GENERALIZED ELECTRICAL LIE ALGEBRAS

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Abstract. We generalize the electrical Lie algebras originally introduced by Lam and Pylyavskyy in several ways. To each Kac-Moody Lie algebra $\mathfrak{g}$ we associate two types (vertex type and edge type) of the generalized electrical algebras. The electrical Lie algebras of vertex type are always subalgebras of $\mathfrak{g}$ and are flat deformations of the nilpotent Lie subalgebra of $\mathfrak{g}$. In many cases including sl_n, so_n, and sp_{2n} we find new (edge) models for our generalized electrical Lie algebras of vertex type. Finding an edge model in general is an interesting an open problem.

Contents

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1. Introduction and main results

The electrical Lie algebras were introduced by T. Lam and P. Pylyavskyy in [8] and further studied by Yi Su in [10]. An infinitely generated electrical Lie algebra of type A also appeared in the context of categorification in representation theory [2]. The aim of this paper is to generalize the notion of electrical Lie

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algebra into a multiparametric family in any given semisimple or Kac-Moody Lie algebra $\mathfrak{g}$, in such a way that they are flat deformations of the nilpotent part $\mathfrak{n}$ of $\mathfrak{g}$. We also construct various embeddings (vertex and edge models) of the electrical Lie algebras into the corresponding Kac-Moody ones.

We start with the definition of the electrical Lie algebra given below which depends on a set of parameters a_i. If all a_i are equal to the imaginary unit these generators in the case of sl_n give an important representation of the Temperley-Lieb algebra for the zero value of the loop fugacity parameter ([1]) and they also give a solution to the Zamolodchikov tetrahedral equation [9], [6].

Definition 1.1. Let $A = (a_{ij})$ be a (generalized, not necessarily symmetrizable) $I \times I$ Cartan matrix and let $\mathfrak{g} = \mathfrak{g}_A =< e_i, f_i, i \in I >$ be the corresponding Kac-Moody Lie algebra. For any family $\mathbf{a} = (a_i, i \in I) \in \mathbb{C}^I$ let $\mathfrak{g}^{(\mathbf{a})}$ be a Lie subalgebra of $\mathfrak{g}$ generated by

$$u_i := e_i + a_i[e_i, f_i] - a_i^2 f_i \ , i \in I. \tag{1.1}$$

We call $\mathfrak{g}^{(\mathbf{a})}$ a *generalized electrical Lie algebra of type* $\mathfrak{g}$.

In particular, $sl_2^{(\mathbf{a})}$ is generated by a single nilpotent $u = \begin{pmatrix} a & 1 \\ -a^2 & -a \end{pmatrix} \in sl_2$.

This definition makes sense if the ground field $\mathbb{C}$ is replaced with any algebra containing all a_i, e.g., with $\mathbb{C}[a_i, i \in I]$ and it is justified by the following.

Theorem 1.2. For any Kac-Moody Lie algebra $\mathfrak{g}$ and any $\mathbf{a} \in \mathbb{C}^I$ the generalized electrical Lie algebra $\mathfrak{g}^{(\mathbf{a})}$ of type $\mathfrak{g}$ satisfies

$$(ad\ u_i)^{1-a_{ij}}(u_j) = -2\delta_{a_{ij},-1} a_{ji} a_i a_j u_i \tag{1.2}$$

for all distinct $i, j \in I$.

It turns out that these relations are defining. This was conjectured for finite types in [8, Conjecture 5.2], which which was partially confirmed in [10] and [3, 4].

Theorem 1.3. In the notation of Theorem 1.2 the relations (1.2) provide a presentation of $\mathfrak{g}^{(\mathbf{a})}$. Moreover, $\mathfrak{g}^{(\mathbf{a})}$ is a flat deformation of the nilpotent part $\mathfrak{n} =< e_i, i \in I >$ of $\mathfrak{g}$ in the sense that $\mathfrak{g}^{(\mathbf{a})} \cong \mathfrak{n}$ for any $\mathbf{a} \in \mathbb{C}^I$ as a naturally filtered Lie algebra.

We prove Theorems 1.3 in Section 3.2 (along with their generalization to any coefficient algebra R containing all a_i) by utilizing a mini-theory of flat deformations of associative and Lie algebras which we present for the reader's convenience in **Appendix**.

Note that if we view all a_i as formal parameters (after replacing $\mathbb{C}$ with $\mathbb{C}[\mathbf{a}] = \mathbb{C}[a_i, i \in I]$), then $\mathbb{C}[\mathbf{a}] \otimes \mathfrak{g}$ is graded by $\mathbb{Z}_{\geq 0}^I$ via $\deg a_i = \deg e_i = -\deg f_i$, $i \in I$. In particular, u_i is homogeneous with $\deg u_i = \deg a_i$ and both $\mathfrak{g}^{(\mathbf{a})}$ and $U(\mathfrak{g}^{(\mathbf{a})})$ are $\mathbb{Z}_{\geq 0}^I$-graded.

Now we construct a real form $\mathfrak{g}^{(\mathbf{b})}$ for the electrical Lie algebra $sl_n^{(\mathbf{b})}$.

Theorem 1.4. Let $\mathfrak{g} = sl_n$. Then $g_{\mathbf{a}} u_i g_{\mathbf{a}}^{-1} = e_i + b_{i-1} f_{i-1}$ for $i = 1, \ldots, n-1$ in the notation of (1.1) (with the convention $f_0 = 0$, $a_0 = 0$), where we abbreviated $b_j := -a_j a_{j+1}$ and $g_{\mathbf{a}} := e^{a_{n-1} f_{n-1}} \cdots e^{a_2 f_2} e^{a_1 f_1} \in SL_n$.

We prove Theorem 1.4 in Section 3.3.

Remark 1.5. This realization with all $a_i = \sqrt{-1}$ (i.e., all $b_j = 1$) was first discovered by T. Lam and A. Postnikov as it is mentioned in [7].

The realization with all a_i imaginary is a "real form" of the electric Lie algebra of type sl_n. We produce more real forms for $\mathfrak{g}^{(\mathbf{a})}$ below and in Section 2.

More generally, for any $\mathbf{b} = (b_1, \ldots, b_{n-2}) \in \mathbb{C}^{n-2}$ we denote by $sl_n^{(\mathbf{b})}$ the Lie subalgebra of sl_n generated by $u_i := e_i + b_{i-1} f_{i-1}$, $i = 1, \ldots, n-1$ (with the convention $b_0 = 0$, $f_0 = 0$ and establish the following

Theorem 1.6. The Lie algebra $sl_n^{(\mathbf{b})}$ has a presentation

- $[u_i, u_j] = 0$ if $|i-j| > 1$
- $[u_i, [u_i, u_j]] = -2b_{\min(i,j)} u_i$ if $|i-j| = 1$

We prove Theorem 1.6 in Section 3.5 as a particular case of Theorem 2.3.

Clearly, if all $b_i \neq 0$, Theorem 1.6 follows from Theorem 1.4. Note however, that for $n = 5$, $b_1 \neq 0$, $b_2 = 0$, $b_3 \neq 0$, such a tuple (a_1, a_2, a_3, a_4) does not exist and the assertion of Theorem 1.6 does not follow from Theorem 1.4.

Theorem 1.4 suggests an "edge model" of generalized electrical Lie algebras as follows.

Given a Kac-Moody Lie algebra $\mathfrak{g}$ and a tuple $\mathbf{b} = \{b_{ij} = b_{ji} | a_{ij} = -1\}$, we denote by $\mathfrak{g}^{(\mathbf{b})}$ the Lie algebra generated by u_i, $i \in I$ subject to

$$(ad\ u_i)^{1-a_{ij}}(u_j) + 2\delta_{a_{ij},-1} b_{ij} u_i = 0 \tag{1.3}$$

for all distinct $i, j \in I$ where $b_{ij} = 0$ unless $a_{ij} = -1$. We refer to $\mathfrak{g}^{(\mathbf{b})}$ as the generalized electrical Lie algebra of the *edge type* $\mathfrak{g}$ associated with the Lie algebra $\mathfrak{g}$ (so it is logical to think of $\mathfrak{g}^{(\mathbf{a})}$ as the *vertex type*). Ultimately, on Theorems 1.4 and 1.6, we say that an *edge model* of an electric Lie algebra of type $\mathfrak{g}$ is any subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{g}^{(\mathbf{b})}$.

Clearly, if all $b_{ij} = a_{ji} a_i a_j$ whenever $a_{ij} = -1$ in the notation of (1.2), then by Theorem 1.3, $\mathfrak{g}^{(\mathbf{b})} \cong \mathfrak{g}^{(\mathbf{a})}$. Note however, that the generalized electrical Lie algebra $sl_n^{(\mathbf{b})}$ of the edge type sl_n is also embedded into sl_n for any $\mathbf{b} \in \mathbb{C}^n$ (here $b_i = b_{i,i+1} = b_{i+1,i}$ for $i = 1, \ldots, n-2$).

Based on this, we can pose a natural

Problem 1.7. Describe the set $\mathcal{V}_A$ of all tuples $\mathbf{b} = \{b_{ij} = b_{ji} | a_{ij} = -1\}$ such that $\mathfrak{g}^{(\mathbf{b})}$ admits an edge model.

Theorem 1.3 gives an edge model for all $\mathbf{b} = (b_{ij}) \in \mathcal{V}_A$ with $b_{ij} = a_{ji} a_i a_j$.

Using the edge model $sl_n^{(\mathbf{b})}$, we generalize a remarkable observation of Lam and Pylyavskyy from [8] that $sl_n^{(\mathbf{1})} \cong sp_{n-1}$, where $\mathbf{1} = (1, \ldots, 1)$, for all $n \geq 2$ as follows. Recall that the Lie algebra sp_n for n odd was introduced in [5].

Theorem 1.8. $sl_n^{(\mathbf{b})}$ preserves the form $\omega_{\mathbf{b}} := \sum_{k=1}^{n-1}\left(\prod_{i=k}^{n-2}(-b_i)\right) v_k^* \wedge v_{k+1}^*$ in $\mathbb{C}^n = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n$. If n is even then this form is non-degenerate as it follows from [11]. In this case $sl_n^{(\mathbf{b})}$ is the intersection of $sp_{\omega_{\mathbf{b}}} \subset sl_n$ with the annihilator of $v^1 = v_1 - b_1v_3 + b_1b_3v_5 - b_1b_3b_5v_7 + \cdots \in \mathbb{C}^n = V_{\omega_1}$ (where $\{v_1, \ldots, v_n\}$ is the standard basis of $\mathbb{C}^n$), therefore, $sl_n^{(\mathbf{b})}$ is isomorphic to sp_{n-1} when $b_1 \cdots b_{n-2} \neq 0$. If n is odd, then the form $\omega_{\mathbf{b}}$ has a one-dimensional kernel that is an invariant of the action of $sl_n^{(\mathbf{b})}$.

We prove Theorem 1.8 in Section 3.4. For $n = 6$ the form $\Omega_6^{(\mathbf{b})}$ is given in $v_1, \ldots, v_6$ by its Gram matrix

$$\Omega_6^{(\mathbf{b})} = \begin{pmatrix} 0 & b_1b_2b_3b_4 & 0 & 0 & 0 & 0 \\ -b_1b_2b_3b_4 & 0 & -b_2b_3b_4 & 0 & 0 & 0 \\ 0 & b_2b_3b_4 & 0 & b_3b_4 & 0 & 0 \\ 0 & 0 & -b_3b_4 & 0 & -b_4 & 0 \\ 0 & 0 & 0 & b_4 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

In view of Theorem 1.8, the electrical Lie algebra of type sl_n has a trivial center if n is odd and a one-dimensional center if n is even. It is curious that the non-injective homomorphism $sl_{2k}^{(\mathbf{b})} \cong sp_{2k-1} \to sl_{2k-1}$ is far from the natural embedding $sp_{2k-1} \subset sl_{2k-1}$.

We will use other principle unipotents for conjugation of the standard embedding into several new ones.

Example 1.9. Conjugating with $g'_{\mathbf{a}} = e^{a_2f_2}e^{a_1f_1}e^{a_3f_3}$ and $g''_{\mathbf{a}} = e^{a_1f_1}e^{a_3f_3}e^{a_2f_2}$ in SL_4 respectively, gives two new embeddings $sl_4^{(\mathbf{a})} \hookrightarrow sl_4$:

$$g'_{\mathbf{a}}u_i{g'}_{\mathbf{a}}^{-1} = e_i + \delta_{i,2}(b_1f_1 + b_3f_3 + b_1b_3[f_1,[f_3,f_2]]) \ ,$$
$$g''_{\mathbf{a}}u_i{g''}_{\mathbf{a}}^{-1} = e_i + (1-\delta_{i,2})(b_if_2 + b[f_i,f_2])$$

for $i = 1, 2, 3$, where we abbreviated $b_1 = -a_1a_2$, $b_3 = -a_2a_3$, $b = -a_1a_2a_3$.

We generalize the first embedding to all $\mathfrak{g}$ with a *conical* Dynkin diagram which includes all simply-laced $\mathfrak{g}$ (Theorem 2.1) and the second embedding to all $\mathfrak{g}$ with a star-like conical Dynkin diagram, including D_4 and $\hat{D}_4$ (Theorem 2.5). The following result gives such an edge model in types B and C and is of "conjugation type" as Theorem 1.6.

Theorem 1.10. (a) Let $\mathfrak{g} = so_{2n+1}$ with $I = \{1, \ldots, n\}$ and the short root is α_1. Then the assignments $u_i \mapsto e_i + b_{i-1,i}f_{i-1}$, $i = 1, \ldots, n$ define an injective homomorphism of Lie algebras $so_{2n+1}^{(\mathbf{b})} \hookrightarrow so_{2n+1}$ (with the convention $b_{01} = 0$)

(b) Let $\mathfrak{g} = sp_{2n}$ with $I = \{1, \ldots, n\}$ and let the long root be α_n. Then the assignments $u_i \mapsto e_i + b_{i-1,i}f_{i-1} - \delta_{i,n}\frac{1}{2}b_{n,n-1}^2[f_{n-1},[f_{n-1},f_n]]$ for $i = 1, \ldots, n$ define an injective homomorphism of the Lie algebras $sp_{2n}^{(\mathbf{b})} \hookrightarrow sp_{2n}$.

We prove Theorem 1.10 in Section 3.5 by using results of Section 2 where we also construct vertex models for electrical Lie algebras of several other Kac-Moody types, including E_6, E_7, E_8, F_4, G_2 and their affine versions as well (Theorems 2.1 and 2.3) along with some edge models (Theorem 2.9).

Our next result also gives an edge model for dihedral, type D, and affine A types, however, unlike Theorem 1.10, we do not expect any intertwiner from the vertex model.

Theorem 1.11. (a) Let $\mathfrak{g} = \mathfrak{g}_A$ be a Kac-Moody algebra of rank 2 with $I = \{1, 2\}$ such that $a_{21} \leq -2$. Then the assignments $u_1 \mapsto e_1$, $u_2 \mapsto e_2 + b_{12} f_1$, define an injective homomorphism of Lie algebras $\mathfrak{g}^{(\mathbf{b})} \hookrightarrow \mathfrak{g}$.

$b := a_{21}a_1a_2$

(b) Let $\mathfrak{g} = so_{2n}$, $n \geq 3$ with $I = \{1, \ldots, n\}$ and with the branch at $i = n-2$. Then the assignments $u_i \mapsto \begin{cases} e_i + b_{i-1,i} f_{i-1} & \text{if } 1 \leq i \leq n-2 \\ e_i + b_{n-2,i} f_{n-2} & \text{if } i \in \{n-1, n\} \end{cases}$ $i = 1, \ldots, n$, define an injective homomorphism of Lie algebras $so_{2n}^{(\mathbf{b})} \hookrightarrow so_{2n}$.

(c) Let $\mathfrak{g} = \widehat{sl}_n$, the untwisted affine Lie algebra of type $\hat{A}_{n-1}$ with $I = \{0, \ldots, n-1\}$. Then the assignments $u_i \mapsto e_i + b_{i-1,i} f_{i-1}$, where $i-1$ is calculated modulo n for $i \in I$, define an injective homomorphism of the Lie algebras $\widehat{sl}^{(\mathbf{b})} \hookrightarrow \widehat{sl}_n$.

We prove Theorem 1.11 in Section 3.8.

Remark 1.12. The affine electrical Lie algebra of type $\hat{A}_{n-1}$ from the theorem above was first discovered by T.Lam and A.Postnikov, see [7].

Theorem 1.13. For any $\mathbf{b} \in \mathbb{C}^{n-1}$ one has:

(a) $sp_{2n}^{(\mathbf{b})} \cong sl_n^{(b_1,\ldots,b_{n-2})} \ltimes J$, where the first factor is a copy of $sl_n^{(b_1,\ldots,b_{n-2})}$ in $sp_{2n}^{(\mathbf{b})}$ generated by $u_1, \ldots, u_{n-1}$ and J is the Lie ideal of $sp_{2n}^{(\mathbf{b})}$ generated by u_n and $sl_n^{(b_1,\ldots,b_{n-2})}$.

As $sl_n^{(b_1,\ldots,b_{n-2})}$ module J is isomorphic to S^2V where V is the restriction to $sl_n^{(b_1,\ldots,b_{n-2})} \subset sl_n$ of the standard sl_n-module $V_{\omega_1} = \mathbb{C}^n$.

(b) If all $b_i \neq 0$ then, under then there is an isomorphism of Lie algebras $J \cong sl_{n+1}^{(\mathbf{b}')}$ for some choice of the parameters $\mathbf{b}'$ (which we specify in the proof) and the Lie algebra $sp_{2n}^{(\mathbf{b})}$ is naturally isomorphic to $sl_n^{(b_1,\ldots,b_{n-2})} \ltimes sl_{n+1}^{(\mathbf{b}')}$.

We prove Theorem 1.13 in Section 3.6.

Remark 1.14. It is well-known that $\mathfrak{h} \ltimes \mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}$ for any Lie algebra $\mathfrak{g}$ and it subalgebra $\mathfrak{h}$ under the right adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ (namely, the diagonal copy of $\mathfrak{h}$ in $\mathfrak{h} \ltimes \mathfrak{g} \cong$ commutes with $(0, \mathfrak{g})$).

Moreover, applying this to $\mathfrak{g} = sp_n$, $\mathfrak{h} = sp_{n-1}$, Theorems 1.8 and 1.13(b) produce, after an appropriate localization, an isomorphism

$$sp_{2n}^{(\mathbf{b})} \cong sl_n^{(b_1,\ldots,b_{n-2})} \oplus sl_{n+1}^{(b_1,\ldots b_{n-1})}$$

recovering [10, Thm. 2.3.1][1].

We conclude with an illustration of Theorem 1.13 for $n = 3$.

Example 1.15. The Chevalley generators of sp_6 are given in terms of the folding of those for sl_6 under the flip $i \mapsto 6 - i$:

$$\tilde{e}_1 = e_1 + e_5,\ \tilde{e}_2 = e_2 + e_4,\ \tilde{e}_3 = e_3,\ \tilde{f}_1 = f_1 + f_5,\ \tilde{f}_2 = f_2 + f_4,\ \tilde{f}_3 = f_3.$$

The flip-invariant copy of $sl_6^{(\mathbf{b})}$ is generated by the elements (Theorem 2.3 with $k = 3$)

$$\tilde{u}_1 = e_1,\ \tilde{u}_2 = e_2 + b_1 f_1,\ \tilde{u}_3 = \tilde{e}_3 + b_2\tilde{f}_2 - b_2^2[\tilde{f}_2, [\tilde{f}_2, \tilde{f}_3]]/2$$
$$\tilde{u}_4 = e_4 + b_1 f_5,\ \tilde{u}_5 = e_5$$

the generators of $sp_6^{(\mathbf{b})}$ are

$$u_1 = \tilde{u}_1 + \tilde{u}_5,\ u_2 = \tilde{u}_2 + \tilde{u}_4,\ u_3 = \tilde{u}_3.$$

Introduce the elements of $sp_6^{(\mathbf{b})}$

$$w_3 = u_3,\ w_2 = [u_2, [u_2, u_3]],\ w_1 = [u_1, [u_1, w_2]].$$

These elements on the one hand generate the ideal J and on the other hand a copy of $sl_4^{(\mathbf{b}')}$ where $\mathbf{b}' = \{-32b_1^2b_2^2, -8b_2^2\}$. This shows a splitting of $sp_6^{(\mathbf{b})}$ into a semidirect product as claimed in the statement of the theorem.

Moreover, as Remark 1.14 claims there is copy of $sl_3^{(\mathbf{b}')\setminus -8b_2^2}$ that commutes with the above copy of $sl_4^{(\mathbf{b}')}$. It is generated by the elements

$$v_1 = -8b_1b_2u_1 + w_1,\ v_2 = 4b_2u_2 + w_2$$

After the appropriate localization

$$sp_6^{(\mathbf{b})} \cong sl_2^{(b_1)} \oplus sl_3^{(b_1,b_2)}$$

where the generators of the summands are

$$\{u_1 - \frac{w_1}{8b_1b_2},\ u_2 + \frac{w_2}{4b_2}\} \text{ and } \{w_3,\ -\frac{w_2}{8b_2},\ \frac{w_1}{4b_1b_2}\}$$

respectively.

2. Other vertex and edge models of electrical Lie algebras

In this section, we generalize Theorem 1.4, Example 1.9 and Theorems 1.10, 1.11 to other semisimple and Kac-Moody algebras.

To any (generalized) Cartan matrix $A = (a_{ij}, i, j \in I)$ we assign its graph $\Gamma(A)$ on the vertex set I such that (ij) is an edge iff $a_{ij} < 0$.

We say that a rooted tree is *conical* if the only possible branch is at the root (which we always denote by 0).

For example, a linear graph with the set of vertex $[-m, n]$, $m, n \geq 0$ is a conical tree. Also, all Dynkin diagrams of types ADE are conical rooted trees.

[1]The electric algebras of B_n type in [10] correspond to C_n type in our approach.

Any rooted tree Γ with root 0 is naturally a layered partial order $\prec$ on Γ with the maximal element 0 and whose minimal elements are the leaves. Furthermore, for any $i \in \Gamma \setminus \{0\}$ denote by $i^+ \in I$ the parent of i in $\prec$, that is, the smallest element j such that $i \prec j$.

Likewise, if Γ is a conical tree, for any non-leaf $i \in \Gamma \setminus \{0\}$ denote by $i^- \in I$ the only son of i in $\prec$.

Theorem 2.1. Suppose that $\Gamma(A)$ is a conical tree with the root 0 and $a_{ij} \in \{0,-1\}$ for all $i, j \in I$ such that $j \neq 0$. Then $(Ad\ g_{\mathbf{a}})(u_i) =$

$$\begin{cases} e_i + b_{i^-} f_{i^-} & \text{if } i \neq 0 \\ e_0 - a_0\mathbf{f} - \frac{a_0^2}{2}[\mathbf{f},[\mathbf{f},f_0]] - \sum\limits_{k\geq 3} \frac{a_0^2}{k!}(ad\ \mathbf{f} + a_0[f_0,\mathbf{f}])^{k-1}([\mathbf{f},[\mathbf{f},f_0]]) & \text{if } i = 0 \end{cases}$$

for $i \in \Gamma(A)$, where we abbreviated $b_{i^-} := -a_i a_{i^+}$ for $i \in I$ (with the convention $b_{i^-} = 0$ for any leaf i of Γ), $\mathbf{f} := \sum\limits_{j \in I : j^+ = 0} a_j f_j$ and $g_{\mathbf{a}} = e^{a_0 f_0} \prod\limits_{i \neq 0} e^{a_i f_i}$, where the product is decreasing (i.e., i^+ always precedes i).

Proof. We need the following

Proposition 2.2. Suppose that $\Gamma(A)$ is a conical tree with the root 0 and such that $a_{ij} \in \{0,-1\}$ for all $i, j \in I$ such that $0 \notin \{i,j\}$. Then

$$(Ad\ g_{\mathbf{a}})(u_i) = \begin{cases} e_i + b_{i^-} f_{i^-} & \text{if } i \neq 0 \\ e_0 + a_0[\hat{\mathbf{f}}, h_0 + a_0 f_0] - a_0^2 \sum\limits_{k\geq 2} \frac{1}{k!}(ad\ \hat{\mathbf{f}})^k(f_0) & \text{if } i = 0 \end{cases}$$

for $i \in \Gamma(A)$, where we abbreviated $b_{i^-} := -a_i a_{i^+}$ for $i \in I$ (with the convention $b_{i^-} = 0$ for any leaf i), $\mathbf{f} := \sum\limits_{j \in I : j^+ = 0} a_j f_j$ and $g_{\mathbf{a}} = e^{a_0 f_0} \prod\limits_{i \neq 0} e^{a_i f_i}$, where the product is decreasing (i.e., i^+ always precedes i), and $\hat{\mathbf{f}} = (Ad\ e^{a_0 f_0})(\mathbf{f}) = \sum\limits_{k \geq 0} \frac{a_0^k}{k!}(ad\ f_0)^k(\mathbf{f})$.

Proof. Indeed, if $i \neq 0$, then $g_{\mathbf{a}} = g' g''$ where $g' = e^{a_0 f_0} \prod\limits_{i \in I'} e^{a_i f_i}$ and $g'' = \prod\limits_{j \in I''} e^{a_j f_j}$, both products decreasing where $I'' = \{j : j \preceq i\}$, $I' = (I \setminus \{0\}) \setminus I''$.

As in the proof of Theorem 1.4, $(Ad\ g'')(u_i) = e_i + b_{i^-} f_{i^-}$ hence

$$(Ad\ g_{\mathbf{a}})(u_i) = (Ad\ g')(e_i + b_{i^-} f_{i^-}) = e_i + b_{i^-} f_{i^-}$$

because both e_i and f_{i^-} are fixed by $Ad\ g'$.

This proves the first case.

Now let $i = 0$. Then

$$(Ad\ g_{\mathbf{a}})(u_0) = (Ad\ e^{a_0 f_0} \prod_{i \neq 0} e^{a_i f_i})(u_0)$$

$$= (Ad\ e^{a_0 f_0} \prod_{i \in I : i^+ = 0} e^{a_i f_i})(u_0) = (Ad\ e^{a_0 f_0} e^{\mathbf{f}})(u_0)$$

because $[f_i, f_j] = 0$ if $i^+ = j^+ = 0$. Furthermore,

$$(Ad\ e^{\mathbf{f}})(u_0) = (Ad\ e^{\mathbf{f}})(e_0 + a_0 h_0 - a_0^2 f_0) = e_0 + a_0 (Ad\ e^{\mathbf{f}})(h_0) - a_0^2 (Ad\ e^{\mathbf{f}})(f_0)$$
$$= u_0 + a_0[\mathbf{f}, h_0] - a_0^2((Ad\ e^{\mathbf{f}})(f_0) - f_0)$$

because $[\mathbf{f}, h_0] = \sum_{i\in I: i^+=0} a_i a_{0i} f_i$. Therefore, $[\mathbf{f}, [\mathbf{f}, h_0]] = 0$. Finally,

$$(Ad\ g_{\mathbf{a}})(u_0) = (Ad\ e^{a_0 f_0})(u_0 + a_0[\mathbf{f}, h_0] - a_0^2((Ad\ e^{\mathbf{f}})(f_0) - f_0))$$
$$= e_0 + a_0[\hat{\mathbf{f}}, h_0 + 2a_0 f_0] - a_0^2((Ad\ e^{\hat{\mathbf{f}}})(f_0) - f_0)$$

because $(Ad\ e^{a_0 f_0})(h_0) = h_0 + 2a_0 f_0$.

This proves the second case. The proposition is proved. □

Since $a_{0i} = -1$ whenever $i^+ = 0$ hence $[f_0, [f_0, \mathbf{f}]] = 0$, we obtain

$$\hat{\mathbf{f}} = \mathbf{f} + a_0[f_0, \mathbf{f}]\ .$$

Also, $[\mathbf{f}, h_0] = -\mathbf{f}$ and $[[f_0, \mathbf{f}], h_0] = [f_0, \mathbf{f}]$, therefore,

$$[\hat{\mathbf{f}}, h_0 + a_0 f_0] = [\mathbf{f} + a_0[f_0, \mathbf{f}], h_0 + a_0 f_0] = -\mathbf{f}\ .$$

Finally, note that

$$[\hat{\mathbf{f}}, f_0] = [\mathbf{f} + a_0[f_0, \mathbf{f}], f_0] = [\mathbf{f}, f_0]$$

and

$$[\hat{\mathbf{f}}, [\mathbf{f}, f_0]] = [\mathbf{f} + a_0[f_0, \mathbf{f}], [\mathbf{f}, f_0]]] = [\mathbf{f}, [\mathbf{f}, f_0]]]$$

The theorem is proved. □

Theorem 2.1 covers the cases when $\mathfrak{g}$ is of types E_6, E_7, E_8, and F_4.

Viewing the Dynkin diagram of sl_n as a conical tree with a "root" $0 = k \in \{1, \ldots, n-1\}$ we obtain the following corollary of Theorem 2.1, which generalizes Theorem 1.4.

Theorem 2.3. For any $k \in \{1, \ldots, n-1\}$ the assignments

$$u_i \mapsto \begin{cases} e_i + b_{i-1} f_{i-1} & \text{if } i < k \\ e_i + b_i f_{i+1} & \text{if } i > k \\ e_k + b_{k-1} f_{k-1} + b_k f_{k+1} + b_{k-1} b_k [f_{k-1}, [f_{k+1}, f_k]] & \text{if } i = k \end{cases}$$

define an injective homomorphism $sl_n^{(\mathbf{b})} \hookrightarrow sl_n$ (where $b_i := -a_i a_{i+1}$).

We prove Theorem 2.3 in Section 3.5.

The following is an immediate corollary of Theorem 2.1.

Corollary 2.4. Let $\Gamma(A)$ be a conical star tree (i.e., every non-root is a leaf) with $a_{i0} = a_{0i} = -1$ for all $i \in I \setminus \{0\}$. Then the assignments

$$u_i \mapsto e_0 - a_0 \delta_{i,0}(\mathbf{f} - \frac{a_0}{2}[\mathbf{f}, [\mathbf{f}, f_0]] - a_0 \sum_{k\geq 3} \frac{1}{k!}(ad\ \mathbf{f} + a_0[f_0, \mathbf{f}])^{k-1}([\mathbf{f}, [\mathbf{f}, f_0]]))\ ,$$

where $\mathbf{f} := \sum_{j\neq 0} a_j f_j$, define an injective homomorphism $\mathfrak{g}^{(\mathbf{a})} \hookrightarrow \mathfrak{g}$.

Theorem 2.5. Suppose that $\Gamma(A)$ is a conical tree with the root 0, J_+ is a set of leaves of $\Gamma(A)$ attached to the 0, and $a_{ij} \in \{0, -1\}$ for all $i, j \in I \setminus J_+$ such that $j \neq 0$ and (i.e., $J_+^+ = \{0\}$) such that $a_{i0} = -1$ for all $i \in J_+$. Then

$$(Ad\ g_{\mathbf{a}})(u_i) = \begin{cases} e_i + b_i f_i' - \frac{b_i^2}{2}[[f_i, f_i'], f_i'] \\ -a_i^2 \sum\limits_{k\geq 3} \frac{a_0^k}{k!}(ad\ (f_i' + a_i[f_i, f_i']))^{k-2}([[f_i, f_i'], f_i']) & \text{if } i \in J_+ \\ e_0 - a_0\mathbf{f} - \frac{a_0^2}{2}[\mathbf{f}, [\mathbf{f}, f_0']] \\ -a_0^2 \sum\limits_{k\geq 3} \frac{1}{k!}((ad\ \mathbf{f} + a_0[f_0', \mathbf{f}])^{k-1}([\mathbf{f}, [\mathbf{f}, f_0']])) & \text{if } i = 0 \\ e_i + b_{i^-} f_{i^-} & \text{otherwise} \end{cases}$$

for $i \in \Gamma(A)$, where we abbreviated $b_{i^-} := -a_i a_{i^+}$ for $i \in I \setminus J_+$, $b_i := -a_0 a_i$, $i \in J_+$ (with the convention $b_{i^-} = 0$ for any leaf i), $\mathbf{f}_+ := \sum\limits_{j\in J_+} a_j f_j$, $\mathbf{f} := \sum\limits_{j\in I\setminus J_+ : j^+=0} a_j f_j$, $f_0' := \sum\limits_{k\geq 0} \frac{1}{k!}(ad\ \mathbf{f}_+)^k(f_0)$, $f_i' := \sum\limits_{k\geq 0} \frac{1}{k!}(ad\ (\mathbf{f}_+ - a_i f_i))^k(f_0)$, $i \in J_+$, and $g_{\mathbf{a}} = e^{\mathbf{f}_+} e^{a_0 f_0} \prod\limits_{i\in I\setminus(\{0\}\cup J_+)} e^{a_i f_i}$, where the product is decreasing (i.e., i^+ always precedes i).

Proof. The case $i \neq 0$, $i \notin J_+$ follows from the proof of Theorem 2.1.

Now let $i \in J_+$. Note that $(Ad\ e^{a_j f_j})(u_i) = u_i$ for all $j \neq 0$, $j \notin J_+$. Also, $i \in J_+$:

$$(Ad\ e^{a_i f_i} e^{a_0 f_0})(u_i) = (Ad\ e^{a_i f_i} e^{a_0 f_0} e^{-a_i f_i})(e_i) = (Ad\ e^{a_0(f_0 + a_i[f_i, f_0])})(e_i)$$

Taking into account that

$$\begin{aligned}(ad\ ((f_0 + a_i[f_i, f_0]))(e_i) &= a_i[[f_i, f_0], e_i] \\ &= a_i[[f_i, e_i], f_0] = -a_0 a_i [h_i, f_0] = -a_i f_0\end{aligned}$$

and

$$(ad\ (f_0 + a_i[f_i, f_0]))^{k-1}(f_0) = a_i[[f_i, f_0], f_0]\ ,$$

we obtain

$$(Ad\ e^{a_i f_i} e^{a_0 f_0})(u_i) = e_i - a_0 a_i f_0 + \sum_{k\geq 2} \frac{a_0^{k-1}}{k!}(ad\ (f_0 + a_i[f_i, f_0]))^{k-1}(a_{i0} a_0 a_i f_0)$$

$$= e_i + b_i f_0 - \frac{b_i^2}{2}[[f_i, f_0], f_0] - a_0 a_i^2 \sum_{k\geq 3} \frac{a_0^{k-1}}{k!}(ad\ (f_0 + a_i[f_i, f_0]))^{k-2}([[f_i, f_0], f_0])\ .$$

Finally,

$$Ad\ g_{\mathbf{a}}(u_i) = (Ad\ e^{\mathbf{f}_+} e^{a_0 f_0})(u_i) = (Ad\ e^{\mathbf{f}_+ - a_i f_i})((Ad\ e^{a_i f_i} e^{a_0 f_0})(u_i))$$

$$e_i + b_i f_{0i}' - \frac{b_i^2}{2}[[f_i, f_{0i}'], f_{0i}'] - a_i^2 \sum_{k\geq 3} \frac{a_0^k}{k!}(ad\ (f_{0i}' + a_i[f_i, f_{0i}']))^{k-2}([[f_i, f_{0i}'], f_{0i}'])\ .$$

Now consider the remaining case $i = 0$. Then

$$Ad\ g_{\mathbf{a}}(u_0) = (Ad\ e^{\mathbf{f}_+} e^{a_0 f_0} e^{\mathbf{f}} e^{-a_0 f_0})(e_0) = (Ad\ e^{\mathbf{f}_+} e^{\hat{\mathbf{f}}})(e_0) = Ad\ e^{\mathbf{f}_+}(e^{\mathbf{f}}(e_0))$$

where $\hat{\mathbf{f}} = (Ad\ e^{a_0 f_0})(\mathbf{f}) = \mathbf{f} + a_0[f_0, \mathbf{f}]$.

Finally, using the argument from the proof of Theorem 2.1 for $i = 0$ and replacing f_0 with $f_0' = Ad\ e^{\mathbf{f}_+}(f_0)$, we finish the proof of Theorem 2.5.

□

The following is an immediate corollary of Theorem 2.5.

Corollary 2.6. In the assumptions of Theorem 2.5 assume additionally, that $I \setminus J_+$ is of type A. Then

(a) the assignments

$$u_i \mapsto \begin{cases} e_i + b_{i^-} f_{i^-} & \text{if } i \notin J_+ \\ e_i + b_i f_0 + b_i \sum\limits_{k \geq 1} \frac{1}{k!}(ad\ (\mathbf{f}_+ - a_i f_i))^k (f_0) & \text{if } i \in J_+ \end{cases}$$

define an injective homomorphism $\mathfrak{g}^{(\mathbf{a})} \hookrightarrow \mathfrak{g}$.

(b) Suppose that $I = \{1, \ldots, n\}$, $\{1, \ldots, n-1\}$ is of type A_{n-1}, and $a_{in} = a_{ni} = \delta_{ik}$. Then the assignments

$$u_i \mapsto \begin{cases} e_i + b_{i+1} f_{i+1} & \text{if } 1 \leq i < k \\ e_k + b_{k-1,k} f_{k-1} + b_{k,k+1} f_{k+1} & \\ +b_{k-1,k} b_{k,k+1} [f_{k-1}, [f_{k+1}, f_k + a_n [f_n, f_k]]] & \text{if } i = k \end{cases}$$

where $b_{ij} := a_{ij} a_i a_j$, define an injective homomorphism $\mathfrak{g}^{(\mathbf{a})} \hookrightarrow \mathfrak{g}$.

Theorem 2.7. Let $\mathfrak{g} = \mathfrak{g}_A$ be a Kac-Moody algebra with $I = \{1, \ldots, r\}$ such that the Cartan matrix A of $\mathfrak{g}$ satisfies $a_{ij} a_{ji} \neq 0$ iff $|i - j| = 1$ and $a_{ij} \leq a_{i-1,j-1}$ for all distinct $i, j = 2, \ldots, r$.

Suppose also that the following three conditions hold.

- Either $a_{12} = a_{21} = -1$ or $a_{21} < -1$.
- Either $a_{i-1,i} a_{i,i+1} > 1$ or $a_{i-1,i} = a_{i,i+1} = a_{i,i-1} = -1$ for $i = 2, \ldots, r$.
- Either $a_{i,i-1} a_{i+1,i} > 1$ or $a_{i+1,i} = a_{i,i-1} = a_{i,i+1} = -1$ for $i = 2, \ldots, r$.

Then the assignments $u_i \mapsto \begin{cases} e_1 & \text{if } i = 1 \\ e_i + b_{i-1} f_{i-1} & \text{if } 2 \leq i \leq r \end{cases}$ define a homomorphism of Lie algebras $\mathfrak{g}'^{(\mathbf{b})} \to \mathfrak{g}$, where we abbreviated $b_j := a_{j+1,j} a_j a_{j+1}$ for $j = 1, \ldots, r-1$, similarly to Theorem 1.10 and A' is $I \times I$ the Cartan matrix of $\mathfrak{g}'$ given by $a'_{12} = a_{12}$ $a'_{21} = a_{21}$ and $a'_{ij} = \min(a_{i-1,j-1}, a_{ij})$ for $i, j = 2, \ldots, r$

Theorem 2.8. Suppose that $I = \{1, \ldots, r+d\}$, $r \geq 1$, $d \geq 1$ such that $a_{ij} a_{ji} \neq 0$ for distinct i, j iff either $i, j \leq r$ and $|i - j| = 1$ or $\min(i, j) = r$, the restriction of A to $\{1, \ldots, r, r+i\}$ satisfies the assumptions of Theorem 2.7 for $i = 1, \ldots, d$ and $a_{r+i,r+j} = 0$ for all distinct $i, j = 1, \ldots, d$. Then the assignments

$$u_i \mapsto \begin{cases} e_1 & \text{if } i = 1 \\ e_i + b_{i-1} f_{i-1} & \text{if } 2 \leq i \leq r \\ e_i + b_i f_r & \text{if } r < i \leq r + d \end{cases}$$

define a homomorphism of Lie algebras $\mathfrak{g}''^{(\mathbf{b})} \to \mathfrak{g}$, where we abbreviated $b_i := a_{ir}a_ra_i$ for $i = r+1, \ldots, r+d$ and A'' is the $I \times I$ Cartan matrix of $\mathfrak{g}''$ with $a''_{ij} = a'_{ij}$ whenever $i, j = 1, \ldots, r$, $a''_{ij} = \min(a_{i-1,j-1}, a_{ij})$ whenever $\min(i,j) = r$, $i \neq j$ and the remaining $a''_{ij} = 0$.

We prove Theorems 2.7 and 2.8 in Section 3.7.

Taking $\mathfrak{g}'' = \mathfrak{g}$ in Theorem 2.8, we obtain the following

Theorem 2.9. Suppose that $I = \{1, \ldots, r+d\}$, $a_{ij}a_{ji} \neq 0$ for distinct i, j iff either $i, j \leq r$ and $|i-j| = 1$ or $\min(i,j) = r$ (as in Theorem 2.8) and the Cartan matrix of $\mathfrak{g}$ satisfies

- Either $a_{12} = a_{21} = -1$ or $a_{21} < -1$.
- Either $a_{i-1,i} = a_{i,i+1} = a_{i,i-1} = -1$ or $a_{i,i+1} < -1$ and $a_{i,i+1} \leq a_{i-1,i}$ for $i = 2, \ldots, r$
- Either $a_{i+1,i} = a_{i,i-1} = a_{i,i+1} = -1$ or $a_{i+1,i} < -1$ and $a_{i+1,i} \leq a_{i,i-1}$ for $i = 2, \ldots, r$.
- $a_{ij} \leq a_{i-1,j-1}$ whenever $\min(i,j) = r$, $i \neq j$.

Then the subalgebra generated by $u_i := \begin{cases} e_1 & \text{if } i = 1 \\ e_i + b_{i-1}f_{i-1} & \text{if } 2 \leq i \leq r \\ e_i + b_i f_r & \text{if } r < i \leq r+d \end{cases}$

is naturally isomorphic to $\mathfrak{g}^{(\mathbf{b})}$ for any $\mathbf{b} = (b_1, \ldots, b_{n-1})$.

We prove Theorem 2.9 in Section 3.8.

As in the proof of Theorem 1.3, all results of this section are valid if one replaces $\mathbb{C}$ with and $\mathbb{C}$-algebra containing all a_i (resp. all b_{ij}).

3. Proof of main results

3.1. **Proof of Theorem1.2.** Let $e_{i^rj} := \frac{1}{r!}(ad\ e_i)^r(e_j)$, $f_{i^rj} := \frac{1}{r!}(ad\ f_i)^r(f_j)$, and $u_{i^rj} := \sum\limits_{k=0}^{r} a_i^{r-k}\binom{a_{ij}+r-1}{r-k}(e_{i^kj} - (-1)^r a_i^{2k}a_j^2 f_{i^kj})$ for any distinct $i, j \in I$ and $r \geq 1$.

We need the following result.

Proposition 3.1. $[u_i, u_{i^rj}] = (r+1)u_{i^{r+1}j}$ for all distinct $i, j \in I$ and $r \geq 1$.

Proof. The following is immediate.

Lemma 3.2. $[f_i, e_{i^kj}] = -(a_{ij}+k-1)e_{i^{k-1}j}$, $[e_i, f_{i^kj}] = -(a_{ij}+k-1)f_{i^{k-1}j}$ for all $k \geq 0$ (with the convention $f_{i^{-1}j} = e_{i^{-1}j} = 0$).

Thus, we have for $r \geq 1$:

$$[e_i, u_{i^rj}] = \sum_{k=0}^{r} a_i^{r-k}\binom{a_{ij}+r-1}{r-k}([e_i, e_{i^kj}] - (-1)^r a_i^{2k}a_j^2[e_i, f_{i^kj}])$$

$$= \sum_{k=0}^{r} a_i^{r-k}\binom{a_{ij}+r-1}{r-k}((k+1)e_{i^{k+1}j} + (-1)^r(a_{ij}+k-1)a_i^{2k}a_j^2 f_{i^{k-1}j})\ ,$$

$$[h_i, u_{i^r j}] = \sum_{k=0}^{r} a_i^{r-k} \binom{a_{ij}+r-1}{r-k} ([h_i, e_{i^k j}] - (-1)^r a_i^{2k} a_j^2 [h_i, f_{i^k j}])$$

$$= \sum_{k=0}^{r} a_i^{r-k} \binom{a_{ij}+r-1}{r-k} (2k + a_{ij})(e_{i^k j} + (-1)^r a_i^{2k} a_j^2 f_{i^k j}) \ ,$$

$$[f_i, u_{i^r j}] = \sum_{k=0}^{r} a_i^{r-k} \binom{a_{ij}+r-1}{r-k} ([f_i, e_{i^k j}] - (-1)^r a_i^{2k} a_j^2 [f_i, f_{i^k j}])$$

$$= \sum_{k=0}^{r} a_i^{r-k} \binom{a_{ij}+r-1}{r-k} (-(a_{ij}+k-1) e_{i^{k-1} j} - (-1)^r (k+1) a_i^{2k} a_j^2 f_{i^{k+1} j})$$

Therefore,

$$[u_i, u_{i^r j}] = \sum_{k=0}^{r+1} c_k a_i^{r+1-k} (e_{i^k j} + (-1)^r a_i^{2k} a_j^2 f_{i^k j})$$

where $c_{r+1} = r + 1$ and

$$c_k = (r - y)\binom{x}{y+1} + (x + r + 1 - 2y)\binom{x}{y} + (x - y + 1)\binom{x}{y-1}$$

$$= (r+1)\binom{x+1}{y+1} - (y+1)\binom{x}{y+1} + (x-2y)\binom{x}{y} + y\binom{x}{y} = (r+1)\binom{x+1}{y+1}$$

because $\binom{x}{y+1} + \binom{x}{y} = \binom{x+1}{y+1}$, $(y+1)\binom{x}{y+1} = (x-y)\binom{x}{y}$, $(x-y+1)\binom{x}{y-1} = y\binom{x}{y}$, where we abbreviated $x := a_{ij} + r - 1$, $y := r - k$.

This proves Proposition 3.1. □

Furthermore,

$$[u_i, u_j] = [e_i + a_i h_i - a_i^2 f_i, e_j + a_j h_j - a_j^2 f_j] = e_{ij} + a_i^2 a_j^2 f_{ij}$$

$$-a_j a_{ji} e_i + a_i a_{ij} e_j + a_j^2 a_i a_{ij} f_j - a_i^2 a_j a_{ji} f_i = u_{ij} - a_j a_{ji}(e_i + a_i^2 f_i)$$

Taking into account that

$$[u_i, e_i + a_i^2 f_i] = [e_i + a_i h_i - a_i^2 f_i, e_i + a_i^2 f_i] = 2a_i u_i$$

and using Proposition 3.1 with $r = 2$, we obtain

$$(ad\ u_i)^2(u_j) = 2u_{i^2 j} - 2a_i a_j a_{ji} u_i \ .$$

In turn, this and Proposition 3.1 imply by an induction in $r \geq 3$ that

$$(ad\ u_i)^r(u_j) = r! u_{i^r j}$$

for all $r \geq 3$.

Finally, if $r = 1 - a_{ij}$ then $\binom{a_{ij}+r-1}{r-k} = 0$ for all $k < r$ and $e_{i^r j} = f_{i^r j} = 0$ which implies that $u_{i^r j} = 0$.

Theorem 1.2 is proved. □

3.2. **Proof of Theorem 1.3.** We apply the results of **Appendix** to our situation as follows.

Let R be any $\mathbb{C}$-algebra, fix a family $\mathbf{a} = (a_i, i \in I)$ of elements of R.

Denote $A := R \otimes U(\mathfrak{g})$. By definition, the subalgebra B of A generated by $X := \{u_i = e_i + a_i h_i - a_i^2 f_i, i \in I\}$ over R is $U(\mathfrak{g}^{(\mathbf{a})})$. According to (1.2), the kernel of the canonical homomorphism $\varphi_{X,B} : R\langle X\rangle \twoheadrightarrow B$ contains the set $Y := \{(ad\ u_i)^{1-a_{ij}}(u_j) + 2\delta_{a_{ij},-1} a_{ji} a_i a_j u_i : i, j \in I, i \neq j\}$.

Let us define a filtration on A by setting $\deg e_i = 1$, $\deg f_i = \deg h_i = 0$ for $i \in I$. Clearly, grA is generated by e_i, f_i, h_i subject to the usual Serre relations except for $[e_i, f_i] = h_i$ which is replaced by $[e_i, f_i] = 0$. In particular, the subalgebra B_0 of grA generated by $e_i, i \in I$ is naturally isomorphic to $R \otimes U(\mathfrak{n})$.

Then the natural inclusion of R-algebras $B \subset A$ is compatible with the filtration and defines a natural inclusion of graded algebras $grB \subset B_0$ because $gr\ u_i = e_i$ for $i \in I$ (that is, $grX = \{e_i, i \in I\}$).

Clearly, $gr\ Y = \{(ad\ e_i)^{1-a_{ij}}(e_j) : i, j \in I, i \neq j\}$ and is the presentation of B_0, i.e., $gr\ Y$ generates the kernel of the canonical homomorphism $\varphi_{grX,grB}$: $R\langle grX\rangle \twoheadrightarrow grB = B_0$.

Thus, all conditions of Proposition 4.1 are met. Thus, the proposition guarantees that the relations (1.2) give a presentation of $B = U(\mathfrak{g}^{(\mathbf{a})})$ as an associative algebra over R hence of the Lie algebra $\mathfrak{g}^{(\mathbf{a})}$ over R.

The theorem is proved. □

3.3. **Proof of Theorem 1.4.** Clearly, $u_i = (Ad\ e^{-a_i f_i})(e_i)$ for all $i \in I = \{1, \ldots, n-1\}$.

Then for $i = 1$ we have

$$(Ad\ g_{\mathbf{a}})(u_1) = (Ad\ e^{a_{n-1} f_{n-1}} \cdots e^{a_2 f_2})(e_1) = e_1\ .$$

Now let $i \geq 2$. Then

$$(Ad\ e^{a_i f_i} \cdots e^{a_2 f_2} e^{a_1 f_1})(u_i) = (Ad\ e^{a_i f_i} e^{a_{i-1} f_{i-1}} e^{-a_i f_i})(e_i)$$

$$= (Ad\ e^{a_{i-1}(f_{i-1} + a_i [f_i, f_{i-1}])})(e_i) = (Ad\ e^{a_{i-1} a_i [f_i, f_{i-1}]})(e_i)$$

$$= e_i + a_{i-1} a_i [[f_i, f_{i-1}], e_i] = e_i - a_{i-1} a_i f_{i-1}\ .$$

Here we used the facts that $(Ad\ e^{a f_j})(e_i) = e_i$ if $i \neq j$, $(Ad\ e^{a f_j})(f_i) = f_i$ whenever $|i-j| > 1$ and $(Ad\ e^{a_i f_i})(f_{i-1}) = f_{i-1} + a_i [f_i, f_{i-1}]$ because $a_{i,i-1} = -1$. This is based on the following general fact

$$(Ad\ e^x)(y) = e^{ad\ x}(y)\ . \tag{3.4}$$

Finally,

$$(Ad\ g_{\mathbf{a}})(u_i) = (Ad\ e^{a_{n-1} f_{n-1}} \cdots e^{a_{i+1} f_{i+1}})(e_i - a_{i-1} a_i f_{i-1}) = e_i - a_{i-1} a_i f_{i-1}\ .$$

The theorem is proved. □

3.4. **Proof of Theorem 1.8.** Recall that sl_n acts on V^* via

$$e_i(v_j^*) = \delta_{ij}v_{i+1}^*,\ f_i(v_j^*) = \delta_{i+1,j}v_i^*$$

where $\{v_i^*\}$ is the basis of V^* dual to $\{v_i\}$. Therefore, a direct calculation shows that the generators $u_i = e_i + b_{i-1}f_{i-1}$ for $i = 1, \ldots, n-1$ of $sl_n^{(\mathbf{b})}$ preserve

$$\omega_{\mathbf{b}} := \sum_{k=1}^{n-1}\left(\prod_{i=k}^{n-2}(-b_i)\right) v_k^* \wedge v_{k+1}^* \tag{3.5}$$

It is immediate that $v^1 \in V$ is invariant under the restriction from sl_n-action on $V := V_{\omega_1}$ to the $sl_n^{(\mathbf{b})}$-action on V. This, in particular, implies that the orthogonal complement $V^1 \subset V^*$ to v^1 is a (co-dimension 1) $sl_n^{(\mathbf{b})}$-submodule of V^*.

It is easy to see that $\omega_{\mathbf{b}} \in \Lambda^2 V^*$ in fact belongs to $\Lambda^2 V^1$ for odd n. Clearly, in this case vectors $w_i = \begin{cases} b_i v_i^* + v_{i+2}^* & \text{if } i \text{ is odd} \\ v_i^* & \text{if } i \text{ is even} \end{cases}$ $i = 1, \ldots, n-1$ form a basis of V^1 and $-\omega_{\mathbf{b}} = w_1 \wedge w_2 + w_3 \wedge w_4 \cdots w_{n-2} \wedge w_{n-1}$ and it of rank $n-1$ if $b_1 \cdots b_{n-2} \neq 0$. Since $\dim sl_n^{(\mathbf{b})} = \dim sp_{n-1}$, this implies that $sl_n^{(\mathbf{b})} \cong sp_{n-1}$, and thus finishes the proof for odd n.

Let n be even. It well-known that $\det M = \prod_{i=1}^{k} m_{2i,2i-1}m_{2i-1,2i}$ for any tridiagonal $2k \times 2k$-matrix M with $m_{ii} = 0$.

This implies that $\omega_{\mathbf{b}}$ is of rank n if $b_1 \cdots b_{n-2} \neq 0$. Therefore, in this case, the annihilator of $\omega_{\mathbf{b}}$ in sl_n is isomorphic to sp_n. Therefore, $sl_n^{(\mathbf{b})}$ identifies with a Lie subalgebra of sp_n.

Since V is a simple sp_n-module, then for any nonzero $v \in V$ the annihilator of v is isomorphic to the maximal parabolic subalgebra of sp_n, i.e., to sp_{n-1}.

Since $sl_n^{(b)}$ also annihilates $v^1 \in V$, $sl_n^{(b)}$ identifies with a Lie subalgebra of sp_{n-1}. Then the dimension argument once again implies that $sl_n^{(\mathbf{b})} \cong sp_{n-1}$ as well for even n.

The theorem is proved. □

3.5. **Proof of Theorems 1.6, 2.3, and 1.10.** Note that Theorem 1.6 is a particular case of Theorem 2.3(a).

Let $\hat{\hat{R}} = \mathbb{C}[a_i, i \in I]$ an let $\hat{R}$ be its subalgebra generated by all $b_{ij} := a_{ji}a_ia_j$ whenever $a_{ij} = -1$ in the assumptions of Theorems 1.10 and 2.3. Clearly, $\hat{R} = \mathbb{C}[b_{ij}]$.

Indeed, let $u_i = u_i^{\mathbf{b}} \in \hat{R} \otimes \mathfrak{g}$, $i \in I$ be the elements as in (the right hand sides of) Theorems 2.3 and 1.10. Let $v_{ij}^{\mathbf{b}} \in \hat{R} \otimes \mathfrak{g}$ be the left hand side of (1.3) in each case.

Lemma 3.3. $v_{ij}^{\mathbf{b}} = 0$ in each case of Theorems 1.10 and 2.3.

Proof. Theorem 2.1 taken in $\hat{\hat{R}} \otimes sl_n$ and $0 = k \in A_{n-1}$ (so that A_{n-1} is the rooted conical tree with sub-trees $\{1,\ldots,k-1\}$ and $\{k+1,\ldots,n-1\}$) implies Lemma 3.3 in the assumptions of Theorem 2.3 for $\hat{R} \otimes sl_n$.

Likewise, Theorem 2.1 taken in $\hat{\hat{R}} \otimes so_{2n+1}$ and $0 = n \in B_n$ (so that B_n is the rooted conical tree with a single sub-tree $\{1,\ldots,k-1\}$) implies Lemma 3.3 in the assumptions of Theorem 1.10(a) in $\hat{R} \otimes so_{2n+1}$.

Finally, Theorem 2.1 taken in $\hat{\hat{R}} \otimes sp_{2n}$ and $0 = n-1 \in C_n$, $J_+ = \{n\}$ (so that C_n is the rooted conical tree with a single sub-tree $\{1,\ldots,n-2\}$ and J_+) implies Lemma 3.3 in the assumptions of Theorem 1.10(b) in $\hat{R} \otimes sp_{2n}$ (with $b_i = b_{i-1,i}$ for $i < n$ and $b_n = b_{n,n-1}$).

The lemma is proved. □

Furthermore, we proceed similarly to the proof of Theorem 1.3. Indeed, let R be any $\mathbb{C}$-algebra containing all b_{ij}.

Denote $A := R \otimes U(\mathfrak{g})$. By definition, the subalgebra B of A generated by $X := \{u_i, i \in I\}$ over R is $U(\mathfrak{g}^{(\mathbf{b})})$.

The canonical specialization homomorphism $\hat{R} \twoheadrightarrow R$ implies that Lemma 3.3 holds over R as well, that is, u_i satisfy the relations of Theorems 1.10 and 2.3, that is, in the notation of Proposition 4.1, the kernel of the canonical homomorphism $\varphi_{X,B} : R\langle X\rangle \twoheadrightarrow B$ contains the set $Y := \{v_{ij}^{\mathbf{b}}, i,j \in I\}$.

Let us define a filtration on A same way as in the proof of Theorem 1.3.

In particular, the subalgebra B_0 of $gr\ A$ generated by $e_i, i = 1,\ldots,n-1$ is naturally isomorphic to $R \otimes U(\mathfrak{n})$, where $\mathfrak{n} =< e_i, i \in I >$ is the nilpotent Lie subalgebra of $\mathfrak{g}$.

Then the natural inclusion of R-algebras $B \subset A$ is compatible with the filtration and defines a natural inclusion of graded algebras $grB \subset B_0$ because $gr\, u_i = e_i$ for $i \in I$ (that is, $grX = \{e_i, i \in I\}$).

Clearly, $grY = \{(ad\ e_i)^{1-a_{ij}}(e_j) : i \neq j\}$ and is the presentation of B_0, i.e., grY generates the kernel of the canonical homomorphism $\varphi_{grX,grB}$: $R\langle grX\rangle \twoheadrightarrow grB = B_0$.

Thus, all conditions of Proposition 4.1 are met. Thus, the proposition guarantees that the relations of Theorems 1.10 and 2.3 give a presentation of $B = U(\mathfrak{g}^{(\mathbf{b})})$ as an associative algebra over R hence, of the Lie algebra $\mathfrak{g}^{(\mathbf{b})}$ over R.

Theorems 1.6, 2.3, and 1.10 are proved. □

3.6. **Proof of Theorem 1.13.**

Proof. The generators of $sp_{2n}^{(\mathbf{b})}$ are as in Theorem 1.10

$$u_i = \tilde{e}_i + b_{i-1}\tilde{f}_{i-1} - \delta_{i,n}\frac{1}{2}b_{n-1}^2[\tilde{f}_{n-1},[\tilde{f}_{n-1},\tilde{f}_n]]$$

(a) It is clear that the set of elements

$$\{u_1,\ldots,u_{n-1}\}$$

generates a subalgebra of $sp_{2n}^{(\mathbf{b})}$ isomorphic to $sl_n^{(b_1,\ldots,b_{n-2})}$. Define J as an ideal of $sp_{2n}^{(\mathbf{b})}$ generated by the set $[sl_n^{(b_1,\ldots,b_{n-2})}, u_n]$.

Recall that sp_{2n} contains sl_n and splits as a sl_n module

$$sp_{2n} \cong V_1 \oplus sl_n \oplus V_2$$

where the sl_n modules V_1 and V_2 are $\mathbb{C}e_n + [sl_n, e_n]$ and $\mathbb{C}f_n + [sl_n, f_n]$ respectively. Note that $V_1 = S^2 V_{\omega_1}$. As a direct calculation shows the projection of J to V_1 is an isomorphism of vector spaces and moreover it is an $sl_n^{b_1,\ldots b_{n-2}}$ module isomorphism where V_1 is a module over $sl_n^{b_1,\ldots b_{n-2}}$ since $sl_n^{b_1,\ldots b_{n-2}} \subset sl_n$.

(b) Introduce the set of elements u'_i, $1 \le i \le n$, as follows:

$$u'_n = u_n,\ \ u'_{n-1} = [u_{n-1}, [u_{n-1}, u_n]]$$

and

$$u'_{n-k} = Ad^2_{u_{n-k}} \circ Ad^2_{u_{n-k+1}} \circ \cdots \circ Ad^2_{u_{n-1}}(u_n)$$

There is an isomorphism of Lie algebras $\phi : sl_{n+1}^{(\mathbf{b}')} \to J$ such that $\phi(v_i) = u'_i$, where v_i are the generators of $sl_{n+1}^{(\mathbf{b}')}$ and $\mathbf{b}' = (b'_1, \ldots, b'_{n-1})$

$$b'_k = -2^{2(n-k)+1} \prod_{i=k}^{n-1} b_i^2$$

$k = 1, \ldots n-1$.

Moreover $[u'_{n-k}, u'_{n-k\pm1}] = (-2)^k \left(\prod_{i=1}^{k-1} b_{n-i}\right) [u_{n-k}, u'_{n-k\pm1}]$ for $1 \le i \le n-1$ and hence

$$[u'_{n-k} + (-2)^{k+1} \left(\prod_{i=1}^{k} b_{n-i}\right) u_{n-k}, u'_{n-k\pm1}] = 0$$

Observe finally, that the elements

$$u''_{n-k} = u'_{n-k} + (-2)^{k+1} \left(\prod_{i=1}^{k} b_{n-i}\right) u_{n-k}$$

form a copy of $sl_n^{(b_1,\ldots,b_{n-2})}$ after the appropriate localization.

This finishes the proof.

□

3.7. **Proof of Theorems 2.7 and 2.8.** We need the following fact.

Proposition 3.4. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra with $I = \{1, \ldots, r\}$ and $u_i := e_i + b_{i-1} f_{i-1}$ (with $b_0 = 0$). Then for $k \ge 2$ one has:

(a) $(ad\ u_1)^k (u_2) = (ad\ e_1)^k (e_2) - 2\delta_{k,2} b_1 u_1$,

$(ad\ u_2)^k (u_1) = (ad\ e_2)^k (e_1) - 2 b_1 \delta_{k,2} (u_2 + (a_{12} + 1) e_2)$.

(b) $(ad\ u_i)^k (u_{i+1}) = (ad\ e_i)^k (e_{i+1}) + b_{i-1}^k b_i (ad\ f_{i-1})^k (f_i) - 2 b_i \delta_{k,2} (u_i - b_{i-1}(a_{i,i-1} + 1) f_{i-1})$,

$(ad\ u_{i+1})^k (u_i) = (ad\ e_{i+1})^k (e_i) + b_i^k b_{i-1} (ad\ f_i)^k (f_{i-1}) - 2 b_i \delta_{k,2} (u_{i+1} - b_{i+1}(a_{i,i+1} + 1))$.

(c) $[u_i, u_j] = 0$ whenever $|i - j| > 1$.

Proof. Prove (a). Indeed,

$$[u_1, u_2] = [e_1, e_2 + b_1 f_1] = e_{12} + b_1 h_1$$

$$(ad\ u_1)^2(u_2) = [e_1, e_{12} + b_1 h_1] = (ad\ e_1)^2(2_2) - 2b_1 e_1 = (ad\ e_1)^2(2_2) - 2b_1 u_1$$

$$(ad\ u_2)^2(u_1) = [e_2 + b_1 f_1, e_{21} - b_1 h_1] = (ad\ e_2)^2(e_1) + b_1 a_{12} e_2 + b_1 a_{12} e_2 - 2b_1^2 f_1$$

$$= (ad\ e_2)^2(e_1) - 2b_1 u_2 + 2b_1(a_{12} + 1)e_2$$

In particular,

$$(ad\ u_1)^3(u_2) = (ad\ e_1)^3(e_2)\ ,$$

$$(ad\ u_2)^3(u_1) = (ad\ e_2)^3(e_1) + b_1[f_1, (ad\ e_2)^2(e_1)] = (ad\ e_2)^3(e_1)\ .$$

By taking commutators repeatedly with u_1 and u_2 respectively, we finish the proof of (a).

Prove (b). $[u_i, u_j] = [e_i + t_k f_k, e_j + t_\ell f_\ell] = e_{ij} + t_k t_\ell f_{k\ell} + \delta_{i\ell} t_i h_i - \delta_{jk} t_j h_j$, where $e_{ij} := [e_i, e_j]$, $f_{k\ell} = [f_k, f_\ell]$. Then

$$(ad\ u_i)^2(u_j) = [e_i + t_k f_k, e_{ij} + t_k t_\ell f_{k\ell} + \delta_{i\ell} t_i h_i - \delta_{jk} t_j h_j]$$

$$= e_{iij} + t_k^2 t_\ell f_{kk\ell} + \delta_{i\ell} t_i t_k a_{ik} f_k - \delta_{i\ell} t_i (2e_i - a_{ik} t_k f_k) + \delta_{jk} a_{ji} t_j e_i + \delta_{jk} t_j (a_{ji} e_i - 2t_j f_j)$$

$$= e_{iij} + t_k^2 t_\ell f_{kk\ell} + t_k(2\delta_{i\ell} t_i a_{ik} - 2\delta_{jk} t_j) f_k + 2(\delta_{jk} a_{ji} t_j - \delta_{i\ell} t_i) e_i$$

if $k \neq i$, $\ell \neq j$ where $e_{ab} := [e_a, e_b]$, $f_{ab} := [f_a, f_b]$, $e_{aab} := [e_a, e_{ab}]$, $f_{aab} := [f_a, f_{ab}]$.

Therefore, we obtain

$$(ad\ u_i)^3(u_j) = (ad\ e_i)^3(e_j) + t_k^3 t_\ell (ad\ f_k)^3(f_\ell)$$

because $[e_i, f_{kk\ell}] = 0$, $[f_k, e_{iij}] = 0$.

By taking commutators repeatedly with u_i, we obtain by induction in r:

$$(ad\ u_i)^r(u_j) = (ad\ e_i)^r(e_j) + t_k^{r-1} t_\ell (ad\ f_k)^r(f_\ell)\ . \tag{3.6}$$

Taking $j = i+1, k = i-1, \ell = i$, $r \mapsto k$, we finish the proof of the first part of (b). Taking $i \mapsto i+1, j = k = i, \ell = i-2$, $r \mapsto k$, we finish the proof of the first part of (b) (all $t_m = b_m$).

This finishes the proof of (b).

Part (c) is immediate.

The proposition is proved. □

Now we can finish proof of Theorem 2.7. Indeed, in view of Proposition 3.4(a) one can take $a'_{12} = a_{12}$, $a'_{21} = a_{21}$

Also, if $i > 1$, then in view of Proposition 3.4(b), one can take $a'_{i,i+1} = \min(a_{i,i+1}, a_{i-1,i})$ and $a'_{i+1,i} = \min(a_{i+1,i}, a_{i,i-1})$.

Finally, in view of Proposition 3.4(c), one can take $a'_{ij} = a_{ij} = 0$ whenever $|i - j| > 1$.

Thus, Theorem 2.7 is proved. □

In the notation of Theorem 2.8, denote

$$u_i' := \begin{cases} e_i + a_{i,i-1}a_{i-1}a_i f_{i-1} & \text{if } 1 \le i \le r \\ e_i + a_{ir}a_r a_i f_r & \text{if } r < i \le r+d \end{cases} .$$

Clearly, the relations for the pairs (u_i', u_{i+1}'), $i = 1, \ldots, r-1$ follow from Theorem 2.7.

It is also clear that $[u_i', u_j'] = 0$ whenever $i = 1, \ldots, r$, $j = r+1, \ldots, r+d$ and $r < i, j \le r+d$.

Theorem 2.8 is proved. □

3.8. **Proof of Theorems 1.11 and 2.9.** We retain the notation of Section 3.5. Theorem 2.8 implies that that the homomorphism of Theorem 2.9 is well-defined. Applying the argument of Section 3.5 once again, we obtain the desired assertion.

Thus, Theorem 2.9 is proved. □

Now prove Theorem 1.11.

Taking $\mathfrak{g} = \mathfrak{g}'' = \mathfrak{g}$ in Theorem 2.9 with $r = 2$ we obtain the assertion of (a).

Taking $\mathfrak{g} = \mathfrak{g}'' = so_{2n}$ in Theorem 2.9 with the branch at the vertex $r = n-2$, $d = 2$ of the Dynkin diagram $I = \{1, \ldots, n\}$ we obtain the assertion of (b).

Prove (c). Let $\mathfrak{g} = \widehat{sl}_n$, $I = \{1, \ldots, n\}$, 1 and n are neighbors. Taking $sl_n \subset \mathfrak{g}$ corresponding to $I \setminus \{n\} = \{1, \ldots, n-1\}$ gives us relations of $sl_n^{(\mathbf{b})}$ for $u_1, \ldots, u_{n-1}$. The remaining relations of $\mathfrak{g}^{(\mathbf{b})}$ can be obtained by the cyclic symmetry $i \mapsto i+1 \mod n$.

Applying the argument of Section 3.5 once again, we see that the subalgebra generated by u_i, $i \in I$ is isomorphic to $\mathfrak{g}^{(\mathbf{b})}$, i.e., is its edge model.

Theorem 1.11 is proved. □

4. Appendix: Flat deformations

For any ring R and any set X denote by $R\langle X\rangle$ the free R-algebra freely generated by X.

Let B be an algebra over R and $X \subset B$ be a generating set. Denote by $\varphi_{X,B}$ the canonical surjection $R\langle X\rangle \twoheadrightarrow B$, denote $J_X := Ker\ \varphi_{X,B}$. The natural filtration on $R\langle X\rangle$ with $\deg x = 1$ for all $x \in X$ induces a natural filtration on B, so let grB be the associated graded algebra of B. Clearly, $grR\langle X\rangle = R\langle X\rangle$. For any subset $S \subset B$ denote by grS the image of S in grB.

If A is a filtered algebra and J is a two-sided ideal of A, then
grA/J is canonically isomorphic to grA/grJ. In particular,

$$grB = R\langle grX\rangle / grJ_X \ .$$

Denote by $\varphi_{grX,grB}$ the canonical surjection $R\langle grX\rangle \twoheadrightarrow grB = R\langle grX\rangle/grJ_X$.

Proposition 4.1. Let B be an algebra over R generated by $X \subset B$. Suppose that Y is a subset of $R\langle X\rangle$ such that

- Y is contained in the kernel of $\varphi_{X,B}$.

• The kernel of $\varphi_{grX,grB}$ is generated by grY.

Then $B = R\langle X\rangle/\langle Y\rangle$, where $\langle Y\rangle$ denotes the two-sided ideal of $R\langle X\rangle$ generated by Y.

Proof. Indeed, for any subset $Y \subset R\langle X\rangle$, the inclusion $grY \subset gr\langle Y\rangle$ implies the inclusion of ideals $\langle grY\rangle \subset gr\langle Y\rangle$. Now let Y be as in the assumptions of the proposition. Then the inclusion $Y \subset J_X$ implies the inclusion of ideals $\langle Y\rangle \subset J_X$ hence $gr\langle Y\rangle \subset grJ_X$ (if the latter inclusion is an equality, then so is the former). Taking into account that $grJ_X = \langle grY\rangle$, we obtain $gr\langle Y\rangle \subset \langle grY\rangle$. These two opposite inclusions imply that $gr\langle Y\rangle = grJ_X$. Therefore, $J_X = \langle Y\rangle$.

The proposition is proved. □

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