GENERALIZED ADJOINT ACTIONS

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Abstract. The aim of this paper is to generalize the classical formula $e^x y e^{-x} = \sum_{k\geq 0} \frac{1}{k!}(ad\ x)^k(y)$. We also obtain combinatorial applications to q-exponentials, q-binomials, and Hall-Littlewood polynomials.

1. Notation and main results

One of the most fundamental tools in Lie theory, the adjoint action of Lie groups on their Lie algebras, is based on the following formula:

$$e^x y e^{-x} = e^{ad\ x}(y) = \sum_{k\geq 0} \frac{1}{k!}(ad\ x)^k(y)\ , \tag{1.1}$$

where $(ad\ x)^k(y) = [x, [x, \ldots, [x, y], \ldots]]$ and $[a, b] = ab - ba$.

The aim of this paper is to generalize (1.1) by replacing e^t with any formal power series

$$f = f(t) = 1 + \sum_{k\geq 1} a_k t^k \tag{1.2}$$

over a field $\Bbbk$.

For any formal power series (1.2) over $\Bbbk$ define polynomials

$$P_k(t) = P_{f,k}(t) = (-1)^k \det \begin{pmatrix} 1 & a_1 t & a_2 t^2 & \ldots & a_k t^k \\ 1 & a_1 & a_2 & \ldots & a_k \\ 0 & 1 & a_1 & \ldots & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \ldots & 1 & a_1 \end{pmatrix}$$

for $k = 0, 1, 2, \ldots$. (with the convention that $P_0(t) = 1$). Clearly, $P_k(1) = 0$ for $k \geq 1$.

The following result is, probably, well-known (for readers' convenience, we prove it in Section 2).

Theorem 1.1. $\frac{f(tx)}{f(x)} = \sum_{k\geq 0} P_{f,k}(t) \cdot x^k$. *In particular,* $P_{f,k}(st) = \sum_{i=0}^{k} P_{f,i}(s) P_{f,k-i}(t) t^i$.

Furthermore, for any algebra $\mathcal{A}$ over $\Bbbk$, a subset $\mathbf{q} = \{q_1, \ldots, q_k\} \subset Z(\mathcal{A})$, $x, y \in \mathcal{A}$, and $k \geq 1$ define

$$(ad\ x)^{\mathbf{q}}(y) = [x, [x, \ldots, [x, y]_{q_1}, \ldots]_{q_{k-1}}]_{q_k}$$

where $[a, b]_q = ab - qba$. It is easy to see that

$$(ad\ x)^{\mathbf{q}}(y) = \sum_{j=0}^{k} (-1)^j e_j(q_1, \ldots, q_k) \cdot x^{k-j} y x^j\ ,$$

where $e_j(q_1, \ldots, q_k)$ is the j-th elementary symmetric function.

Theorem 1.2. *Let $\mathcal{A}$ be $\Bbbk$-algebra and suppose that f is any power series* (1.2) *with $a_k \neq 0$ for $k \geq 1$. Then*

$$f(x) y f(x)^{-1} = y + \sum_{k\geq 1} a_k (ad\ x)^{\mathbf{q}_k}(y)\ , \tag{1.3}$$

for any $x, y \in \mathcal{A}$, where $\mathbf{q_k} = \{q_{1k}, \ldots, q_{kk}\}$ is the set of roots of $P_{f,k}(t)$.

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Remark 1.3. A formula for $f(x)yf(x)^{-1}$ without assumption that $a_k \neq 0$ is given in Proposition 2.3.

It is easy to see that if $a_k = \frac{1}{k!}$ for all k, then $P_k(t) = \frac{(t-1)^k}{k!}$ which immediately recovers (1.1). Suppose now that $a_k = \frac{1}{[k]_q!}$ for all k, where $k_q! = [1]_q \cdots [k]_q$ is the q-factorial and $[\ell]_q = 1 + q + \cdots + q^{\ell-1}$. We will show (Proposition 2.5) that $P_{f,k}(t) = \dfrac{(t-1)(t-q)\cdots(t-q^{k-1})}{[k]_q!}$ for $f(t) = e_q^t = \sum_{k\geq 0} \dfrac{t^k}{[k]_q!}$, therefore, recover the following famous result (see e.g., [3]).

Theorem 1.4. *Let* $e_q^x = \sum_{k\geq 0} \dfrac{x^k}{[k]_q!}$ *be the q-exponential. Then* $e_q^x \cdot y \cdot (e_q^x)^{-1} = \sum_{k\geq 0} \dfrac{1}{[k]_q!} (ad\ x)^{\{1,q,\ldots,q^{k-1}\}}(y)$.

On the other hand, combining Theorem 1.1 and Proposition 2.5, we recover the following well-known properties of q-exponentials and q-binomials:

$$e_q^{q^n x} = e_q^x \left(1 + \sum_{k=1}^{n} \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{[k]_q!} x^k\right)$$

for $n \geq 0$, in particular,

$$e_q^{qx} = e_q^x \cdot (1 + (q-1)x)$$

and

$$1 + \sum_{k=1}^{n} \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{[k]_q!} x^k = \prod_{i=1}^{n} (1 + (q-1)q^{i-1}x)\ .$$

We conclude with a curious observation that the polynomials $P_{f,k}(t)$ are related to the Hall-Littlewood symmetric polynomials.

Proposition 1.5. *Suppose that* $f(t) = \prod_{k\geq 1}(1 - x_k t)$. *Then*

$$P_{f,k}(t) = Q_{(k)}(\mathbf{x}; t)$$

for all $k \geq 0$, *where* $\mathbf{x} = \{x_k, k \geq 0\}$ *is viewed an infinite set of variables,* $Q_\lambda(\mathbf{x}; t)$ *is Hall-Littlewood polynomial (*[2, Section 3.2]*), and* (k) *is a horizontal row of length* k. *In particular,*

$$Q_{(k)}(\mathbf{x}; t) = (-1)^k \det \begin{pmatrix} 1 & -e_1 t & e_2 t^2 & \ldots & (-1)^k e_k t^k \\ 1 & -e_1 & e_2 & \ldots & (-1)^k e_k \\ 0 & 1 & -e_1 & \ldots & (-1)^{k-1} e_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \ldots & 1 & -e_1 \end{pmatrix}$$

for all $k \geq 0$, *where* $e_k = e_k(\mathbf{x})$ *is the* k*-th elementary symmetric function.*

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2. Proofs

Proof of Theorem 1.1. We need the following well-known fact (attributed to Wronski, see e.g., [1]).

Lemma 2.1. *Let* f *be any formal power series* (1.2). *Then* $\dfrac{1}{f(t)} = 1 + \sum_{k\geq 1} D_k(f) t^k$, *where*

$$D_k(f) = (-1)^k \det \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_k \\ 1 & a_1 & a_2 & \ldots & a_{k-1} \\ 0 & 1 & a_1 & \ldots & a_{k-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \ldots & 1 & a_1 \end{pmatrix}$$

(with the convention $D_0(f) = 1$*).*

The following generalization of Lemma 2.1 is, apparently, well-known (for readers' convenience we prove it here).

Lemma 2.2. *Let* $f(t) = 1 + \sum_{k\geq 1} a_k t^k$, $g(t) = 1 + \sum_{k\geq 1} b_k t^k$ *be formal power series. Then*

$$\frac{g(t)}{f(t)} = \sum_{k\geq 0} D_k(g,f)t^k \ ,$$

where $D_k(g,f) = (-1)^k \det \begin{pmatrix} 1 & b_1 & \dots & b_{k-1} & b_k \\ 1 & a_1 & \dots & a_{k-1} & a_k \\ 0 & 1 & \dots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$ *(with the convention* $D_0(g,f) = 1$*).*

Proof. Indeed, using Lemma 2.1, we obtain (with the convention $b_0 = 1$):

$$\frac{g(t)}{f(t)} = g(t)\cdot\frac{1}{f(t)} = \left(\sum_{i\geq 0} b_i t^i\right)\left(\sum_{j\geq 0} D_j(f)t^j\right) = \sum_{k\geq 0} d_k t^k$$

where

$$d_k = \sum_{i=0}^{k} b_i D_{k-i}(f) = (-1)^k \det \begin{pmatrix} b_0 & b_1 & \dots & b_{k-1} & b_k \\ 1 & a_1 & \dots & a_{k-1} & a_k \\ 0 & 1 & \dots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

by Cramer rule because $b_i D_{k-i}(f) = (-1)^k \det \begin{pmatrix} 0 & 0 & \dots & b_i & \dots & 0 \\ 1 & a_1 & \dots & a_i & \dots & a_k \\ 0 & 1 & \dots & a_{i+1} & \dots & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 & a_2 \\ 0 & 0 & \dots & \dots & 1 & a_1 \end{pmatrix}$.

The lemma is proved. □

Then taking $b_k = a_k t^k$ for $k \geq 1$ in Lemma 2.2, we obtain the first assertion of Theorem 1.1.

To prove the second assertion of the theorem, compute $\frac{f(stx)}{f(x)}$ in two ways, using the first assertion:

$$\frac{f(stx)}{f(x)} = \sum_{k\geq 0} P_{f,k}(st)\cdot x^k$$

and $\dfrac{f(stx)}{f(x)} = \dfrac{f(stx)}{f(tx)}\cdot\dfrac{f(tx)}{f(x)} = \left(\sum_{i\geq 0} P_{f,i}(s)\cdot(tx)^i\right)\left(\sum_{j\geq 0} P_{f,j}(t)\cdot x^j\right)$. Comparing the coefficients of x^k in both series, we obtain the second assertion of Theorem 1.1. □

Proof of Theorem 1.2. We need the following result.

Proposition 2.3. *For any power series* f *as in* (1.2) *one has:*

(a) $P_{f,k}(t) = \sum_{j=0}^{k} a_{k-j} D_j(f)\cdot t^j$ *for all* $k \geq 0$.

(b) $f(x)yf(x)^{-1} = y + z_1 + z_2 + \cdots$, *where* $z_k = \sum_{i=0}^{k} a_i D_{k-i}(f)\cdot x^i y x^{k-i}$ *for all* $k \geq 1$.

Proof. Prove (a). Indeed, using Lemma 2.1, we obtain:

$$\frac{f(tx)}{f(x)} = \sum_{i,j\geq 0}\left(a_i t^i x^i\right)\left(D_j(f)x^j\right) = \sum_{k\geq 0}\left(\sum_{i=0}^{k} a_i D_{k-i}(f)\cdot t^i\right) \ .$$

This together with Theorem 1.1 proves (a).

Prove (b) now. Indeed,

$$f(x)yf(x)^{-1} = \sum_{i,j\geq 0} \left(a_i x^i\right) y \left(D_j(f)x^j\right) = \sum_{k\geq 0} \left(\sum_{j=0}^{k} a_i D_{k-i}(f) \cdot x^i y x^{k-i} \right)$$

This proves (b).

□

Now we can finish the proof of Theorem 1.2. Indeed, suppose that $P_{f,k}(t)$ is factored as

$$P_{f,k}(t) = a_k(t - q_{11}) \cdots (t - q_{1k}) \ .$$

Then, by Proposition 2.3(a), $a_i D_{k-i} = a_k(-1)^{k-i} e_{k-i}(q_{1i}, \ldots, q_{1k})$ for $i = 0, \ldots, k$. Therefore, in the notation of Proposition 2.3(b),

$$z_k = \sum_{i=0}^{k} a_k(-1)^{k-i} e_{k-i}(q_{11}, \ldots, q_{1k}) \cdot x^i y x^{k-i} = a_k(ad\ x)^{\mathbf{q}_k}(y)$$

for all $k \geq 1$, which together with Proposition 2.3(b) verifies (1.3).

Theorem 1.2 is proved. □

Proposition 2.4. $P_{e_q^t,k}(t) = \dfrac{(t-1)(t-q)\cdots(t-q^{k-1})}{[k]_q!}$ *for all* $k \geq 1$.

Proof. It suffices to show that $P_{e_q^t,k}(q^a) = 0$ for all $0 \leq a < k$. We proceed by induction in such pairs (a, k). If $a = 0$, then we have nothing to prove since $P_{f,k}(1) = 0$ for all f.

Using Theorem 1.1, we obtain:

$$P_{f,k}(q^a) = \sum_{i=0}^{k} P_{f,i}(q^b) P_{f,k-i}(q^{a-b}) q^{(a-b)i} \ .$$

Taking $f = e_q^t$, $1 \leq b \leq a < k$, and using the inductive hypothesis, this gives $P_{f,k}(q^a) = 0$ for any $1 \leq a < k$.

The proposition is proved. □

Corollary 2.5. *For all* $k \geq 1$ *one has:* $\det \begin{pmatrix} 1 & \frac{t}{[1]_q} & \frac{t^2}{[2]_q} & \cdots & \frac{t^k}{[k]_q} \\ 1 & \frac{1}{[1]_q} & \frac{1}{[2]_q} & \cdots & \frac{1}{[k]_q} \\ 0 & 1 & \frac{1}{[1]_q} & \cdots & \frac{1}{[k-1]_q} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & \frac{1}{[1]_q} \end{pmatrix} = \dfrac{(1-t)(q-t)\cdots(q^{k-1}-t)}{[k]_q!}.$

Proof of Proposition 1.5. Indeed, if $f(t)$ is as in Proposition 1.5, then

$$\frac{f(tu)}{f(u)} = \prod_{k\geq 1} \frac{1 - x_k tu}{1 - x_k u} = \sum_{k\geq 0} Q_{(k)}(\mathbf{x}; t) u^k$$

by [2, Equations (2.10) and (2.13)]. This and Theorem 1.1 imply that $P_{f,k} = Q_{(k)}(\mathbf{x}; t)$ for all $k \geq 0$, which proves the first assertion of Proposition 1.5.

To prove the second assertion, note that $a_k = (-1)^k e_k(\mathbf{x})$ for all $k \geq 0$ because of the well-known formula (see e.g., [2, Section 1.2]):

$$\prod_{k\geq 1}(1 - x_k t) = \sum_{k\geq 0}(-1)^k e_k(\mathbf{x}) t^k \ .$$

This and the first assertion of Proposition 1.5 imply the second assertion of the proposition. □

References

[1] A. Inselberg, On determinants of Toeplitz-Hessenberg matrices arising in power series, *J. Math. Anal. Appl.* **63** (1978), no. 2, 347–353.

[2] I. Macdonald, *Symmetric functions and Hall polynomials* Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[3] A. Volkov, Beyond the "pentagon identity," *Lett. Math. Phys.* **39** (1997), no. 4, 393–397.

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