matrix. Then \( \det G_{\alpha} = q^n \prod_k \Phi_k(q) a_k \) where \( a_k \in \mathbb{Z} \) and \( n \in \mathbb{Z} \).

Proof. It well-known ([19]) that the specialization of the form \( \langle \cdot, \cdot \rangle \) for any \( q = \zeta \), where \( \zeta \in \mathbb{C}^\times \) is not a root of unity, is well defined and non-degenerate. Thus, \( \det G_{\alpha} \) is a rational function of \( q \) whose zeroes and poles are roots of unity and zero. This implies \( \det G_{\alpha} = q^n \prod_k \Phi_k(q) a_k \) where \( c \in \mathbb{Q} \) and \( n, a_k \in \mathbb{Z} \). It remains to prove that \( c = 1 \). To prove this claim, note that by [19, Theorem 14.2.3] and properties of \( \langle \cdot, \cdot \rangle \), for any \( b \in B^\text{can} \), there exists \( b \in q^{-1/2}b \) such that for all \( b, b' \in B^\text{can} \),
\[
\langle b, b' \rangle = \delta b b' + q^{-1}Z[q^{-1}] .
\]
This in turn implies that for \( B^-_{\alpha} = \{ b : b \in B^\text{can}, \deg b = \alpha \} \), \( \det G_{\alpha} \in 1 + q^{-1}Z[q^{-1}] \). Since \( \det G_{\alpha} \) is, up to a power of \( q \), independent of the choice of basis \( B^-_{\alpha} \), it follows that \( c = 1 \). \( \square \)

Now, let \( B_{-\alpha} \) be any basis of \( (U^-_{Z})_{\alpha} = \{ u \in U^- : \deg u = \alpha \} \) and let \( B_0 \) be the dual basis of \( B_{-\alpha} \) with respect to \( \langle \cdot, \cdot \rangle \). Then the Gram matrix \( G_{-\alpha} = (\langle b, b' \rangle)_{b, b' \in B_{-\alpha}} \) satisfies \( G_{-\alpha} = G_{\alpha}^{-1} \) over \( \mathbb{Q}(q) \). As \( \langle zU^-, zU^+ \rangle \subset R' \), all entries of \( G_{\alpha} \) are in \( R' \), while \( \det G_{\alpha}^{-1} \) is in \( R' \) by Lemma 3.10. Therefore, all entries of \( G_{-\alpha} \) are in \( R' \).

To prove the second inclusion note that by Proposition 3.6, we have for all \( b_\pm \in B_{0\pm} \)
\[
\langle q^{-\frac{1}{2}}(\deg b_-) b_-, q^{-\frac{1}{2}}(\deg b_+)+\frac{1}{2}(\deg b_-) \rangle (b_-, b_+) \in R'
\]
since \( \langle b_-, b_+ \rangle = 0 \) implies that \( \deg b_+ = \deg b_- \). \( \square \)

For \( \mathfrak{g} \) semisimple we can strengthen Proposition 3.9 as follows.

Theorem 3.11. If \( \mathfrak{g} \) is semisimple then \( \langle B_{-\alpha}, B_{\alpha} \rangle \subset \langle U^-_{Z}, U^-_{Z} \rangle = \mathbb{Z}[q, q^{-1}] \).

We prove this Theorem in Section 5. We expect that the converse is also true: if \( \langle U^-_{Z}, U^+_{Z} \rangle \subset \mathbb{Z}[q, q^{-1}] \) then \( \mathfrak{g} \) is semisimple (see Lemma 4.14 and Example 4.16).

Remark 3.12. We can conjecture that \( \langle U^-_{Z}, U^+_{Z}(w) \rangle \subset \mathbb{Z}[q, q^{-1}] \) where \( w \in W \) and \( U^+_{Z}(w) \) is the corresponding Schubert cell.

3.3. Proofs of Theorems 1.3, 1.5 and 1.10. First, we need a stronger version of Proposition 2.7.

Proposition 3.13. Let \( d : B_{n-} \times B_{n+} \to \mathbb{Z}[\nu + \nu^{-1}], \nu = q^{\frac{1}{2}}, \) be defined as in Proposition 2.7. Then for all \( b_\pm \in B_{n\pm}, d_{b_- b_+} = \prod_{k \geq 3} (q^{-\frac{1}{2}}\nu^k) \Phi_k(q) a_k \), \( a_k \in \mathbb{Z}_{\geq 0} \), and, in particular, is monic. Moreover, in \( U_q(\widehat{\mathfrak{g}}) \) we have
\[
d_{b_- b_+}(b_+ b_--b_- b_+) \in \sum_{(\alpha, \alpha \pm, b_\pm) \in \Gamma \setminus \{(0, 0)\}} \mathbb{Z}[q, q^{-1}] d_{b_- b_+} K_{\alpha, \alpha \pm} \otimes b_- b_+ .
\]
Proof. By (2.7)

$$b_+b_- = \sum_{b'\pm_1 \in B_{\pm}, \alpha_\pm \in \Gamma: \deg b'_\pm + \alpha_+ + \alpha_- = \deg b_\pm} F_{b'_\pm, b'_\mp, \alpha_+, \alpha_-}^{b_-, b_+} K_{\alpha_-, \alpha_+} b'_- b'_+$$

where by (2.8), (3.6), Lemma A.30 and (A.37)

$$F_{b'_\pm, b'_\mp, \alpha_+, \alpha_-} = \sum_{b'\pm_1 \in B_{\pm}, \deg b'\pm_1 = \alpha_\pm} \chi^{\frac{1}{2}}(b'_\pm, b'_\mp) \chi^{\frac{1}{2}}(b'_\mp, b'_\pm) C_{b_\pm}^{b'_\pm, b'_\mp, b''_\pm} \tilde{C}_{b_\mp}^{b'_\mp, b'_\pm, b''_\mp} (b'_\mp, S^{-1}(b''_\pm)) (b''_\mp, b'_\pm)$$

$$= \text{sgn}(\alpha_+) q^{-\gamma(\alpha_+)} \chi^{\frac{1}{2}}(\alpha_- \deg b'_\pm) \chi^{\frac{1}{2}}(\alpha_+ \deg b'_\mp) \sum_{b'\pm_1 \in B_{\pm}, \deg b'\pm_1 = \alpha_\pm} \tilde{C}_{b_\pm}^{b'_\pm, b'_\mp, b''_\pm} (b'_\mp, b''_\pm) (b''_\mp, b'_\pm)$$

Since $\tilde{C}_{b_\pm}^{b'_\pm, b'_\mp, b''_\pm} \in \mathbb{Z}[q, q^{-1}]$, by Proposition 3.9 we have $\tilde{F}_{b'_\pm, b'_\mp, \alpha_+, \alpha_-} \in \mathbb{Z}[q, q^{-1}, \Phi_k^{-1}: k > 2]$. Thus, by Lemma 2.8 we can choose $d_{b_-, b_+}$ to satisfy the first assertion. Since by (3.7)

$$b_1 b_- = \sum_{b'\pm_1 \in B_{\pm}, \alpha_\pm \in \Gamma: \deg b'\pm_1 + \alpha_+ + \alpha_- = \deg b_\pm} F_{b'_\pm, b'_\mp, \alpha_+, \alpha_-}^{b_-, b_+} K_{\alpha_-, \alpha_+} \circ b'_- b'_+,$$

the second assertion is now immediate. □

Proofs of Theorems 1.3, 1.5. We apply Theorem 2.9 with the data from §3.1:

- $k = \mathbb{Q}(\nu)$, $R_0 = \mathbb{Z}[\nu, \nu^{-1}]$, $\nu = q^{\frac{1}{2}}$
- $\Gamma = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}(\alpha_i + \alpha_{i+1})$
- $K_{0, \alpha_i} = 0$, $K_{\alpha_i, 0} = 0$
- $V^+ = \bigoplus_{i \in I} \mathbb{K} E_i$, $V^- = \bigoplus_{i \in I} \mathbb{K} F_i$
- $\gamma$ is determined by $E_i = E_i$, $F_i = F_i$, $i \in I$
- $\chi(\alpha_i, \alpha_j) = q^{\delta_{ij}}$, $\langle E_i, F_j \rangle = \delta_{ij}(q_i - q_j^{-1})$, $i, j \in I$

Then $U_q(\hat{g}) = U_q(V^-, V^+)$ while $\mathcal{H}_q^+(\hat{g})$ identifies with the subalgebra of $\mathcal{H}_q^+(V^-, V^+)$ generated by $K_{\alpha_i}$, $E_i$, $F_i$, $i \in I$. Applying Theorem 2.9(a), we obtain elements $b_- \circ b_+ \in \mathcal{H}_q^+(\hat{g})$ which proves Theorem 1.3. Theorem 1.5 then follows from Theorem 2.9(b).

It remains to prove that all coefficients in the decompositions of invariant bases with respect to the initial ones in Theorems 1.3 and 1.5 are polynomials in $q$ or $q^{-1}$ and not just in $q^{\frac{1}{2}}$. But this is immediate from Proposition 3.13. □

Proof of Theorem 1.10. This is immediate from Proposition 2.11 since the anti-involution $^t$ of $U_q(\hat{g})$ satisfies $B_{n_\pm^t} = B_{n_\mp}$ □

3.4. Colored Heisenberg and quantum Weyl algebras and their bases. Let $\hat{H}_q(\hat{g})$ be the $\mathbb{K}$-algebra generated by $U_q^\pm$ and $L_i^\pm$, $i \in I$ where

$$L_i E_i = q_i^{\frac{1}{2}(\epsilon(a_i))} E_i L_i, \quad L_i F_i = q_i^{-\frac{1}{2}(\epsilon(a_i))} F_i L_i, \quad [E_i, F_j] = \delta_{ij} \epsilon_i L_i^2 (q_i^{-1} - q_i).$$

Note that $\hat{H}_q(\hat{g})$ admits a $\mathbb{K}$-anti-linear anti-involution $^\gamma$ extending $^\gamma: U_q^\pm \to U_q^\pm$ and satisfying $L_i^{\pm 1} = L_i^{-1}$ and an anti-involution $^t$ extending the anti-isomorphisms $U_q^\pm \to U_q^\pm$ discussed above and preserving the $L_i^{\pm 1}$, $i \in I$. The following is obvious.

Lemma 3.14. (a) The assignments $E_i \mapsto E_i$, $F_i \mapsto F_i$, $K_{\epsilon_i} \mapsto L_i^2$, $K_{-\epsilon_i} \mapsto 0$, $i \in I$ define a homomorphism of algebras $\psi^\epsilon: U_q(\hat{g}) \to \hat{H}_q(\hat{g})$. □
(b) $\hat{H}_q^e(g)$ is generated by $\text{Im}\,\psi^e$ and $L_i^{-1}$, $i \in I$ and has the triangular decomposition $\hat{H}_q^e(g) = U_q^- \otimes \mathcal{L} \otimes U_q^+$, where $\mathcal{L}$ is the subalgebra generated by $L_i^{\pm 1}$, $i \in I$.

(c) $\psi^e$ commutes with $\bar{\cdot}$ and $t$.

(d) The set $B_{n^-} \cdot \bullet B_{n^+} := \psi^e(B_{n^-} \cdot B_{n^+})$ is linearly independent and $L \cdot B_{n^-} \cdot \bullet B_{n^+}$, where $L$ is the multiplicative subgroup of $\mathbb{L}$ generated by the $L_i^{\pm 1}$, $i \in I$, is a basis of $\hat{H}_q^e(g)$.

Note that $\hat{H}_q^e(g)$ is graded by the group $Q := \mathbb{Z}^I$ with $\deg Q E_i = \deg Q F_i = \deg Q L_i = \alpha_i = -\deg Q L_i^{-1}$, where $\{\alpha_i\}_{i \in I}$ is the standard basis of $\mathbb{Z}^I$. Let $\hat{H}_q^e(g)_0$ be the subalgebra of elements of degree 0.

**Lemma 3.15.** There exists a unique projection $\tau : \hat{H}_q^e(g) \rightarrow \hat{H}_q^e(g)_0$ commuting with $\bar{\cdot}$ such that $\tau(x) \in q^{\frac{1}{2}x} \prod_{i \in I} L_i^{-n_i} x$ for $x$ homogeneous with $\deg Q x = \sum_{i \in I} n_i \alpha_i$.

Let $A_q^e(g)$ be the k-algebra with presentation (1.6). The following Lemma is easily checked.

**Lemma 3.16.** The algebra $A_q^e(g)$ admits an anti-linear anti-involution $\bar{\cdot}$ defined on generators by $\bar{x}_i = x_i$, $\bar{y}_i = y_i$, and an anti-involution $t$ defined by $x_i^t = y_i$, $y_i^t = x_i$.

**Proposition 3.17.** The assignments $x_i \mapsto q^{\frac{1}{2}x_i} L_i^{-1} E_i$, $y_i \mapsto q^{\frac{1}{2}y_i} F_i L_i^{-1}$ define an isomorphism of algebras $j_e : A_q^e(g) \rightarrow \hat{H}_q^e(g)_0$ which commutes with $\bar{\cdot}$ and $t$. Moreover, $A_q^e(g)$ has a triangular decomposition $A_q^e(g) = U_q^{e,+} \oplus U_q^{e,-}$ where $U_q^{e,+}$ (respectively, $U_q^{e,-}$) is the subalgebra of $A_q^e(g)$ generated by the $x_i$ (respectively, $y_i$), $i \in I$.

**Proof.** Let $X_i = L_i^{-1} E_i$, $Y_i = F_i L_i^{-1}$. Then in $\hat{H}_q^e(g)$ we have

$$0 = \sum_{r+s=1-a_{ij}} (-1)^r E_i^{(s)} E_j^{(r)} = \sum_{r+s=1-a_{ij}} (-1)^r (L_i X_i^{(s)} L_j X_j (L_i X_i)^{(r)})$$

$$= \sum_{r+s=1-a_{ij}} (-1)^r q_i^{a_{ij}[(s)-]} L_i^{a_{ij}} X_i^{(s)} L_j X_j L_i^{a_{ij}} X_j (L_i X_i)^{(r)}$$

$$= L_i^{1-a_{ij}} L_j \sum_{r+s=1-a_{ij}} (-1)^r q_i^{a_{ij}[(s)+]} t_{a_{ij}} X_i^{(s)} X_j (L_i X_i)^{(r)}$$

$$= q_i^{a_{ij}[(s)+]} L_i^{1-a_{ij}} L_j \sum_{r+s=1-a_{ij}} (-1)^r q_i^{a_{ij}[(s)+]} X_i^{(s)} X_j (L_i X_i)^{(r)}.$$}

This implies that

$$\sum_{r+s=1-a_{ij}} (-1)^r q_i^{a_{ij}[(s)+]} X_i^{(s)} X_j (L_i X_i)^{(r)} = 0.$$}

Thus, the $X_i$ satisfy the defining identity of $A_q^e(g)$. Since $Y_i = X_i^t$, the identity for the $Y_i$ is now immediate. The remaining identities are trivial. Thus, $j_e$ is a well-defined homomorphism of algebras $A_q^e(g) \rightarrow \hat{H}_q^e(g)_0$ and its image clearly lies in $\hat{H}_q^e(g)_0$. Since the defining relations of $U_q^{e,+}$ are the only relations in the subalgebra of $\hat{H}_q^e(g)_0$ generated by the $\{X_i\}_{i \in I}$, it follows that the restrictions of $j_e$ to $U_q^{e,+}$ are injective. Since the corresponding subalgebras quasi-commute, the assertion follows.

Now we have all necessary ingredients to prove Theorem 1.19.

**Proof of Theorem 1.19.** It follows from Lemma 3.15 and Proposition 3.17 that $\tau(B_{n^-} \cdot \bullet B_{n^+})$ is a basis of $\hat{H}_q^e(g)_0$. Then $B_{n^-} \cdot \bullet B_{n^+} := j_e^{-1} \tau(B_{n^-} \cdot \bullet B_{n^+})$ is the desired basis of $A_q^e(g)$.
3.5. Invariant quasi-derivations. Following Lemma A.34 and also [19, Proposition 3.1.6], define \( k \)-linear endomorphisms \( \partial_i, \partial_i^{op}, i \in I \) of \( U_q^+ \) by
\[
[F_i, x^+] = (q_i - q_i^{-1})(K_{i,+} \circ \partial_i(x^+) - K_{i,-} \circ \partial_i^{op}(x^+)), \quad x^+ \in U_q^+.
\] (3.8)

Then
\[
[x^-, E_i] = (q_i - q_i^{-1})(K_{i,+} \circ \partial_i(x^{-t}) - K_{i,-} \circ \partial_i^{op}(x^{-t})), \quad x^- \in U_q^-.
\]

Lemma 3.18. For all \( x^+, y^+ \in U_q^+ \), \( i \in I \) we have
(a) \( \partial_i(x^+) = \partial_i(x^+), \partial_i^{op}(x^+) = \partial_i^{op}(x^+) \)
(b) \( \partial_i(x^{+a}) = (\partial_i^{op}(x^+))^a \)
(c) \( \partial F_i(x^+) = (q_i - q_i^{-1})q_i^{\frac{1}{2}\alpha_i^\gamma(x^+ - \alpha_i)} \partial_i(x^+), \partial F_i^{op}(x^+) = (q_i - q_i^{-1})q_i^{\frac{1}{2}\alpha_i^\gamma(x^+ - \alpha_i)} \partial_i^{op}(x^+) \)
(d) \( \partial_i, \partial_i^{op} \) are quasi-derivations. Namely, for \( x^+, y^+ \in U_q^+ \) homogeneous we have
\[
\partial_i(x^+ y^+) = q_i^{\frac{1}{2}\alpha_i^\gamma(deg y^+)} \partial_i(x^+) y^+ + q_i^{\frac{1}{2}\alpha_i^\gamma(deg x^+)} x^+ \partial_i(y^+),
\]
\[
\partial_i^{op}(x^+ y^+) = q_i^{\frac{1}{2}\alpha_i^\gamma(deg y^+)} \partial_i^{op}(x^+) y^+ + q_i^{\frac{1}{2}\alpha_i^\gamma(deg x^+)} x^+ \partial_i^{op}(y^+).
\] (3.9)

(e) \( \partial_i \partial_j^{op} = \partial_j^{op} \partial_i \) for all \( i, j \in I \)

Proof. Parts (a)–(b) are immediate consequences of (3.8). Part (c) follows from (3.8) and (A.41). Then (d) is a consequence of (A.33) while (e) is immediate from (c) and Lemma A.9(c).

In particular, the operators \( \partial_i, \partial_i^{op} \) are locally nilpotent hence we can define a function \( \ell_i : U_q^+ \to \mathbb{Z}_{\geq 0} \)
\[
\ell_i(x^+) = \max\{k \in \mathbb{Z}_{\geq 0} : \partial_i^k(x^+) \neq 0\}, \quad x^+ \in U_q^+.
\]

Corollary 3.19. If \( x^+, y^+ \in U_q^+ \) are homogeneous then for all \( n \geq 0 \)
\[
(\partial_i^n)(x^+ y^+) = \sum_{a+b=n} q_i^{\frac{1}{2}\alpha_i^\gamma (a deg y^+ - b deg x^+)} \partial_i^{(a)}(x^+) \partial_i^{(b)}(y^+)
\]
\[
(\partial_i^{op})^n(x^+ y^+) = \sum_{a+b=n} q_i^{\frac{1}{2}\alpha_i^\gamma (-a deg y^+ + b deg x^+)} (\partial_i^{op})^{(a)}(x^+) \partial_i^{(b)}(y^+)
\] (3.10)

In particular,
\[
(\partial_i^{(top)})(x^+ y^+) = q_i^{\frac{1}{2}\alpha_i^\gamma (\ell_i(x^+) deg y^+ - \ell_i(y^+) deg x^+)} (\partial_i^{(top)})(x^+) (\partial_i^{(top)})(y^+),
\]
where \( (\partial_i^{(top)})(x^+) = \partial_i^{(\ell_i(x^+))}(x^+) \).

We define \( \partial_i^{-}, \partial_i^{op} : U_q^- \to U_q^- \) by \( \partial_i^{-}(x) = \partial_i(x^t)^t \) and \( \partial_i^{op}(x) = \partial_i^{op}(x^t)^t \), \( x \in U_q^- \). Then \( \ell_i : U_q^- \to \mathbb{Z}_{\geq 0} \) and \( (\partial_i^{(top)})^{-} \) are defined accordingly. We will sometimes use the notation \( \partial_i^{-}, \partial_i^{(top)} \) for \( \partial_i^{-}, \partial_i^{op} \).

Lemma 3.20. For all \( x, y \in U_q^- \) and \( k \in \mathbb{Z}_{\geq 0} \)
\[
\langle (\partial_i^{(top)})^k(x), y \rangle = \langle x, F_i^{(k)}(y) \rangle, \quad \langle (\partial_i^{op})^k(x), y \rangle = \langle x, F_i^{(k)}(y) \rangle
\] (3.11)

Proof. It is sufficient to show that \( (q_i - q_i^{-1})\langle \partial_i(x^t)^t, y \rangle = \langle x, F_i y \rangle \). Then an obvious induction yields \( (q_i - q_i^{-1})^n\langle (\partial_i^{(top)})^n(x), y \rangle = \langle x, F_i^n y \rangle \) and the assertion follows. We have
\[
(q_i - q_i^{-1})\langle \partial_i(x^t)^t, y \rangle = (q_i - q_i^{-1})q_i^{-\frac{1}{2}\alpha_i^\gamma(deg y)} \langle y, \partial_i(x^t)^t \rangle = (q_i - q_i^{-1})q_i^{-\frac{1}{2}\alpha_i^\gamma(deg y)} \langle y, F_i^{op}(x^t)^t \rangle
\]
\[
= q_i^{-\frac{1}{2}\alpha_i^\gamma(deg y)} \langle y, F_i^{op}(x^t)^t \rangle
\]
\[
= q_i^{-\frac{1}{2}\alpha_i^\gamma(deg y) - \frac{1}{2}\alpha_i^\gamma(deg y)} \langle F_i y, x^t \rangle = q_i^{-\frac{1}{2}\alpha_i^\gamma(deg y + \alpha_i)} \langle F_i y, x^t \rangle = \langle x, F_i y \rangle.
\]

The second identity follows from the first since \( \langle x^*, y \rangle = \langle x, y^* \rangle \).
Example 3.21. Recall ([19, 14.5.3]) that $F_i^{(n)} \in \mathcal{B}_{\text{can}}$ for all $i \in I$, $n \in \mathbb{Z}_{\geq 0}$. Clearly, $\langle x, F_i^{(n)} \rangle = 0$ unless $x \in \mathbb{k} F_i^{n}$. Since $\partial_i(E_i^n) = (n)_{q_i} E_i^{n-1}$, it follows from Lemma 3.20 that $\langle F_i^n, F_i^{(n)} \rangle = 1$, hence $F_i^{n} = \delta_{F_i^{(n)}} \in \mathcal{B}_{n_i}$.

We will need some properties of $\mathcal{B}_{n_\pm}$ with respect to $\partial_i^\pm$ which we gather in the following proposition.

Proposition 3.22. Let $b_\pm \in \mathcal{B}_{n_\pm}$. Then

(a) For all $r \in \mathbb{Z}_{\geq 0}$,

$$
\langle (\partial_i^-)^{(r)}(b_-) \rangle = \sum_{b'_- \in \mathcal{B}_{n_-}} \tilde{C}_{b'_-, b_-}^F \langle F_i^{(r)}(b_-) \rangle = \sum_{b'_- \in \mathcal{B}_{n_-}} \tilde{C}_{b'_-, b_-}^\partial \langle \partial_i^- \rangle^{(r)}(b_-),
$$

(b) For all $r \in \mathbb{Z}_{\geq 0}$,

$$
\langle (\partial_i^+)^{(r)}(b_+) \rangle = \sum_{b'_+ \in \mathcal{B}_{n_+}} \tilde{C}_{b'_+, b_+}^F \langle F_i^{(r)}(b_+) \rangle = \sum_{b'_+ \in \mathcal{B}_{n_+}} \tilde{C}_{b'_+, b_+}^\partial \langle \partial_i^+ \rangle^{(r)}(b_+),
$$

where $\tilde{C}_{b'_-, b_-}^\partial, \tilde{C}_{b'_+, b_+}^\partial \in \mathbb{Z}[q, q^{-1}]$ are defined in Corollary 3.7. Thus, in particular, $\langle (\partial_i^\pm)^{(r)}(\mathcal{B}_{n_\pm}) \rangle \subseteq \mathbb{Z}[q, q^{-1}]\mathcal{B}_{n_\pm}$.

Proof. To prove (a), note that by Lemma 3.20, Remark 3.8 and Example 3.21 we have for any $b', b' \in \mathcal{B}_{\text{can}}$

$$
\langle (\partial_i^-)^{(r)}(b_-) \rangle = \sum_{b'_- \in \mathcal{B}_{\text{can}}} \langle (\partial_i^-)^{(r)}(b_-), b'_- \rangle \delta_{b'_-} = \sum_{b'_- \in \mathcal{B}_{\text{can}}} \langle \delta_{b_-}, F_i^{(r)}(b'_-) \rangle \delta_{b'_-} = \sum_{b'_- \in \mathcal{B}_{\text{can}}} \tilde{C}_{b'_-, b_-}^\partial \langle \partial_i^- \rangle^{(r)}(b_-).
$$

The remaining identities are proved similarly.

To prove (b), note that since $B_{n_+} = B_{n_+}^t$ and $B_{n_-} = B_{n_-}^t$, it suffices to prove that $\langle (\partial_i^-)^{(top)}(b_-) \rangle \in B_{n_-}$. Following [19, §14.3], denote $B_{i;i \geq r} = \mathcal{B}_{\text{can}} \cap F_i^{\infty} U_q^{-}$ and $B_{i;r} = B_{i;i \geq r} \setminus B_{i;i \geq r+1}$. It follows from [19, §14.3] that for all $i \in I$, $\mathcal{B}_{\text{can}} = \bigcup_{r \geq 0} B_{i;r}$. Let $b \in \mathcal{B}_{\text{can}}$ and let $n = \ell_i(b)$, $u = (\partial_i^-)^{(top)}(b) = (\partial_i^-)^{(n)}(b)$) Then $u \in \text{ker} \partial_i$ which, by Lemma 3.20, is orthogonal to $\mathcal{B}_{i;\text{can}}$, $s > 0$. Thus, we can write

$$
u = \sum_{b'_- \in \mathcal{B}_{\text{can}}} \langle u, b'_- \rangle \delta_{b'_-} = \sum_{b'_- \in \mathcal{B}_{\text{can}}} \langle \delta_{b_-}, F_i^{(n)}(b'_-) \rangle \delta_{b'_-}.
$$

By [19, Theorem 14.3.2], for each $b' \in \mathcal{B}_{\text{can}}$ there exists a unique $\pi_{i;n}(b') \in \mathcal{B}_{\text{can}}$ such that $F_i^{(n)}(b' - \pi_{i;n}(b')) \in \sum_{r < n} \mathbb{Z}[q, q^{-1}]\mathcal{B}_{\text{can}}$. Using Lemma 3.20, we conclude that for any $b' \in \mathcal{B}_{i;r}$ with $r > n$, $\langle \delta_{b_-}, F_i^{(r)}(b'_-) \rangle = \langle (\partial_i^-)^{(r)}(b_-), U_q^{-} \rangle = 0$. Thus,

$$
u = \sum_{b'_- \in \mathcal{B}_{i;0}} \langle \delta_{b_-}, \pi_{i;n}(b'_-) \rangle \delta_{b'_-}.
$$

Note that, since $u \neq 0$, we cannot have $\langle \delta_{b_-}, \pi_{i;n}(b'_-) \rangle = 0$ for all $b'_- \in \mathcal{B}_{i;0}$. Since $\langle \delta_{b_-}, b'_- \rangle = \delta_{b_-, b'_-}$, we conclude that there exists a unique $b' \in \mathcal{B}_{i;0}^{\text{can}}$ such that $\pi_{i;n}(b') = b$ and then $u = (\partial_i^-)^{(top)}(b) = \delta_{b'_-}$. Since $\pi_{i;n} : \mathcal{B}_{i;0}^{\text{can}} \to \mathcal{B}_{i;n}^{\text{can}}$ is a bijection by [19, Theorem 14.3.2], the assertion follows. □
Let $b_+ \in B_{n+}$ and let $r \leq \ell_i(b_+)$. By the above Proposition, there exists a unique $b'_+ \in B_{n+}$ such that $\ell_i(b'_+) = r$ and $\partial_i^{(r)}(b'_+) = \partial_i^{(\text{top})}(b_+)$. This implies that for each $b_+ \in B_{n+}$ and each $0 \leq r \leq \ell_i(b_+)$ there exists a unique element of $B_{n+}$, denoted $\tilde{\partial}_i^{(r)}(b_+)$, such that $\ell_i(\tilde{\partial}_i^{(r)}(b_+)) = \ell_i(b_+) - r$ and

$$\partial_i^{(r)}(b_+) - \left( \frac{\ell_i(b_+)}{r} \right)_q \tilde{\partial}_i^{(r)}(b_+) \in \sum_{b'_+ \in B_{n+} : \ell_i(b'_+) < \ell_i(b_+) - r} \mathbb{Z}[q,q^{-1}] b'_+. \quad (3.12)$$

The correspondence $b_+ \mapsto \tilde{\partial}_i^{(r)}(b_+)$ is a bijection. In particular, $\partial_i^{(\text{top})}(b_+) = \tilde{\partial}_i^{(\ell_i(b_+))}(b_+)$. Moreover, using [18, 5.3.8-5.3.10] we obtain

$$\partial_i^{(r)}(b_+) - \left( \frac{\ell_i(b_+)}{r} \right)_q \tilde{\partial}_i^{(r)}(b_+) \in \sum_{b'_+ \in B_{n+} : \ell_i(b'_+) < \ell_i(b_+) - r} q \mathbb{Z}[q][b'_+]. \quad (3.13)$$

**Example 3.23.** We now discuss the construction of elements of the form $F_i^{r_i} \bullet b_+$, $i \in I$, $r \in \mathbb{Z}_{\geq 0}$, $b_+ \in B_{n+}$. We need the following

**Lemma 3.24.** For all $x_+ \in U_q^+$, $i \in I$ and $r \in \mathbb{Z}_{\geq 0}$

$$x^+ F_i^r = \sum_{r', r'' \geq 0, r' + r'' \leq r} (-1)^{r'_i} q \binom{r}{2} \binom{r'}{2} (r' + r'')_q \binom{r}{r'} \sum_{r'' \geq 0} K_{r''} K_{r'''} K_{r''}^{r'''} \partial_i^{(r')} \partial_i^{(r''')} \partial_i^{(r''')} \partial_i^{(r''')} (x^+).$$

**Proof.** Since by (A.36)

$$\Delta(1 \otimes \Delta)(F_i^r) = \sum_{r', r'' \geq 0, r' + r'' = r} q \binom{r}{2} \binom{r'}{2} (r' + r'')_q \binom{r}{r'} K_{r''} K_{r'''} K_{r''}^{r'''} \partial_i^{(r')} \partial_i^{(r''')} \partial_i^{(r''')} \partial_i^{(r''')} (x^+),$$

we have by Proposition A.36, Lemma A.30, (A.6) and Lemma 3.18(c)

$$x^+ F_i^r = \sum_{r', r'' \geq 0, r' + r'' \leq r} (-1)^{r'_i} q \binom{r}{2} \binom{r'}{2} (r' + r'')_q \binom{r}{r'} K_{r''} K_{r'''} K_{r''}^{r'''} \partial_i^{(r')} \partial_i^{(r''')} \partial_i^{(r''')} \partial_i^{(r''')} (x^+).$$
In particular, if \( b_+ \in B_{\mathfrak{a}_+} \cap \ker \partial_i^{op} \) then we have
\[
F_i^r b_+ = F_i^r b_+ + \sum_{r'=0}^r (-1)^{r'} q_i^{-(r')} \langle r' \rangle q_i^r \sum_{b'_+} C_{b'_+}^{F_i^r, b_+} K_i^{r'} \phi (F_i^{r-r'} b'_+) \]
This implies that \( F_i^r \cdot b_+ = F_i^r \circ b_+ \) is the unique \( \tilde{\sigma} \)-invariant element of \( U_q(\mathfrak{g}) \) of the form
\[
F_i^r b_+ + \sum_{r'=1}^{\min(r, \ell_i(b_+))} \sum_{b'_+} C_{b'_+}^{+} K_i^{r'} \phi (F_i^{r-r'} b'_+), \quad C_{b'_+}^{+} \in qZ[q]. \tag{3.14}
\]
Similarly, if \( b_+ \in \ker \partial_i \) then \( F_i^r \circ b_+ = F_i^{r} b_+ \) and \( F_i^r \cdot b_+ \) is the unique \( \tilde{\sigma} \)-invariant element of \( U_q(\mathfrak{g}) \) of the form
\[
F_i^r b_+ + \sum_{r'=1}^{\min(r, \ell_i(b_+))} \sum_{b'_+} C_{b'_+}^{-} K_i^{r'} \phi (F_i^{r-r'} b'_+), \quad C_{b'_+}^{-} \in q^{-1}Z[q^{-1}]. \tag{3.15}
\]
The coefficients \( C_{b'_+}^{+}, C_{b'_+}^{-} \) can be expressed inductively, but in general it is not possible to write an explicit formula for them.

4. Examples of double canonical bases

4.1. Double canonical basis of \( U_q(\mathfrak{sl}_2) \). In this section we explicitly compute the double canonical basis in \( \mathcal{H}_q^+(\mathfrak{g}) \) and \( U_q(\mathfrak{g}) \) for \( \mathfrak{g} = \mathfrak{sl}_2 \).

Lemma 4.1. In \( \mathcal{H}_q^+(\mathfrak{sl}_2) \) we have
\[
F^{m-} \circ E^{m+} = \sum_{j \geq 0} (-1)^j q^j |m-|_{m-}+1, j \left[ \begin{array}{c} m \nonumber \\
q^{-2} \end{array} \right] K_j^+ \circ F^{m-j} E^{m-j}, \quad m_+ \in \mathbb{Z}_{\geq 0}. \tag{4.1}
\]

Proof. For \( m_+ \in \mathbb{Z}_{\geq 0} \) denote the right hand side of (4.1) by \( b_{m_+, m_+} \). Let \( C_+ = b_{1, 1} = FE - qK_+ \). Observe that \( C_+ \) is central in \( \mathcal{H}_q^+(\mathfrak{g}) \), since \( C_+ F = F(FE + (q^{-1} - q)K_+) - qK_F = FC_+ \) and similarly \([E, C_+] = 0\). Furthermore, \( \mathcal{C}_+ = EF - q^{-1}K_+ = FE + (q^{-1} - q)K_+ - q^{-1}K_+ = C_+ \). We have
\[
b_{m, m} C_+ = \sum_{j \geq 0} (-1)^j q^j \left[ \begin{array}{c} m \\
j \end{array} \right] q^2 K_j^+ F^{m-j} E^{m-j} (FE - qK_+)
\]
\[
= \sum_{j \geq 0} (-q)^j \left[ \begin{array}{c} m \\
j \end{array} \right] K_j^+ (F^{m-1-j} E^{m+1-j} - q^{2(m-j)+1} K_+ F^{m-j} E^{m-j})
\]
\[
= \sum_{j \geq 0} (-q)^j \left[ \begin{array}{c} m \\
j \end{array} \right] + q^{2(m-j)+1} \left[ \begin{array}{c} m \\
-j \end{array} \right] q^2 K_j^+ F^{m-1-j} E^{m+1-j} = b_{m, m+1}.
\]
Therefore, \( b_{m, m} = C_+^m, m \in \mathbb{Z}_{\geq 0} \), whence for all \( m_+ \in \mathbb{Z}_{\geq 0} \)
\[
b_{m_- m_+} = \sum_{j \geq 0} (-q)^j \left[ \begin{array}{c} m \\
j \end{array} \right] F^{m_- m_-} + (K_j^+ \circ F^{m-j} E^{m-j}) E^{[m_- m_-]} \phi = F^{[m_- m_-]} + C_+ E^{[m_- m_-]},
\]
where \( m = \min(m_+, m_-) \) and \( |a_+| = \max(0, a) \). Since \( C_+ \) is \( \tilde{\sigma} \)-invariant and central, it follows that \( b_{m_- m_+} = b_{m_- m_+} \). By definition, \( b_{m_- m_+} - F^{m_-} E^{m_+} \in \sum_{j>0} qZ[q] K_j^+ \circ F^{m-j} E^{m-j} \), and the assertion follows by Theorem 1.3. \( \square \)
Thus, the double canonical basis of $\mathcal{H}_q^+(\mathfrak{sl}_2)$ is
\[
\mathbf{B}_+^\circ = \{ K^a_q \circ F^m_0 C^0_m E^m_0 : a, m, m_0 \in \mathbb{Z}_{\geq 0}, \min(m_+, m_-) = 0 \}.
\]
Let $C^{(0)} = 1$, $C^{(1)} = C = C_+ - q^{-1}K_- = FE - qK_+ - q^{-1}K_-$ and define inductively
\[
C^{(m+1)} = CC^{(m)} - K_+K_-C^{(m-1)}, \quad m \geq 1.
\] (4.2)
Note that $C$ is central in $U_q(\hat{g})$ and $\hat{\gamma}$-invariant, hence $C^{(m)} = C$. It follows directly by induction on $m$ that

**Lemma 4.2.** For all $m, k \geq 0$
\[
F^k C^{(m)} = \sum_{a,b \geq 0} (-1)^{a+b} q^{(k+1)(a-b)} \left[ \begin{array}{c} m-a \\ b \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} m-b \\ a \end{array} \right]_q q^2 K^a_q K^b_q \circ F^{m+k-a-b} E^{m-a-b}
\]
\[
C^{(m)} E^k = \sum_{a,b \geq 0} (-1)^{a+b} q^{(k+1)(a-b)} \left[ \begin{array}{c} m-a \\ b \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} m-b \\ a \end{array} \right]_q q^2 K^a_q K^b_q \circ F^{m-a-b} E^{m+k-a-b}
\]
\[
(4.3)
\]

**Proposition 4.3.** For all $m \geq 0$,
\[
C^{(m)} = \sum_{0 \leq i \leq j, i+j \leq m} (-1)^i q^{-j-i^2} \left[ \begin{array}{c} m-i \\ j \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} j \\ i \end{array} \right]_{q^{-2}} K^i_+ K^j_- F^{m-i-j} \circ E^{m-i-j}, \quad m \geq 0.
\] (4.4)

In particular, $C^{(m)} = F^m \bullet E^m \in \mathbf{B}_+^\circ$.

**Proof.** Let $\iota : \mathcal{H}_q^+(\hat{g}) \to U_q(\hat{g})$ be the natural inclusion of vector spaces. One can show by induction on $k$ that in $U_q(\hat{g})$
\[
\iota(C^k) = \iota(C^{k+1} - q^{-2k-1}K_- \iota(C^k) + (1 - q^{-2k}) K_- K_+ \iota(C^{k-1}).
\] (4.5)

Denote by $X_m$ the right hand side of (4.4). It follows from (4.5) that
\[
X_m C - K_+K_-X_{m-1}
\]
\[
= \sum_{i,j \geq 0} (-1)^i q^{-j-i^2} \left[ \begin{array}{c} m-i \\ j \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} j \\ i \end{array} \right]_{q^{-2}} (K^i_+ K^j_- \iota(C^{m+1-i-j})
\]
\[
- q^{-2(m-i-j)-1} K^i_+ K^{i+1}_- \iota(C^{m-i-j}) + (1 - q^{-2(m-i-j)}) K^i_+ K^{i+1}_- \iota(C^{m-i-j})
\]
\[
+ \sum_{i,j \geq 0} (-1)^{j+1} q^{-j-i^2} \left[ \begin{array}{c} m-1-i \\ j \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} j \\ i \end{array} \right]_{q^{-2}} K^{i+1}_+ K^j_- \iota(C^{m-i-j})
\]
\[
= \sum_{i,j \geq 0} (-1)^i q^{-j-i^2} \left[ \begin{array}{c} m-i \\ j \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} j \\ i \end{array} \right]_{q^{-2}} (K^i_+ K^j_- \iota(C^{m+1-i-j}) - q^{-2(m-i-j)-1} K^i_+ K^{i+1}_- \iota(C^{m-i-j})
\]
\[
+ \sum_{i,j \geq 0} (-1)^{j+1} q^{-j-i^2} q^{-2(m-i)} \left[ \begin{array}{c} m-1-i \\ j \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} j \\ i \end{array} \right]_{q^{-2}} K^{i+1}_+ K^j_- \iota(C^{m-i-j})
\]
\[
= \sum_{i,j \geq 0} (-1)^j q^{-j-i^2} \left[ \begin{array}{c} m-i \\ j \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} j \\ i \end{array} \right]_{q^{-2}} + q^{-2(m+1-i-j)} \left[ \begin{array}{c} m-i \\ j-1 \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} j-1 \\ i \end{array} \right]_{q^{-2}} K^i_+ K^j_- \iota(C^{m+1-i-j})
\]
\[
+ q^{-2(j-i)} \left[ \begin{array}{c} j-1 \\ i-1 \end{array} \right]_{q^{-2}} \iota(C^{m+1-i-j})
\]
\[
= \sum_{i,j \geq 0} (-1)^j q^{-j-i^2} \left[ \begin{array}{c} m+1-i \\ j \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} j \\ i \end{array} \right]_{q^{-2}} K^i_+ K^j_- \iota(C^{m+1-i-j}) = X_{m+1}
\]
since
\[
\begin{bmatrix}
j - 1 \\
i
\end{bmatrix}_{q^{-2}} + q^{-2(j-i)} \begin{bmatrix}
j - 1 \\
i - 1
\end{bmatrix}_{q^{-2}} = \begin{bmatrix}
j \\
i
\end{bmatrix}_{q^{-2}}.
\]
\[q^{-2(m-i-j+1)} \begin{bmatrix}
m - i \\
j - 1
\end{bmatrix}_{q^{-2}} + \begin{bmatrix}
m - i \\
j
\end{bmatrix}_{q^{-2}} = \begin{bmatrix}
m + 1 - i \\
j
\end{bmatrix}_{q^{-2}}.
\]

Thus, \(X_m\) satisfies the recurrence relation (4.2). Since \(X_0 = 1\) and \(X_1 = C\), we conclude that \(X_m = C^{(m)}\) for all \(m \geq 0\). The second assertion is now immediate by Theorem 1.5 since \(C^{(m)} = C^{(m)}\) and by (4.4),
\[
C^{(m)} - F^m \circ E^m \in \sum_{j > 0}^{0 \leq i \leq \min(j, m-j)} q^{-1}Z[q^{-1}]K^j F^{m-i} \circ E^{m-i-j}.
\]

\[\square\]

**Corollary 4.4.** For all \(m_-, m_+ \geq 0\),
\[
F^{m_-} \bullet E^{m_+} = \sum_{0 \leq i \leq j} (-1)^j q^{-j-i} (j-i)(m_+ - m_-) \begin{bmatrix}
m - i \\
j
\end{bmatrix}_{q^{-2}} \begin{bmatrix}
j \\
i
\end{bmatrix}_{q^{-2}} K^i F^{m_- - i} \circ E^{m_+ - i-j}.
\]

where \(m = \min(m_+, m_-)\).

CombiningLemma 4.2 and Proposition 4.3 and using (3.4) we obtain the following identity.

**Proposition 4.5.** For all \(m, a, b \geq 0\) with \(a + b \leq m\) we have in \(Z[\nu]\)
\[
\sum_{r=0}^{\min(a,b)} (-1)^r \nu^r \frac{[m-r]!}{[a-r]! [b-r]! [r]!} = \nu^{ab} [m - a - b]_\nu \begin{bmatrix}
m - b \\
a
\end{bmatrix}_\nu \begin{bmatrix}
m - a \\
b
\end{bmatrix}_\nu.
\]

Our preceding computations, together with Theorem 1.5, immediately yield the following

**Proposition 4.6.** For all \(m_\pm \in Z_{\geq 0}\),
\[
F^{m_-} \bullet E^{m_+} = F^{m_- - m} C^{(m)} E^{m_+ - m}
\]
\[
= \sum_{0 \leq a+b \leq m} (-1)^{a+b} q^{[(m_+ + m_- + 1) - (a+b)]} \begin{bmatrix}
m - a \\
b
\end{bmatrix}_{q^{-2}} \begin{bmatrix}
m - b \\
a
\end{bmatrix}_{q^{-2}} K^a_+ K^b_- \circ F^{m_- - a-b} E^{m_+ - a-b}
\]

where \(m = \min(m_+, m_-)\). Thus, the double canonical basis in \(U_q(\widehat{sl}_2)\) is given by
\[
\{K^a_+ K^a_- \circ F^{m} C^{(m)} E^{m} : a_\pm, m_\pm, m_0 \in Z_{\geq 0}, \min(m_+, m_-) = 0\}.
\]

An easy induction shows that
\[
C^{(a)} C^{(b)} = \sum_{j=0}^{\min(a,b)} (K^- K^+_j) C^{(a+b-2j)}
\]

**Lemma 4.7.** For all \(n \geq 0\) we have
\[
F^n E^n = \sum_{r=0}^{n} \left( \sum_{j=0}^{r} c^{(n)}_{r,j} K^j_+ K^r_+ \right) C^{(n-r)} E^m F^n = \sum_{r=0}^{n} \left( \sum_{j=0}^{r} c^{(n)}_{r,j} K^r_+ K^j_+ \right) C^{(n-r)}
\]

where \(c^{(n)}_{0,0} = 1, \overline{c^{(n)}_{r,j}} = c^{(n)}_{r,-j} \in Z_{\geq 0}[q, q^{-1}]\). In particular, Conjecture 1.21 holds for \(g = sl_2\).
Proof. The induction base is clear since $FE = C + qK_+ + q^{-1}K_-$. Thus, $c_{0,0}^{(1)} = 1$ and $c_{1,0}^{(1)} = q = c_{1,1}^{(1)}$.

For the inductive step we have

$$F^{n+1}E^{n+1} = \sum_{r=0}^{n} F\left( \sum_{j=0}^{r} c_{r,j}^{(n)} K_j^r K_{r-j}^{-} \right) C^{(n-r)} E = \sum_{r=0}^{n} \left( \sum_{j=0}^{r} c_{r,j}^{(n)} q^{2(r-2j)} K_j^r K_{r-j}^{-} \right) FEC^{(n-r)}$$

$$= \sum_{r=0}^{n} \left( \sum_{j=0}^{r} c_{r,j}^{(n)} q^{2(r-2j)} K_j^r K_{r-j}^{-} \right) \left( C + qK_+ + q^{-1}K_- \right) C^{(n-r)}$$

$$= \sum_{r=0}^{n} \left( \sum_{j=0}^{r} c_{r,j}^{(n)} q^{2(r-2j)} K_j^r K_{r-j}^{-} \right) C^{(n-r+1)} + \sum_{r=0}^{n} \left( \sum_{j=0}^{r} c_{r,j}^{(n)} q^{2(r-2j)} K_j^{r+1} K_{r-j}^{-} \right) C^{(n-r-1)}$$

$$+ \sum_{r=0}^{n} \left( \sum_{j=0}^{r} c_{r,j}^{(n)} q^{2(r-2j)+1} K_j^r K_{r-j}^{-} \right) C^{(n-r)} + \sum_{r=0}^{n} \left( \sum_{j=0}^{r} c_{r,j}^{(n)} q^{2(r-2j)-1} K_j^{r-1} K_{r-j}^{-} \right) C^{(n-r)}$$

$$= \sum_{r=0}^{n} \left( \sum_{j=0}^{r} q^{2(r-2j)} \left( c_{r,j}^{(n)} + c_{r-2,j-1}^{(n)} + q^{-1} c_{r-1,j}^{(n)} + q c_{r-1,j-1}^{(n)} \right) \right) K_j^r K_{r-j}^{-} C^{(n-r)}$$

where we use the convention that $c_{r,j}^{(n)} = 0$ if $r < 0$, $j < 0$, $j > r$ or $r > n$. Set

$$c_{r,j}^{(n+1)} = q^{2(r-2j)} \left( c_{r,j}^{(n)} + c_{r-2,j-1}^{(n)} + q^{-1} c_{r-1,j}^{(n)} + q c_{r-1,j-1}^{(n)} \right)$$

Then $c_{0,0}^{(n+1)} = 1$ and $c_{r,j}^{(n+1)} \in \mathbb{Z}_{\geq 0}[q,q^{-1}]$. Also

$$c_{r,r+j}^{(n+1)} = q^{2(r+j)} \left( c_{r-r,j}^{(n)} + c_{r-2,r-j-1}^{(n)} + q c_{r-1,r-j}^{(n)} + q^{-1} c_{r-1,r-j-1}^{(n)} \right) = c_{r,j}^{(n+1)}.$$

This proves the inductive step. The second formula follows from the first by applying $\sim$. □

Remark 4.8. One can prove, using the above computation, an even stronger statement, namely that for any two elements $b$, $b'$ of $\mathcal{B}_{\tilde{sl}_2}$, $bb'$ decomposes as a linear combination of elements of the same basis with coefficients being Laurent polynomials in $q$ with positive coefficients. However, this fact is special for $\mathfrak{sl}_2$ and is unlikely to hold in greater generality.

4.2. Action on a double basis for $\mathfrak{sl}_2$. We now consider the action of $U_q(\mathfrak{g})$ on the double canonical basis of $U_q(\tilde{\mathfrak{sl}}_2)$. To preserve $\sim$-invariance, it is necessary to consider its twisted version given by

$$F_i(x) := q_i^{\lambda_i}(x) \frac{1}{x_i} F_i x - q_i^{\lambda_i}(x) - x F_i, \quad E_i(x) := K_i^{-1} \lambda_i \frac{1}{x_i} E_i x - q_i^{\lambda_i} x E_i,$$

(4.8)

for any $\lambda \in \mathbb{Z}^I$ (cf. [13]). We denote the corresponding operators by $E_\lambda, F_\lambda$.

Lemma 4.9. Let $\lambda, \alpha_\pm \in \mathbb{Z}$. Then for all $m_+ > m_-$

$$F_\lambda(K^a_- K^a_+ \circ F^{m_+} \bullet E^{m_-}) = \left( \frac{1}{2} \lambda + \alpha_+ - \alpha_+ + 2m_+ - 2m_- \right)_q K^{a+1} K_{a-} \circ F^{m_-} E^{m_-}$$

$$+ \left( \frac{1}{2} \lambda + \alpha_+ - \alpha_- + m_+ - m_- \right)_q K^{a-1} K_{a+} \circ F^{m_-} E^{m_-}$$

(4.9)

$$+ \left( \frac{1}{2} \lambda + \alpha_+ - \alpha_- + m_- - m_+ \right)_q K^{a-1} K_{a+} \circ F^{m_+} E^{m_+} + \left( \frac{1}{2} \lambda + \alpha_+ - \alpha_- \right)_q K^{a-1} K_{a+} \circ F^{m_-} E^{m_-}$$

where we use the convention that $F^r \bullet E^s = 0$ if $r < 0$ or $s < 0$, while for $m_+ \leq m_-$

$$F_\lambda(K^a_- K^a_+ \circ F^{m_-} \bullet E^{m_+}) = \left( \frac{1}{2} \lambda + \alpha_+ - \alpha_- + m_+ - m_- \right)_q K^{a-} K^{a+} \circ F^{m_-} E^{m_-}$$

(4.10)
Furthermore, for all \( m_+ \geq m_- \)

\[
E_\lambda(K_n^- K_+^a \circ F^{m_-} \cdot E^{m_+}) = \left( \frac{1}{2} \lambda + a_+ - a_- \right)_q K_n^- K_+^{a_+ - 1} \circ F^{m_-} \cdot E^{m_+ - 1} + \left( \frac{1}{2} \lambda + a_+ - a_- \right)_q K_n^- K_+^{a_+} \circ F^{m_-} \cdot E^{m_+} \tag{4.11}
\]

while for all \( m_+ < m_- \)

\[
E_\lambda(K_n^- K_+^a \circ F^{m_-} \cdot E^{m_+}) = \left( \frac{1}{2} \lambda + a_+ - a_- + m_- - m_+ \right)_q K_n^- K_+^{a_+ - 1} \circ F^{m_-} \cdot E^{m_+ - 1} + \left( \frac{1}{2} \lambda + a_+ - a_- \right)_q K_n^- K_+^{a_+ - 1} \circ F^{m_-} \cdot E^{m_+ - 1} + \left( \frac{1}{2} \lambda + a_+ - a_- + m_- - m_+ \right)_q K_n^- K_+^{a_+ - 1} \circ F^{m_-} \cdot E^{m_+} \cdot E^{m_+ - 1} \cdot E^{m_+} \tag{4.12}
\]

**Proof.** It is an easy consequence of (4.3) that

\[
FE^k = CE^{k-1} + q^k K_+ \circ E^{k-1} + q^{-k} K_- \circ E^{k-1},
\]

\[
E^k F = CE^{k-1} + q^{-k} K_+ \circ E^{k-1} + q^k K_- \circ E^{k-1}, \quad k \geq 1
\]

We also have

\[
F_\lambda(K_+^a K_-^a \circ x) = K_+^a K_-^a \circ F_{\lambda+2a_+ -2a_-}(x),
\]

so it is sufficient to prove all identities for \( a_+ = a_- = 0 \). Suppose first that \( m_+ > m_- \). Then \( F^{m_-} \cdot E^{m_+} = C^{(m_+)} E^{m_+ - m_-} \) and

\[
F_\lambda(C^{(m_-)} E^{m_+ - m_-}) = q^{\frac{1}{2} \lambda+m_+ - m_-} C^{(m_-)} E^{m_+ - m_-} - q^{-\frac{1}{2} \lambda - (m_+ - m_-)} C^{(m_-)} E^{m_+ - m_-} F
\]

\[
= \left( \frac{1}{2} \lambda + m_+ - m_- \right)_q C^{(m_-)} E^{m_+ - m_- - 1} + \left( \frac{1}{2} \lambda + 2(m_+ - m_-) \right)_q K_+ \circ C^{(m_-)} E^{m_+ - m_- - 1}
\]

and it remains to use (4.2). If \( m_+ \leq m_- \) then \( F^{m_-} \cdot E^{m_+} = F^{m_- - m_+} C^{(m_+)} \) and

\[
F_\lambda(F^{m_-} \cdot E^{m_+}) = (q^{\frac{1}{2} - (m_- - m_+)} - q^{\frac{1}{2} + m_+ - m_-}) F^{m_- - m_+ - 1} C^{(m_+)} = \left( \frac{1}{2} \lambda + m_+ - m_- \right)_q F^{m_- - 1} \cdot E^{m_+}
\]

The identities involving \( E_\lambda \) are proved similarly. \( \square \)

**Corollary 4.10.** If \( k_+ = \frac{1}{2} + a_+ - a_- \max(0, m_- - m_+) \) is a non-negative integer then \( k_+ = \max \{ k \geq 0 : E_\lambda(K_n^- K_+^a \circ F^{m_-} \cdot E^{m_+}) \neq 0 \} \). Similarly, if \( k_- = \frac{1}{2} + a_+ - a_- + m_+ - m_- + \max(0, m_+ - m_-) \) is a non-negative integer then \( k_- = \max \{ k \geq 0 : E_\lambda(K_n^- K_+^a \circ F^{m_-} \cdot E^{m_+}) \neq 0 \} \).

**Proof.** We prove only the first statement, the proof of the second one being similar. If \( m_- \leq m_+ \) then by an obvious induction we obtain

\[
E_\lambda(F^{m_-} \cdot E^{m_+}) = \left( \frac{1}{2} \right)_q \cdots \left( \frac{1}{2} - s + 1 \right)_q K_+^s \circ F^{m_-} \cdot E^{m_+ + s}
\]

which is zero if and only if \( \lambda \in 2\mathbb{Z}_{\geq 0} \) and \( s \geq \frac{1}{2} + 1 \). If \( m_+ > m_- \) then each term in the right hand side of (4.12) is of the form \( K \circ F^a \cdot E^b \) with \( a - b = m_- - m_+ - 1 \) and the term with the largest coefficient is \( F^{m_- - 1} \cdot E^{m_+} \). Thus,

\[
E_\lambda(F^{m_- - m_+} \cdot E^{m_+}) = \left( \frac{1}{2} + m_- - m_+ \right)_q \cdots \left( \frac{1}{2} + 1 \right)_q F^{m_-} \cdot E^{m_+} + \cdots
\]

where the remaining terms are of the form \( K \circ F^a \cdot E^a \circ F^a \cdot E^a \) with the coefficients being of the form \( \prod_{j=0}^s (\frac{1}{2} + k - j)_q \) with \( k < m_- - m_+ \). It follows that \( E_\lambda(F^{m_-} \cdot E^{m_+}) = 0 \) only if \( \frac{1}{2} \lambda + m_- - m_+ \in \mathbb{Z}_{\geq 0} \) and \( s > \frac{1}{2} \lambda + m_- - m_+ \).

Define

\[
\varepsilon^\lambda(F^{m_-} \cdot E^{m_+}) = \frac{1}{2} + \max(0, m_- - m_+)
\]

Then we obtain the following
Corollary 4.11. For all \( \lambda \in \mathbb{Z} \), \( m_+ \in \mathbb{Z}_{\geq 0} \)
\[
E_\lambda(F^{m_-} \bullet E^{m_+}) = \langle \varepsilon^\lambda(F^{m_-} \bullet E^{m_+}) \rangle_q b + \sum_{b' : \varepsilon^\lambda(b') < \varepsilon^\lambda(F^{m_-} \bullet E^{m_+})} c_{b'} \bullet b'
\]
where \( b = \begin{cases} F^{m_- - 1} \bullet E^{m_+}, & m_- > m_+ \\ K^{-1}_+ \circ F^{m_-} \bullet E^{m_+ + 1}, & m_- \leq m_+. \end{cases} \)

4.3. Some elements in double canonical bases in ranks 2 and 3. We will need explicit formulae for some elements of dual canonical bases for computational purposes. We already listed the most obvious ones in Example 3.21.

Example 4.12. It easy to see, extending [19, §14.5.4], that the elements \( F_i^{(s)} F_j^{(1)} F_k^{(n-s)} \), \( 0 \leq s \leq n \leq -a_{ij} \) are in \( \mathbb{B}^{\text{can}} \) and form a basis of the homogeneous component of \( U^{-}_{\mathbb{C}} \) of degree \( na_{-i} + \alpha_{-j} \). Let \( F_{i^* j^* r^*} = \delta_{i^*}^{(r)} F_j^{(1)} F_k^{(s)} \), \( r, s \geq 0 \), \( r + s \leq -a_{ij} \) and let \( E_{i^* j^* r^*} = (F_{i^* j^* r^*})^{t_i} \). We summarize their properties in the following Lemma, which is proved by direct computations based on Lemma 3.20.

Lemma 4.13. (a) For all \( k, l \geq 0 \), \( k + l < -a_{ij} \) we have
\[
F_i F_q F_{k + 1} F_{ij + 1} + q^{a_{ij} - 1} F_{k + 1} F_{ij} F_i = q^{-1} F_{k + 1} F_{ij + 1} + q F_{k + 1} F_{ij + 1} + q^{a_{ij} - 1} F_{k + 1} F_{ij + 1}
\]
(b) For all \( r, s \geq 0 \), \( r + s \leq -a_{ij} \) we have
\[
\Delta(F_{i^* j^* r^*}) = \sum_{r', r'' = r} q_i^{q_i + (r'' + s - a_{ij})} (r') q_i^r r'' F_i^{r'} \otimes F_{i^* j^* r''} + \sum_{s' + s'' = s} q_i^{s''(s' + r + 1)} (s') q_i^r F_{i^* j^* r''} \otimes F_{i^* j^* r''}
\]
(c) For all \( s, r, s', r' \geq 0 \), \( s + r = s' + r' \leq -a_{ij} \), we have
\[
\langle F_{i^* j^* r^*}, E_{i^* j^* r^*} \rangle = (-1)^{s + s'} q_i^{s' + (r' - r)(a_{ij} + 1)} p_{s', r, r'}(q) \frac{1}{\prod_{l=0}^{s + r - 1} q_i^{a_{ij} - l} - q_i^{a_{ij} - r}}
\]
where
\[
p_{s', r, r'}(q) = \sum_{l=0}^{\min(s', r)} q_i^{(r' + s' + 2a_{ij} - 2)} \binom{s'}{l} \binom{r'}{r - l} q_i^{r'} \in \mathbb{Z}_{\geq 0}[q, q^{-1}].
\]

The following Lemma provides a partial converse to Theorem 3.11.

Lemma 4.14. Suppose that \( \langle b_-, b_+ \rangle \in \mathbb{Z}[q, q^{-1}] \) for all \( b_\pm \in \mathbb{B}_{n+} \). Then for every \( i \neq j \), \( a_{ij}a_{ji} < 4 \).

Proof. We may assume without generality that \( a_{ij} \neq a_{ji} \) and \( |a_{ij}| \geq |a_{ji}| \) hence \( d_i \leq d_j \). Then by Lemma 4.13, \( \langle F_{ij}, E_{ij} \rangle = (q_j - q^{-1}) (q_i - q_i^{-1}) (a_{ij} - a_{ij}^{-1}) \) which can only be in \( \mathbb{Z}[q, q^{-1}] \) if \( d_j / |a_{ij}| \leq d_i + d_j \). Therefore, \( |a_{ij}| \leq 1 + d_i / d_j < 2 \), hence \( a_{ij} = -1 \) and \( d_j = -d_i a_{ij} \). Suppose that \( |a_{ij}| \geq 4 \). Applying Lemma 4.13 again we obtain
\[
\langle F_{i^* j^*}, E_{i^* j^*} \rangle = \frac{(q_j - q^{-1}) (q_i - q_i^{-1}) (q_{a_{ij}} - q_{-a_{ij}}^{-1})}{q_{a_{ij}}^{a_{ij}} - q_{-a_{ij}}^{-a_{ij}} (q_{a_{ij}}^{a_{ij}} + 1 - a_{ij}^{-1})} = \frac{(q_i - q_i^{-1}) (q_{a_{ij}}^{a_{ij}} - q_{-a_{ij}}^{-a_{ij}})}{q_{a_{ij}}^{a_{ij}} - q_{-a_{ij}}^{-a_{ij}} (q_{a_{ij}}^{a_{ij}} + 1)}.
\]
This cannot be a Laurent polynomial if \( |a_{ij}| > 4 \) by the degree considerations, while for \( a_{ij} = -4 \) we have \( \langle F_{i^* j^*}, E_{i^* j^*} \rangle = (q^4 - 1)/(q^4 + q^2 + 1) \notin \mathbb{Z}[q, q^{-1}] \). Thus, \( |a_{ij}| \leq 3 \).

From now on, given \( f = \sum a_{ij} \nu^j \in \mathbb{Z}[r, \nu^{-1}] \), let \( [f]_+ = \sum_{j > 0} a_{ij} \nu^j \) and \( [f]_- = \sum_{j < 0} a_{ij} \nu^j \). We will now consider some examples in rank 2.
First, assume that $a_{ij} = -1$ (in particular, this includes all subdiagrams of rank 2 for a semisimple and all affine cases except those of rank 2). Then $d_j = dd_i$, $a_{ij} = -d$ and by Lemma 4.13

$$[E_{is}, F_{is}] = [E_{jis}, F_{jis}] = -\frac{(s)_q!}{(d-1)_q \cdots (d-s+1)_q} (K^{s}_{i+1} K_{j+i} - K^{s}_{i} K_{j-i}),$$

hence $d_{E_{is}, E_{jis}} = d_{F_{is}, E_{jis}}$ and $F_{is} \bullet E_{is} - d_{F_{is}, E_{jis}} F_{is} E_{jis} = F_{jis} \bullet E_{jis} - d_{F_{is}, E_{jis}} F_{jis} E_{jis}$, while

$$F_{ij} \bullet E_{ij} = F_{ij} E_{ij} - q_i K_{i+1} K_{j+i} - q_i^{-1} K_{i} K_{j-i},$$

$$F_{is} \bullet E_{is} = F_{is} E_{is} - q_i^d K_{i+1} K_{j+i} - q_i^{-d} K_{i} K_{j-i},$$

and for $d > 2$

$$F_{is} \bullet E_{is} = \begin{cases} (d-1)_q F_{is} E_{is} - q_i^2 K_{i+1} K_{j+i} - q_i^{-2} K_{i} K_{j-i}, & d \text{ even} \\ \left(\frac{d}{2} (d-1)\right)_q F_{is} E_{is} - q_i^2 K_{i+1} K_{j+i} - q_i^{-2} K_{i} K_{j-i}, & d \text{ odd} \end{cases}$$

while for $d > 3$

$$F_{is} \bullet E_{is} = \begin{cases} (d-1)_q F_{is} E_{is} - q_i^3 K_{i+1} K_{j+i} - q_i^{-3} K_{i} K_{j-i}, & d = 0 \pmod 3 \\ \left(\frac{d}{2} (d-1)\right)_q F_{is} E_{is} - q_i^3 K_{i+1} K_{j+i} - q_i^{-3} K_{i} K_{j-i}, & d = 1 \pmod 6 \\ \left(\frac{d}{2} (d-2)\right)_q F_{is} E_{is} - q_i^3 K_{i+1} K_{j+i} - q_i^{-3} K_{i} K_{j-i}, & d = 2 \pmod 6 \\ \left(\frac{d}{2} (d-1)\right)_q F_{is} E_{is} - q_i^3 K_{i+1} K_{j+i} - q_i^{-3} K_{i} K_{j-i}, & d = 4 \pmod 6 \\ \left(\frac{d}{2} (d-2)\right)_q F_{is} E_{is} - q_i^3 K_{i+1} K_{j+i} - q_i^{-3} K_{i} K_{j-i}, & d = 5 \pmod 6. \end{cases}$$

Note that if $d \leq 2$ then $F_{is} \in B_{m_+}$; for $d = 2$ we also have $F_{is} \in B_{m_-}$ for all $k \in \mathbb{Z}_{\geq 0}$. Then we can use (4.6) to compute $F^{m_+}_{ij} \bullet E^{m_-}_{ij}$ (respectively, $F^{m_+}_{is} \bullet E^{m_-}_{is}$) for all $m_+ \in \mathbb{Z}_{\geq 0}$. Similarly, we obtain

$$F_{ij} \circ E_{ij} = F_{ij} E_{ij} - q_i^d K_{i+1} K_{j+i} + \left(q_i^d - [q_i^d]_+ \right) K_{i+1} K_{j+i},$$

$$F_{ij} \bullet E_{ij} = F_{ij} E_{ij} - q_i^{-1} K_{i} K_{j-i} F_j \circ E_j + q_i^{-1-d} K_{i} K_{j-i}$$

$$F_{ij} \circ E_{ij} = F_{ij} E_{ij} - q_i K_{i+1} K_{j+i} + [q_i^{-d}] K_{i+1} K_{j+i},$$

$$F_{ij} \bullet E_{ij} = F_{ij} E_{ij} - q_i K_{i+1} K_{j+i} - [q_i^{-1-d}] K_{i} K_{j-i}$$

If $d_i = d_j$, $a_{ij} = a_{ji} = -a$ we obtain

$$[E_{is}, F_{is}] = [E_{jis}, F_{jis}] = -\frac{(1)_q (s)_q!}{(a)_q \cdots (a-s+1)_q} (K^{s}_{i+1} K_{j+i} - K^{s}_{i} K_{j-i}),$$

and so for all $1 \leq s \leq a$

$$F_{is} \bullet E_{is} - \left(\frac{a}{s}\right)_q F_{is} E_{is} = -q_i K^{a}_{i+1} K_{j+i} - q_i^{-1} K^{a}_{i} K_{j-i} = F_{jis} \bullet E_{jis} - \left(\frac{a}{s}\right)_q F_{jis} E_{jis}.$$

We also have

$$F_{ij} \circ E_{ij} = (a)_q F_{ij} E_{ij} - q_i a K_{i+1} K_{j+i} E_{ij} + \left(q_i a - [q_i^{-1}]_+ \right) K_{i} K_{j+i},$$

$$F_{ij} \bullet E_{ij} = F_{ij} E_{ij} - q_i^{-a} K_{i} K_{j-i} E_{ij} - [q_i^{-a}]_+ (K_{i} K_{j+i} + K_{i} K_{j-i}).$$

Furthermore, for $a > 1$

$$F_{ij} \circ E_{j+i} = \left(\frac{a}{2}\right)_q F_{ij} E_{j+i} - (q_i^a + [q_i^{-2}]_+) a K_{i} K_{j+i} E_{ij} + [q_i^{-a}] (1 + [q_i^{-a}]_+) K_{i+1} K_{j+i} F_{ij} E_{ij}$$

$$- (q_i^a - [q_i^{-a}]_+) K_{i+1} K_{j+i} K_{i+1} K_{j+i},$$

$$F_{ij} \bullet E_{j+i} = F_{ij} E_{j+i} - q_i^{-1} K_{i} K_{j-i} E_{ij} - \left(\frac{a-1}{2}\right)_q a^{-1} K_{i} K_{j-i} F_{ij} E_{ij}.$$
as well as three more elements obtained from these by applying the automorphism which interchanges \(F_i\) and \(F_j\). The corresponding elements of \(\mathcal{B}_{\alpha-\alpha}^{\text{an}}\) are, respectively,

\[
F_{ij} = (\frac{q^2}{2}q^2F_iF_j - (q^2 + (2)q)F_iF_j + (q - q^{-1})(F_iF_j^2 + F_jF_i^2) + ((2)q + 2q^{-3})(F_iF_jF_iF_j - q^{-2}(2)qF_iF_j^2)/(q - q^{-1})(q^2 - q^{-2})(q^4 - q^{-4})
\]

\[
F_{ijij} = (q^2F_i^2 - F_j^2F_i + (3)q(F_i^2F_jF_i - F_jF_iF_jF_i))/((q - q^{-1})(q^2 - q^{-2})(q^4 - q^{-4}))
\]

\[
F_{ijiz} = q^4(2)qF_iF_jF_iF_j - (q^2 + (2)q)F_iF_j + (q - q^{-1})(F_iF_j + F_jF_i) + q^4(2)qF_iF_jF_iF_j/(q - q^{-1})(q^2 - q^{-2})(q^4 - q^{-4})
\]

Set \(F_{\alpha} = F_{\alpha}^{\pm}\). Since \(d_{F_i,F_j} = (2)q\) by the previous example we have

\[
F_{ij}^{\pm} = (2)qF_iF_j + (q - q^{-1})K_{-i}K_{-j}F_iF_j + (q - q^{-1})K_{-i}K_{-j}F_jF_i - 2q^2K_{+i}K_{+j}K_{+i}K_{+j}
\]

Similarly,

\[
F_{ij}^{\pm} = (2)qF_iF_j + (q - q^{-1})K_{-i}K_{-j}F_iF_j + (q - q^{-1})K_{-i}K_{-j}F_jF_i - 2q^2K_{+i}K_{+j}K_{+i}K_{+j}
\]
\[(q^{-5} - q^{-3})K_{-i}K_{-j}F_{ji} \circ E_{ij} + 2q^{-3}K_{-i}K_{-j}K_{+i}F_{j} \circ E_{j} \]
\[(q^{-3} - q^{-5})K_{-i}K_{-j}K_{+i}F_{i} \circ E_{i} + (q^{-6} - q^{-4} - q^{-2})K_{-i}K_{-j}K_{+j}K_{-j} - q^{-4}K_{-i}K_{+i}K_{+j} \]
\[F_{ji}j \circ E_{jjji} = (2)q(4)qF_{ji}jE_{jii} + (q - q^{-3})(2)qK_{-i}K_{-j}F_{ji}E_{ji} - 2q^{2}K_{-i}K_{+j} \]
\[F_{ji}j \bullet E_{jjji} = F_{ji}j \circ E_{jjji} + (q^{-1} - q^{-3})K_{-i}K_{-j}F_{ji} \circ E_{ji} - 2q^{2}K_{-i}K_{+j}K_{+j} - q^{-4}K_{-i}K_{-j}K_{+i}K_{+j}. \]

**Example 4.16.** Let \(\mathfrak{g} = \mathfrak{sl}_3\), that is, \(I = \{1, 2, 3\}\) and \(a_{ij} = a_{ji} = -1\) for all \(i \neq j\). For \(\{i, j, k\} = \{1, 2, 3\}\) let

\[F_{ijk} = ((q - q^{-1})(q^{3} - q^{-3}))^{-1}\left((2)qF_{i}F_{j}F_{k} - F_{j}F_{k}F_{i} - F_{k}F_{i}F_{j}\right) + q^{-4}\left((2)qF_{i}F_{j}F_{k} - F_{i}F_{j}F_{k} - F_{j}F_{k}F_{i}\right). \]

Then \(F_{ijk} = \delta_{F_{i}^{(1)}F_{j}^{(1)}F_{k}^{(1)}}\). We have

\[F_{ijk} \bullet E_{ijk} = (3)qF_{ijk}E_{ijk} - q^{-2}K_{-i}K_{-j}K_{+k} - q^{-2}K_{-i}K_{+j}K_{-k} \]
\[F_{ijk} \bullet E_{ikj} = (3)qF_{ijk}E_{ikj} - qK_{+i}K_{+j}K_{-k} - q^{-1}K_{-i}K_{-j}K_{-k} \]
\[F_{ijk} \bullet E_{jik} = (3)qF_{ijk}E_{jik} - qK_{+i}K_{+j}K_{-k} - q^{-1}K_{+i}K_{+j}K_{-k} \]
\[F_{ijk} \bullet F_{jki} = (3)qF_{ijk}E_{jki} - q^{3}K_{+j}F_{i}E_{i} + (q^{-4} - q^{2})K_{-i}K_{+j}K_{+k} \]
\[F_{ijk} \bullet E_{kji} = (3)qF_{ijk}E_{kji} - qK_{+i}K_{+j}K_{-k} + q^{4}K_{-i}K_{-j}F_{i}E_{i} + (q - q^{-5})K_{+i}K_{+j}K_{+k} \]
\[F_{ijk} \bullet E_{kji} = F_{ijk} \circ E_{kji} - q^{-3}K_{+i}F_{j} \circ E_{j} - q^{-2}K_{+i}K_{+j}F_{j} \circ E_{j} - q^{-4}K_{+i}K_{+j}K_{+j}F_{j} \circ E_{k} + (q^{-1} - q^{-5})K_{+i}K_{+j}K_{+j}K_{-j}K_{-k} \]
\[F_{ijk} \bullet E_{kji} = F_{ijk} \circ E_{kji} - q^{3}K_{+i}K_{+j}F_{j} \circ E_{j} - q^{2}K_{+i}K_{+j}K_{+j}F_{j} \circ E_{k} + (q^{-4} - q^{-2})K_{+i}K_{-j}K_{+j}K_{+k} \]
\[F_{ijk} \bullet F_{kji} = (3)qF_{ijk}E_{kji} - q^{-3}K_{+i}K_{+j}F_{j} \circ E_{j} - q^{2}K_{+i}K_{+j}K_{+j}F_{j} \circ E_{k} + (q^{-4} - q^{-2})K_{+i}K_{-j}K_{+j}K_{+k}. \]

These examples show that we can have \(\mathfrak{d}_{\alpha, \beta, \nu} \neq 1\) even if all subdiagrams of rank 2 of the Dynkin diagram of \(\mathfrak{g}\) are of finite type.

**4.4. Reshetikhin–Semenov-Tian-Shanvyksy map.** Define a pairing \(\langle \cdot, \cdot \rangle : U^{-} \otimes U^{+} \to k\) by \(\{u_{-}, u_{+}\} = \langle u_{-}, u_{+}\rangle\), \(u_{-} \in U_{+}^{L}\). It follows from Proposition 3.20 that

\[\{u_{-}, u_{+}\} = \{u_{-}, u_{+}^{t}\}, \quad \{u_{-}, u_{+}\} = \{u_{-}, E_{i}^{(1)}(1)u_{+}\} = \{u_{-}, E_{i}^{(1)}(1)u_{+}\}. \]

Let \(\Lambda\) be a fixed weight lattice for \(\mathfrak{g}\) containing \(\Gamma\) and let \(\pi : \tilde{\Gamma} \to \Lambda\) be the homomorphism of monoids defined by \(\pi(\alpha_{\pm}) = \pm \alpha_{i}\). Given \(u \in U_{q}(\mathfrak{g})\) homogeneous, let \(\deg_{\Gamma} u = \pi(\deg_{k} u)\). Note that \(\{u_{-}, u_{+}\} \neq 0\) implies that \(\deg_{\Gamma} u = - \deg_{k} u_{+}, u_{-} \in U_{q}^{L}\).

Extend the \(\alpha_{i}^{\vee}\) to elements of Hom_{\mathbb{Z}}(\Lambda, \mathbb{Z}). Let \(\tilde{U} q(\mathfrak{g})\) be the algebra \(\tilde{U} q(\mathfrak{g})\) extended by adjoining elements of the form \(K_{\alpha_{-}, \mu}, \mu \in \Lambda\). Thus, \(\tilde{U} q(\mathfrak{g})\) is generated by the \(U_{q}^{L}\) and \(K_{\alpha_{-}, \mu}, \mu \in \Lambda, \alpha_{\pm} \in \mathbb{Z}\Gamma\) such that for all \(i \in I\)

\[K_{\alpha_{-}, \mu, \alpha_{+}} = q_{i}^{\alpha_{i}^{\vee}(2\mu + \alpha_{+} - \alpha_{-})}E_{i}K_{\alpha_{-}, \mu, \alpha_{+}} \quad K_{\alpha_{-}, \mu, \alpha_{+}} = q_{i}^{-\alpha_{i}^{\vee}(2\mu + \alpha_{+} - \alpha_{-})}F_{i}K_{\alpha_{-}, \mu, \alpha_{+}}. \]

It should be noted that \(\tilde{U} q(\mathfrak{g}) = \tilde{U} q(\mathfrak{g})\) if \(2\Lambda = \mathbb{Z}\Gamma\).

Recall that a \(U_{q}(\mathfrak{g})\)-module \(V\) is called lowest weight of lowest weight \(-\mu \in \Lambda\) if there exists \(v_{-\mu} \in V \setminus \{0\}\) such that \(V = U(\mathfrak{g})v_{-\mu}, U^{-}v_{-\mu} = 0\) and \(K_{i}v_{-\mu} = q_{i}^{\alpha_{i}^{\vee}(\mu)}v_{-\mu}, i \in I\). Clearly, a
lowest weight module is graded by $\Gamma$ and we denote by $|v|$ the degree of a homogeneous element $v$ of $V$; then $K_i v = q_i^{\alpha_i^\vee (-\mu + |v|)} v$, $i \in I$.

Let $V$ be a lowest weight module of lowest weight $-\mu \in \Lambda$. Let $\langle \cdot, \cdot \rangle_V$ be a symmetric pairing $V \otimes V \to k$ such that $(xu | v)_V = (u | x^t v)_V$ for all $x \in U_q(\mathfrak{g})$, $u, v \in V$. The radical of such a pairing is clearly a submodule of $V$ hence for $V$ simple it is non-degenerate. Since $\langle u | v \rangle_V \neq 0$ implies that $|u| = |v|$ for $u, v \in V$ homogeneous and homogeneous components of $V$ are finite dimensional, it follows that if $\langle \cdot, \cdot \rangle_V$ is non-degenerate then any basis of $V$ admits a dual basis with respect to $\langle \cdot, \cdot \rangle_V$.

Let $B_\pm$ be a homogeneous bases of $U_q^\pm$. Define a map $\Xi : V \otimes V \to \tilde{U}_q(\mathfrak{g})$ by

$$
\Xi(v \otimes v') := q^{\frac{1}{2}\gamma(|v'|)+\frac{1}{2}\gamma(|v|)} \sum_{b_\pm \in B_\pm} q^n(b_+) \langle b_+ v', b_-^t v \rangle_V (K_{[b_+,v'],0} \circ b_-) (K_{0,2\mu-|v'|} \circ b_+),
$$

(4.14)

for all $v, v' \in V$, where $\{b_\pm\}_{b_\pm \in B_\pm} \subset U_q^\pm$ denotes the dual basis to $B_\pm$ with respect to the pairing $\{\cdot, \cdot\}$. Thus, $\{b_+, b_-\} = \delta_{\pm, \gamma'}$, $\{b'_\pm, b'\} = \delta_{\gamma', \gamma}$. Note that the sum in (4.14) is finite since $|xv| = |v| + \deg_{\Gamma} x$ for any $v \in V$, $x \in U_q^-$ homogeneous, $\deg_{\Gamma} x \in -\Gamma$, there are finitely many $\nu \in \Gamma$ such that $|v - \nu| \in \Gamma$ and all homogeneous components of $U_q^-$ are finite dimensional.

**Proposition 4.17 (Theorem 1.25).** Let $V^\#$ be $V$ with the left action of $U_q(\mathfrak{g})$ defined by $x \triangleright v = S(x)^t v$, $x \in U_q(\mathfrak{g})$, $v \in V$. Then $\Xi : V^\# \otimes V \to \tilde{U}_q(\mathfrak{g})$ is a homomorphism of left $U_q(\mathfrak{g})$-modules where $V^\# \otimes V$ is endowed with a $U_q(\mathfrak{g})$-module structure via the comultiplication and the $U_q(\mathfrak{g})$ action on $\tilde{U}_q(\mathfrak{g})$ is the adjoint one.

**Remark 4.18.** The formulae in Theorem 1.25 are obtained from the action defined above. The module $V^\#$ is highest weight of highest weight $\mu$.

**Proof.** Let $v, v' \in V$ be homogeneous and set $\xi = |v|, \xi' = |v'|$. We also abbreviate $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_V$, $|x| = \deg_{\Gamma} x$ for $x \in U_q(\mathfrak{g})$ homogeneous and set $\kappa(\xi, \xi) = \frac{1}{2} \gamma(\xi) - \frac{1}{2} \gamma(\xi')$. Since $\langle b_+ v' | b_-^t v \rangle \neq 0$ implies that $|b_-| + |b_\pm| = \xi' - \xi$, it follows that $K_i \Xi(v \otimes v') = q_i^{\alpha_i^\vee (\xi' - \xi)} \Xi(v \otimes v') = \Xi(K_i^{-1} v \otimes K_i v')$.

Furthermore,

$$
q^{\kappa(\xi', \xi)} \Xi(E_i^{(1)}(v \otimes v')) = q^{\kappa(\xi', \xi)} \Xi(v \otimes E_i^{(1)}(v')) - q^{\kappa(\xi', \xi)} \Xi(K_{+i}^{-1} F_i^{(1)}(v) \otimes K_{+i}(v'))
$$

$$
= q^{\kappa(\xi' + \alpha_i, \xi) - \kappa(\xi', \xi)} \sum_{b_\pm \in B_\pm} q^{n(b_+)} \langle b_+ E_i^{(1)}(v') | b_-^t (v) \rangle (K_{\xi' - |b_+| + \alpha_i, 0} \circ b_-) (K_{0,2\mu - \xi' - \xi} \circ b_+)
$$

$$
- q^{\kappa(\xi', \xi - \alpha_i) - \kappa(\xi', \xi)} q_i^{\alpha_i^\vee (\xi' - \xi)} \sum_{b_\pm \in B_\pm} q^{n(b_+)} \langle b_+^t (v') | (E_i^{(1)}(b_-) | b_-) (K_{\xi' - |b_-|, 0} \circ b_-) (K_{0,2\mu - \xi' - \xi} \circ b_+)
$$

$$
= q_i^{\frac{1}{2} \alpha_i^\vee (\xi')} \sum_{b_\pm \in B_\pm} q^{n(b_+)} \langle E_i^{(1)}(b_+^t (v') | b_-^t (v)) (K_{\xi' - |b_+| + \alpha_i, 0} \circ b_-) (K_{0,2\mu - \xi' - \xi} \circ b_+)
$$

$$
+ q_i^{\frac{1}{2} \alpha_i^\vee (\xi')} \sum_{b_\pm \in B_\pm} q^{n(b_+)} \langle K_{+i} \circ \partial_i^{-1} (b_+) (v') | b_-^t (v)) (K_{\xi' - |b_+| + \alpha_i, 0} \circ b_-) (K_{0,2\mu - \xi' - \xi} \circ b_+)
$$

$$
- q_i^{\frac{1}{2} \alpha_i^\vee (\xi')} \sum_{b_\pm \in B_\pm} q^{n(b_+)} \langle K_{-i} \circ \partial_i^{-1} (b_+) (v') | b_-^t (v)) (K_{\xi' - |b_+| + \alpha_i, 0} \circ b_-) (K_{0,2\mu - \xi' - \xi} \circ b_+)
$$

$$
- q_i^{\frac{1}{2} \alpha_i^\vee (2\xi' - \xi) + 1} \sum_{b_\pm \in B_\pm} q^{n(b_+)} \langle b_+^t (v') | (E_i^{(1)}(b_-) | b_-) (K_{\xi' - |b_-|, 0} \circ b_-) (K_{0,2\mu - \xi' - \xi} \circ b_+)
$$

$$
= q_i^{\frac{1}{2} \alpha_i^\vee (\xi')} \sum_{b_\pm \in B_\pm} q^{n(b_+)} \langle b_+^t (v') | (b_- E_i^{(1)} | b_-) (K_{\xi' - |b_+| + \alpha_i, 0} \circ b_-) (K_{0,2\mu - \xi' - \xi} \circ b_+)
$$

$$
= q_i^{\frac{1}{2} \alpha_i^\vee (\xi')} \sum_{b_\pm \in B_\pm} q^{n(b_+)} \langle b_+^t (v') | (b_- E_i^{(1)} | b_-) (K_{\xi' - |b_+| + \alpha_i, 0} \circ b_-) (K_{0,2\mu - \xi' - \xi} \circ b_+)$$
for all 

\[ q^{-\kappa(\xi',\xi)}[E_i^{(1)}, \Xi(v \otimes v')]K_{+i}^{-1} \]

\[
= q_{+i}^{\frac{1}{2} \alpha_i'^{(2\mu'_-\xi)}} \sum_{b_+ \in B_+} q^{\eta(b_+)} \langle b_+(v') | (E_i^{(1)})^{t}(v) (K_{\xi'\mu_+} \circ b_+) \rangle (K_{0,2\mu'-\xi} \circ b_+)
\]

\[ + \sum_{b_+ \in B_+} q^{\eta(b_+)} \frac{1}{2} \alpha_i'^{(2\mu'-\xi'\mu_+)}} \sum_{b_+ \in B_+} q^{\eta(b_+)} \langle b_+(v') | (E_i^{(1)})^{t}(v) (K_{\xi'\mu_+} \circ b_+) \rangle (K_{0,2\mu'-\xi'\mu_+} \circ b_+) \]

\[ - \sum_{b_+ \in B_+} q^{\eta(b_+)} \frac{1}{2} \alpha_i'^{(2\mu'-\xi'\mu_-)}} \sum_{b_+ \in B_+} q^{\eta(b_+)} \langle b_+(v') | (E_i^{(1)})^{t}(v) (K_{\xi'\mu_-} \circ b_+) \rangle (K_{0,2\mu'-\xi'\mu_-} \circ b_+). \]

On the other hand,

\[
q^{-\kappa(\xi',\xi)}[E_i^{(1)}, \Xi(v \otimes v')]K_{+i}^{-1} \]

\[
= q_{+i}^{\frac{1}{2} \alpha_i'^{(2\mu'-\xi)}} \sum_{b_+ \in B_+} q^{\eta(b_+)} \langle b_+(v') | (E_i^{(1)})^{t}(v) (K_{\xi'\mu_+} \circ b_+) \rangle (K_{0,2\mu'-\xi} \circ b_+)
\]

\[ - \sum_{b_+ \in B_+} q^{\eta(b_+)} \frac{1}{2} \alpha_i'^{(2\mu'-\xi'\mu_-)}} \sum_{b_+ \in B_+} q^{\eta(b_+)} \langle b_+(v') | (E_i^{(1)})^{t}(v) (K_{\xi'\mu_-} \circ b_+) \rangle (K_{0,2\mu'-\xi'\mu_-} \circ b_+). \]

Furthermore, since

\[
u_+ = \sum_{b_+ \in B_+} \{ b_+, u_+ \} b_+ = \sum_{b_- \in B_-} \{ b_-, u_+ \} b_-, \quad u_- = \sum_{b_+ \in B_+} \{ u_-, b_+ \} b_+ = \sum_{b_- \in B_-} \{ u_-, b_- \} b_-
\]

for all \( u_+ \in U^+ \), we obtain, using (4.13)

\[
q^{-\kappa(\xi',\xi)}[E_i^{(1)}, \Xi(v \otimes v')]K_{+i}^{-1} \]

\[
= \sum_{b_+, b'_+ \in B_+, \, b_- \in B_-} q^{\eta(b_+)} \frac{1}{2} \alpha_i'^{(2\mu'-\xi'\mu_-)}} \sum_{b_+ \in B_+} q^{\eta(b_+)} \langle b_+(v') | (E_i^{(1)})^{t}(v) (K_{\xi'\mu_-} \circ b_+) \rangle (K_{0,2\mu'-\xi'\mu_-} \circ b_+). \]
Lemma 4.19. Since each homogeneous piece of a lowest weight module is finite dimensional. Extend the pairing $B$ uniquely defined by this condition. Extend the pairing $\langle \cdot, \cdot \rangle$.

The computation for the action of $F_i^{(1)}$ is similar and is omitted.  

Let $\rho$ be an element of $\Lambda$ satisfying $\alpha^\vee_i(\rho) = 1$ for all $i \in I$. If $\mathfrak{g}$ is finite dimensional then $\rho$ is uniquely defined by this condition. Extend the pairing $\cdot : \Gamma \times \Gamma \to \mathbb{Z}$ to a pairing $\Lambda \times \Lambda \to \mathbb{Q}$.

Lemma 4.19. Let $V$ be a lowest weight module of lowest weight $-\mu \in \Lambda$ and suppose that the pairing $\langle \cdot, \cdot \rangle_V$ is non-degenerate. Then the canonical invariant $1_V$ in $V \# \otimes \tilde{V}$ is given by

$$1_V = q^{2\rho \mu} \sum_{\nu \in \mathcal{H}_V} q^{-2\nu(\nu')} \nu \otimes \tilde{\nu}$$

where $\mathcal{H}_V$ is a homogeneous basis of $V$ and $\{\tilde{\nu}\}_{\nu \in \mathcal{H}_V} \subset V$ is its dual basis with respect to the pairing $\langle \cdot | \cdot \rangle_V$.

Proof. Note that for any $u \in V$ we have $u = \sum_{b \in \mathcal{H}_V} \langle u, b \rangle_V b = \sum_{b \in \mathcal{H}_V} \langle b, u \rangle_V \tilde{b}$. This sum is finite since each homogeneous piece of a lowest weight module is finite dimensional. Since $|u| = |\nu|$ for
all \( v \in \mathcal{B}_V \), \( K_i(1_V) = 1_V \). Furthermore,

\[
E_i(q^{-2\rho \mu} 1_V) = \sum_{b \in \mathcal{B}_V} q^{-2\eta(|\bar{b}|)} b \otimes E_i \bar{b} - \sum_{b \in \mathcal{B}_V} q^{2\eta(|\bar{b}|)} K_i^{-1} F_i b \otimes K_i \bar{b}
\]

\[
= \sum_{b, b' \in \mathcal{B}_V} q^{-2\eta(|\bar{b}'|)} \langle b', E_i \bar{b} \rangle V b \otimes \bar{b}' - \sum_{b \in \mathcal{B}_V} q^{-2\eta(|\bar{b}|)} q_i F_i b \otimes \bar{b}
\]

\[
= \sum_{b, b' \in \mathcal{B}_V} q^{-2\eta(|\bar{b}'|)} q_i^2 \langle F_i b, \bar{b}' \rangle V b' \otimes \bar{b}' - \sum_{b \in \mathcal{B}_V} q^{-2\eta(|\bar{b}|)} q_i^2 F_i b \otimes \bar{b} = 0
\]

and similarly \( F_i(1_V) = 0 \). \( \square \)

### 4.5. Towards Conjecture \ref{conj:4.2}.26.

**Example 4.20.** Let \( g = \mathfrak{sl}_2 \) and let \( V \) be the \((m + 1)\)-dimensional \( U_q(\mathfrak{g}) \)-module with its standard basis \( v_a = E^{(a)} v_0 \), \( 0 \leq a \leq m \). Then \( E^{(a)} v_b = (a+b) v_{b+a} \), \( F^{(a)} v_b = (-1)^a (m+b-a) v_{b-a} \), \( 0 \leq b \leq m \) where we set \( v_k = 0 \) if \( k < 0 \) or \( k > m \). Denote \( \{ v^a \}_{0 \leq a \leq m} \) the dual basis of \( V \) with respect to the pairing \( \langle \cdot, \cdot \rangle_m := \langle \cdot, \cdot \rangle_V \). Then we have

\[
\Xi(v^a \otimes v_b) = q^{\binom{b}{2} - \binom{a}{2}} \sum_{k = \max(0, b-a)}^b q^k \langle v^a, E^{(a-b+k)} F^{(k)} v_b \rangle m (K^{-b-k}_- \circ F^{a-b+k})(K^m_- \circ E^k)
\]

\[
= q^{\binom{b}{2} - \binom{a}{2}} \sum_{k = \max(0, b-a)}^b (-1)^k q^k \binom{m-b+k}{k} \binom{a}{b-k} q^{K^{-b-k}_- \circ F^{a-b+k}}(K^m_- \circ E^k),
\]

whence we obtain, using \ref{eq:3.4} and \ref{eq:4.3}

\[
\Xi(1_V) = \sum_{a=0}^m q^{m-2a} \Xi(v^a \otimes v_a)
\]

\[
= \sum_{0 \leq k \leq a \leq m} (-1)^k q^{k+m-2a-m} \binom{m-a+k}{k} \binom{a}{k} q^{K^{-a-k}_- \circ F^k}(K^m_- \circ E^k)
\]

\[
= \sum_{0 \leq k \leq a \leq m} (-1)^k q^{k(1+m-k-2a)} \binom{m-a+k}{k} \binom{a}{k} q^{K^{-a-k}_- K^m_- \circ F^k E^k}
\]

\[
= \sum_{r, s \geq 0, r+s \leq m} (-1)^{m-r-s} q^{(r+s-m-1)(r-s)} \binom{m-r}{s} \binom{m-s}{r} q^{K^r_- K^s_- \circ F^{m-r-s} E^{m-r-s}} = (-1)^m C(m).
\]

This proves Conjecture \ref{conj:4.2} for \( g = \mathfrak{sl}_2 \).

**Example 4.21.** Let \( g = \mathfrak{sl}_{n+1} \) and let \( V \) be the simple lowest weight module of lowest weight \( -\omega_1 \). Its standard basis is \( v_i \), \( 0 \leq i \leq n \), where \( E^{(1)}_i v_j = \delta_{i,j+1} v_{j+1} \) and \( F^{(1)}_i v_j = -\delta_{i,j} v_{j-1} \). Denote \( \alpha_{i,j} = \sum_{k=i}^j \alpha_k \). Then \( |v_i| = \alpha_{1,i} \) and \( \gamma(|v_i|) = -i + 1 - \delta_{i,0} \), \( 0 \leq i \leq n \). Let \( \{ v^j \}, 0 \leq j \leq n \) be the dual basis of \( V \) with respect to the pairing \( \langle \cdot, \cdot \rangle_V \). We have

\[
\Xi(v^i \otimes v_j) = q^{i-j+\delta_{i,j-1} \delta_{j,a}} \sum_{b \in B, 1 \leq k \leq j+1} q^{i-j-k+1} \langle v^i, b \rangle \cdots F^{(1)} v_j \rangle m (K_{a_{1,k-1},0} \circ b_-)(K_{0,2\omega_1-a_{1,j}} \circ E_{[k,j]^*})
\]

\[
= q^{i-j+\delta_{i,j-1} \delta_{j,a}} \sum_{k=1}^{\min(i,j)+1} (-q)^{i-j-k+1} (K_{a_{1,k-1},0} \circ F_{[k,i]})(K_{0,2\omega_1-a_{1,j}} \circ E_{[k,j]^*})
\]
A.36 \otimes \Delta V \text{ we claim that } F \otimes F_{[i,j]}^* \text{ is } \phi \text{-invariant. For, it is easy to show by induction on } j - i \text{ that }

\Delta(E_{[i,j]}^*) = \sum_{k=i-1}^{j} q^{2i_k} \alpha_{i,k} \delta_{i+1,k}^* \otimes E_{[i+1,j]}^* \quad \Delta(F_{[i,j]}) = \sum_{k=i-1}^{j} q^{2i_k} \alpha_{i,k} \delta_{i,k} \otimes F_{[k,j]},

which in particular implies that 

\langle F_{[i,j]}, E_{[a,b]} \rangle = \delta_{i,a} \delta_{j,b} (q - q^{-1})^{j-i+1}, \quad \langle F_{[i,j]}, E_{[a,b]}^* \rangle = \delta_{i,a} \delta_{j,b} (q - q^{-1})^{j-i+1}(q - q^{-1})^{j-i+1}.

By Proposition A.36 we obtain

E_{[i,j]}^* F_{[i,j]} = \sum_{i-1 \leq r \leq k \leq j} (-1)^{j-k+r-i-\delta_{i,r}} q^{j-k-1-r+i-\delta_{r,i+1}+\delta_{j,k}} (q - q^{-1})^{2-\delta_{r,i+1}+\delta_{j,k}} \times

K_{-i} \cdots K_{-r} K_{+1} \cdots K_{+j} F_{[r+1,k]} E_{[r+1,k]}^*.

Therefore,

\Xi(1_V) = \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} q^{n+i+j} K_{-i} \cdots K_{-j} K_{+1} \cdots K_{+n} E_{[i+1,j]}^* F_{[i+1,j]}

= \sum_{0 \leq i \leq j \leq n} (-1)^{1-j-r-i} q^{n+2(i+j) - k-r-i-\delta_{i,r}+\delta_{j,k}} (q - q^{-1})^{2-\delta_{i,r}+\delta_{j,k}} \times

K_{-i} \cdots K_{-r} K_{+1} \cdots K_{+n} F_{[r+1,k]} E_{[r+1,k]}^*.

Since

\left( \sum_{i=0}^{r} (-1)^{\delta_{i,r}} q^{2i-\delta_{i,r}} (q - q^{-1})^{1-\delta_{i,r}} \right) \left( \sum_{j=k}^{n} q^{2j+\delta_{j,k}} (q - q^{-1})^{1-\delta_{j,k}} \right) = -q^{2n}. 


This computation also shows that the image of $(-1)^nK_{0,-\omega_1+\omega_n}\Xi(1_V)$ in $\mathcal{H}_q^+(\mathfrak{g})$ is $\bar{\omega}$-invariant. Together with Theorems 1.3 and 1.8 this implies that

$$F_{[a,b]} \circ E_{[a,b]}^* = \sum_{j=a-1}^{b} (-q)^{b-j}K_{+j+1} \cdots K_b F_{[a,j]} E_{[a,j]}^*.$$  \hfill (4.17)

Then

$$\sum_{i=0}^{n} (-q)^{-i}K_{-i} \cdots K_{-i} F_{[i+1,n]} \circ E_{[i+1,n]}^*$$

$$= \sum_{0 \leq i \leq j \leq n} (-1)^{n-i-j} q^{n-i-j} K_{-i} F_{[i+1,j]} \cdots K_{-j} F_{[i+1,j]} E_{[i+1,j]}^* = (-1)^n K_{0,-\omega_1+\omega_n}\Xi(1_V),$$

and (4.16) follows by Theorem 1.5. In particular, we obtain an explicit formula for $F_{[i,j]} \circ E_{[i,j]}^*$ and $F_{[i,j]} \cdot E_{[i,j]}^*$, $1 \leq i \leq j \leq n$.

**Example 4.22.** Let $\mathfrak{g} = \mathfrak{sp}_4$ and let $V(-\omega_1)$ be the lowest weight module of the lowest weight $-\omega_1$. Its standard basis is $\{v_i\}_{0 \leq i \leq 3}$ with the non-trivial actions being

$$E_{1}^{(1)} v_0 = v_1, \quad E_{2}^{(1)} v_1 = v_2, \quad E_{1}^{(1)} v_2 = v_3, \quad F_{1}^{(1)} v_3 = -v_2, \quad F_{2}^{(1)} v_2 = -v_1, \quad F_{1}^{(1)} v_1 = -v_0.$$  

Denote $\{v_i^j\}_{0 \leq i \leq 3}$ the dual basis of $V(-\omega_1)$ with respect to the pairing $\langle -,- \rangle_{V(-\omega_1)}$. Then

$$\Xi(1_{V(-\omega_1)}) = q^4 \Xi(v^0 \otimes v_0) + q^2 \Xi(v^1 \otimes v_1) + q^{-2} \Xi(v^2 \otimes v_2) + q^{-4} \Xi(v^3 \otimes v_3)$$

$$= q^4 K_{+1}^2 K_{+2}^2 + q^2 (K_{-1} K_{+1} K_{+2} - q K_{+1} K_{+2} \circ F_0 E_1)$$

$$+ q^{-2} (K_{-1} K_{-2} K_{+1} - q^2 K_{-1} K_{+1} \circ F_0 E_2 + q^3 K_{+1} \circ F_0 E_{21})$$

$$+ q^{-4} (K_{-1} K_{-2} K_{-3} - q K_{-1} K_{-2} \circ F_1 E_1 + q^3 K_{-1} \circ F_0 E_{21} - q^4 F_0 E_{21} E_{121}).$$

It is easy to check that $\Xi(1_{V(-\omega_1)}) = -F_{121} \bullet E_{121}$ since

$$F_{121} \circ E_{121} = F_{121} E_{121} - q K_{+1} F_{12} E_{21} + q^3 K_{+1} K_{+2} F_{0} E_1 - q^4 K_{+1}^2 K_{+2}$$

$$F_{121} \bullet E_{121} = F_{121} \circ E_{121} - q^{-1} K_{-1} F_{0} E_{21} \circ E_{12} - q^{-3} K_{-1} K_{-2} F_0 E_1 + q^{-4} K_{+1}^2 K_{-2}.$$

Similarly, for the lowest weight module $V(-\omega_2)$ of lowest weight $-\omega_2$. Its standard basis $\{v_i\}_{0 \leq i \leq 4}$ satisfies

$$E_{2}^{(1)} v_0 = v_1, \quad E_{1}^{(1)} v_1 = v_2, \quad E_{1}^{(1)} v_2 = (2) q v_3, \quad E_{2}^{(1)} v_3 = v_4$$

$$F_{2}^{(1)} v_1 = -v_0, \quad F_{1}^{(1)} v_2 = -2 q v_1, \quad F_{1}^{(1)} v_3 = -v_2, \quad F_{2}^{(1)} v_4 = -v_3$$

and

$$\Xi(1_{V(-\omega_2)}) = q^6 \Xi(v^0 \otimes v_0) + q^2 \Xi(v^1 \otimes v_1) + \Xi(v^2 \otimes v_2) + q^{-2} \Xi(v^3 \otimes v_3) + q^{-6} \Xi(v^4 \otimes v_4)$$

$$= q^6 K_{+1}^2 K_{+2}^2 + q^2 (K_{-2} K_{+1} K_{+2} - q^2 K_{+1} K_{+2} \circ F_2 E_{2})$$

$$+ (K_{-1} K_{-2} K_{+1} K_{+2} - (q + q^3) K_{-1} K_{+1} K_{+2} \circ F_1 E_{1} + (q^3 + q^5) q K_{+1} K_{+2} \circ F_1 E_{12})$$

$$+ q^{-2} (K_{-1} K_{-2} K_{+1} K_{+2} - (q + q^3) K_{-1} K_{+1} K_{+2} \circ F_1 E_{1} + q^2 K_{-1} K_{+2} \circ F_1 E_{1} + q^2 K_{-1} K_{+2} \circ F_1 E_{12} - q^4 K_{+1} K_{+2} \circ F_1 E_{12})$$

$$+ q^{-4} (K_{-1} K_{-2} K_{+1} K_{+2} - q^2 K_{-1} K_{+1} K_{+2} \circ F_2 E_{2} + q^2 (2) q K_{+1} K_{+2} \circ F_2 E_{21} + q^4 K_{-2} \circ F_1 E_{21}$$

$$+ q^2 K_{+1} K_{+2} F_2 E_{2} + q^6 K_{+1} K_{+2}^2$$

where $E_{2112} = E_2 E_{112} - q^2 E_{112}^2$ and $F_{2112} = E_{2112}^*$. Since

$$F_{2112} \circ E_{2112} = F_{2112} E_{2112} - q^2 K_{+2} F_{211} E_{112} - (q^5 + q^3) K_{+1} K_{+2} F_{21} E_{12}$$

$$- q^4 K_{+1} K_{+2} F_2 E_{2} + q^6 K_{+1} K_{+2}^2$$

...
The assignments (b) extends to an of canonical basis $k$ of $U$.

Proof. 5.1. Invariant braid group action on Drinfeld double. Denote by $U'_q(\hat{g})$ the quotient of $k[z_i^{\pm 1} : i \in I] \otimes_k U_q(\hat{g})$ by the ideal generated by $z_i^2 \otimes 1 - 1 \otimes K_{i+i}$. It is easy to see that $\bar{\gamma}$ extends to a $Q$-linear anti-involution of $U'_q(\hat{g})$ by replacing $z_i \mapsto \bar{z}_i$. Then it is immediate that the set

$$B'_g = \{(\prod_{i \in I} z_i^{a_i}) b : b \in B_{\hat{g}}, a_i \in \mathbb{Z}\}$$

is a $\bar{\gamma}$-invariant basis of $U'_q(\hat{g})$. In the sequel we use the presentation of $U_q(g)$ obtained from (1.2) and (1.3) by replacing $K_{i+i}$ with $K_i^{\pm 1}$. The following Lemma is immediate.

Lemma 5.1. (a) The assignments $E_i \mapsto E_i$, $F_i \mapsto F_i$, $K_{i+i} \mapsto K_i^{\pm 1}$, $z_i \mapsto 1$ extends to a surjective homomorphism of algebras $\phi : U'_q(\hat{g}) \rightarrow U_q(g)$.

(b) The assignments $E_i \mapsto E_i z_i^{-1}$, $F_i \mapsto F_i$, $K_i^{\pm 1} \mapsto K_i z_i^{-1}$ extends to an injective homomorphism of algebras $\iota : U_q(g) \rightarrow U'_q(\hat{g})$ which splits $\phi$.

Clearly, there exists a unique anti-involution $\bar{\gamma}$ on $U'_q(\hat{g})$ which commutes with $\iota$ and $\phi$. It is also easy to see that there exists a unique basis $B_g$ of $U_q(g)$ such that $\iota(B_g) = B'_g \cap \iota(U_q(g))$. Clearly $B_g = \phi(B'_g)$ and each element of $B_g$ is fixed by $\bar{\gamma}$. From now on we refer to $B_g$ as the double canonical basis of $U_q(g)$.

Given $\alpha_\pm \in \Gamma$, set $\text{Ad}^\pm K_{\alpha_-\alpha_+}(x) = \chi^\pm((\alpha_-\alpha_+), \text{deg}_F x) x$ for $x \in U_q(g)$ homogeneous. Let $Q$ be the free abelian group generated by the $\alpha_i$, $i \in I$ and let $\hat{Q} = Q \oplus \hat{Q}$. Then $\Gamma$ (respectively, $\hat{\Gamma}$) is a submonoid of $Q$ (respectively, $\hat{Q}$). Extend $\alpha_i^\gamma \in \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{Z})$ to elements of $\text{Hom}_\mathbb{Z}(Q, \mathbb{Z})$ in a natural way. The Weyl group $W$ of $\hat{g}$ acts on $Q$ and hence on $\hat{Q}$ via $s_i(\alpha) = \alpha - \alpha_i^\gamma(\alpha)\alpha_i$, $i \in I$.

Lemma 5.2. In the presentation (1.2)–(1.3) of $U'_q(\hat{g})$, we have $T_i(z_j) = z_j z_i^{-a_{ij}}$, $i, j \in I$

and

$$T_i(K_{\pm i}) = \begin{cases} K_{\pm i} z_i^{-2}, & i = j \\ K_{\pm j} K_{\pm i}^{-a_{ij}}, & i \neq j \end{cases}$$

Moreover, the $T_i$ satisfy the braid relations, commute with $\bar{\gamma}$ and satisfy $T_i \ast = \ast T_i^{-1}$, $T_i \circ \iota = \iota \circ T_i^{-1}$.

Proof. Recall that our presentation of $U_q(g)$ is obtained from the standard one by rescaling $E_i \mapsto (q_i^{-1} - q)^{-1} E_i$, $F_i \mapsto (q_i - q_i^{-1})^{-1} F_i$ for all $i \in I$. In this presentation the symmetries $T'_i$, $T''_{i-1}$ of $U_q(g)$ defined in [19, §37.1.3] are given by $T'_i(K_j) = K_j K_i^{-a_{ij}} = T''_{i-1}(K_j)$,

$$T''_{i-1}(E_j) = \begin{cases} F_i K_i^{-1}, & i = j \\ \sum_{r+s = -a_{ij}} (-1)^r q_i^r E_i^{(r)} E_j E_i^{(s)}, & i \neq j \end{cases}$$
\[ T_{i,-1}''(F_j) = \begin{cases} 
 K_i E_i, & i = j \\
 \sum_{r+s=-a_{ij}} (-q_i)^{-r} F_i^{(r)} F_j^{(s)}, & i \neq j 
 \end{cases} \]

and

\[ T_{i,1}'(E_j) = \begin{cases} 
 K_i F_i, & i = j \\
 \sum_{r+s=-a_{ij}} (-1)^s q_i^r E_i^{(r)} E_j^{(s)}, & i \neq j 
 \end{cases} \]

\[ T_{i,1}'(F_j) = \begin{cases} 
 E_i K_i^{-1}, & i = j \\
 \sum_{r+s=-a_{ij}} (-q_i)^{-r} F_i^{(s)} F_j^{(r)}, & i \neq j 
 \end{cases} \]

By [19, Proposition 37.1.2] \( T_{i,1}', T_{i,-1}'' \) are automorphisms of \( U_q(g) \) while by [19, Theorem 39.4.3] they satisfy the braid relations of the braid group of \( g \). Also, \( T_{i,-1}' = (T_{i,1}'')^{-1} \). It is easy to see that \( T_{i,-1}'(E_j), T_{i,-1}''(E_j) \) and \( T_{i,1}'(E_j), T_{i,-1}'(F_j), i \neq j \), are given on \( U_q(\tilde{g}) \) by the same formula as on \( U_q(g) \). Furthermore we have

\[ z_i T_{i,-1}''(E_i) = T_{i,-1}''(E_i z_i^{-1}) = F_i K_{-i} z_i^{-1}, \]

\[ z_i T_{i,1}'(E_i) = T_{i,1}'(E_i z_i^{-1}) = K_{-i} F_i z_i^{-1} \]

whence \( T_{i,-1}''(E_i) = F_i K_{-i} z_i^{-2} \) and \( T_{i,1}'(E_i) = K_{-i} z_i^{-2} F_i \). Similarly, \( T_{i,-1}'(F_i) = K_{-i} z_i^{-1} E_i z_i^{-1} = K_{-i} z_i^{-2} E_i \) and \( T_{i,1}'(F_i) = F_i K_{-i} z_i^{-2} \). Finally,

\[ z_i T_{i,-1}''(K_{\pm i}) = T_{i,1}'(K_{\mp i} z_i^{-1}) = K_{\mp i} z_i^{-1} = z_i T_{i,1}'(K_{\pm i}) \]

whence \( T_{i,-1}''(K_{\pm i}) = K_{\pm i} z_i^{-2} = T_{i,1}'(K_{\pm i}) \), while \( z_i^{-1} z_i^{-a_{ij}} T_{i,-1}''(K_{\pm j}) = K_{\pm j} z_i^{-1} (K_{\pm i} z_i^{-1})^{-a_{ij}} \).

Define \( T_i(x) = T_{i,-1}''(Ad^{\pm} K_i(x)) = Ad^{\pm} K_i T_{i,-1}'(x), x \in U_q(\tilde{g}) \). Then we have \( T_i^{-1}(x) = T_{i,1}'(Ad^{\pm} K_i(x)) \). It is easy to see that \( T_i \) is given on generators by the formulae from Lemma 5.2.

For example, \( T_i(E_i) = q_i K_{-i} z_i^{-2} F_i = K_{-i} z_i^{-2} F_i \) and \( F_i \). Thus, in particular, \( T_i \) is an automorphism of \( U_q(\tilde{g}) \). Clearly, \( T_{i,1}'(E_i) = T_{i,1}'(E_i) \), while for \( j \neq i \)

\[ T_i(E_j) = \sum_{r+s=-a_{ij}} (-1)^s q_i^{-s} q_j^{b_{ij}} E_i^{(s)} E_j^{(r)} E_i^{(r)} E_j^{(s)} = T_i(E_j) \]

where we used that \( E^{(k)} = (-1)^k E^{(k)} \). The remaining identities are checked similarly. The identities involving * and ′ can be checked using the explicit formulas for \( T_i^{-1} = T_i'' \circ Ad^{\pm} \tilde{K}_{-i} \).

It remains to prove that \( T_i \) satisfy the braid relations, For, let \( w \) be an element of the Weyl group of \( g \) and let \( w = s_{i_1} \cdots s_{i_r} \) be its reduced decomposition. It is sufficient to prove that \( T_{i_1} \circ \cdots \circ T_{i_r} \) depends only on \( w \) and not on the reduced decomposition. This holds for Lusztig symmetries \( T_{i,1}' \), \( T_{i,-1}' \) by [19, §39.4.4], whence for each \( w \in W \) one has a well-defined automorphism \( T_{w,-1}' \) of \( U_q(g) \) satisfying \( T_{w,-1}' = T_{i,-1}' \circ \cdots \circ T_{i_r,-1}' \). We have

\[ T_{i_1} \circ \cdots \circ T_{i_r}(x) = Ad^{\pm} K_{\sum_{j=1}^r s_{i_1} \cdots s_{i_{j-1}}(a_{ij})} \circ T_{i_r,-1}' \circ \cdots \circ T_{i_1,-1}' = Ad^{\pm} K_{\sum_{j=1}^r s_{i_1} \cdots s_{i_{j-1}}(a_{ij})} \circ T_{w,-1}' \]

It is well-known that \( \sum_{j=1}^r s_{i_1} \cdots s_{i_{j-1}}(a_{ij}) = \sum_{\bf{\beta}} \in R_+ \cap \bf{w}(-R_+) \bf{\beta} \), where \( R_+ \subset Q \) denotes the set of positive roots of \( g \), depends only on \( w \) and not on its reduced decomposition. Therefore, the right hand side depends only on \( w \).

**Proof of Theorem 1.13.** Note that \( U_q(\tilde{g}) \) embeds into \( U_q(\tilde{g}) \) via \( E_i \mapsto E_i, F_i \mapsto F_i, K_{\pm i} \mapsto K_{\pm i}, K_{-i}^{-1} \mapsto K_{\mp i} z_i^{-2} \) for all \( i \in I \). All assertions of Theorem 1.13 are then immediate consequences of Lemma 5.2. \( \square \)
In particular, for each \( w \in W \), we have a unique automorphism \( T_w \) of \( U_q(\hat{g}) \) such that \( T_{s_i} = T_i \) and \( T_w = T_{w',w''} \) for any reduced decomposition \( w = w'w'' \). It follows from Lemma 5.2 that for all \( x \in U_q(\hat{g}) \)

\[
T_w(x) = T_w(\bar{x}), \quad T_w(x^*) = (T_{w^{-1}}(x))^*, \quad T_w(x^t) = (T_{w^{-1}}(x))^t.
\]

Furthermore, we have for \( x \in U_q(\hat{g}) \) homogeneous

\[
T_w(x) = \chi^\frac{1}{2}((\langle w,0 \rangle, w \deg_w x) T^\prime_{w,-1}(x) = \chi^\frac{1}{2}((0, \langle w^{-1} \rangle), \deg_w x) T^\prime_{w,-1}(x).
\]

where \( \langle w \rangle = \sum_{\beta \in R^+ \cap w(-R_+)} \beta \) and the action of \( W \) on \( \Gamma \) is extended to \( \hat{\Gamma} \) diagonally.

### 5.2. Elements \( T_w \), quantum Schubert cells and their bases

Let \( \mathfrak{g} \) be any Kac-Moody Lie algebra. Given \( w \in W \) define

\[
U_q^+(w) = T_w(KU_q^-) \cap U_q^+.
\]

Let \( i = (i_1, \ldots, i_m), m = \ell(w) \), be such that \( w = s_{i_1} \cdots s_{i_m} \) is a reduced decomposition. Then for \( a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0} \) define

\[
E^a_i := E^{a_1}_{i_1} E^{a_2}_{i_2} \cdots E^{a_m}_{i_m}.
\]

It follows from [19] and (5.1) that for all \( w \in W, i \in I \) such that \( \ell(ws_i) = \ell(w) + 1 \), we have

\[
T_w(E_i), T^{-1}_{w^{-1}}(E_i) \in U_q^+, \quad T_w(F_i), T^{-1}_{w^{-1}}(F_i) \in U_q^-.
\]

Thus, the \( E^a_i \in U_q^+ \). It follows from [19, Proposition 40.2.1] that the \( E^a_i \) are linearly independent. Let \( U^+_q(w,1) \) be the \( \mathbb{k} \)-subspace of \( U_q^+ \) spanned by the \( E^a_i, a \in \mathbb{Z}_{\geq 0}^m \). Let \( U^+_q(w) = T_w(U^+_q) \cap U_q^+ \).

**Conjecture 5.3** ([25, Proposition 2.10] and [17, Theorem 1.1]). For any \( \mathfrak{g} \) we have a unique (tensor) factorization \( U_q^+ = U_q^+(w) \cdot U_q^+(w') \). In particular, \( U_q^+(w,1) = U^+_q(w) \).

We retain an elementary proof for the special case of \( \mathfrak{g} \) semisimple here for reader’s convenience, since the arguments in [17, 25] are rather long and non-trivial.

**Proposition 5.4.** If \( \mathfrak{g} \) is semisimple then \( U^+_q(w) = U^+_q(w,1) \).

**Proof.** We need the following

**Lemma 5.5.** For any Kac-Moody Lie algebra \( \mathfrak{g} \), \( U^+_q(w,1) \subset U_q^+(w) \).

**Proof.** Since \( U^+_q(w,1) \) is contained in the subalgebra of \( U_q^+ \) generated by the \( T_{uv}(E_i), 1 \leq r \leq m \), where \( u_r = s_1 \cdots s_{r-1} \), it suffices to prove that \( T^{-1}_{uv}(T_{uw}(E_i)) \in KU_q^- \), \( 1 \leq r \leq m \). Indeed, write \( w = u_r s_i v_r \) where \( v_r = s_{r+1} \cdots s_m \). Since \( \ell(w) = \ell(u_r) + \ell(v_r) + 1 \) we have by (5.4)

\[
T^{-1}_{uv}(T_{uw}(E_i)) = T^{-1}_{v_r}(T^{-1}_{w^{-1}}(E_i)) = T^{-1}_{v_r}(K^{-1}_r \circ F_i) \in KU_q^-.
\]

To prove the inclusion \( U^+_q(w) \subset U^+_q(w,1) \) for \( \mathfrak{g} \) semisimple, let \( w_o \) be the longest element in \( W \) and set \( w' = w^{-1} w_o \). Since \( \ell(w) + \ell(w') = \ell(w_o) \), we can choose a reduced word \( i_o \) for \( w_o \) which is the concatenation of reduced words \( i \) and \( i' \) for \( w \) and \( w' \) respectively. Then by [19, Corollary 40.2.2], monomials \( E^{a_i}_{i_o}a \in \mathbb{Z}_{\geq 0}^m, a' \in \mathbb{Z}_{\geq 0}^{(w')^t} \) form a basis of \( U_q^+ \). Observe that \( E^{a_i}_{i_o} \circ T_{u_r}(E^{a_i}_{i_o}) \in U^+_q(w)T_w(E^{a_i}_{i_o}) \). Let \( u \in U^+_q(w) \). Then we can write \( u = \sum_{a \in \mathbb{Z}_{\geq 0}^{(w')}} aT_w(E^{a_i}_{i_o}) \), where \( a \in U^+_q(w) \).

Then

\[
T^{-1}_{w^{-1}}(u) = \sum_{a \in \mathbb{Z}_{\geq 0}^{(w')}} T^{-1}_{w^{-1}}(aT_w(E^{a_i}_{i_o})).
\]

By definition of \( U^+_q(w) \), \( T^{-1}_{w^{-1}}(aT_w(E^{a_i}_{i_o})) \in KU_q^- \). Note that the triangular decomposition \( U_q(\hat{g}) = K \otimes U_q^- \otimes U_q^+ \) implies that the \( E^{a_i}_{i_o} \) are linearly independent over \( KU_q^- \). Therefore, \( T^{-1}_{w^{-1}}(u) \in KU_q^- \) if and only if \( c_{a'} = 0 \) unless \( a' = 0 \).
5.3. **Proof of Theorem 3.11.** We will often need the following identity, which is an immediate consequence of (A.35) (cf. [19, Lemma 1.4.4])

$$\langle F^r_1, E^r_1 \rangle = q_i(z)^{(r)}q_i^{-1}, \quad r \in \mathbb{Z}_{\geq 0}, \ i \in I.$$ (5.5)

Let $w \in W$ and let $w = s_{i_1} \cdots s_{i_m}$ be its reduced decomposition. Denote $i = (i_1, \ldots, i_m) \in I^m$ and set $w_r = s_{i_r} \cdots s_{i_m}, 0 \leq r \leq m$. Given $\mathbf{a} \in \mathbb{Z}_+^{I_m}$, let $\mu_i(\mathbf{a}) := q^{-\frac{1}{2}}d_{ai}(w_r^{-1})^\ast \alpha_i$ (cf. (5.2)). Let $U^+_Z(w, 1) = U^+_q(w, 1) \cap U^-_Z$. We need the following Lemma.

**Lemma 5.6.** The elements $\{\mu_i(\mathbf{a}) F_i^a : \mathbf{a} \in \mathbb{Z}_+^{I_m}\}$ (respectively, $\{\mu_i(\mathbf{a}) F_i^a : \mathbf{a} \in \mathbb{Z}_+^{I_m}\}$) form a $\mathbb{Z}[q, q^{-1}]$-basis of $U^+_Z(w, 1)$ (respectively, $U^-_Z(w, 1)$). Moreover,

$$\langle \mu_i(\mathbf{a}) F_i^a, \mu_i(\mathbf{a})' E_i^a' \rangle \in \mathbb{Z}[q, q^{-1}]$$

and equals zero unless $\mathbf{a} = \mathbf{a}'$.

**Proof.** Set

$$E_i^a = E_i^{(a_1)}T^u_{w_1} \cdots T^u_{w_m} \langle F_i^{(a_m)} \rangle.$$  

Then by (5.2), $\bar{E}_i^a = \mu_i(\mathbf{a})^{-1}\langle \prod_{i=1}^{m}(a_i)! \rangle^{-1}E_i^a$. We also set $\bar{E}_i^a = \bar{E}_i^a$. It follows from [19, Proposition 41.1.4] that the monomials $\{\bar{E}_i^a\}_{a \in \mathbb{Z}_+^{I_m}}$ (respectively, $\{\bar{F}_i^a\}_{a \in \mathbb{Z}_+^{I_m}}$) form a $\mathbb{Z}[q, q^{-1}]$-basis of $zU^+_Z(w, 1)$ (respectively, $zU^-_Z(w, 1)$), where $zU^+_Z(w, 1) = zU^+_U(w, 1)$. Moreover, it follows from [19, Proposition 38.2.3 and (5.5)] that

$$\langle \bar{F}_i^a, \bar{E}_i^a \rangle = \delta_{a, a'} q \sum_{r=1}^{m} \eta(w_{r-1}(a_r)) \prod_{i=1}^{r} \langle F_i^{(a_r)} \rangle \langle E_i^{(a_r)} \rangle = \delta_{a, a'} q \sum_{r=1}^{m} \eta(w_{r-1}(a_r)) \prod_{i=1}^{r} \langle q \rangle.$$  

This implies that

$$\langle \bar{F}_i^a, \mu(\mathbf{a})' E_i^a \rangle \in \delta_{a, a'} q \mathbb{Z}$$

and so $\{\mu(\mathbf{a}) E_i^a : \mathbf{a} \in \mathbb{Z}_+^{I_m}\}$ (respectively, $\{\mu(\mathbf{a}) F_i^a : \mathbf{a} \in \mathbb{Z}_+^{I_m}\}$) is a $\mathbb{Z}[q, q^{-1}]$-basis of $U^+_Z$ (respectively, of $U^-_Z$). Finally,

$$\langle \mu(\mathbf{a}) F_i^a, \mu(\mathbf{a})' E_i^a \rangle \in \mu(\mathbf{a})^2 \delta_{a, a'} q \mathbb{Z}.$$  

Since $\mu(\mathbf{a})^2 \in q \mathbb{Z}$, the last assertion follows. □

**Proof of Theorem 3.11.** Suppose that $g$ is semisimple and that $w = w_0$ is the longest element in $W$. Then $U^+_Z(w_0) = U^+_Z$ and by Lemma 5.6, $U^+_Z$ admit a pair of bases $B^+_\pm$ such that $(B^-_\pm, B^+_\pm) \subset \mathbb{Z}[q, q^{-1}]$. Thus, $U^-_Z = \mathbb{Z}[q, q^{-1}]$. The same argument as in the proof of Proposition 3.9 shows that $(B^-_\pm, B^+_\pm) \subset \mathbb{Z}[q, q^{-1}]$. □

5.4. **Braid group action for $U_q(sl_2)$.** Retain the notation from §4.1.

**Lemma 5.7.** We have, for all $a_\pm \in \mathbb{Z}$, $m_\pm \in \mathbb{Z}_{\geq 0}$

$$T(K^a_-K^a_+ \circ F^m_- \cdot E^m_+) = K^{-a_-m_-}K^{a_+m_+} \circ F^m_+ \cdot E^m_-.$$  

**Proof.** We claim that $T(C^{(r)}) = (K_+K_-)^{-r}C^{(r)}$. Indeed, this is obvious for $r = 0$ and easily seen to hold for $r = 1$. Then by induction hypothesis we have

$$T(C^{(r+1)}) = T(C)T(C^{(r)}) - (K_+K_-)^{-1}T(C^{(r-1)}) = (K_+K_-)^{-r-1}(CC^{(r)} - K_+K_-C^{(r-1)})$$

$$= (K_+K_-)^{-r-1}C^{(r+1)}.$$  

Since $T(E^m_+) = K^{m_+} \circ F^m_+$, $T(F^m_-) = K^{m_-} \circ E^m_-$, and the $C^{(r)}$, $r \geq 0$ are central we obtain, setting $m = \min(m_-, m_+)$

$$T(K^a_-K^a_+ \circ F^m_- \cdot E^m_+) = K^{-a_-m_-}K^{a_+m_+} \circ (K^{m_-} \circ E^m_-)C^{(m)}(K^{m_+} \circ F^m_+).$$
\[= K_{-a-m}K_{+a+m} \circ F^{m+} \bullet E^{-m}. \]

5.5. **Braid group action on elements of** \(B_{n+}\). Retain the notation of \(\S 3.5\). It follows from Proposition 3.22(b) that for any element \(b_+ \in B_{n+}\) and \(r \in \mathbb{Z}_{\geq 0}\) there exists a unique \(b'_+ \in B_{n+}\) such that \(\partial_i^{\text{top}}(b_+) = \partial_i^{\text{top}}(b'_+)\) and \(\ell_i(b'_+) = \ell_i(b_+) + r\). We denote this element by \(\tilde{\partial}_i^{-r}(b_+)\).

Observe that
\[
\tilde{\partial}_i^{-r}(b_+) = \tilde{\partial}_i^{-r-\ell_i(b_+)}\partial_i^{\text{top}}(b_+). \tag{5.6}
\]

**Proposition 5.8.** For all \(b_+ \in B_{n+}\) and \(r \in \mathbb{Z}_{\geq 0}\), \(i \in I\) we have
\[
E_i^r \circ b_+ - \tilde{\partial}_i^{-r}(b_+) \in \sum_{b'_+ \in B_{n+}: \ell_i(b'_+) < r} \mathbb{Z}[q]b'_+, \tag{5.7}
\]
where for any \(x \in U_q^+\) and \(r \in \mathbb{Z}_{\geq 0}\) homogeneous we denote
\[
E_i^r \circ x := q_i^{-\frac{1}{2}r}\alpha_i(x)E_i^r x, \quad \alpha_i(x) := \alpha_i(\deg x).
\]

**Proof.** First, note that for \(b_+ \in \ker \partial_i\), \(\ell_i(E_i^r \circ b_+) = r = \ell_i(\tilde{\partial}_i^{-r}(b_+))\) and by Corollary 3.19
\[
\partial_i^{(r)}(E_i^r \circ b_+ - \tilde{\partial}_i^{-r}(b_+)) = \partial_i^{\text{top}}(b_+) - \partial_i^{\text{top}}(b_+) = 0,
\]
hence by Proposition 3.22(a) and Corollary 3.7
\[
E_i^r \circ b_+ - \tilde{\partial}_i^{-r}(b_+) \in \sum_{b'_+ \in B_{n+}: \ell_i(b'_+) < r} \mathbb{Z}[q, q^{-1}]b'_+. \tag{5.8}
\]

Given \(\lambda = (\lambda_i)_{i \in I} \in \mathbb{Z}^I\) and \(i \in I\), define \(\mathbb{C}\)-linear operators on \(U_q^+\) by
\[
F_{i; \lambda}(x) = q_i^{-\lambda_i+\frac{1}{2}\alpha_i(x)}E_i x - q_i^{-\frac{1}{2}\alpha_i(x)}x E_i q_i^{\lambda_i+\lambda_i} - q_i^{\lambda_i-\alpha_i(x)}x,
\]
where
\[
K_i; \lambda(x) = q_i^{\lambda_i-\alpha_i(x)}x.
\]

The following result is well-known (see e.g. [3, Section 3] and also Proposition A.35).

**Lemma 5.9.** For any \(\lambda \in \mathbb{Z}^I\), the assignments \(E_i \mapsto \partial_i, F_i \mapsto F_{i; \lambda}, K_i \mapsto K_{i; \lambda}\) define a structure of a \(U_q(\mathfrak{g})\)-module on \(U_q^+\). Moreover, the submodule \(\mathcal{Y}_\lambda\) of \(U_q^+\) generated by 1 is simple and if \(\lambda \in \mathbb{Z}_{\geq 0}^I\) then \(\mathcal{Y}_\lambda = \{ x \in U_q^+ : \ell_i(x) \leq \lambda_i \}\) and is integrable.

**Remark 5.10.** Here we use the “standard” generators of \(U_q(\mathfrak{g})\).

We need the following technical fact which is easy to check by induction.

**Lemma 5.11.** For all \(\lambda \in \mathbb{Z}^I\), \(r \in \mathbb{Z}_{\geq 0}\) and \(x \in U_q^+\) homogeneous
\[
q_i^{-(\lambda_i-\alpha_i(x)-1)-(\frac{1}{2})r}F_{i; \lambda}(x) = (1 - q_i^2)^{-r}q_i^{-\frac{1}{2}r}\alpha_i(x)\sum_{k=0}^{r} (-1)^k q_i^{k(2\lambda_i-\alpha_i(x)-2r+2+k(k-1))}q_i^k E_i^{r-k}E_i^k.
\]

This immediately implies that
\[
E_i^r \circ x = q_i^{-(\lambda_i-\alpha_i(x))-(\frac{1}{2})r+1}(1 - q_i^2)^{r}F_{i; \lambda}(x)
+ q_i^{-\frac{1}{2}r}\alpha_i(x)\sum_{k=1}^{r} (-1)^{k+1} q_i^{k(2\lambda_i-\alpha_i(x)+k-2r+1)}q_i^k E_i^{r-k}E_i^k. \tag{5.9}
\]

Let \(b_+ \in B_{n+} \cap \ker \partial_i\). It follows by an obvious induction from [18, Proposition 5.3.1] that
\[
q_i^{r\varphi_i(b_+)-(\frac{1}{2})r+1}(1 - q_i^2)^{r}F_{i; \lambda}(b_+) = \prod_{t=0}^{r-1} (1 - q_i^{2\varphi_i(b_+)-1})q_i^{-\frac{1}{2}r}\sum_{b'_+ \in B_{n+} : \ell_i(b'_+) < r} q_i^{q_i^r[b'_+]},
\]
where \(\varphi_i(b_+) = \prod_{t=0}^{r-1} (1 - q_i^{2\varphi_i(b_+)})\).
where \( \varphi_{i}^{}(b_{+}) = \lambda_{i}^{} - \alpha_{i}^{\gamma}(b_{+}). \) Combining this identity with (5.9) we obtain

\[
E_{i}^{r} \circ b_{+} = \sum_{l=0}^{r-1} (1 - 2(\lambda_{l}^{} - \alpha_{l}^{\gamma}(b_{+}) - t)) \partial_{l}^{-r}(b_{+}) \\
+ q_{i}^{-2\alpha_{l}^{\gamma}(b_{+})} \sum_{k=1}^{r} (-1)^{k+1} q_{i}^{k(2\lambda_{l}^{} - \alpha_{l}^{\gamma}(b_{+}) + k - r - 1)} \left( \begin{array}{c} r \\ k \end{array} \right) E_{i}^{-k} b_{+} E_{i}^{k} + \sum_{b_{+}' \in B_{n}^{+}} q Q[q] b_{+}'.
\]

(5.10)

By Corollary 3.7 we have for all \( 1 \leq k \leq r \)

\[
q_{i}^{-\frac{1}{2} \alpha_{l}^{\gamma}(b_{+}) + k(2\lambda_{l}^{} - \alpha_{l}^{\gamma}(b_{+}) + k - r)} E_{i}^{-k} b_{+} E_{i}^{k} = q_{i}^{2\lambda_{l}^{} \sum_{b_{+}' \in B_{n}^{+}} C_{b_{+}',r,k}^{l} b_{+}'}.
\]

Since only finitely many terms in this sum are non-zero, there exists \( \lambda_{i} \in \mathbb{Z}_{\geq 0}, \lambda_{i} \geq \alpha_{l}^{\gamma}(b_{+}) + r \)

such that \( q_{i}^{2\lambda_{l}^{} \sum_{b_{+}' \in B_{n}^{+}} C_{b_{+}',r,k}^{l} b_{+}'} q Z[q] \) for all \( b_{+} \in B_{n}^{+}, 1 \leq k \leq r. \) Therefore, it follows from (5.10) that

\[
E_{i}^{r} \circ b_{+} - \partial_{i}^{-r}(b_{+}) \in \sum_{b_{+}' \in B_{n}^{+}} q Z[q] b_{+}'.
\]

(5.11)

It remains to apply (5.8).

Corollary 5.12. For any \( b_{+} \in B_{n}^{+} \) we have

\[
b_{+} - E_{i}^{\ell_{i}(b_{+})} \circ \partial_{i}^{(\text{top})}(b_{+}) \in \sum_{b_{+}' \in B_{n}^{+}, \ker \partial_{i}, 0 \leq \ell_{i}(b_{+})} q Z[q] E_{i}^{r} \circ b_{+}'.
\]

(5.11)

Proof. It follows from the Theorem that the elements \( \{ E_{i}^{r} \circ b_{+} : b_{+} \in B_{n}^{+} \cap \ker \partial_{i}, r \geq 0 \} \) form a \( Z[q] \)-basis of the lattice \( Z[q] B_{n}^{} \) and the transfer matrix is unitriangular with off-diagonal elements in \( q Z[q]. \) Then the inverse matrix has the same property.

We can now prove the following

Theorem 5.13. For all \( b_{+} \in B_{n}^{+}, i \in I \)

\[
T_{i}(b_{+}) = K_{i}^{-\ell_{i}(b_{+})} \circ F_{i}^{\ell_{i}(b_{+})} \circ T_{i}^{\text{top}}(\partial_{i}^{(\text{top})}(b_{+})), \quad T_{i}^{-1}(b_{+}) = K_{i}^{-\ell_{i}(b_{+})} \circ F_{i}^{\ell_{i}(b_{+})} \circ T_{i}^{-1}(\partial_{i}^{(\text{top})}(b_{+})).
\]

In particular, all elements of \( B_{n}^{} \) are tame.

Proof. We only prove the first identity, the proof of the second one being similar. We need the following crucial corollary of [20, Theorem 1.2].

Proposition 5.14. \( T_{i} \) induces a bijection \( B_{n}^{} \cap \ker \partial_{i} \rightarrow B_{n}^{} \cap \ker \partial_{i}^{\text{op}}. \)

Proof. It follows from [19, Lemma 38.1.3 and Proposition 38.1.6] that \( T_{i}^{\text{op}} \) induces an isomorphism of algebras \( \ker \partial_{i} = \ker \partial_{i}^{\text{op}} = \ker \partial_{i} \). Moreover [20, Theorem 1.2] implies that if \( b \in B_{n}^{\text{can}} \cap \ker \partial_{i}^{\text{op}} \) then \( T_{i}^{\text{op}}(b_{i}^{\text{op}}) \in (B_{n}^{\text{can}})^{\text{st}} \cap \ker \partial_{i}^{\text{op}}. \) Now, let \( b_{+} = \partial_{b} = b' \in B_{n}^{} \cap \ker \partial_{i} \) and \( b' \in B_{n}^{\text{can}} \cap \ker \partial_{i}^{\text{op}}. \) Then it follows from (5.2) and [19, Proposition 38.2.1] that \( \delta_{b,b'} = \delta_{b}^{(\text{op})} = q^{\frac{1}{2} \nu} (T_{i}(\delta_{b}^{(\text{op})})^{\text{st}}, T_{i}^{-1}(b_{\text{op}}^{(\text{st})})) = q^{\frac{1}{2} \nu} (T_{i}(\delta_{b}^{(\text{op})})^{\text{st}}, b_{\text{op}}^{\text{st}}), \) where \( b_{\text{op}}^{\text{st}} = b_{\text{op}} \cap \ker \partial_{i}^{\text{op}} \) and \( \nu \in \mathbb{Z} \) depends only on the degree of \( b. \) This implies that \( T_{i}(\delta_{b}^{(\text{st})}) = q^{\frac{1}{2} \nu} \delta_{b'}. \) But since \( T_{i} \) commutes with \( \tilde{\gamma} \), it follows that \( \nu = 0. \)

We have, for any \( r > 0, b_{+}' \in B_{n}^{} \cap \ker \partial_{i} \)

\[
T_{i}(E_{i}^{r} \circ b_{+}') = q_{i}^{-\frac{1}{2} \gamma}(\deg b_{+}') T_{i}(E_{i}^{r}) T_{i}(b_{+}') = q_{i}^{-\frac{1}{2} \gamma}(\deg b_{+}') K_{i}^{-r} \circ T_{i}(b_{+}').
\]

\[
= q_{i}^{-\frac{1}{2} \gamma}(\deg b_{+}') -\frac{1}{2} \gamma(s_{i}(\deg b_{+}')) K_{i}^{-r} \circ (F_{i}^{r} T_{i}(b_{+})) = K_{i}^{-r} \circ (F_{i}^{r} T_{i}(b_{+})).
\]
Then applying $T_i$ to (5.11) yields

$$T_i(b_+) = K_{i+1}^{-\ell_i(b_+)} \circ F_i^{\ell_i(b_+)} T_i(\partial_i^{(top)}(b_+)) + \sum_{0<r<\ell_i(b_+)} D_{b_+}^{b_+} K_{i+1}^{-r} \circ (F_i^r T_i(b'_+))$$

$$= K_{i+1}^{-\ell_i(b_+)} \circ (F_i^{\ell_i(b_+)} T_i(\partial_i^{(top)}(b_+)) + \sum_{0<r<\ell_i(b_+)} D_{b_+}^{b_+} K_{i+1}^{-r} \circ (F_i^{\ell_i(b_+)-r} T_i(b'_+)).$$

Since $\bar{\partial}$ commutes with the $T_i$, this element is $\bar{\partial}$-invariant. Since all $D_{b_+}^{b_+}$ belongs to $q\mathbb{Z}[q]$ and $T_i(b'_+) \in B_{a+} \cap \ker \partial_+^{op}$ for all $b_+ \in B_{a+} \cap \ker \partial$, $T_i(b'_+) = K_{i+1}^{-\ell_i(b_+)} \circ F_i^{\ell_i(b_+)} \circ T_i(\partial_i^{(top)}(b_+))$ by Theorem 1.3. But since for $b_+ \in \ker \partial_+^{op}$, $F_i^{\ell_i(b_+)} b_+$ in $U_q(\mathfrak{g})$ and in $H_q^+(\mathfrak{g})$ coincide, it follows that $F_i^{\ell_i(b_+)} \circ T_i(\partial_i^{(top)}(b_+)) = F_i^{\ell_i(b_+)} \circ T_i(\partial_i^{(top)}(b_+))$ by Theorem 1.5.

**Example 5.15.** We now use the above Theorem to compute $F_i^r \cdot b_+$, $r \geq 0$, $b_+ \in B_{a+} \cap \ker \partial_+^{op}$ for $g = s_{13}$. In this case $B_{a+}$ consists of elements

$$b_+(a) := q^{\frac{1}{2}(a_1-a_2)(a_1-a_2-1)} E_{a_1} E_{a_2} E_{a_1 a_2} E_{a_1 a_2}, \quad a = (a_1, a_2, a_1, a_2) \in \mathbb{Z}_{\geq 0}^4, \min(a_1, a_2) = 0.$$ Then

$$B_{a+} \cap \ker \partial_+^{op} = \{b_+(0, a_2, a_1, 0) : a_2, a_1 \in \mathbb{Z}_{\geq 0}\}.$$ Since $T_1^{-1}(E_2) = E_{21}$, $T_1^{-1}(E_{12}) = E_2$ we have $T_1^{-1}(b_+(0, a_2, a_1, 0)) = b_+(0, a_2, a_1, 0)$. Then $F_i^r \cdot b_+(0, a_2, a_1, 0) = K_{i+1}^r \circ T_i(\partial_i^{(top)}(b_+(0, a_2, a_1, 0)))$. Since

$$\ell_1(b_+(0, a_1', a_1', a_1')) = a_1', \quad \partial_1^{(top)}(b_+(0, a_1', a_1', a_1')) = b_+(0, a_1' + a_1', 0, a_1')$$

we conclude that

$$\bar{\partial}_i^{-r}(b_+(0, a_1, 0, a_2)) = \begin{cases} b_+(0, a_1 - r, r, a_2), & 0 \leq r \leq a_1 \\ b_+(r - a_1, 0, a_1, a_2), & r > a_1. \end{cases}$$

Since

$$b_+(a_1', a_2', a_1', a_2', a_1') = \sum_{t=0}^{a_1'} (-1)^t q^{t(a_1' + a_2' + 1)} \begin{bmatrix} a_1' \\ t \end{bmatrix} E_1^{a_1' + a_2' - t} \circ b_+(0, a_2' + a_1' - t, 0, a_2' + t),$$

we obtain

$$F_i^r \cdot b_+(0, a_2, a_1, 0) = \sum_{t=0}^{\min(r, a_1)} (-1)^t q^{t(a_1-a_2-r+1)} \begin{bmatrix} \min(r, a_1) \\ t \end{bmatrix} K_1^t \circ F_i^{r-t} b_+(0, a_2 + t, a_1 - t, 0).$$

Then it is easy to see that $T_2(F_i^r \cdot b_+(0, a_2, a_1, 0)) = K_{i+2}^{-a_2} \circ b_-(0, a_2, 0, r) \cdot E_1^{a_2} = (K_{i+2}^{-a_2} \circ F_i^{a_2} \cdot b_+(0, a_2, 0, r))^t$. In a similar fashion, using $T_i^{-1}$ we obtain

$$F_i^r \cdot b_+(0, a_2, 0, a_21) = \sum_{t=0}^{\min(r, a_2)} (-1)^t q^{-t(a_2-r+1)} \begin{bmatrix} \min(r, a_2) \\ t \end{bmatrix} K_1^t \circ F_i^{r-t} b_+(0, a_2 + t, 0, a_2 - t).$$
5.6. Wild elements of a double canonical basis. Assume that $a_{ij} = a_{ji} = -a, d_i = d_j = 1, a \geq 2$ and consider elements $F_{ij} \bullet E_{ji}$ computed in §4.3. Then for $a = 2$ we have

$$T_1(F_{ij} \bullet E_{ij}) = K_{-1}^{-1} F_{ij} \bullet E_{ij1} + K_{-1}^{-1} F_{ij2} \bullet E_{ij2},$$

while for $a = 3$

$$T_1(F_{ij} \bullet E_{ij}) = (3)q K_{-1}^{-1} F_{ij3} + (2)q K_{-1}^{-1} F_{ij3} \bullet E_{ij3} + (2)q F_{ij2} \bullet E_{ij2} + (2)q K_{+1} K_{+1} F_{ij} \bullet E_{ij} + K_{-1} K_{-1}^{-1} K_{+1} K_{+1}. $$

**APPENDIX A. DRINFELD AND HEISENBERG DOUBLES**

A.1. Nichols algebras. Let $\mathbb{k}$ be a field, let $V$ be a $\mathbb{k}$-vector space and let $\Psi = \Psi_V : V \otimes V \to V \otimes V$ be a braiding, that is, $\Psi$ is invertible and $\Psi_{1,2} \Psi_{2,3} \Psi_{1,2} = \Psi_{2,3} \Psi_{1,2} \Psi_{2,3}$ as endomorphisms of $V \otimes^3$, where

$$\Psi_{i,i+1} = \text{id}_V \otimes \text{id}_V \otimes \text{id}_V \in \text{End}_k(V \otimes^n), \quad 1 \leq i < n.$$ 

Define $[n]_\Psi, [n]_\Psi! \in \text{End}_k(V \otimes^n)$, $n \in \mathbb{Z}_{\geq 0}$, by

$$[n]_\Psi = \text{id}_V \otimes \cdots \otimes \text{id}_V \Psi_n \cdots \otimes \text{id}_V \Psi_1,$$

$$[n]_\Psi! = ([1]_\Psi \otimes \cdots \otimes [1]_\Psi) \circ ([2]_\Psi \otimes \cdots \otimes [2]_\Psi) \circ \cdots \circ [n]_\Psi.$$ 

In particular, $[0]_\Psi! = 1$ and $[1]_\Psi! = \Psi_1$. Furthermore, given $\sigma$ in the symmetric group $S_n$, let $\Psi_\sigma = \Psi_{\sigma(1), \sigma(2)} \cdots \Psi_{\sigma(n-1), \sigma(n)}$ where $\sigma = (i_1, i_2, \cdots, i_r) \in S_n$ is a reduced expression. A standard argument shows that $\Psi_\sigma$ depends only on $\sigma$ and not on the reduced expression. In particular, if $\ell(\sigma) + \ell(\tau) = \ell(\sigma \tau)$, where $\ell(\sigma)$ denotes the length of any reduced expression of $\sigma$ as a product of elementary transpositions, then $\Psi_\sigma \Psi_\tau = \Psi_{\sigma \tau}$. Then

$$[n]_\Psi! = \sum_{\sigma \in S_n} \Psi_\sigma.$$ 

Let $\sigma_0 : (1, \ldots, n) \mapsto (n, \ldots, 1)$ be the longest element of $S_n$. Since $\ell(\sigma) + \ell(\sigma^{-1}) = \ell(\sigma) + \ell(\sigma^{-1}) = \ell(\sigma_0)$, it follows that $\Psi_{\sigma} \Psi_{\sigma^{-1}} = \Psi_{\sigma_0} = \Psi_{\sigma_0^{-1}} \Psi_{\sigma}$. This implies that

$$[n]_\Psi! = [\Psi_{\sigma_0}]_\Psi [n]_\Psi! = [n]_\Psi!.$$ 

Also, by [7, Proposition 5.5] or [8, Proposition 4.17], $[n]_{\Psi_\tau} = \Psi_{\tau \sigma_0} \Psi_{\tau} \Psi_{\sigma_0}$ for all $\tau \in S_n$, hence

$$[n]_{\Psi_\tau}! = \Psi_{\sigma_0} [n]_{\Psi_\tau}! = [n]_{\Psi_\tau}!.$$ 

Let $r, s > 0$. Then the element $\Psi_\sigma \in \text{End}_k(V \otimes^{r+s})$ where $\sigma : (1, \ldots, r + s) \mapsto (s + 1, \ldots, r + s, 1, \ldots, s)$ defines a braiding $\Psi_{V \otimes^r V \otimes^s}$. The tensor algebra $T(V) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V \otimes^n$ of $V$, where $V \otimes^0 = k$, is the free associative algebra generated by $V$. The braiding $\Psi$ extends to a braiding $\Psi_{T(V)} : T(V) \otimes T(V) \to T(V) \otimes T(V)$ via $\Psi_{T(V)}|_{V \otimes^r V \otimes^s} = \Psi_{V \otimes^r V \otimes^s}$. Then $T(V) \otimes T(V)$ can be endowed with a braided algebra structure via $m_{T(V)} : T(V) \otimes T(V) \to T(V)$ is the multiplication map.

The Woronowicz symmetrizer $\text{Wor}(\Psi) : T(V) \to T(V)$ is the linear map defined by

$$\text{Wor}(\Psi)|_{V \otimes^n} = [n]_\Psi!.$$ 

It turns out (cf. [2, 15]) that $\ker \text{Wor}(\Psi)$ is a bi-ideal of $T(V)$. Note that (A.1) implies that $\ker \text{Wor}(\Psi) = \ker \text{Wor}(\Psi^{-1})$.

**Definition A.1.** The quotient $T(V)/\ker \text{Wor}(\Psi)$ is called the Nichols-Woronowicz algebra $B(V, \Psi)$ of $(V, \Psi)$.
The algebra $\mathcal{B}(V, \Psi)$ is thus a braided bialgebra, where the braiding $\Psi_{\mathcal{B}(V, \Psi)}$ on $\mathcal{B}(V, \Psi) \otimes \mathcal{B}(V, \Psi)$ is induced by $\Psi_{T(V)}$. By construction, $\mathcal{B}(V, \Psi)$ is $\mathbb{Z}_{\geq 0}$-graded, $\mathcal{B}'(V, \Psi)$ being the canonical image of $V^{\otimes r}$. Since $V \cap \ker \text{Wor}(\Psi) = 0$, $V$ identifies with its canonical image in $\mathcal{B}(V, \Psi)$ and can be shown to coincide with the space of primitive elements in $\mathcal{B}(V, \Psi)$.

The braided antipode $S_\Psi$ on $T(V)$ is defined by $S_\Psi|_{V^{\otimes n}} = (-1)^n \Psi_{\sigma_o}$ where $\sigma_o : (1, \ldots, n) \mapsto (n, \ldots, 1)$ is the longest permutation in $S_n$. It satisfies the usual properties, namely

$$m \circ (S_\Psi \otimes 1) \circ \Delta = \varepsilon, \quad \Delta \circ S_\Psi = (S_\Psi \otimes S_\Psi) \circ \Psi_{T(V)} \circ \Delta, \quad S_\Psi \circ m = m \circ \Psi_{T(V)} \circ (S_\Psi \otimes S_\Psi) \quad (A.3)$$

where $m = m_{T(V)}$ (see for example [15, §9.4.6]). By (A.1), $S_\Psi$ preserves ker Wor(\Psi) hence factors through to a map $S_\Psi : \mathcal{B}(V, \Psi) \to \mathcal{B}(V, \Psi)$ satisfying (A.3).

### A.2. Bar and star involutions

Let $\tilde{\cdot} : \mathbb{k} \to \mathbb{k}$ be a field involution and fix an additive involutive map $\tilde{\cdot} : V \to V$ satisfying $\tilde{\tilde{x}} = x \cdot \bar{x}$, $v \in V$, $x \in \mathbb{k}$ (we will call such a map anti-linear). There is a unique anti-linear algebra homomorphism $\tilde{\cdot} : T(V) \to T(V)$ whose restriction to $\mathbb{k}$ and $V$ coincides with the corresponding $\tilde{\cdot}$. We say that $\Psi$ is unitary if $\tilde{\cdot} \circ \Psi = \Psi^{-1}$. If $\Psi$ is unitary then, by (A.1), ker Wor(\Psi) = ker Wor(\Psi), hence $\tilde{\cdot}$ factors through to an anti-linear algebra involution of $\mathcal{B}(V, \Psi)$.

**Proposition A.2.** $\Psi_{T(V)}(\tilde{\cdot} \otimes \tilde{\cdot}) \circ \Delta = \Delta \circ \tilde{\cdot}$. Moreover, the same identity holds for $\mathcal{B}(V, \Psi)$.

**Proof.** Let $v \in V^{\otimes n}, n \geq 0$. We prove that $\Psi_{T(V)}(\tilde{u}(1) \otimes \tilde{u}(2)) = \Delta(v)$ by induction on $n$. The identity is clear for $u \in V$. Furthermore, take $u \in V^{\otimes r}, v \in V^{\otimes s}$. Then

$$\Delta(\tilde{u}v) = \Delta(v) \Delta(\tilde{u}) = \Psi_{T(V)}(\tilde{u}(1) \otimes \tilde{u}(2)) \Psi_{T(V)}(\tilde{v}(1) \otimes \tilde{v}(2)) = (m_{T(V)} \otimes m_{T(V)})(1 \otimes \Psi_{T(V)} \otimes 1)(\Psi_{T(V)} \otimes \Psi_{T(V)})(\tilde{u}(1) \otimes \tilde{u}(2) \otimes \tilde{v}(1) \otimes \tilde{v}(2))$$

On the other hand,

$$\Psi_{T(V)}(\tilde{\cdot} \otimes \tilde{\cdot}) \Delta(uv) = \Psi_{T(V)}(\tilde{\cdot} \otimes \tilde{\cdot})(m_{T(V)} \otimes m_{T(V)})(1 \otimes \Psi_{T(V)} \otimes 1)(\tilde{u}(1) \otimes \tilde{u}(2) \otimes v(1) \otimes v(2)) = \Psi_{T(V)}(m_{T(V)} \otimes m_{T(V)})(1 \otimes \Psi_{T(V)}^{-1} \otimes 1)(\tilde{u}(1) \otimes \tilde{u}(2) \otimes \tilde{v}(1) \otimes \tilde{v}(2)).$$

So, the first assertion follows from the commutativity of the diagram

$$\begin{array}{cc}
U_1 \otimes U_2 \otimes U_3 \otimes U_4 & U_1 \otimes U_3 \otimes U_2 \otimes U_4 \\
\Psi_{U_1 \otimes U_2 \otimes U_3 \otimes U_4} & \\
U_3 \otimes U_4 \otimes U_1 \otimes U_2 & U_3 \otimes U_1 \otimes U_4 \otimes U_2
\end{array}$$

where $U_i = V^{\otimes r_i}, r_1 + r_2 = r, r_3 + r_4 = s$. The second assertion is immediate. \qed

Let $\tau_n \in \text{End}(V^{\otimes n})$ be the map satisfying $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_1, v_i \in V$. We say that $\Psi$ is self-transposed if $\Psi = \tau_2 \Psi \tau_2$. Define $* \in \text{End} T(V)$ by $v^*|_{V^{\otimes n}} = \tau_n$. Then $*$ is the unique anti-automorphism of $T(V)$ whose restriction to $\mathbb{k}$ and $V$ is the identity. Since for a self-transposed $\Psi$ we have $\tau_n \Psi_{i,i+1} \tau_n = \Psi_{n-i,n-i+1}, 1 \leq i \leq n-1$, it follows that

$$\tau_n \Psi_{\sigma} \tau_n = \Psi_{\sigma_o \sigma_o}, \quad \sigma \in S_n. \quad (A.4)$$

This implies that $[n]_{\Psi^!} \circ \tau_n = \tau_n \circ [n]_{\Psi^!}$ hence $*$ preserves ker Wor(\Psi) and so factors through to an anti-automorphism of $\mathcal{B}(V, \Psi)$.

**Lemma A.3.** Suppose that $\Psi$ is self-transposed. Then $\Delta \circ * = * \circ \Delta_{\text{op}}$ on $T(V)$, where $\Delta_{\text{op}}(u) = \tilde{u}(2) \otimes \tilde{u}(1)$ in Sweedler’s notation, $u \in T(V)$. Moreover, the same identity holds on $\mathcal{B}(V, \Psi)$. 
Proof. The assertion clearly holds for \( v \in V \). Let \( u \in V^\otimes r \), \( v \in V^\otimes s \). By the induction hypothesis,
\[
\Delta((uv)^*) = \Delta(v^*)\Delta(u^*) = (u^*_2 \otimes u^*_1)(u_2^* \otimes u_1^*)
\]
\[
= (m_{T(V)} \otimes m_{T(V)})(u^*_2 \otimes \Psi_T(V)(v^*_1 \otimes u^*_2)) \otimes u^*_1
\]
\[
= (m_{T(V)} \otimes m_{T(V)})(u^*_2 \otimes \Psi_T(V)(u_2^* \otimes u_1^*)) \otimes u^*_1.
\]
By (A.4) we have \( \Psi_T(V) \circ * = * \circ \Psi_T(V) \), hence
\[
\Delta((uv)^*) = (m_{T(V)} \otimes m_{T(V)})(u^*_2 \otimes \Psi_T(V)(u^*_2 \otimes u^*_1))
\]
\[
= (* \circ *) \circ ((m_{T(V)} \otimes m_{T(V)})(u_2^* \otimes \Psi_T(V)(u_2^* \otimes u_1^*)) \otimes u_1^*)) = * \circ \Delta^{op}(uv). \quad \Box
\]

If \( \Psi \) is both self-transposed and unitary we can define \( : = \cdot \circ \cdot \), which is the unique anti-linear anti-involution of \( T(V) \) and \( B(V, \Psi) \) whose restriction to \( V \) coincides with \( \cdot \). Clearly, \( \cdot \circ \cdot \circ \cdot = \Psi^{-1} \).
We also have
\[
\Delta \circ : = \Psi_T(V) \circ (\cdot \circ \cdot \circ \cdot) \circ \Delta^{op} = (\cdot \circ \cdot \circ \cdot) \circ \Psi_{(1)}^{-1}(V) \circ \Delta^{op} \quad (A.5)
\]

A.3. Pairing and quasi-derivations. Let \( V^* \) be another \( k \)-vector space with a braiding \( \Psi^* : V^* \otimes V^* \rightarrow V^* \otimes V^* \). Suppose that there exists a pairing \( \langle \cdot, \cdot \rangle : V^* \otimes V \rightarrow k \) and let \( \langle \cdot, \cdot \rangle' \) be the natural pairing \( T(V^*) \otimes T(V) \rightarrow k \) defined by
\[
\langle f_1 \otimes \cdots \otimes f_r, v_1 \otimes \cdots \otimes v_r \rangle' = \prod_{k=1}^r \langle f_k, v_k \rangle, \quad f_k \in V^*, \ v_k \in V, \ 1 \leq k \leq r,
\]
while \( \langle (V^*)^\otimes r, V^\otimes s \rangle = 0 \) if \( r \neq s \). If \( \Psi^* \) is the adjoint of \( \Psi \) with respect to \( \langle \cdot, \cdot \rangle|_{V^* \otimes 2 \otimes V^\otimes 2} \) define \( \langle \cdot, \cdot \rangle : T(V^*) \otimes T(V) \rightarrow k \) by
\[
\langle f, u \rangle = \langle f, \text{Wor}(\Psi)(u) \rangle = \langle \text{Wor}(\Psi^*)(f), u \rangle', \quad f \in T(V^*), \ u \in T(V).
\]
The following Lemma is standard.

Lemma A.4. Suppose that \( \Psi^* \) is the adjoint of \( \Psi \). Then
(a) for all \( f, f' \in T(V^*) \), \( v, v' \in T(V) \) we have
\[
\langle ff', v \rangle = \langle f, u_1 \rangle \langle f', u_2 \rangle, \quad \langle f, vv' \rangle = \langle f, u_1 \rangle \langle f, v' \rangle,
\]
where \( \Delta(v) = u_1 \otimes u_2 \) and \( \Delta(f) = f_1 \otimes f_2 \) in Sweedler’s notation.
(b) Let \( \cdot, \cdot : V^* \otimes V \rightarrow k \) be non-degenerate. Then ker Wor(\( \Psi \)) = \{ \( v \in T(V) : \langle T(V^*), v \rangle = 0 \} \), and ker Wor(\( \Psi^* \)) = \{ \( f \in T(V^*) : \langle f, T(V) \rangle = 0 \} \). In particular, \( \cdot, \cdot \) induces a non-degenerate pairing \( \cdot, \cdot : B(V^*, \Psi^*) \otimes B(V, \Psi) \rightarrow k \) satisfying (a).

Remark A.5. The same construction works if \( \Psi^* \) is the adjoint of \( \Psi^{-1} \).

Lemma A.6. \( \langle f, S_{\Psi}(u) \rangle = \langle S_{\Psi^*}(f), u \rangle \) for all \( f \in T(V^*), \ u \in T(V) \) (respectively, \( f \in B(V^*, \Psi^*), \ u \in B(V, \Psi) \)).

Proof. We may assume, without loss of generality, that \( f \in V^* \otimes \odot n, \ u \in V \otimes \odot n \). Let \( \sigma \) be the longest permutation in \( S_n \). Then
\[
\langle f, S_{\Psi}(u) \rangle = (-1)^n \langle f, [n]_{\Psi}^{\Psi_{\sigma}[u]} \rangle' = (-1)^n \langle f, \Psi_{\sigma}[n]_{\Psi}^{\Psi_{\sigma}[u]}(u) \rangle' = (-1)^n \langle \Psi_{\sigma}^{-1}(f), u \rangle = \langle S_{\Psi^*}(f), u \rangle,
\]
where we used (A.2). The assertion for Nichols algebras is now immediate. \( \Box \)

Proposition A.7. (a) Suppose that \( \Psi \) is unitary and \( \langle \cdot, \cdot \rangle \) satisfies \( \langle f, v \rangle = -\langle f, v \rangle, \ f \in V^*, \ v \in V \). Then for all \( f \in T(V^*), \ v \in T(V) \) we have \( \langle f, \bar{v} \rangle = \langle f, S_{\Psi}^{-1}(v) \rangle \) and \( S_{\Psi}^{-1}(\bar{v}) = S_{\Psi}(v) \).
(b) Suppose that \( \Psi \) is self-transposed. Then \( \langle f^*, v \rangle = \langle f, v^* \rangle \) for all \( f \in T(V^*), \ v \in T(V) \).
(c) Suppose that the assumptions of (a) and (b) hold. Then \( \langle f, S_{\Psi}^{-1}(v) \rangle = \langle S_{\Psi}(v), f \rangle \) for all \( f \in T(V^*), \ v \in T(V) \).
(d) Identities (a)–(c) hold in corresponding Nichols algebras.

Proof. To prove (a) we use induction on the degree in $T(V)$. The induction base is given by the assumption. Suppose that the identity is established for all $f \in V^{\otimes r}, v \in V^{\otimes r}, r < n$. Note that the induction hypothesis implies
\[
\langle f \otimes \bar{g}, \bar{u} \otimes \bar{v} \rangle = \langle f \otimes g, (S^{-1}_\Psi \otimes S^{-1}_\Psi)(u \otimes v) \rangle, \quad f, g \in V^{\otimes r}, u \in V^{\otimes s}, v \in V^{\otimes s}, 0 < r+s \leq n.
\]
Furthermore, $S^{-1}_\Psi = S_{\Psi^{-1}}$. Hence for all $u \in V^{\otimes n}, f \in V^{\otimes r}, g \in V^{\otimes s}$ with $0 < r+s = n$ we have
\[
\langle fg, \bar{u} \rangle = \langle f \otimes g, \Delta(\bar{u}) \rangle = \langle f \otimes g, \Psi_{T(V)}(\hat{\otimes} \hat{\Delta}(u)) \rangle = \langle f \otimes g, \Psi_{T(V)}^{-1}(\hat{\otimes} \hat{\Delta}(u)) \rangle = \langle f \otimes g, \Delta(S_{\Psi^{-1}}(u)) \rangle = \langle fg, S^{-1}_\Psi(u) \rangle,
\]
where we used (A.3) and Proposition A.2. The identity $S^{-1}_\Psi(v) = \tilde{S}_\Psi(v)$ is a direct consequence of the unitarity of $\Psi$ and the definition of $S_{\Psi}$.

To prove (b), we also use induction on the degree in $T(V)$. The induction base is obvious. Suppose that the identity is established for all $f \in V^{\otimes r}, v \in V^{\otimes r}, r < n$. Then for all $u \in V^{\otimes n}, f \in V^{\otimes r}, g \in V^{\otimes s}$ with $0 < r+s = n$ we have
\[
\langle (fg)^*, u \rangle = \langle g^* \otimes f^*, \Delta(u) \rangle = \langle g^*, \Psi_{\otimes 1}(f^*, \Psi_{\otimes 2}(u)) \rangle = \langle g, \Psi_{\otimes 1}(f, \Psi_{\otimes 2}(u)) \rangle = \langle (fg)^*, u \rangle,
\]
where we used Lemma A.3.

To prove (c) note that by (a) and (b) we have
\[
\langle f, g \rangle = \langle f, \Delta^{-1}(g^*) \rangle = \langle f, (S^{-1}_\Psi(g^*))^* \rangle.
\]
Let $f \in V^{\otimes n}, g \in V^{\otimes n}$. Then $S^{-1}_\Psi(g^*)^* = (-1)^n \tau_n \Psi_{\otimes 1}(\tau_n(v)) = (-1)^n \Psi_{\otimes 1}(v) = S^{-1}_\Psi(v)$, where we used (A.4). Part (d) is immediate. \qed

Suppose that for every $n > 0$, there exists an invertible $L_n \in \text{End}(V^{\otimes n})$ such that $L_n^2 = (-1)^n S_{\Psi} \circ *, L_n \circ \hat{\circ} = \circ \circ L_n^{-1}$ and $L_n \circ ^* = ^* \circ L_n$. Let $L \in \text{End}(T(V))$ be the linear operator defined by $L|_{V^{\otimes n}} = L_n$ and define $(\cdot, \cdot) : B(V^*, \Psi^*) \otimes B(V, \Psi) \rightarrow k$ by
\[
(f, v) = \langle f, L^{-1}(v) \rangle.
\]

**Lemma A.8.** Suppose that $\Psi$ is self-transposed and unitary. Then for all $f \in B^r(V^*, \Psi^*), v \in B^s(V, \Psi)$ we have
\[
\langle f, \tilde{v} \rangle = (-1)^r \delta_{r,s}(f, v).
\]

Proof. Let $f \in B^r(V^*, \Psi^*), v \in B^s(V, \Psi)$, the case $r \neq s$ being trivial. Then
\[
\langle f, \tilde{v} \rangle = \langle f, L^{-1}(\tilde{v}) \rangle = (-1)^r \langle f, L^{-2}((L_r(v^*))^*) \rangle = (-1)^r \langle f, L^{-1}(v) \rangle = (-1)^r (f, v). \quad \Box
\]

Given $f \in B(V^*, \Psi^*), v \in B(V, \Psi)$ define $k$-linear operators $\partial_f, \partial_f^{op} : B(V, \Psi) \rightarrow B(V, \Psi), \partial_v, \partial_v^{op} : B(V^*, \Psi^*) \rightarrow B(V^*, \Psi^*)$ by
\[
\begin{align*}
\partial_f(g) &= \langle g(1), \Psi_{\otimes 2}(v) \rangle, & \partial_f^{op}(g) &= \langle \Psi_{\otimes 1}(v), g(2) \rangle, & f, g \in B(V^*, \Psi^*) \\
\partial_v(u) &= \langle u(1), \Psi_{\otimes 2}(v) \rangle, & \partial_v^{op}(u) &= \langle \Psi_{\otimes 1}(v), u(2) \rangle, & u, v \in B(V, \Psi).
\end{align*}
\]
Then for all $f, g \in B(V^*, \Psi^*), u, v \in B(V, \Psi)$
\[
\begin{align*}
\langle f, uv \rangle &= \langle \partial_v(f), u \rangle = \langle \partial_v^{op}(f), v \rangle \\
\langle fg, u \rangle &= \langle f, \partial_v(u) \rangle = \langle g, \partial_f^{op}(u) \rangle.
\end{align*}
\]
The definitions immediately imply that if \( f \in \mathcal{B}(V^*, \Psi^*) \), \( v \in \mathcal{B}(V, \Psi) \) are homogeneous then \( \partial_f, \partial_f^{op}, \partial_v, \partial_v^{op} \) are homogeneous. Moreover, if say \( f \in \mathcal{B}^r(V^*, \Psi^*) \), \( v \in \mathcal{B}^k(V, \Psi) \) then \( \partial_f(v), \partial_f^{op}(v) \in \sum_{r=0}^{k-r} \mathcal{B}^r(V, \Psi) \) and \( \partial_v(f), \partial_v^{op}(f) \in \sum_{r=0}^{r-k} \mathcal{B}^r(V^*, \Psi^*) \). Thus, \( \partial_f, \partial_f^{op}, \partial_v, \partial_v^{op} \) are locally nilpotent.

**Lemma A.9.** (a) The assignment \( v \mapsto \partial_v, v \in \mathcal{B}(V, \Psi) \) (respectively, \( f \mapsto \partial_f, f \in \mathcal{B}(V^*, \Psi^*) \)) defines a homomorphism of algebras \( \mathcal{B}(V, \Psi) \to \text{End}_k \mathcal{B}(V^*, \Psi^*) \) (respectively, \( \mathcal{B}(V^*, \Psi^*) \to \text{End}_k \mathcal{B}(V, \Psi) \)).

(b) The assignment \( v \mapsto \partial_v^{op}, v \in \mathcal{B}(V, \Psi) \) (respectively, \( f \mapsto \partial_f^{op}, f \in \mathcal{B}(V^*, \Psi^*) \)) defines an anti-homomorphism of algebras \( \mathcal{B}(V, \Psi) \to \text{End}_k \mathcal{B}(V^*, \Psi^*) \) (respectively, \( \mathcal{B}(V^*, \Psi^*) \to \text{End}_k \mathcal{B}(V, \Psi) \)).

(c) For all \( u, v \in \mathcal{B}(V, \Psi), f, g \in \mathcal{B}(V^*, \Psi^*) \) we have \( \partial_v \partial_v^{op} = \partial_v^{op} \partial_v \) and \( \partial_f \partial_f^{op} = \partial_f^{op} \partial_f \).

**Proof.** We have for all \( u \in \mathcal{B}(V, \Psi), f, g \in \mathcal{B}(V^*, \Psi^*) \)

\[
\partial_f \partial_g(u) = \langle g, u_2(1) \rangle \partial_f(u_1(1)) = \langle g, u_2(1) \rangle \langle f, u_2(2) \rangle = \langle f g, u_2(1) \rangle = \partial_f g(u),
\]

and

\[
\partial_f \partial_g^{op}(u) = \partial_f \langle g, u_2(1) \rangle = \langle g, u_2(1) \rangle \langle f, u_2(2) \rangle = \langle fg, u_2(1) \rangle = \partial_f \partial_g(u).
\]

The corresponding statements for operators \( \partial_v, v \in \mathcal{B}(V, \Psi) \) are checked similarly. \( \square \)

**A.4. Double smash products.** Let \( H \) be a bialgebra with the comultiplication \( \Delta_H \) and the counit \( \varepsilon_H \). Denote by \( H^{cop} \) the bialgebra \( H \) with the opposite comultiplication. Suppose that \( C \) is an \( H^{cop} \otimes H \)-module algebra. In other words, we have two commuting left actions \( \triangleright \) and \( \triangleleft \) of \( H \) on \( C \) satisfying

\[
h \triangleright (c c') = (h_1) \triangleright c(h_2) \triangleright c', \quad h \triangleleft (c c') = (h_2) \triangleleft c(h_1) \triangleleft c', \quad h \in H, c, c' \in C,
\]

where \( \Delta_H(h) = (h_1) \otimes (h_2) \). Define \( \mathcal{D}_{H,H^{cop}}(C) \) as \( C \otimes H \) with the product

\[
(c \otimes 1) \cdot (1 \otimes h) = c \otimes h, \quad (h \otimes 1) \cdot (1 \otimes c) = (h_1) \triangleright (h_3) \triangleleft c = (h_2) \otimes h.
\]

**Proposition A.10.** \( \mathcal{D}_{H,H^{cop}}(C) \) is an associative algebra. Moreover, \( H \) and \( C \) identify with sub-algebras of \( \mathcal{D}_{H,H^{cop}}(C) \).

**Proof.** The only non-trivial identity to check is \( h \cdot (c c') = (h \cdot c) \cdot c' \) for all \( h \in H, c, c' \in C \). We have

\[
(h \cdot c) \cdot c' = (h_1) \triangleright (h_3) \triangleleft c \cdot (h_2) \cdot c' = (h_1) \triangleright h_2 (h_3) \triangleleft c \cdot (h_2) \triangleright h_2 \triangleleft c',
\]

\[
= (h_1) \triangleright ((h_4) \triangleleft c \cdot (h_5) \triangleright c')) \cdot (h_2) \triangleright h_2 = (h_1) \triangleright (h_3) \triangleleft (c c') \cdot h_2 = h \cdot (c \cdot c'). \quad \square
\]

**Remark A.11.** Note that if \( \triangleright \) (respectively, \( \triangleleft \)) is trivial, that is \( h \triangleright c = \varepsilon_H(h) c \) (respectively, \( h \triangleleft c = \varepsilon_H(h) c \)), then \( \mathcal{D}_{H,H^{cop}}(C) = C \otimes H \) (respectively, \( C \otimes H^{cop} \)).

Suppose now that \( C \) is a bialgebra and that \( \Delta_C, \varepsilon_C \) are homomorphisms of \( H^{cop} \otimes H \)-modules, where \( H^{cop} \otimes H \) acts naturally on \( C \otimes C \) and \( k \). Thus, \( \Delta_C(h \triangleright h' \triangleleft c) = (h' \triangleleft c_{(1)}) \otimes (h \triangleright c_{(2)}) \) and \( \varepsilon_C(h \triangleright h' \triangleleft c) = \varepsilon_C(c) \varepsilon_C(h) \varepsilon_C(h') \) for all \( c \in C, h, h' \in H \).

**Proposition A.12.** Suppose that the actions \( \triangleright, \triangleleft \) satisfy

\[
h_2 \triangleright c_{(1)} \otimes h_1 \triangleleft c_{(2)} = \varepsilon_H(h) \Delta(c), \quad c \in C, h \in H.
\]

Then \( \mathcal{D}_{H,H^{cop}}(C) \) is a bialgebra with the comultiplication and the counit defined by \( \Delta(c \cdot h) = \Delta_C(c) \cdot \Delta_H(h) \) and \( \varepsilon(c \cdot h) = \varepsilon_C(c) \varepsilon_H(h) \), \( c \in C, h \in H, \) and \( C, H^{cop} \) identify with its sub-bialgebras. If both \( C \) and \( H \) are Hopf algebras and

\[
S_C(h \triangleright h' \triangleleft c) = S^{-2}_H(h') \triangleright h \triangleleft S_C(c), \quad c \in C, h, h' \in H.
\]
then $D_{H,H^{\text{cop}}}(C)$ is a Hopf algebra with the antipode defined by $S(c \cdot h) = S_H^{-1}(h) \cdot S_C(c)$ and $C$, $H^{\text{cop}}$ identify with its Hopf subalgebras.

Proof. We need to check that $\Delta(h \cdot c) = \Delta^H_H(h) \cdot \Delta_C(c)$ for all $c \in C$, $h \in H$. Indeed

$$
\Delta(h \cdot c) = \Delta((h_1) \triangleright h_3 \triangleright c) \cdot (h_2) = \Delta_C(h_1) \triangleright h_4 \triangleright c) \cdot (h_3 \otimes h_2) = (h_4) \triangleright (c_1 \otimes (h_1) \triangleright c_2) \cdot (h_3 \otimes h_2) \cdot (h_2) = h_2 \cdot c_1 \otimes h_1 \cdot c_2 = \Delta^H_H(h) \cdot \Delta_C(c).
$$

The property of $\varepsilon$ is obvious. For the antipode, we have

$$
S(h \cdot c) = S_H^{-1}(h_2) \cdot S_C(h_1) \triangleright h_3 \triangleright c = S_H^{-1}(h_2) \cdot S_H^{-2}(h_3) \triangleright h_1 \triangleright S_C(c) = S_H^{-1}(h_4) \triangleright S_H^{-2}(h_5) \triangleright S_H^{-1}(h_2) h_1 \triangleright S_C(c) \cdot S_H^{-1}(h_3) = S_C(c) \cdot S_H^{-1}(h).
$$

Denote $H^{\text{op}}$ the opposite algebra and coalgebra of $H$. Note that we can endow $H^{\text{op}} \otimes C^{\text{op}}$ with an associative algebra structure via

$$
c \cdot h = h_2 \cdot (h_1) \triangleright h_3 \triangleright c).
$$

Denote the resulting algebra $D_{H^{\text{op}},H^{\text{cop}}}(C^{\text{op}})$. The following proposition is immediate.

**Proposition A.13.** The map $\tau : C \otimes H \to H \otimes C$, $c \otimes h \mapsto h \otimes c$ is an isomorphism of algebras $D_{H,H^{\text{cop}}}(C)^{\text{op}} \to D_{H^{\text{op}},H^{\text{cop}}}(C^{\text{op}})$. Moreover, if (A.8) and (A.9) hold then $\tau$ is an isomorphism of Hopf algebras $D_{H,H^{\text{cop}}}(C)^{\text{op}} \to D_{H^{\text{op}},H^{\text{cop}}}(C^{\text{op}})$.

Let $\overline{\cdot}$ be a field involution on $k$ and suppose that it extends to an anti-linear anti-involutions of algebras $C$ and $H$. Assume that $\overline{\cdot}$ is an anti-linear involution of coalgebras for $H$. Note that then we have $S_H(h) = S_H^{-1}(h)$, $h \in H$. Extend $\overline{\cdot}$ to an anti-linear map $D_{H,H^{\text{cop}}}(C) \to D_{H,H^{\text{cop}}}(C)$ by

$$
\overline{c \cdot h} = \overline{h} \cdot \overline{c}.
$$

**Lemma A.14.** Suppose that

$$
\overline{h_2} \triangleright \overline{h_1} \triangleright \overline{c} = \varepsilon_H(\overline{h}) \overline{c} = \overline{h_1} \triangleright \overline{h_2} \triangleright \overline{c}, \quad h \in H, c \in C. \quad (A.10)
$$

Then $\overline{\cdot}$ is an anti-linear anti-involution of the algebra $D_{H,H^{\text{cop}}}(C)$.

Proof. We have

$$
\overline{h \cdot c} = \overline{(h_1) \triangleright (h_3) \triangleright c \cdot h_2} = \overline{h_2} \cdot \overline{(h_1) \triangleright (h_3) \triangleright c} = \overline{h_2} \cdot \overline{(h_4) \triangleright (h_5) \triangleright c} \cdot \overline{h_3} = \varepsilon_H(\overline{h_4}) \overline{h_2} \cdot \overline{(h_1) \triangleright c} \cdot \overline{h_3} = \varepsilon_H(\overline{h_1}) \overline{\varepsilon_H(h_2) \cdot \overline{c} \cdot h_2} = \overline{c} \cdot \overline{h}.
$$

This shows that $\overline{\cdot}$ is a well-defined anti-linear anti-involution of $D_{H,H^{\text{cop}}}(C)$. \qed

**Remark A.15.** It is easy to check that (A.10) holds if

$$
\overline{h \cdot c} = S_H^{-1}(\overline{h}) \cdot \overline{c}, \quad \overline{h \cdot c} = S_H(\overline{h}) \cdot \overline{c}, \quad h \in H, c \in C. \quad (A.11)
$$

A.5. Bialgebra pairings and doubles of bialgebras. We will now consider a special case of the double smash product construction. Given bialgebras $H$ and $C$, $\phi \in \text{Hom}_k(C \otimes H, k)$ is said to be a bialgebra pairing if for all $c, c' \in C$ and $h, h' \in H$

$$
\phi(cc', h) = \phi(c, h_1) \phi(c', h_2), \quad \phi(c, hh') = \phi(c_1, h) \phi(c_2, h'), \quad \phi(c, 1) = \varepsilon_C(c), \quad \phi(1, h) = \varepsilon_H(h).
$$

If both $C$ and $H$ are Hopf algebras, a bialgebra pairing $\phi$ is called a Hopf pairing if

$$
\phi(S_C(c), h) = \phi(c, S_H(h)), \quad c \in C, h \in H.
$$
Given a bialgebra pairing \( \phi : C \otimes H \to \mathbb{k} \), define
\[
h \triangleright c = c(1)\phi(c(2), h), \quad c \triangleright h = c(2)\phi(c(1), h), \quad c \triangleright h = h(1)\phi(c, h(2)), \quad h \triangleright c = h(1)\phi(c, h(1)) \tag{A.12}
\]
The following is easily checked.

**Lemma A.16.** Let \( \phi, \phi' \) be two bialgebra pairings \( C \otimes H \to \mathbb{k} \). Then \( \triangleright, \triangleright \) define a structure of an \( H \)- (respectively, a \( C \))- bimodule algebra on \( C \) (respectively, on \( H \)). Moreover,
\[
\Delta_C(h \triangleright c \triangleright h') = (c(1) \triangleright h') \otimes (h \triangleright c(2)) \tag{A.13}
\]

Given two bialgebra pairings \( \phi_+, \phi_- : C \otimes H \to \mathbb{k} \) define \( \mathcal{D}_{\phi_+, \phi_-}(C, H) \) as \( \mathcal{D}_{H,H^{op}}(C) \) where \( h \triangleright c = h \triangleright c \) and \( h \triangleright c = c \leq S^{-1}_H(h) \). Thus, in \( \mathcal{D}_{\phi_+, \phi_-}(C, H) \) we have
\[
h \cdot c = c(2) \cdot h(2)\phi_-(c(1), S^{-1}_H(h(3)))\phi_+(c(3), h(1)) = (h(1) \triangleright c \triangleright S^{-1}_H(h(3))) \cdot h(2) = c(2) \cdot (S^{-1}_C(c(1)) \triangleright h \triangleright c(3)) \tag{A.14}
\]

We abbreviate \( \mathcal{D}_\phi(C, H) = \mathcal{D}_{\phi_+, \phi_-}(C, H) \)

**Proposition A.17.** Let \( H \) be a Hopf algebra, \( C \) be a bialgebra and \( \phi, \phi_\pm : C \otimes H \to \mathbb{k} \) be bialgebra pairings.

(a) \( \mathcal{D}_{\phi_+, \phi_-}(C, H) \) is an associative algebra and \( C, H \) identify with its subalgebras.
(b) \( \mathcal{D}_\phi(C, H) \) is a bialgebra and \( C, H^{op} \) identify with its sub-bialgebras. Moreover, if \( C \) is a Hopf algebra and \( \phi \) is a Hopf pairing then \( \mathcal{D}_\phi(C, H) \) is a Hopf algebra.

**Proof.** Part (a) is immediate from Proposition A.10. To prove (b) note that by (A.13) we only need to check that (A.8) and (A.9) hold. Indeed
\[
h(2) \triangleright c(1) \otimes c(2) \leq S^{-1}_H(h(1)) = \phi(c(2), h(2))\phi(c(3), S^{-1}_H(h(1)))c(1) \otimes c(4)
\]

Finally, to prove (A.9) note that
\[
\begin{align*}
S_C(h \triangleright c \triangleright S^{-1}_H(h')) &= \phi(c(3), h)\phi(c(1), S^{-1}_H(h'))SC(c(2)) \\
&= \phi(SC(c(3)), S^{-1}_H(h'))\phi(SC(c(1)), S^{-2}_H(h'))SC(c(2)) = S^{-2}_H(h') \triangleright SC(c) \triangleright S^{-1}_H(h).
\end{align*}
\]

Note the following useful identity in \( \mathcal{D}_{\phi_+, \phi_-}(C, H) \)
\[
c \cdot h = h(2) \cdot (S^{-1}_H(h(1)) \triangleright c \triangleright h(3)) = \phi(c(2), h) \Delta(c) \tag{A.15}
\]

The following is a straightforward consequence of (A.12) and (A.14).

**Proposition A.18.** \( H \) is a left (respectively, right) \( \mathcal{D}_\phi(C, H) \)-module algebra via \( c \triangleright h' = c \triangleright h' \) and \( h \triangleright h' = h(2)h'S^{-1}_H(h(1)) \) (respectively, via \( h' \triangleleft c = h' \triangleleft c \) and \( h' \triangleleft h = S^{-1}_H(h(2))h'h(1) \) ), \( c \in C, h, h' \in H \). Moreover, if \( C \) is a Hopf algebra then \( C \) is a left (respectively, right) \( \mathcal{D}_\phi(C, H) \)-module algebra via \( h \triangleright c' = h \triangleright c' \), \( c \triangleright c' = c(2)c'S^{-1}_C(c(1)) \) (respectively, via \( c' \triangleleft c = c' \triangleleft c \), \( c \triangleleft = S^{-1}_C(c(2))c'c(1) \) ), \( c, c' \in C, h \in H \).

The compatibility conditions from Lemma A.14 read
\[
\frac{c(1)\phi_+(c(2), h(2))\phi_+(c(3), h(1))}{c(1)} = \varepsilon_H(h) = \frac{c(3)\phi_-(c(2), h(2))\phi_-(c(1), S^{-1}_H(h(2)))}{c(3)} \tag{A.16}
\]

and are satisfied if
\[
\phi_\pm(c, S^{-1}_H(h)) = \phi_\pm(c, S^{-1}_H(h)), \quad c \in C, h \in H.
\]
A.6. **Bosonization of Nichols algebras.** Suppose that $V$ is a left Yetter-Drinfeld module over a Hopf algebra $H$ with the comultiplication $\Delta_H$ and the antipode $S_H$. That is, $V$ is a left $H$-module with the action denoted by $\triangleright$ and a left $H$-comodule with the co-action $\delta : V \to H \otimes V$. We use the Sweedler-type notation $\delta(v) = v^{(1)} \otimes v^{(0)}$. The action and co-action are compatible, that is
\[
\delta(h \triangleright v) = h^{(1)}v^{(-1)}S_H(h^{(3)}) \otimes h^{(2)} \triangleright v^{(0)}, \quad h \in H, \, v \in V, \tag{A.17}
\]
where $(\Delta_H \otimes 1)\Delta_H = h^{(1)} \otimes h^{(2)} \otimes h^{(3)}$.

The category $\mathcal{YD}_H$ of left Yetter-Drinfeld modules over $H$ is a braided tensor category with the braiding $\Psi : V \otimes W \to W \otimes V$ being given by
\[
\Psi_{V,W}(v \otimes w) = v^{(-1)} \triangleright w \otimes v^{(0)}, \quad v \in V, \, w \in W. \tag{A.18}
\]
Note that
\[
\Psi_{V,W}^{-1}(w \triangleright v) = v^{(0)} \otimes S_H^{-1}(v^{(-1)}) \triangleright w. \tag{A.19}
\]

In particular, $T(V)$ is a braided Hopf algebra in the category $\mathcal{YD}_H$. We will denote the corresponding Nichols algebra by $B(V)$.

Consider now the algebra $T(V) \rtimes H = T(V) \otimes H$ with the cross-relation
\[
h \cdot u = (h^{(1)} \triangleright u) \cdot h^{(2)}. \tag{A.20}
\]

It has a co-algebra structure defined by
\[
\Delta(v) = v \otimes 1 + \delta(v), \quad \Delta(h) = \Delta_H(h), \quad \varepsilon(v) = 0, \quad \varepsilon(h) = \varepsilon_H(h), \quad v \in V, \, h \in H. \tag{A.21}
\]

It is easy to check, using (A.17), that this comultiplication and counit extend to homomorphisms of respective algebras.

**Lemma A.19.** Let $u \in T(V)$. Then $\Delta(u) = u^{(1)} u^{(-1)} (1 \otimes u^{(2)}), \quad$ where $\Delta(u) = u^{(1)} \otimes u^{(2)}$.

**Proof.** For $v \in V$ there is nothing to prove. Suppose that the identity holds for all $u \in V^\otimes r$, $r < n$. Let $u \in V^\otimes r$, $v \in V^\otimes s$, $r, s > 0$, $r + s = n$. Then
\[
\Delta(uv) = \Delta(u) \Delta(v) = (u^{(1)} \otimes 1) \Psi(u^{(2)} \otimes v^{(1)}) (1 \otimes v^{(2)}) = \Psi(v^{(1)} u^{(2)} \triangleright v^{(1)}) \otimes u^{(0)} w^{(2)},
\]
whence
\[
\Delta(uv) = u^{(1)} u^{(-1)} (1 \otimes u^{(2)}) \Psi(u^{(2)} \otimes v^{(1)}) (1 \otimes v^{(2)}) = u^{(1)} u^{(2)} v^{(1)} v^{(2)} \otimes u^{(0)} v^{(2)}
\]
\[
= u^{(1)} u^{(-2)} \triangleright v^{(1)} u^{(2)} \otimes u^{(0)} v^{(2)} = u^{(1)} u^{(-1)} (1 \otimes u^{(2)}) \otimes u^{(0)} v^{(2)}, \quad \square
\]

Denote by $S$ the braided antipode on $T(V)$ corresponding to the braiding $\Psi_{V,V}$. Note that $S$ is a morphism in the category $\mathcal{YD}_H$ hence commutes with the action and the co-action of $H$. Define $S : T(V) \times H \to T(V) \times H$ by
\[
S(uh) = S_H(u^{(-1)}h) S(u^{(0)}). \tag{A.22}
\]

**Lemma A.20.** $S$ is an antipode for $T(V) \times H$. Moreover, $S$ is invertible and
\[
S^{-1}(uh) = S_H^{-1}(u^{(-1)}h) S_H^{-1}(u^{(0)}), \quad u \in T(V), \, h \in H. \tag{A.23}
\]

**Proof.** By definition, we have $S(uh) = S(h)S(u), \, u \in T(V), \, h \in H$. Furthermore, using (A.17), we obtain
\[
S(hu) = S((h^{(1)} \triangleright u) h^{(2)}) = S_H(h^{(1)} \triangleright u)(-1) h^{(2)} S((h^{(1)} \triangleright u)^{(0)})
\]
\[
= S_H(h^{(1)} u^{(-1)} S_H(h^{(3)}) h^{(4)}) S(h^{(2)} \triangleright u^{(0)}) = S_H(h^{(1)} u^{(-1)}) S(h^{(2)} \triangleright u^{(0)})
\]
\[
= S_H(u^{(-1)}) S_{h^{(1)}}(h^{(2)} \triangleright S(u^{(0)})) = S_H(u^{(-1)})(S_{h^{(2)}} h^{(3)} \triangleright S(u^{(0)})) S(h^{(1)})
\]
\[
= S_H(u^{(-1)}) S(u^{(0)}) S(h) = S(u) S(h).
\]
To prove that $S$ is an anti-endomorphism of $T(V) \times H$, it remains to show that $S(uv) = S(v)S(u)$ for all $u, v \in T(V)$. Indeed,

$$S(uv) = S_H((uv)^{(1)})S((uv)^{(0)}) = S_H(v^{(-1)})S_H(u^{(-1)})((S(u^{(0)})^{(-1)} \triangleright S(v^{(0)}))(S(u^{(0)}))^{(0)}$$

$$= S_H(v^{(-1)})S_H(u^{(-2)})(u^{(-1)} \triangleright S(v^{(0)}))S(u^{(0)})$$

$$= S_H(v^{(-1)})(S_H(u^{(-2)})u^{(-1)} \triangleright S(v^{(0)}))S_H(u^{(-3)})S(u^{(0)})$$

$$= S_H(v^{(-1)})S(v^{(0)})S_H(u^{(-1)})S(u^{(0)}) = S(v)S(u).$$

We have

$$m(S \otimes 1)\Delta(u) = S(u_{(1)}u_{(2)})u_{(0)} = S_H(u_{(1)}u_{(2)})S(u_{(0)}) = S_H((\Delta(u))^{(-1)})\varepsilon(u^{(0)}) = \varepsilon(u).$$

On the other hand,

$$m(1 \otimes S)\Delta(u) = u_{(1)}u_{(2)}^{-1}S(u_{(2)}) = u_{(1)}u_{(2)}^{-1}S_H(u_{(2)})S(u_{(0)}) = u_{(1)}\varepsilon_H(u_{(2)})S(u_{(2)})$$

$$= u_{(1)}S(u_{(2)}) = \varepsilon(u).$$

Define $\tilde{S} : T(V) \times H \to T(V) \times H$ by $\tilde{S}(uh) = S_H^{-1}(h)S_H^{-1}(u^{(0)})S_H^{-1}(u^{(-1)})$. Then we have

$$\tilde{S}(hu) = \tilde{S}((h(1) \triangleright u)h(2)) = S_H^{-1}(h(2))S_H^{-1}(h(1) \triangleright u)(h^{(-1)})$$

$$= S_H^{-1}(h(3)h(2) \triangleright u(0))S_H^{-1}(h(1)u(-1)S_H^{-1}(h(3)))$$

$$= S_H^{-1}(h(3))h(2)S_H^{-1}(u(0))S_H^{-1}(u(-1))S_H^{-1}(h(1))$$

Now

$$S\tilde{S}(uh) = S(S_H^{-1}(h)S_H^{-1}(u^{(0)})S_H^{-1}(u^{(-1)})) = u^{(-1)}S(S_H(u^{(0)}))h$$

$$= u^{(-2)}S_H(u^{(-1)})u^{(0)}h = \varepsilon_H(u^{(-1)})u^{(0)}h = uh,$$

while

$$\tilde{S}S(uh) = \tilde{S}(S_H(u^{(-1)}h)S(u^{(0)})) = \tilde{S}(S(u^{(0)})u^{(-1)}h$$

$$= u^{(0)}S_H^{-1}(u(-1))u^{(-2)}h = u^{(0)}\varepsilon_H(u^{(-1)})h = uh.$$}

Thus, $\tilde{S}$ is the inverse of $S$. 

Observe that ker Wor($\Psi$) is a bi-ideal in $T(V) \times H$. In particular, we can consider the quotient of $T(V) \times H$ by that ideal which is isomorphic to $B(V) \times H$. Clearly, Lemma A.19 and A.20 hold in $B(V) \times H$.

Let $\gamma$ be a field involution on $k$ and fix its extension to $V$ as in §A.2. Suppose that $\overline{h \triangleright v} = S_H^{-1}(h) \triangleright \overline{v}$ and that $(\gamma \otimes \gamma) \circ \delta \circ \gamma = \delta$. 

Lemma A.21. Suppose that $\Psi$ is self-transposed. Then $\Psi$ is also unitary, that is $\gamma \otimes \gamma \circ \Psi = \Psi^{-1} \circ \gamma$. 

Proof. Since $\Psi$ is self-transposed, it follows that

$$u^{(-1)} \triangleright v \otimes u^{(0)} = v^{(0)} \otimes v^{(-1)} \triangleright u$$

Applying $\gamma \otimes \gamma$ to both sides yields

$$(\gamma \otimes \gamma) \circ \Psi(u \otimes v) = \overline{u^{(-1)} \triangleright v \otimes u^{(0)}} = \overline{v^{(0)} \otimes v^{(-1)} \triangleright u} = \overline{v^{(0)} \otimes S_H^{-1}(v^{(-1)}) \triangleright \overline{v}} = \Psi^{-1}(\overline{v} \otimes \overline{v})$$

where we used (A.19).
Thus, if (A.24) holds, \(B(V)\) admits the anti-linear anti-involution \(\bar{\cdot}\). Then by Lemma A.14, (A.11) and Remark A.11, \(\bar{\cdot}\) extends uniquely to an anti-linear anti-involution on \(B(V) \rtimes H\) such that \(v \cdot h = h \cdot \bar{v}, \ v \in V, \ h \in H\). Thus, we obtain the following

**Lemma A.22.** Suppose that \(\Psi : V \otimes V \rightarrow V \otimes V\) is self-transposed, \(\bar{\cdot}\) commutes with the co-action on \(V\) and \(h \triangleright v = S^{-1}_H(h) \triangleright v\). Then \(\bar{\cdot}\) extends to an anti-linear algebra anti-involution of \(B(V) \rtimes H\).

A.7. Drinfeld double. Let \(C, H\) be Hopf algebras and fix a Hopf pairing \(\xi : C \otimes H \rightarrow k\). Let \(V\) (respectively, \(V^*\)) be an object in \(H\)-\text{YD}\ (respectively, in \(\text{YD}\)). Then we have a right \(C\)-module (respectively, \(H\)-module) structure on \(V\) (respectively, \(V^*\)) defined by

\[
\langle f, h \triangleright v \rangle = \langle f \triangleleft h, v \rangle, \quad \langle c \triangleright f, v \rangle = \langle f, v \rangle, \quad f, v \in V, \ c \in C, \ h \in H.
\]

Assume that a pairing \(\langle \cdot, \cdot \rangle : V^* \otimes V \rightarrow k\) satisfies

\[
\langle f, h \triangleright v \rangle = \langle f \triangleleft h, v \rangle, \quad \langle c \triangleright f, v \rangle = \langle f, v \rangle, \quad f, v \in V, \ c \in C, \ h \in H.
\]

**Lemma A.23.** Suppose that (A.26) holds. Then the braiding \(\Psi^*\) is the adjoint of \(\Psi\) with respect to \(\langle \cdot, \cdot \rangle^* : V^* \otimes V \rightarrow k\) in the notation of §A.3.

**Proof.** We need to show that for all \(f, g \in V^*, \ u, v \in V\)

\[
\langle \Psi^*(f \otimes g), u \otimes v \rangle^* = \langle f \otimes g, \Psi(u \otimes v) \rangle
\]

which, by the definition of \(\Psi\) and \(\Psi^*\) is equivalent to

\[
\langle f(-1) \triangleright g, u \rangle \langle f(0), v \rangle = \langle f, u(-1) \triangleright v \rangle \langle g, u(0) \rangle
\]

But, using (A.25) and (A.26) we obtain

\[
\langle f(-1) \triangleright g, u \rangle \langle f(0), v \rangle = \langle g, u \triangleleft f(-1) \rangle \langle f(0), v \rangle = \langle g, u(0) \rangle \langle f(-1), u(-1) \rangle \langle f(0), v \rangle
\]

\[
= \langle g, u(0) \rangle \langle f \triangleleft u(-1) \rangle \langle v \rangle = \langle g, u(0) \rangle \langle f, u(-1) \triangleright v \rangle.
\]

Thus, we can define the pairing \(\langle \cdot, \cdot \rangle : T(V^*) \otimes T(V) \rightarrow k\) as in §A.3. Note that (A.26) holds for all \(f \in T(V^*), \ v \in T(V)\). Clearly, we can replace \(T(V), T(V^*)\) by the corresponding Nichols algebras.

It should be noted that \(V\) is not an \(H\)-\(C\) bimodule with respect to the actions \(\triangleright\) and \(\triangleleft\). Given \(c \in C, \ h \in H\) define, for all \(f, v \in V^*

\[
v \triangleleft (c \cdot h) = S^{-1}_H(h) \triangleright (v \triangleleft c), \quad (c \cdot h) \triangleright f = c \cdot h \triangleright (f \triangleleft S^{-1}_H(h)). \tag{A.27}
\]

**Lemma A.24.** \(V\) (respectively, \(V^*\)) is a right (respectively, left) Yetter-Drinfeld module over \(D_\xi(C, H)\), with the right coaction on \(V\) defined by \(\delta_H(v) = v^{(0)} \otimes v^{(-1)}\), the left coaction on \(V^*\) defined by \(\delta(f) = f(-1) \otimes f(0)\) and the left (right) action defined by (A.27).

**Proof.** Let \(c \in C, \ h \in H\) and \(v \in V\). By definition, we have \(v \triangleleft (c \cdot h) = (v \triangleleft c) \cdot h\). On the other hand,

\[
(v \triangleleft h) \triangleleft c = \xi(c, (S^{-1}_H(h) \triangleright v)^{-1}) (S^{-1}_H(h) \triangleright v) = \xi(c, S^{-1}_H(h(3)) v^{(-1)} h^{(1)} S^{-1}_H(h(2)) \triangleright v^{(0)})
\]

\[
= \xi(c(1), S^{-1}_H(h(3))) \xi(c(3), h^{(1)}) S^{-1}_H(h(2)) \triangleright v^{(0)}
\]

\[
= \xi(c(1), S^{-1}_H(h(3))) \xi(c(3), h^{(1)}) S^{-1}_H(h(2)) \triangleright (v \triangleleft c(2))
\]

\[
= \xi(c(1), S^{-1}_H(h(3))) \xi(c(3), h^{(1)}) S^{-1}_H(h(2)) (v \triangleleft (c_2 \cdot h_2)) = v \triangleleft (h \cdot c).
\]

Thus, (A.27) defines a right \(D_\xi(C, H)\)-module structure on \(V\). It remains to verify that this action is compatible with the right co-action. Recall that \(H^{\text{cop}}\) identifies with a sub-bialgebra of \(D_\xi(C, H)\), hence we only need to check the compatibility condition for \(c \in C\). We have

\[
(v^{(0)} \triangleleft c(2)) \otimes S_C(c(1)) v^{(-1)}(3) = (v^{(0)} \triangleleft c(2)) \otimes S_C(c(1)) c(4) v^{(-2)}(3) \xi(c(3), S^{-1}_H(v^{(-1)})) \xi(c(5), v^{(-3)}))
\]
The algebra \( V \) fixes pairings \( U \) relative to \( C, V, H \) are isomorphic to \( \mathcal{D}_\xi(C, H) \times B(V) \) (respectively, \( \mathcal{B}(V^*) \times B(V) \)).

Proposition A.26. The algebra \( \mathcal{U}_\xi(V^*, V, C, H) \) is isomorphic to the braided double \( \mathcal{B}(V^*) \times \mathcal{D}_\xi(C, H) \times \mathcal{B}(V) \) in the sense of [2] and admits a triangular decomposition. In particular, if \( \langle \cdot, \cdot \rangle \) equals zero, \( \mathcal{U}_\xi(V^*, V, C, H) \) is the Heisenberg double.

Proof. Define \( \beta : V^* \otimes V \to \mathcal{D}_\xi(C, H) \) by \( \beta = \beta_+ - \beta_- \) where \( \beta_+(f, v) = f^{(-1)}(f^{(0)}, v)^+ \) and \( \beta_-(f, v) = v^{(-1)}(f, v^{(0)})^- \), \( f \in V^*, v \in V \). Then in \( \mathcal{U}_\xi(V^*, C, V, H) \), we have \( [v, f] = \beta(f, v) \). Thus, \( \mathcal{U}_\xi(V^*, C, V, H) \) is a braided double. By [2, Theorem A], it remains to prove that

\[
\beta_+(f, v) \in \mathcal{D}_\xi(C, H), \quad x \in V, \quad f \in V^*.
\]

(A.28)

Using Lemma A.24, we obtain

\[
\beta_+(x_1 \triangleright f, v)x_2 = (x_1 \triangleright f)(x_2 \triangleright (x_1 \triangleright f)(0), v)^+ x_2 = x_1 f^{(-1)}(f^{(0)}, v) x_2 = x_1 \beta_+(f, v \triangleright x_2),
\]

while

\[
x_1 \beta_-(f, v \triangleleft x_2) = \langle f, v^{(0)} \triangleleft x_3 \rangle \triangleright x_1 \triangleright (x_3 \triangleright x_2)^{(-1)} x_4 = \langle f, v^{(0)} \triangleleft x_3 \rangle v^{(-1)} x_4 = \langle f, v^{(0)} \triangleleft x_3 \rangle v^{(-1)} x_2 = x_1 \beta_-(f, v \triangleright x_2).
\]

We now obtain another presentation of \( \mathcal{U}_\xi(V^*, C, V, H) \). Given a pairing \( \langle \cdot, \cdot \rangle : V^* \otimes V \to k \) satisfying (A.26), define \( \phi : \mathcal{B}(V^*) \otimes C \otimes \mathcal{B}(V) \otimes H \to k \) by \( \phi(f c, v h) = \langle f, v \rangle \xi(c, h) \).

Lemma A.27. \( \phi \) is a Hopf pairing.

Proof. We have

\[
\phi((f c)(1), v h) \phi((f c)(2), v' h') = \langle f c(1), v \rangle \langle f c(2), v' \rangle \xi(f c(1), h) \xi(c, h'),
\]

where we used (A.26). Similarly,

\[
\phi(f c, (v h)(1)) \phi(f' c', (v h)(2)) = \langle f, c(1) \rangle \langle f' c', (v h)(2) \rangle \xi(c, v) \xi(c(1), h(1)) \xi(c', h(2))
\]

\[
= \langle f, \xi(1) \rangle \langle f' c', \xi(1) \rangle \xi(c, v) \xi(c(1), h(1)) \xi(c', h(2)) = \langle f, \xi(1) \rangle \langle f' c', \xi(1) \rangle \xi(c', h(2))
\]

\[
= \langle f, \xi(1) \rangle \langle c(1) \triangleright f' c', \xi(1) \rangle \xi(c(1), h(1)) \xi(c', h(2)) = \langle f, \xi(1) \rangle \langle f' c', \xi(1) \rangle \xi(c(1), h(1)) \xi(c', h(2))
\]

\[
= \phi(f c, (v h)(1)) \phi(f' c', (v h)(2)) = \phi((f c) \cdot (f' c'), (v h)).
\]
Clearly, $\phi(fc, 1) = \varepsilon(f)\varepsilon_C(c)$ while $\phi(1, vh) = \varepsilon(v)\varepsilon_H(h)$. Finally, we have

\[
\phi(S(fc), vh) = \phi(S_C(f(-1)c)S(f(0)), vh) = \phi(S_C(f(-1)c_1)S(f(0)), vh)
\]

\[
= \langle S_C(f(-1)c_2), v \rangle \xi(S_C(f(-2)c_1), h)
\]

\[
= \langle f(0), S(v(0)) \rangle \xi(f(-1)c_2, S_H(v(-1)))\xi(f(-2)c_1, S_H(h))
\]

\[
= \langle f(0), S(v(0)) \rangle \xi(f(-1)c, S_H(v(-1))) = \langle f(0), S(v(0)) \rangle \xi(f(-1)c, S_H(v(-1)))
\]

\[
= \langle f, S_H(v(-1)h_2) \rangle \mathcal{S}(v_0(S_H(v(-1))) = \phi(fc, S_H(v(-1)h_2))
\]

\[
= \phi(fc, S_H(v(-1)h_2)) \mathcal{S}(v_0(S_H(v(-1))) = \phi(fc, S(vh)).
\]

\[\square\]

**Theorem A.28.** The algebra $\mathcal{U}_C(V^*, C, V, H)$ is isomorphic to $\mathcal{D}_{\phi_+, \phi_-}(B(V^*) \rtimes C, B(V) \rtimes H)$ where $\phi_{\pm}(fc, vh) = \langle f, v \rangle \xi(c, h)$. In particular, for all $v \in B(V)$, $f \in B(V^*)$ we have in $\mathcal{U}_C(V^*, C, V, H)$

\[
v \cdot f = f(0) \mathcal{S}(v(0)) \mathcal{S}(v(1)) = \phi(f(0), \mathcal{S}(v(0)) \mathcal{S}(v(1)) = \phi(f(0), v(0) \mathcal{S}(v(0)) \mathcal{S}(v(1))
\]

Moreover, if $\langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_-$ then $\mathcal{U}_C(V^*, C, V, H)$ is a Hopf algebra with the comultiplication defined by $\Delta(f) = f \otimes 1 + f(-1) \otimes f(0)$, $\Delta(v) = 1 \otimes v + v(0) \otimes v(-1)$, $\Delta(c) = c_c, S_H(c_1)$, $\Delta(h) = \Delta_H^0(h)$, $v \in V$, $f \in V^*$, $c \in C$, $h \in H$.

**Proof.** Let $\mathcal{D} = \mathcal{D}_{\phi_+, \phi_-}(B(V^*) \rtimes C, B(V) \rtimes H)$. Clearly, the subalgebra of $\mathcal{U} := \mathcal{U}_C(V^*, C, V, H)$ generated by $V^*$ and $C$ identifies with $B(V^*) \rtimes C$. Likewise, the subalgebra generated by $V$ and $H$ identifies with $B(V) \rtimes H$ since

\[
(h_1) \triangleright v \cdot h_2 = h_3 \cdot ((h_1) \triangleright v) \triangleright h_2 = h_3 \cdot (S_H^1(h_2)h_1) \triangleright v = h \cdot v, \quad h \in H, v \in V.
\]

Furthermore, in $\mathcal{D}$ we have, for all $v \in V$, $f \in V^*$, $c \in C$ and $h \in H$

\[
v \cdot c = c_2v(2)\phi_-(c_1), S_H^1(v(3)) = c_1v(0)\phi_+(c_1, v(-1)) = c_1v(0)\phi_+(c_2, v(-1))
\]

while

\[
h \cdot f = f(2)h(2)\phi_-(f(1), S_H^1(h(3)))\phi_+(f(3), h(1)) = f(0)h(1)\phi_-(f(1), S_H^1(h(2))) = (h_2) \triangleright f \cdot h_1
\]

and

\[
v \cdot f = f(2)v(2)\phi_-(f(1), S_H^1(v(3)))\phi_+(f(3), v(1)) = v(-1)\phi_-(f, S_H^1(v(0))) + f \cdot v + f(-1)\phi_+(f(0), v)
\]

\[
= f \cdot v + f(-1)(f(0), v) - v(-1)(f(0), v)
\]

Thus, all relations between generators of $\mathcal{D}$ hold in $\mathcal{U}$, hence we have a homomorphism of algebras $\mathcal{D} \rightarrow \mathcal{U}$, which is clearly an isomorphism of vector spaces.

It remains to prove (A.29). Observe that Lemma A.19 implies that

\[
(\Delta \otimes 1)\Delta(v) = \Delta(v_1(v_2(v_3))) = (v_1) \mathcal{S}(v_2) = v(0) \mathcal{S}(v(1)) \mathcal{S}(v(2)) \mathcal{S}(v(3)) \mathcal{S}(v_0) \mathcal{S}(v_1) \mathcal{S}(v_2) \mathcal{S}(v_3)
\]

and similarly for $(\Delta \otimes 1)\Delta(f)$. Then by (A.14) and Lemma A.20 we have

\[
v \cdot f = f(2)v(2)\phi_-(f(1), S_H^1(v(3)))\phi_+(f(3), v(1))
\]

\[
= f(0)\mathcal{S}(v(0)) S_H(v(-1)) = f(0)\mathcal{S}(v(0)) S_H(v(-1)) = \mathcal{S}(v(0)) S_H(v(-1)) = \mathcal{S}(v(0)) S_H(v(-1))
\]
The identity (A.29) can be also written in the following form

\[ v \cdot f = f_0^0 \langle f_2^0, v^{(-2)}(1) \rangle \xi \left( f_2^0, v^{(-2)}(1) \right) - \langle f_2^0, v^{(3)} \rangle \xi \left( f_2^0, v^{(3)} \right) - \langle f_2^0, v^{(2)} \rangle \xi \left( f_2^0, v^{(2)} \right) - \langle f_2^0, v^{(1)} \rangle \xi \left( f_2^0, v^{(1)} \right) - \langle f_2^0, v^{(0)} \rangle \xi \left( f_2^0, v^{(0)} \right) . \]

Note that if \( \langle \cdot, \cdot \rangle_+ = 0 \) on \( V^* \otimes V \), we obtain

\[ v \circ f = f_0^0 \langle f_2^0, v^{(-2)}(1) \rangle \xi \left( f_2^0, v^{(-2)}(1) \right) + \langle f_2^0, v^{(3)} \rangle \xi \left( f_2^0, v^{(3)} \right) + \langle f_2^0, v^{(2)} \rangle \xi \left( f_2^0, v^{(2)} \right) + \langle f_2^0, v^{(1)} \rangle \xi \left( f_2^0, v^{(1)} \right) . \]

We denote the corresponding braided double \( B(V^*) \times C \times B(V) \) by \( \mathcal{H}_+(V^*, C, V) \). Similarly, if \( \langle \cdot, \cdot \rangle_+ = 0 \) on \( V^* \otimes V \) we have

\[
\begin{align*}
\langle v, f \rangle &= f_0^0 \langle f_2^0, v^{(-2)}(2) \rangle \xi \left( f_2^0, v^{(-2)}(2) \right) - \langle f_2^0, v^{(3)} \rangle \xi \left( f_2^0, v^{(3)} \right) - \langle f_2^0, v^{(2)} \rangle \xi \left( f_2^0, v^{(2)} \right) - \langle f_2^0, v^{(1)} \rangle \xi \left( f_2^0, v^{(1)} \right) - \langle f_2^0, v^{(0)} \rangle \xi \left( f_2^0, v^{(0)} \right) .
\end{align*}
\]

The corresponding braided double is denoted \( \mathcal{H}_-(V^*, H, V) \). Clearly, \( \mathcal{H}_\pm(V^*, C, V) \) naturally identify with subspaces of \( \mathcal{U}_\xi(V^*, C, V) \).

Consider also some special cases of (A.29). If \( f \in V^* \) we have

\[
\begin{align*}
\langle v, f \rangle &= f_0^0 \langle f_2^0, v^{(-2)}(1) \rangle \xi \left( f_2^0, v^{(-2)}(1) \right) - \langle f_2^0, v^{(3)} \rangle \xi \left( f_2^0, v^{(3)} \right) - \langle f_2^0, v^{(2)} \rangle \xi \left( f_2^0, v^{(2)} \right) - \langle f_2^0, v^{(1)} \rangle \xi \left( f_2^0, v^{(1)} \right) - \langle f_2^0, v^{(0)} \rangle \xi \left( f_2^0, v^{(0)} \right) .
\end{align*}
\]

Similarly, if \( v \in V \) we have

\[
\begin{align*}
\langle v, f \rangle &= f_0^0 \langle f_2^0, v^{(-2)}(2) \rangle \xi \left( f_2^0, v^{(-2)}(2) \right) - \langle f_2^0, v^{(3)} \rangle \xi \left( f_2^0, v^{(3)} \right) - \langle f_2^0, v^{(2)} \rangle \xi \left( f_2^0, v^{(2)} \right) - \langle f_2^0, v^{(1)} \rangle \xi \left( f_2^0, v^{(1)} \right) - \langle f_2^0, v^{(0)} \rangle \xi \left( f_2^0, v^{(0)} \right) .
\end{align*}
\]

Let \( \xi \) be a field involution of \( k \). Suppose that it extends to \( V, V^*, C \) and \( H \) and that \( \xi \) satisfies

\[
\xi((\overline{c}, h)) = \xi(c, S_H^{-1}(h))
\]

and \( c \mapsto f = S_C^{-1}(\overline{c}) \mapsto \overline{f}, h \mapsto v = S_H^{-1}(\overline{h}) \mapsto \overline{v} \). Then \( \xi \) extends to an anti-linear algebra anti-involution and coalgebra involution of \( \mathcal{D}_\xi(C, H) \). Moreover, we have

\[
\boxed{h \mapsto f = \xi(f, S_H^{-1}(h)) \mapsto \xi(f, H) \mapsto f = S_H^{-1} h \mapsto \overline{f}}.
\]

Since \( \mathcal{H}_\text{cop} \) identifies with a sub-bialgebra of \( \mathcal{D}_\xi(C, H) \), it follows that for all \( x \in \mathcal{D}_\xi(C, H) \) we have \( x \mapsto f = S^{-1}(\overline{x}) \mapsto \overline{f} \). Assuming that \( V^* \) is self-transposed, it follows from Lemma A.22 that \( \xi \) extends to an anti-linear algebra anti-involution of \( B(V^*) \times \mathcal{D}_\xi(C, H) \). Similarly, \( \overline{v} \mapsto \overline{x} = \overline{\overline{\overline{v}}} \mapsto S^{-1}(\overline{\overline{\overline{v}}}) \) for all \( x \in \mathcal{D}_\xi(C, H) \) and \( v \in V \), whence \( \xi \) extends to an anti-linear algebra anti-involution of \( \mathcal{D}_\xi(C, H) \times B(V) \).

**Proposition A.29.** Suppose that \( \langle f, v \rangle_\pm = -\langle \overline{f}, \overline{v} \rangle_\pm, f \in V^*, v \in V \). Then \( \xi \) extends to an anti-linear algebra anti-involution of \( \mathcal{U}_\xi(V^*, C, V, H) \).
Proof. Define \( \tilde{\zeta} \) on \( \mathcal{U} = \mathcal{U}_c(V^*, C, V, H) \) by \( \overline{f \cdot x \cdot v} = \overline{\tau \cdot \tau \cdot \overline{f}} \), \( x \in \mathcal{D}_c(C, H) \), \( v \in \mathcal{B}(V) \), \( f \in \mathcal{B}(V^*) \). Since the restrictions of \( \tilde{\zeta} \) to \( \mathcal{D}_c(C, H) \times \mathcal{B}(V) \) and \( \mathcal{B}(V^*) \times \mathcal{D}_c(C, H) \) are well-defined anti-linear algebra anti-involutions, it remains to prove that \( [\overline{v}, \overline{f}] = [\overline{\tau}, \overline{\tau}] \) for all \( v \in V, \ f \in V^* \). Indeed,

\[
[f, v] = -\overline{f^{(1)}}(\overline{f^{(2)}}, \overline{\tau}^+) + \overline{v^{(1)}}(\overline{f}, \overline{v^{(2)}}) = \overline{f^{(1)}}(\overline{f^{(2)}}, v) - v^{(-1)}(\overline{f}, v^{(2)}) = [\overline{v}, \overline{f}].
\]

\( \square \)

A.8. Diagonal bradings. We now consider an important special case of the constructions discussed above. Let \( \Gamma \) be an abelian monoid and fix its bicharacter \( \chi : \Gamma \times \Gamma \to k^* \). Let \( V = \bigoplus_{\alpha \in \Gamma} V_\alpha \) be a \( \Gamma \)-graded vector space over \( k \). Define a braiding \( \Psi : V \otimes V \to V \otimes V \) by \( \Psi(v \otimes v') = \chi(\alpha, \alpha') v' \otimes v \), where \( v \in V_\alpha, v' \in V_{\alpha'} \). Furthermore, let \( V^* = \bigoplus_{\alpha \in \Gamma} V_\alpha^* \) be another \( \Gamma \)-graded vector space over \( k \) and let \( \langle \cdot, \cdot \rangle : V^* \otimes V \to k \) be any pairing satisfying \( \langle V_\alpha^*, V_{\alpha'} \rangle = 0 \) if \( \alpha \neq \alpha' \in \Gamma \). Then \( \langle \cdot, \cdot \rangle \) is non-degenerate provided that its restrictions to \( V_\alpha^* \otimes V_\alpha \) are non-degenerate for all \( \alpha \in \Gamma \). If \( \Psi^* \) is the adjoint of \( \Psi \) with respect to the form \( \langle \cdot, \cdot \rangle \) in the notation of §A.3 then it is easy to see that \( \Psi^*(f \otimes f') = \chi(\alpha', \alpha) f' \otimes f \), \( f \in V_\alpha^*, f' \in V_{\alpha'}^* \). Henceforth we will assume that \( \langle \cdot, \cdot \rangle \) is non-degenerate and denote \( \Gamma_0 = \{ \alpha \in \Gamma : V_\alpha \neq 0 \} = \{ \alpha \in \Gamma : V_\alpha^* \neq 0 \} \). We will always assume that \( \Gamma \) is generated by \( \Gamma_0 \).

The algebras \( T(V), B(V, \Psi), T(V^*) \) and \( B(V^*, \Psi^*) \) are naturally \( \Gamma \)-graded. By abuse of notation we write \( \chi(x, y) = \chi(\deg x, \deg y) \) where \( x, y \) are homogeneous elements of \( T(V), B(V, \Psi) \) or \( T(V^*), B(V^*, \Psi^*) \) and \( \deg x \) denotes the degree of \( x \) with respect to \( \Gamma \). Note that if \( u \in B(V, \Psi) \) is homogeneous and \( \Delta(u) = \mathcal{U}(u_1) \otimes \mathcal{U}(u_2) \) in Sweedler’s notation then \( \deg u = \deg \mathcal{U}(u_1) + \deg \mathcal{U}(u_2) \).

Henceforth, if \( u, v \in T(V) \) are homogeneous then \( \Psi(u \otimes v) = \chi(u, v) v \otimes u \) hence \( \Delta(uv) = \chi(u_2) \mathcal{U}(u_1) \mathcal{U}(u_2) \).

Lemma A.30. For \( f \in B(V^*, \Psi^*), v \in B(V, \Psi) \) homogeneous, \( \langle f, v \rangle = 0 \) unless \( \deg f = \deg v \).

Proof. This statement clearly holds for \( f \in V^*, v \in V \). Let \( f \in B^{r-1}(V, \Psi^*) \) and \( v \in B^r(V, \Psi) \) be homogeneous. Then for all \( \alpha \in \Gamma_0, F_\alpha \in V_\alpha^*, \langle F_\alpha f, v \rangle = \langle F_\alpha, \mathcal{U}(u_1) \rangle \langle f, \mathcal{U}(u_2) \rangle \) which is zero unless \( \deg \mathcal{U}(u_1) = \alpha \) and \( \deg \mathcal{U}(u_2) = \deg f \), whence \( \deg v = \alpha + \deg f \). Since \( B^r(V^*, \Psi^*) \subset \sum_{\alpha \in \Gamma_0} V_\alpha^* B^{r-1}(V^*, \Psi^*) \), the assertion follows. \( \square \)

Lemma A.31. For all \( f, g \in B(V^*, \Psi^*), u, v \in B(V, \Psi) \) homogeneous

\[
\partial_f(uv) = \chi(f_1, v) \chi(f_1, f_2)^{-1} \partial_{f_1}(u) \partial_{f_2}(v)
\]

\[
\partial_g^{op}(uv) = \chi(u, f_2) \chi(f_1, f_2)^{-1} \partial_{f_1}^{op}(u) \partial_{f_2}^{op}(v),
\]

\[
\partial_a(fg) = \chi(g, u_1) \chi(u_2, u_1)^{-1} \partial_{u_1}(f) \partial_{u_2}(g)
\]

\[
\partial_g^{op}(fg) = \chi(u_2, f) \chi(u_2, u_1)^{-1} \partial_{u_1}(f) \partial_{u_2}^{op}(g).
\]

In particular, for all \( E_\alpha \in V_\alpha, F_\alpha \in V_\alpha^* \)

\[
\partial_{F_\alpha}(uv) = \chi(\alpha, \deg v) \partial_{F_\alpha}(u)v + u \partial_{F_\alpha}(v),
\]

\[
\partial_{op}^{op}(uv) = \partial_{op}^{op}(u)v + \chi(\deg u, \alpha) u \partial_{op}^{op}(v) \quad (A.33)
\]

and

\[
\partial_{E_\alpha}(fg) = \chi(\deg g, \alpha) \partial_{E_\alpha}(f)g + f \partial_{E_\alpha}(g),
\]

\[
\partial_{op}^{op}(fg) = \partial_{op}^{op}(f)g + \chi(\alpha, \deg f) f \partial_{op}^{op}(g). \quad (A.34)
\]

Proof. We prove only the first identity; others are proved similarly. We have

\[
\partial_f(uv) = \langle f, \mathcal{U}(u_2) \mathcal{U}(u_1) \rangle \chi(u_2, u_1) = \chi(u_1, u_2) \langle f, \mathcal{U}(u_2) \mathcal{U}(u_1) \rangle \mathcal{U}(u_1) \mathcal{U}(u_1)
\]

\[
= \chi(u_2, u_1) \langle f_1, u_2 \rangle \mathcal{U}(u_1) \mathcal{U}(u_1) = \chi(f_1, v) \chi(f_1, f_2)^{-1} \partial_{f_1}(u) \partial_{f_2}(v),
\]

where we used \( \chi(x, \mathcal{U}(1)) \chi(x, \mathcal{U}(2)) = \chi(x, v) \) for all \( x, v \in B(V, \Psi) \) and Lemma A.30. \( \square \)
An obvious induction together with (A.7) implies then that
\[ \partial E_\alpha(F^r_\alpha) = \partial^op E_\alpha(F^r_\alpha) = \langle F_\alpha, E_\alpha \rangle [r]_{\chi^{-1}_\alpha} F^r_\alpha, \quad \langle F^r_\alpha, E_\alpha \rangle = (\langle F_\alpha, E_\alpha \rangle)^r [r]_{\chi^{-1}_\alpha}, \quad r \in \mathbb{Z}_{\geq 0}, \quad (A.35) \]
where we abbreviate \( \chi_\alpha := \chi(\alpha, \alpha) \). Note also the following identity (cf. [19, Lemma 1.4.2])
\[ \Delta(F^r_\alpha) = \sum_{r'+r''=r} [r']_{\chi_\alpha} F^r_\alpha \otimes F^{r''}_\alpha. \quad (A.36) \]

Clearly, \( \Psi \) is self-transposed provided that \( \chi \) is symmetric, that is \( \chi(\gamma, \gamma') = \chi(\gamma', \gamma) \) for all \( \gamma, \gamma' \in \Gamma \). In that case, if the \( V_\alpha \) are finite dimensional for all \( \alpha \in \Gamma_0, \mathcal{B}(V^*, \Psi^*) \) is isomorphic to \( \mathcal{B}(V, \Psi) \) as a braided bialgebra.

If \( \chi \) is symmetric, let \( v = v_1 \cdots v_r \in \mathcal{B}(V, \Psi) \) where \( v_i \in V_{\alpha_i} \) and so \( \alpha_i \in \Gamma_0 \). The definition of the braided antipode (cf. §A.1) immediately implies that
\[ S(v) = S_\Psi(v) = (-1)^r \left( \prod_{1 \leq i < j \leq r} \chi(\alpha_i, \alpha_j) \right) v^*. \]
If \( \alpha_1 + \cdots + \alpha_r = \alpha'_1 + \cdots + \alpha'_s \) with \( \alpha_i, \alpha'_j \in \Gamma_0, 1 \leq i \leq r, 1 \leq j \leq s \) implies that \( r = s \mod 2 \) and \( \prod_{1 \leq i < j \leq r} \chi(\alpha_i, \alpha_j) = \prod_{1 \leq i < j \leq s} \chi(\alpha'_i, \alpha'_j) \) (which is manifestly the case if \( \Gamma \) is freely generated by \( \Gamma_0 \)) we can define a unique character \( \text{sgn} : \Gamma \rightarrow \{ \pm 1 \} \) with \( \text{sgn}(\alpha) = -1, \alpha \in \Gamma_0 \cup \{ 0 \} \) and a function \( \gamma : \Gamma \rightarrow \mathbb{k}^\times \) satisfying \( \gamma(\alpha) = 1, \alpha \in \Gamma_0 \cup \{ 0 \} \) and
\[ \chi(\alpha, \alpha') = \chi(\alpha', \alpha)^{-1} = \frac{\gamma(\alpha + \alpha')}{\gamma(\alpha) \gamma(\alpha')}, \quad \alpha, \alpha' \in \Gamma. \]
Then for any \( v \in \mathcal{B}(V) \) homogeneous
\[ S(v) = \text{sgn}(v) \gamma(v)v^*, \quad (A.37) \]
where we abbreviate \( \text{sgn}(v) := \text{sgn}(\text{deg} v) \) and \( \gamma(v) := \gamma(\text{deg} v) \). We will say that \( \Gamma \) affords a sign character if there exists a character \( \text{sgn} : \Gamma \rightarrow \{ \pm 1 \} \) satisfying \( \text{sgn}(\alpha) = -1, \alpha \in \Gamma_0 \).

Suppose that \( \gamma : V \rightarrow V \) preserves the \( \Gamma \)-grading. Then the braiding \( \Psi \) is unitary if and only if \( \chi(\alpha, \alpha') = \chi(\alpha', \alpha)^{-1} \). The following is an immediate consequence of (A.37) and Proposition A.7(c,d).

**Proposition A.32.** Suppose that \( \Gamma \) affords a sign character, \( \chi = \chi_\gamma \) with \( \gamma(\alpha) = 1, \alpha \in \Gamma_0 \cup \{ 0 \} \) and \( \chi(\alpha, \alpha') = \chi(\alpha', \alpha)^{-1} \) for all \( \alpha, \alpha' \in \Gamma_0 \). Assume that the pairing \( \langle \cdot, \cdot \rangle : V^* \otimes V \rightarrow \mathbb{k} \) satisfies
\[ \langle F_\alpha, E_\beta \rangle = (F_\alpha, E_\beta), \quad \alpha \in \Gamma_0, E_\alpha \in V_\alpha, F_\alpha \in V^*_\alpha. \]
Then for all \( f \in T(V^*), u \in T(V) \) or \( f \in \mathcal{B}(V^*, \Psi^*), u \in \mathcal{B}(V, \Psi) \) homogeneous we have
\[ \overline{\langle f, u^* \rangle} = \text{sgn}(u) \gamma(u)^{-1} \langle f, u \rangle. \]

Suppose that \( \gamma(\alpha) \) is a square in \( \mathbb{k} \) for all \( \alpha \in \Gamma \) and fix \( \gamma^{\frac{1}{2}} : \Gamma \rightarrow \mathbb{k}^\times \). Set \( \chi^{\frac{1}{2}} = \chi^{\frac{1}{2}}_\gamma \). The operator \( L_n : V^\otimes n \rightarrow V^\otimes n \) defined on \( u \in V^\otimes n \) homogeneous by \( L_n(u) = \gamma(u)^{\frac{n}{2}} u \) clearly satisfies \( L_n^2(u) = (-1)^n \psi_\gamma(u^*) \), commutes with \( \ast \) and is unitary with respect to \( \gamma \). The following is straightforward corollary of Lemma A.8.

**Corollary A.33.** In the assumptions of Proposition A.32, the form \( \langle \cdot, \cdot \rangle : \mathcal{B}(V^*, \Psi^*) \otimes \mathcal{B}(V, \Psi) \rightarrow \mathbb{k} \) is defined for \( u \in \mathcal{B}(V, \Psi) \) homogeneous and \( f \in \mathcal{B}(V^*, \Psi^*) \) by \( (f, u) = (\gamma^{\frac{1}{2}})(u)^{-1} \langle f, u \rangle \) and satisfies
\[ \overline{\langle f, u^* \rangle} = \text{sgn}(u) \langle f, u \rangle \]
and for all \( f, f' \in \mathcal{B}(V^*, \Psi^*), u, u' \in \mathcal{B}(V, \Psi) \) homogeneous
\[ \langle ff', u \rangle = \langle \gamma^{\frac{1}{2}}(f, f') \rangle^{-1} \langle f', u(1) \rangle \langle f, u(2) \rangle, \quad \langle f, uu' \rangle = \langle \gamma^{\frac{1}{2}}(u, u') \rangle^{-1} \langle f(1), u \rangle \langle f(2), u' \rangle. \]
A.9. Drinfeld double in the diagonal case. Let \( H = k[\Gamma \oplus \Gamma] \cong k[\Gamma] \otimes k[\Gamma] \) be the monoidal bialgebra of \( \Gamma \oplus \Gamma \) with a basis \( K_{\alpha,\alpha'} \), \( \alpha,\alpha' \in \Gamma \). Denote by \( H^+ \) (respectively, \( H^- \)) the subalgebra of \( H \) generated by the \( K_{0,\alpha} \) (respectively, \( K_{\alpha,0} \)), \( \alpha \in \Gamma \); clearly, \( H^\pm \cong k[\Gamma] \). Let \( \hat{H} \) (respectively, \( \hat{H}^\pm \)) be localizations of the corresponding algebras at \( K_{\alpha,\alpha'} \) (respectively, \( K_{0,\alpha}, K_{\alpha,0} \)), \( \alpha,\alpha' \in \Gamma \). Then \( \hat{H} \) identifies with \( D\xi_\chi(\hat{H}^-, \hat{H}^+) \) where the Hopf pairing \( \xi_\chi : \hat{H}^- \otimes \hat{H}^+ \to k \) is defined by \( \xi_\chi(K_{0,\alpha}, K_{\alpha,0'}) = \chi(\alpha',\alpha) \).

Let \( V, V^* \) be \( \Gamma \)-graded \( k \)-vector spaces as in §A.8. We regard \( V \) (respectively, \( V^* \)) as left Yetter-Drinfeld \( \hat{H}^- \) (respectively, \( \hat{H}^+ \))-module via
\[
K_{0,\alpha} \triangleright v = \chi(\alpha,\beta)v, \quad \delta(v) = K_{0,\beta} \otimes v
\]
\[
K_{\alpha,0} \triangleright f = \chi(\beta,\alpha)f, \quad \delta(f) = K_{\beta,0} \otimes f, \quad \alpha,\beta \in \Gamma, \quad v \in V_\beta, f \in V^*_\beta.
\]
Then by (A.25) we have
\[
v \in K_{0,\alpha} = \chi(\alpha,\beta)v, \quad f \in K_{0,\alpha} = \chi(\alpha,\beta)f,
\]
and we can regard \( V \) (respectively, \( V^* \)) as a right (respectively, left) Yetter-Drinfeld module over \( \hat{H} \) as in (A.27), with \( v \in K_{\alpha,0} = \chi(\alpha,\beta)^{-1}\chi(\alpha,\alpha)v \) and \( K_{\alpha,0} \triangleright f = \chi(\beta,\alpha)\chi(\alpha',\beta)^{-1}f \).

Let \( \langle \cdot, \cdot \rangle_\pm \) be pairings \( V^* \otimes V \to k \) satisfying the assumptions of §A.8. Clearly, (A.26) holds. Denote by \( \partial_f, \partial^\op_f : B(V) \to B(V) \) and \( \partial_f, \partial^\op_f : B(V^*) \to B(V^*) \), \( v \in B(V), f \in B(V^*) \), the linear operators corresponding to the respective pairings \( \langle \cdot, \cdot \rangle_\pm \), as defined in §A.3. Consider now the algebra \( \hat{U}_\chi(V^*, V) \) which is the subalgebra of \( \hat{U}_\chi(V^*, \hat{H}^-, \hat{H}^+) \) generated by \( V^*, V \) and \( H^\pm \). In particular, we have the following cross-relations
\[
K_{\alpha,\alpha'}E_\beta = \chi(\beta,\alpha)^{-1}\chi(\alpha',\beta)E_\beta K_{\alpha,\alpha'}, \quad K_{\alpha,\alpha'}F_\beta = \chi(\beta,\alpha)\chi(\alpha',\beta)^{-1}F_\beta K_{\alpha,\alpha'},
\]
\[
[E_\alpha, F_\beta] = K_{\beta,0}(F_\beta, E_\alpha) - K_{0,\alpha}(F_\beta, E_\alpha), \quad E_\alpha \in V_\alpha, F_\beta \in V^*_\beta, \alpha,\alpha',\beta \in \Gamma. \tag{A.38}
\]
If \( \langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_- \) then, by Theorem A.28, \( \hat{U}_\chi(V^*, V) \) is a Hopf algebra with the comultiplication defined by
\[
\Delta(F_\alpha) = F_\alpha \otimes 1 + K_{0,\alpha} \otimes F_\alpha, \quad \Delta(E_\alpha) = 1 \otimes E_\alpha + E_\alpha \otimes K_{0,\alpha}. \tag{A.39}
\]
and the antipode
\[
S(F_\alpha) = -K_{0,\alpha}^{-1}F_\alpha, \quad S(E_\alpha) = -E_\alpha K_{0,\alpha}^{-1}. \tag{A.40}
\]
for all \( \alpha \in \Gamma, E_\alpha \in V_\alpha \) and \( F_\alpha \in V^*_\alpha \).

Lemma A.34. For all \( \alpha \in \Gamma, E_\alpha \in V_\alpha, F_\alpha \in V^*_\alpha, v \in B(V), f \in B(V^*) \) we have in \( \hat{U}_\chi(V^*, V) \)
\[
[v, F_\alpha] = K_{0,\alpha}\partial_{F_\alpha}^\op(v) - \partial_{F_\alpha}(v)K_{0,\alpha}, \quad [E_\alpha, f] = \partial_{E_\alpha}(f)K_{0,\alpha} - K_{0,\alpha}\partial_{E_\alpha}^\op(f). \tag{A.41}
\]
Proof. This is immediate from §A.3, (A.30), (A.31) and the fact that if \( \langle f, E_\alpha \rangle_\pm \neq 0 \) (respectively, \( \langle F_\alpha, v \rangle_\pm \neq 0 \)) then \( \delta(f) = K_{0,\alpha} \otimes f \) (respectively, \( \delta(v) = K_{0,\alpha} \otimes v \)).

The following result is an easy consequence of Proposition A.18, Lemma A.19, (A.39) and (A.40).

Proposition A.35. Let \( \langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_- \). Then \( B(V) \times H^+ \) is a left (respectively right) \( \hat{U}_\chi(V^*, V) \)-module algebra via \( F_\alpha \triangleright v = \partial_{F_\alpha}(v)K_{0,\alpha}, v \triangleright E_\alpha = E_\alpha v - K_{0,\alpha}vK_{0,\alpha}^{-1}E_\alpha \) (respectively, \( v \triangleright E_\alpha = \partial_{F_\alpha}(v)K_{0,\alpha}, v \triangleright F_\alpha = F_\alpha v - K_{0,\alpha}F_\alpha K_{0,\alpha}^{-1}F_\alpha \)), \( v \in B(V), \alpha \in \Gamma, E_\alpha \in V_\alpha, F_\alpha \in V^*_\alpha \). Similarly, \( B(V^*) \times H^- \) is a left (respectively right) \( \hat{U}_\chi(V^*, V) \)-module algebra via \( E_\alpha \triangleright f = \partial_{E_\alpha}(f)K_{0,\alpha}, f \triangleright E_\alpha = E_\alpha f - K_{0,\alpha}E_\alpha K_{0,\alpha}^{-1}E_\alpha \), \( f \in B(V^*), \alpha \in \Gamma, E_\alpha \in V_\alpha, F_\alpha \in V^*_\alpha \).

Given \( f \in B(V^*), v \in B(V) \) homogeneous, we obtain by (A.29)
\[
v \cdot f = \oint_{\langle f \rangle} K_{\deg L(3),0} \cdot \oint_{\langle v \rangle} \chi(f_{\langle 2 \rangle}, v_{\langle 3 \rangle})^{-1} \chi(f_{\langle 3 \rangle}, v_{\langle 3 \rangle})^{-1} \langle f_{\langle 1 \rangle}, \bar{S}^{-1}(v_{\langle 3 \rangle}) \rangle - \langle f_{\langle 3 \rangle}, \bar{S}^{-1}(v_{\langle 3 \rangle}) \rangle + \langle f_{\langle 1 \rangle}, v_{\langle 3 \rangle} \rangle \tag{A.42}
\]
Lemma A.37. If \( \langle f_{(2)}, f_{(1)} \rangle \in \mathcal{B}(V^*) \) this can be written in the following form
\[
v \cdot f = (\chi(f_{(2)}, f_{(1)}) \gamma(f_{(3)}, f_{(1)}))^{-1} \langle f_{(1)}, S^{-1}(\varpi(3)) \rangle - \langle f_{(3)}, \varpi(1) \rangle + x
\]
where, as before, we abbreviate \( \chi(x, y) := \chi(dg x, dg y) \). The following Proposition generalizes [19, Proposition 3.1.7] and is an immediate consequence of (A.43) and (A.37).

**Proposition A.36.** Suppose that \( \Gamma \) affords the sign character, \( \chi = \chi_\gamma \) with \( \gamma : \Gamma \to \mathbb{k}^* \) satisfying \( \gamma(\alpha) = 1, \alpha \in \Gamma \cup \{0\} \). Then for all \( f \in \mathcal{B}(V^*), v \in \mathcal{B}(V) \) homogeneous we have in \( \mathcal{U}_\chi(V^*, V) \)
\[
v \cdot f = \text{sgn}(f_{(1)}) (\chi(f_{(2)}, f_{(1)}) \gamma(f_{(3)}, f_{(1)}))^{-1} \langle f_{(1)}, S^{-1}(\varpi(3)) \rangle - \langle f_{(3)}, \varpi(1) \rangle + x
\]

Suppose that \( \langle \cdot, \cdot \rangle_- = 0 \) (respectively, \( \langle \cdot, \cdot \rangle_+ = 0 \)) on \( V^* \otimes V \). Then we obtain for \( v \in \mathcal{B}(V), f \in \mathcal{B}(V^*) \) homogeneous
\[
v \circ_+ f = f_{(1)} K_{\text{deg} f_{(2)}, 0}(f_{(3)}, \varpi(1)) + \varpi(2), \tag{A.44}
v \circ_- f = \text{sgn}(f_{(1)}) (\chi(f_{(2)}, f_{(1)}) \gamma(f_{(3)}, f_{(1)}))^{-1} \langle f_{(1)}, S^{-1}(\varpi(3)) \rangle - \langle f_{(3)}, \varpi(1) \rangle + x
\]
We conclude this section with the following Lemma.

**Lemma A.37.** Retain the assumptions of Proposition A.36.

(a) If \( \chi(\alpha, \beta) = \chi(\alpha, \beta)^{-1} \) for all \( \alpha, \beta \in \Gamma \) and \( \langle f, v \rangle_+ = -\langle f, v \rangle \), \( f \in V^*, v \in V \), then \( \mathcal{U}_\chi(V^*, V) \) admits a unique anti-linear anti-involution extending \( \bar{\gamma} : V \to V, \bar{\gamma} : V^* \to V^* \) and satisfying \( K_{\alpha, \alpha'} = K_{\alpha', \alpha} \). \( \alpha, \alpha' \in \Gamma \).

(b) Suppose that \( \langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_0 \). Then \( \ast \) extends to an anti-involution of \( \mathcal{U}_\chi(V^*, V) \) whose restrictions to \( V, V^* \) are the identity maps while \( K_{\alpha, \alpha'}^\ast = K_{\alpha', \alpha} \). \( \alpha, \alpha' \in \Gamma \).

(c) Any pair of graded isomorphisms \( \xi, \xi' : V^* \to V \) and \( \xi_+, \xi' : V^* \to V^* \) satisfying \( \xi_+ \circ \xi = \text{id}_{V^*} \) gives rise to an anti-involution \( \xi \) of \( \mathcal{U}_\chi(V^*, V) \) satisfying \( \xi(K_{\alpha, \alpha'}) = K_{\alpha', \alpha} \). \( \alpha, \alpha' \in \Gamma \). Moreover, if the assumptions of part (a) hold and \( \xi_\pm \) commute with \( \bar{\gamma} \) then so does \( \xi \).

**Proof.** Part (a) is an immediate consequence of Proposition A.29. Part (b) follows from (A.38). To prove (c), note that \( \xi_\pm \) define isomorphisms of braided bialgebras \( \xi_\pm : \mathcal{B}(V) \to \mathcal{B}(V^*) \) (respectively, \( \xi_- : \mathcal{B}(V^*) \to \mathcal{B}(V) \)) such that \( \xi_+ \circ \xi_- = \text{id}_{\mathcal{B}(V^*)} \) and \( \xi_- \circ \xi_+ = \text{id}_{\mathcal{B}(V)} \). Define \( \xi : \mathcal{U}_\chi(V^*, V) \to \mathcal{U}_\chi(V^*, V) \) by \( \xi(f) = \xi_-(f^\ast) \), \( f \in \mathcal{B}(V^*), \xi(v) = \xi_+(v^\ast), v \in \mathcal{B}(V) \) and \( \xi(h) = h, h \in H_\pm \). It remains to observe that (A.38) are preserved by \( \xi \).

For an anti-involution \( \xi \) commuting with \( \bar{\gamma} \), define a pairing \( \langle \cdot, \cdot \rangle : \mathcal{B}(V^*) \otimes \mathcal{B}(V^*) \to \mathbb{k} \) by
\[
\langle f, g \rangle = \langle f, \xi(g^\ast) \rangle, \quad f, g \in \mathcal{B}(V^*)
\]
in the above notation and that of Corollary A.33. In particular, we have for \( f \in \mathcal{B}(V^*) \) homogeneous
\[
\langle f, \overline{\varpi} \rangle = \text{sgn}(f) \langle f, g \rangle. \tag{A.46}
\]
Since the braiding \( \Psi \) and \( \Psi^* \) are self-transposed in the sense of \( \S 1.2.3 \), \( \langle \cdot, \cdot \rangle \) is symmetric (note that this form is similar to the one defined in [19, \S 1.2.3]).
### List of notation

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