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Hecke-Hopf algebras ☆

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Let W be a Coxeter group. The goal of the paper is to construct new Hopf algebras that contain Hecke algebras $H_{\mathbf{q}}(W)$ as (left) coideal subalgebras. Our *Hecke-Hopf algebras*[1] $\mathbf{H}(W)$ have a number of applications. In particular they provide new solutions of quantum Yang-Baxter equation and lead to a construction of a new family of endo-functors of the category of $H_{\mathbf{q}}(W)$-modules. Hecke-Hopf algebras for the symmetric group are related to Fomin-Kirillov algebras; for an arbitrary Coxeter group W the "Demazure" part of $\mathbf{H}(W)$ is being acted upon by generalized braided derivatives which generate the corresponding (generalized) Nichols algebra.

Contents

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1. Introduction and main results

It is well-known that Hecke algebras $H_q(W)$ of Coxeter groups W do not have interesting Hopf algebra structures since the only available one emerges via a complicated isomorphism with the group algebra of W and, moreover this would make $H_q(W)$ into yet another cocommutative Hopf algebra. The goal of this paper is to show how to extend a Hecke algebra $H_q(W)$ to a (non-cocommutative) Hopf algebra $\mathbf{H}(W)$ that contains $H_q(W)$ as a left coideal subalgebra.

We start with the simplest case when W is the symmetric group S_n generated by s_i, $i=1,\ldots,n-1$ subject to the usual Coxeter relations.

Definition 1.1. For $n\ge 2$ denote by $\mathbf{H}(S_n)$ the $\mathbb{Z}$-algebra generated by s_i and D_i, $i=1,\ldots,n-1$ subject to relations:

- $s_i^2=1$, $s_iD_i+D_is_i=s_i-1$, $D_i^2=D_i$, $i=1,\ldots,n-1$.
- $s_js_i=s_is_j$, $D_js_i=s_iD_j$, $D_jD_i=D_iD_j$ if $|i-j|>1$.
- $s_js_is_j=s_is_js_i$, $D_is_js_i=s_js_iD_j$, $D_js_iD_j=s_iD_jD_i+D_iD_js_i+s_iD_js_i$ if $|i-j|=1$.

Remark 1.2. We will leave as an exercise to the reader to show that the braid relations $D_iD_{i+1}D_i=D_{i+1}D_iD_{i+1}$ and Yang-Baxter relations $D_is_iD_{i+1}s_iD_{i+1}=D_{i+1}s_iD_{i+1}s_iD_i$ hold in $\mathbf{H}(S_n)$.

Theorem 1.3. *For any $n\ge 2$, $\mathbf{H}(S_n)$ is a Hopf algebra over $\mathbb{Z}$ with the coproduct Δ, the counit ε, and antipode anti-automorphism S given respectively by (for $i=1,\ldots,n-1$):*

$$\Delta(s_i)=s_i\otimes s_i,\ \Delta(D_i)=D_i\otimes 1+s_i\otimes D_i,\ \varepsilon(s_i)=1,\ \varepsilon(D_i)=0,$$
$$S(s_i)=s_i,\ S(D_i)=-s_iD_i\ .$$

We prove Theorem 1.3 along with its generalization, Theorem 1.25, in Section 7.7.

[1] In a recent preprint `arXiv:1608.07509` the term *Hopf-Hecke algebras* was used in different context.

Remark 1.4. In fact, the Hopf algebras $\mathbf{H}(S_n)$, $n = 3, 4, 5$ were studied in [1, Section 3.3] (equations (14)-(18) with $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \frac{1}{4}$) in the context of classification of finite-dimensional pointed Hopf algebras, with the presentation similar to that in Remark 1.27 below. It would be interesting to see how would the Hopf algebras $\mathbf{H}(S_n)$, $n \geq 6$ (as well as $\mathbf{H}(W)$, where W is any Coxeter group, see below) fit the classification program of pointed Hopf algebras started in [2] and, conversely, how would a rich theory of pointed Hopf algebras enhance the study of $\mathbf{H}(S_n)$ and their representations.

The algebra $\mathbf{H}(S_n)$ has some additional symmetries.

Theorem 1.5.

(a) *The assignments $s_i \mapsto -s_i$, $D_i \mapsto 1 - D_i$ define an automorphism of $\mathbf{H}(S_n)$.*
(b) *The assignments $s_i \mapsto -s_i$, $D_i \mapsto s_i + D_i$ define an automorphism of $\mathbf{H}(S_n)$.*
(c) *The assignments $s_i \mapsto s_i$, $D_i \mapsto D_i$ define an anti-automorphism of $\mathbf{H}(S_n)$.*

We prove Theorem 1.5 along with its generalization, Theorem 1.37 in Section 7.6.

Define a family of elements $D_{ij} \in \mathbf{H}(S_n)$, $1 \leq i < j \leq n$ by $D_{i,i+1} = D_i$ and $wD_{ij}w^{-1} = D_{w(i),w(j)}$ for any permutation $w \in S_n$ such that $w(i) < w(j)$ (it follows from Definition 1.1 that the elements D_{ij} are well-defined). Denote by $\mathbf{D}(S_n)$ the subalgebra of $\mathbf{H}(S_n)$ generated by all D_{ij}.

Proposition 1.6. *For all $n \geq 2$, $\mathbf{H}(S_n)$ factors as $\mathbf{H}(S_n) = \mathbf{D}(S_n) \cdot \mathbb{Z}S_n$ over $\mathbb{Z}$, i.e., the multiplication map defines an isomorphism of $\mathbb{Z}$-modules $\mathbf{D}(S_n) \otimes \mathbb{Z}S_n \widetilde{\longrightarrow} \mathbf{H}(S_n)$.*

We prove Proposition 1.6 in Section 7.7. The algebra $\mathbf{D}(S_n)$ is can be viewed as a deformed Fomin-Kirillov algebra because of the following result (see also Remark 5.26 for more details).

Proposition 1.7. *For $n \geq 2$ the algebra $\mathbf{D}(S_n)$ is generated by D_{ij}, $1 \leq i < j \leq n$ subject to:*

- *$D_{ij}^2 = D_{ij}$ for all $1 \leq i < j \leq n$.*
- *$D_{ij}D_{k\ell} = D_{k\ell}D_{ij}$ whenever $\{i, j\} \cap \{k, \ell\} = \emptyset$.*
- *$D_{ij}D_{jk} = D_{ik}D_{ij} + D_{jk}D_{ik} - D_{ik}$, $D_{jk}D_{ij} = D_{ij}D_{ik} + D_{ik}D_{jk} - D_{ik}$ for $1 \leq i < j < k \leq n$.*

We prove Proposition 1.7 along with its generalization, Proposition 1.31, in Section 7.7.

Remark 1.8. In Section 5 we construct a (Hopf) algebra of symmetries of $\mathbf{D}(S_n)$ and of its generalizations to arbitrary groups. These Hopf algebras can be viewed as generalizations of Nichols algebras.

Recall that Hecke algebra $H_q(S_n)$ is generated over $\mathbb{Z}[q,q^{-1}]$ by $T_1,\dots,T_{n-1}$ subject to relations:

- Braid relations $T_iT_jT_i = T_jT_iT_j$ if $|i-j|=1$ and $T_iT_j = T_jT_i$ if $|i-j|>1$.
- Quadratic relations $T_i^2 = (1-q)T_i + q$.

Theorem 1.9. *For any $n\ge 2$ the assignment $T_i \mapsto s_i + (1-q)D_i$, $i=1,\dots,n-1$ defines an injective homomorphism of $\mathbb{Z}[q,q^{-1}]$-algebras $\varphi : H_q(S_n) \hookrightarrow \mathbf{H}(S_n)\otimes\mathbb{Z}[q,q^{-1}]$.*

We prove Theorem 1.9 in Section 7.7.

Thus, it is natural to call $\mathbf{H}(S_n)$ the *Hecke-Hopf algebra* of S_n.

Theorem 1.9 implies that any $\mathbf{H}(S_n)\otimes\mathbb{Z}[q,q^{-1}]$-module is automatically an $H_q(S_n)$-module. That is, the tensor category $\mathbf{H}(S_n)\otimes\mathbb{Z}[q,q^{-1}] - Mod$ of $\mathbf{H}(S_n)\otimes\mathbb{Z}[q,q^{-1}]$-modules is equivalent to a sub-category of the (non-tensor) category $H_q(S_n)$-Mod. We can strengthen this by noting that the relations $\Delta(\varphi(T_i)) = s_i\otimes\varphi(T_i) + D_i\otimes(1-q)$ for $i=1,\dots,n-1$ imply the following result.

Corollary 1.10. *In the notation of Theorem 1.3, the image $\varphi(H_q(S_n))\cong H_q(S_n)$ is a left coideal subalgebra in $\mathbf{H}(S_n)$, in particular, the assignment $T_i\mapsto s_i\otimes T_i + D_i\otimes(1-q)$, $i=1,\dots,n-1$, is a (coassociative and counital) homomorphism of $\mathbb{Z}[q,q^{-1}]$-algebras:*

$$H_q(S_n) \to \mathbf{H}(S_n)\otimes H_q(S_n)\ . \tag{1.1}$$

In turn, the coaction (1.1) defines a large family of conservative endo-functors of the category $H_q(S_n) - Mod$.

Corollary 1.11. *For any $\mathbf{H}(S_n)$-module M the assignments $V\mapsto F_M(V) := M\otimes V$ define a family of endo-functors on $H_q(S_n) - Mod$ so that $F_{M\otimes N} = F_M\circ F_N$ for all $M,N\in\mathbf{H}(S_n) - Mod$.*

Remark 1.12. If $q=1$, then $\mathbb{C}S_n$ is a Hopf subalgebra of $\mathbf{H}(S_n)\otimes\mathbb{C}$. Of course, this gives a "classical" analogue $\underline{F}_M : \mathbb{C}S_n - Mod \to \mathbb{C}S_n - Mod$ of the functors F_M. However, we do not expect that, under the equivalence of $\mathbf{H}_q(S_n) - Mod$ with $\mathbb{C}S_n - Mod$, for a generic $q\in\mathbb{C}$, the functors F_M will identify with $\underline{F}_M$.

The following result shows the existence of a large number of finite-dimensional $\mathbf{H}(S_n)$-modules.

Proposition 1.13. *For any $n\ge 2$, the polynomial algebra $\mathbb{Z}[x_1,\dots,x_n]$ is an $\mathbf{H}(S_n)$-module algebra via the natural permutation action of S_n and*

$$D_i \mapsto \frac{1}{1-x_ix_{i+1}^{-1}}(1-s_i)\ ,$$

the i-th Demazure operator. In particular, any graded component of $\mathbb{Z}[x_1,\dots,x_n]$ is an $\mathbf{H}(S_n)$-submodule.

We prove Proposition 1.13 in Section 7.8.

As an application, for any quadratic solution of QYBE we construct infinitely many new quadratic solutions of QYBE (Section 2).

Now we generalize the above constructions to arbitrary Coxeter groups W.

Recall that a Coxeter group W is generated by $s_i, i\in I$ subject to relations $(s_is_j)^{m_{ij}}=1$, where $m_{ij}=m_{ji}\in\mathbb{Z}_{\ge 0}$ are such that $m_{ij}=1$ iff $i=j$.

Definition 1.14. For any Coxeter group $W=\langle s_i|i\in I\rangle$ we define $\hat{\mathbf{H}}(W)$ as the $\mathbb{Z}$-algebra generated by s_i, D_i, $i\in I$ subject to relations:

(i) Rank 1 relations: $s_i^2=1$, $D_i^2=D_i$, $s_iD_i+D_is_i=s_i-1$ for $i\in I$.
(ii) Coxeter relations: $(s_is_j)^{m_{ij}}=1$
(iii) Linear braid relations: $\underbrace{D_is_js_i\cdots s_{j'}}_{m_{ij}}=\underbrace{s_j\cdots s_{i'}s_{j'}D_{i'}}_{m_{ij}}$ for all distinct $i,j\in I$ with

$$m_{ij}\ne 0,\text{ where } i'=\begin{cases} i & \text{if } m_{ij} \text{ is even}\\ j & \text{if } m_{ij} \text{ is odd}\end{cases} \text{ and } \{i',j'\}=\{i,j\}.$$

Example 1.15. The linear braid relation for $W=S_3$ is $D_1s_2s_1=s_2s_1D_2$ and linear braid relations for the dihedral group W of order 8 are $D_1s_2s_1s_2=s_2s_1s_2D_1$ and $D_2s_1s_2s_1=s_1s_2s_1D_2$.

Theorem 1.16. *For any Coxeter group the algebra $\hat{\mathbf{H}}(W)$ is a Hopf algebra with the coproduct Δ, the counit ε, and antipode anti-automorphism S given respectively by (for $i\in I$):*

$$\Delta(s_i)=s_i\otimes s_i,\ \Delta(D_i)=D_i\otimes 1+s_i\otimes D_i,\ \varepsilon(s_i)=1,\ \varepsilon(D_i)=0,$$
$$S(s_i)=s_i,\ S(D_i)=-s_iD_i\ .$$

We prove Theorem 1.16 with its generalization to other groups, Theorem 3.2, in Section 7.1.

Define $\mathcal{S}:=\{ws_iw^{-1}\,|\,w\in W, i\in I\}$. This is the set of all reflections in W. It is easy to see that linear braid relations in $\hat{\mathbf{H}}(W)$ imply that for any $s\in\mathcal{S}$ there is a unique element $D_s\in\hat{\mathbf{H}}(W)$ such that $D_{s_i}=D_i$ for $i\in I$ and $D_{s_iss_i}=s_iD_ss_i$ for any $i\in I$, $s\in\mathcal{S}\setminus\{s_i\}$ (Lemma 7.18).

Let $\hat{\mathbf{D}}(W)$ be the subalgebra of $\hat{\mathbf{H}}(W)$ generated by all D_s, $s\in\mathcal{S}$ and $\mathbf{K}(W):=\bigcap\limits_{w\in W} w\hat{\mathbf{D}}(W)w^{-1}$.

By definition, $\mathbf{K}(W)$ is a subalgebra of $\hat{\mathbf{D}}(W)$ and $w\mathbf{K}(W)w^{-1}=\mathbf{K}(W)$ for all $w\in W$.

Theorem 1.17. *For any Coxeter group W the ideal $\mathbf{J}(W)$ generated by $\mathbf{K}(W) \cap Ker\ \varepsilon$ is a Hopf ideal, therefore, the quotient algebra $\underline{\mathbf{H}}(W) = \hat{\mathbf{H}}(W)/\mathbf{J}(W)$is a Hopf algebra.*

We prove Theorem 1.17 in Section 7.3.

Remark 1.18. In Section 3 we generalize Theorem 1.17 to arbitrary groups W (Theorem 3.5) and in Section 4 we generalize it even further – to the case when W is replaced by an arbitrary Hopf algebra H (Theorem 4.5).

We refer to $\underline{\mathbf{H}}(W)$ as the *lower Hecke-Hopf algebra* of W.

Definition 1.19. Given a Coxeter group W, a commutative unital ring $\Bbbk$, and $\mathbf{q} = (q_i) \in \Bbbk^I$ such that $q_i = q_j$ whenever m_{ij} is odd, a (generalized) Hecke algebra $H_{\mathbf{q}}(W)$ is a $\Bbbk$-algebra generated by T_i, $i \in I$, subject to relations:

- quadratic relations: $T_i^2 = (1 - q_i)T_i + q_i$ for $i \in I$.
- braid relations: $\underbrace{T_iT_j\cdots}_{m_{ij}} = \underbrace{T_jT_i\cdots}_{m_{ij}}$ for all distinct $i, j \in I$.

Main Theorem 1.20. *For any commutative unital ring $\Bbbk$ the assignments $T_i \mapsto s_i + (1 - q_i)D_i$, $i \in I$, define an injective homomorphism of $\Bbbk$-algebras $\underline{\varphi}_W : H_{\mathbf{q}}(W) \to \underline{\mathbf{H}}(W) \otimes \Bbbk$ (whose image is a left coideal subalgebra in $\underline{\mathbf{H}}(W) \otimes \Bbbk$).*

We prove Theorem 1.20 in Section 7.5.

The following is a corollary from the proof of Theorem 1.20 (in the case q_i are integer powers of q, it was proved in [12, Section 3.1]).

Corollary 1.21. *For any commutative unital ring $\Bbbk$ the Hecke algebra $H_{\mathbf{q}}(W)$ is a free $\Bbbk$-module, moreover, the elements T_w, $w \in W$ form a $\Bbbk$-basis in $H_{\mathbf{q}}(W)$.*

Now we will construct a "Hopf cover" $\mathbf{H}(W)$ of $\underline{\mathbf{H}}(W)$ with an easier to control using the following important structural result.

Theorem 1.22. *For any Coxeter group W, the algebra $\hat{\mathbf{D}}(W)$ is generated by all D_s, $s \in \mathcal{S}$ subject to relations $D_s^2 = D_s$, $s \in \mathcal{S}$. Furthermore, $\hat{\mathbf{H}}(W)$ factors as $\hat{\mathbf{H}}(W) = \hat{\mathbf{D}}(W) \cdot \mathbb{Z}W$, i.e., the multiplication map defines an isomorphism of $\mathbb{Z}$-modules $\hat{\mathbf{D}}(W) \otimes \mathbb{Z}W \widetilde{\longrightarrow} \hat{\mathbf{H}}(W)$.*

We prove Theorem 1.22 in Section 7.2.

Remark 1.23. In Section 3 we extend this factorization result to arbitrary groups (Theorem 3.6) and in Section 4 we generalize it even further (Lemma 4.15).

Using Theorem 1.22, we identify $\hat{\mathbf{D}}(W_J)$ with a subalgebra of $\hat{\mathbf{D}}(W)$ for any $J \subset I$ by claiming that $\hat{\mathbf{D}}(W_J)$ is generated by all D_s with $s \in \mathcal{S} \cap W_J$.

For distinct $i, j \in I$ denote by $\mathbf{K}_{ij}(W)$ the set of all elements in $\mathbf{K}(W_{\{i,j\}}) \cap Ker\ \varepsilon \subset \hat{\mathbf{D}}(W_{i,j})$ having degree at most m_{ij}, where we view the free algebra $\hat{\mathbf{D}}(W)$ as naturally filtered by $\deg D_s = 1$ for $s \in \mathcal{S}$ (clearly, $\mathbf{K}_{ij}(W) = \{0\}$ if $m_{ij} = 0$).

Theorem 1.24. *For any Coxeter group W the ideal $\underline{\mathbf{J}}(W)$ generated by all $\mathbf{K}_{ij}(W)$, $i, j \in I$, $i \neq j$, is a Hopf ideal, therefore, the quotient algebra $\mathbf{H}(W) = \hat{\mathbf{H}}(W)/\underline{\mathbf{J}}(W)$ is a Hopf algebra.*

We prove Theorem 1.24 in Section 7.3.

We call $\mathbf{H}(W)$ the *Hecke-Hopf algebra* of W.

When W is *simply-laced*, i.e., $m_{ij} \in \{0, 2, 3\}$ for all distinct $i, j \in I$, we find the presentation of $\mathbf{H}(W)$, thus generalizing that for S_n in Definition 1.1.

Theorem 1.25. *Suppose that W is simply-laced. Then the Hecke-Hopf algebra $\mathbf{H}(W)$ is generated by s_i, D_i, $i \in I$ subject to relations:*

- $s_i^2 = 1$, $s_i D_i + D_i s_i = s_i - 1$, $D_i^2 = D_i$ *for* $i \in I$.
- $s_j s_i = s_i s_j$, $D_j s_i = s_i D_j$, $D_j D_i = D_i D_j$ *if* $m_{ij} = 2$.
- $s_j s_i s_j = s_i s_j s_i$, $D_i s_j s_i = s_j s_i D_j$, $D_j s_i D_j = s_i D_j D_i + D_i D_j s_i + s_i D_j s_i$ *if* $m_{ij} = 3$.

We prove Theorem 1.25 in Section 7.7.

Remark 1.26. In Section 5 we show that by "homogenizing" the relations in Theorem 1.25, one obtains a Hopf algebra $\mathbf{H}_0(W)$ (Definition 5.23) which acts on $\mathbf{D}(W)$ via braided derivatives and thus is closely related to the corresponding Nichols algebra.

Remark 1.27. It follows from Theorem 1.25 that the algebra $\mathbf{H}(W) \otimes \frac{1}{2}\mathbb{Z}$ for simply-laced W has the following presentation in generators s_i and $d_i = D_i + \frac{1}{2}(s_i - 1)$:

- $s_i^2 = 1$, $s_i d_i + d_i s_i = 0$, $d_i^2 = 0$ for $i \in I$.
- $s_j s_i = s_i s_j$, $d_j s_i = s_i d_j$, $d_j d_i = d_i d_j$ if $m_{ij} = 2$.
- $s_j s_i s_j = s_i s_j s_i$, $d_i s_j s_i = s_j s_i d_j$, $d_j s_i d_j = s_i d_j d_i + d_i d_j s_i + \frac{1}{4}(s_i - s_i s_j s_i)$ if $m_{ij} = 3$.

Actually, both $\underline{\mathbf{H}}(W)$ and $\mathbf{H}(W)$ can be factored in the sense of Theorem 1.22 as follows.

Theorem 1.28. *$\underline{\mathbf{H}}(W) = \underline{\mathbf{D}}(W) \cdot \mathbb{Z}W$, $\mathbf{H}(W) = \mathbf{D}(W) \cdot \mathbb{Z}W$ for all Coxeter groups W, where $\underline{\mathbf{D}}(W)$, $\mathbf{D}(W)$ are respectively the images of $\hat{\mathbf{D}}(W)$ under the projections $\hat{\mathbf{H}}(W) \twoheadrightarrow \underline{\mathbf{H}}(W)$, $\hat{\mathbf{H}}(W) \twoheadrightarrow \mathbf{H}(W)$.*

We prove Theorem 1.28 in Section 7.3.

Remark 1.29. It is natural to ask whether $\underline{\mathbf{D}}(W)$ and $\mathbf{D}(W)$ are free as $\mathbb{Z}$-modules.

We extend Proposition 1.7 and provide an explicit description of $\mathbf{D}(W)$ for an arbitrary simply-laced Coxeter group W.

Definition 1.30. Given a Coxeter group W, we say that a pair (s, s') of distinct reflections is *compatible* if there are $i, j \in I$ and $w \in W$ such that $s = ws_iw^{-1}$, $s' = ws_jw^{-1}$, $\ell(ws_i) = \ell(w) + 1$, $\ell(ws_j) = \ell(w) + 1$.

For $w, w' \in W$ denote by $m_{w,w'} \in \mathbb{Z}_{\geq 0}$ the order of ww' in W (if it is infinite, we set $m_{w,w'} = 0$).

Proposition 1.31. *In the assumptions of Theorem 1.25, the algebra $\mathbf{D}(W)$ is generated by D_s, $s \in \mathcal{S}$ subject to relations:*

- $D_s^2 = D_s$ *for all* $s \in \mathcal{S}$.
- $D_sD_{s'} = D_{s'}D_s$ *for all compatible pairs* $(s, s') \in \mathcal{S} \times \mathcal{S}$ *with* $m_{s,s'} = 2$.
- $D_sD_{s'} = D_{ss's}D_s + D_{s'}D_{ss's} - D_{ss's}$ *for all compatible pairs* $(s, s') \in \mathcal{S} \times \mathcal{S}$ *with* $m_{s,s'} = 3$.

We prove Proposition 1.31 in Section 7.7.

Remark 1.32. It would be interesting to find a more explicit characterization of compatible pairs (s, s') with a given $m_{s,s'}$. For instance, we expect that in a simply-laced W each pair (s, s') of reflections with $m_{s,s'} = 2$, i.e., $ss' = s's$, is compatible.

The following is a refinement of Theorem 1.20.

Theorem 1.33. *In the notation of Theorem 1.20, the assignments $T_i \mapsto s_i + (1 - q_i)D_i$, $i \in I$, define an injective homomorphism of algebras $\varphi_W : H_{\mathbf{q}}(W) \hookrightarrow \mathbf{H}(W)$. Moreover, $\underline{\varphi}_W = (\pi_W \otimes 1) \circ \varphi_W$, where $\pi_W : \mathbf{H}(W) \twoheadrightarrow \underline{\mathbf{H}}(W)$ is the canonical surjective homomorphism of Hopf algebras (which is identity on $\mathbb{Z}W$ and $\pi_W(\mathbf{D}(W)) = \underline{\mathbf{D}}(W)$).*

We prove Theorem 1.33 in Section 7.5.

It follows from Theorem 1.25 that both $\pi_{S_2 \times S_2}$ and π_{S_3} are the identity maps and that both definitions of $\mathbf{H}(S_n)$ agree. One can ask whether π_{S_n} is an isomorphism for $n \geq 4$.

It follows from Theorem 1.33 that braid relations

$$\underbrace{D_iD_j\cdots}_{m_{ij}} = \underbrace{D_jD_i\cdots}_{m_{ij}} \tag{1.2}$$

hold in $\mathbf{D}(W)$. In fact, there are other relations in $\mathbf{D}(W)$.

Theorem 1.34. *Given a Coxeter group W, for any distinct $i, j \in I$ with $m := m_{ij} \geq 2$ and any $w \in W$ such that $\ell(ws_i) = \ell(w) + 1, \ell(ws_j) = \ell(w) + 1$ the following relations hold in*

$\mathbf{D}(W)$ *for all divisors* n *of* m, $r \in [1,n]$ *(where we abbreviated* $D_k := D_{w\cdot \underbrace{s_i s_j \cdots}_{2k-1} \cdot w^{-1}} \in \mathbf{D}(W)$, $k = 1, \ldots, m$*):*

(a) *Quadratic-linear relations (for* $1 \le p < \frac{m}{2n}$*):*

$$\sum_{0\le a<b<\frac{m}{n}: b-a=\frac{m}{n}-p} D_{r+an}D_{r+bn} = \sum_{0\le a'<b'<\frac{m}{n}: b'-a'=p} D_{r+b'n}D_{r+a'n} - \sum_{p\le c<\frac{m}{n}-p} D_{r+cn}\ ,$$

$$\sum_{0\le a<b<\frac{m}{n}: b-a=\frac{m}{n}-p} D_{r+bn}D_{r+an} = \sum_{0\le a'<b'<\frac{m}{n}: b'-a'=p} D_{r+a'n}D_{r+b'n} - \sum_{p\le c<\frac{m}{n}-p} D_{r+cn}\ .$$

(b) *Yang-Baxter type relations (for* $0 \le t \le \frac{m}{n}$*):*

$$\overrightarrow{\prod_{t\le a\le \frac{m}{n}-1}} (1 - D_{r+an}) \overrightarrow{\prod_{0\le b\le t-1}} D_{r+bn} = \overleftarrow{\prod_{0\le b\le t-1}} D_{r+bn} \overleftarrow{\prod_{t\le a\le \frac{m}{n}-1}} (1 - D_{r+an})\ .$$

We prove Theorem 1.34 in Section 7.4.

Remark 1.35. After the first version of the present paper was posted to Arxiv, Dr. Weideng Cui informed us that he found a presentation of $\mathbf{D}(W_{\{i,j\}})$ for $m_{ij} \in \{4,6\}$ in [7]. That is, if $m_{ij} = 4$, $\mathbf{D}(W_{\{i,j\}})$ is generated by D_s, $s \in \mathcal{S}$ subject to relations $D_s^2 = D_s$, $s \in \mathcal{S}$, the braid relations (1.2), and the relations from Theorem 1.34. If $m_{ij} = 6$, there are more relations than those prescribed by Theorem 1.34.

Remark 1.36. If W is not crystallographic, i.e., $m_{ij} \in \{5\} \sqcup \mathbb{Z}_{\ge 7}$ for some distinct $i, j \in I$, we expect even more relations in $\mathbf{D}(W)$, e.g., for $m_{ij} = 5$, one can show that:

$$\begin{aligned} &D_1D_2D_3D_4 + (D_1D_2D_4 + D_2D_3D_4 - D_2D_4)(D_5 - 1) \\ &= D_5D_4D_3D_1 + D_5D_3D_2D_1 - D_5D_3D_1 \end{aligned} \tag{1.3}$$

where we abbreviated $D_1 := D_{s_i}$, $D_2 = D_{s_is_js_i}$, $D_3 = D_{s_js_is_js_is_j} = D_{s_is_js_is_js_i}$, $D_4 = D_{s_js_is_j}$, $D_5 = D_{s_j}$ (as in Theorem 1.34). We do not expect (1.3) to follow from the quadratic relations (Theorem 1.34(a)).

Now we establish a number of symmetries of $\mathbf{H}(W)$ and $\mathbf{D}(W)$.

Theorem 1.37. *For any Coxeter group* W *one has:*

(a) *$\mathbf{H}(W)$ and $\underline{\mathbf{H}}(W)$ admit an anti-involution $\bar{\cdot}$ such that $\overline{s}_i = s_i$, $\overline{D}_i = D_i$ for $i \in I$.*
(b) *$\mathbf{D}(W)$ and $\underline{\mathbf{D}}(W)$ admit the following symmetries.*
 (i) *A W-action by automorphisms via $s_i(D_s) = \begin{cases} 1 - D_{s_i} & \text{if } s = s_i \\ D_{s_i s s_i} & \text{if } s = s_i \end{cases}$ for $s \in \mathcal{S}$, $i \in I$.*
 (ii) *An s-derivation d_s (i.e., $d_s(xy) = d_s(x)y + s(x)d_s(y)$) such that $d_s(D_{s'}) = \delta_{s,s'}$, $s, s' \in \mathcal{S}$.*
(c) *$\mathbf{H}(W)$ admits an involution θ such that $\theta(s_i) = -s_i$, $\theta(D_i) = 1 - D_i$ for $i \in I$.*

We prove Theorem 1.37 in Section 7.6.

Remark 1.38. Proposition 7.34 implies that for a finite Coxeter group W, the algebra $\underline{\mathbf{H}}(W)$ also admits an involution θ as in Theorem 1.37(c). Based on a more general argument of Proposition 7.31(b), we can conjecture this for all Coxeter groups. In fact, the "innocently looking" Theorem 1.37(c) is highly nontrivial, in particular, applying θ in the form $\theta = S^{-2}$ (according to Proposition 7.31(a)) to the braid relations (1.2) in $\mathbf{D}(W)$ one can obtain a large number of relations in degrees less than m_{ij}.

Using Theorem 1.33, we can extend Corollaries 1.10 and 1.11 to all Coxeter groups.

Corollary 1.39. *Let W be a Coxeter group. Then:*

(a) *$H_{\mathbf{q}}(W)$ is a left $\mathbf{H}(W)$-comodule algebra via $T_i \mapsto s_i \otimes T_i + D_i \otimes (1 - q_i)$ for $i \in I$.*
(b) *For any $\mathbf{H}(W)$-module M the assignments $V \mapsto F_M(V) := M \otimes V$ define a family of (conservative) endo-functors on $H_{\mathbf{q}}(W) - Mod$ so that $F_{M\otimes N} = F_M \circ F_N$ for all $M, N \in \mathbf{H}(W) - Mod$.*

Furthermore, let us extend Proposition 1.13 to all W. Recall from [11] that an $I \times I$-matrix $A = (a_{ij})$ is a generalized Cartan matrix if $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$ and $a_{ij} \cdot a_{ji} = 0$ implies $a_{ij} = a_{ji} = 0$.

The following is a (conjectural) generalization of Proposition 1.13 to *crystallographic* Coxeter groups W, i.e., such that $m_{ij} \in \{0, 2, 3, 4, 6\}$ for all distinct $i, j \in I$.

Conjecture 1.40. *Let $A = (a_{ij})$, $i, j \in I$ be a generalized Cartan matrix. Let $W = W_A$ be the corresponding crystallographic Coxeter group, i.e., $m_{ij} = \begin{cases} 2 + a_{ij}a_{ji} & \text{if } a_{ij}a_{ji} \leq 2 \\ 6 & \text{if } a_{ij}a_{ji} = 3 \\ 0 & \text{if } a_{ij}a_{ji} > 3 \end{cases}$ for $i, j \in I$, $i \neq j$ and let $\mathcal{L}_I = \mathbb{Z}[t_i^{\pm 1}], i \in I$. Then the assignments*

$$s_i(t_j) := t_i^{-a_{ij}} t_j,\ \ D_i(t_j) := t_j \frac{1 - t_i^{-a_{ij}}}{1 - t_i}\ , \tag{1.4}$$

for $i, j \in I$ turn $\mathcal{L}_I$ into an $\mathbf{H}(W)$-module algebra.

We verified the conjecture in the simply-laced case, i.e., when A is symmetric (Section 7.8). We also verified that $\hat{\mathbf{H}}(W)$, indeed, acts on $\mathcal{L}_I$ via (1.4) (Proposition 7.42) for any A. After the first version of the present paper was posted to Arxiv, Dr. Weideng Cui informed us that he proved the conjecture for $m_{ij} \in \{4, 6\}$ in [7] using his presentation of $\mathbf{H}(W_{\{i,j\}})$ (see Remark 1.35), that is, for all crystallographic Coxeter groups.

It would also be interesting to see if $\mathbf{K}(W)$ annihilates $\mathcal{L}_I$ as well, i.e., if the desired action of $\mathbf{H}(W)$ on $\mathcal{L}_I$ factors through that of $\underline{\mathbf{H}}(W)$.

We conclude Introduction with the observation that all results of this section extend to what we call *extended Coxeter groups* $\hat{W}$. Namely, given a Coxeter group $\langle s_i | i \in I\rangle$, we let $\hat{W}$ be any group generated by $\hat{s}_i$, $i \in I$ such that

- $\hat{s}_i^2$ is central for $i \in I$.
- braid relations: $\underbrace{\hat{s}_i\hat{s}_j\cdots}_{m_{ij}} = \underbrace{\hat{s}_j\hat{s}_i\cdots}_{m_{ij}}$ for all distinct $i, j \in I$.
- The assignments $\hat{s}_i \mapsto s_i$, define a (surjective) group homomorphism $\hat{W} \twoheadrightarrow W$.

Clearly, in any extended Coxeter group W one has a relation $\hat{s}_i^2 = \hat{s}_j^2$ whenever m_{ij} is odd. In particular, $\hat{S}_n$ is a central extension of S_n with the cyclic center.

Then Definitions 1.1 and 1.14 carry over and give $\mathbf{H}(\hat{S}_n)$ and $\hat{\mathbf{H}}(\hat{W})$ with the only modification: the rank 1 relations $s_iD_i + D_is_i = s_i - 1$ are replaced with $\hat{s}_iD_i + D_i\hat{s}_i = \hat{s}_i - \hat{s}_i^2$ because $\hat{s}_i$ is not necessarily an involution. Then Theorems 1.3, 1.5 and Proposition 1.6 hold for $\mathbf{H}(\hat{S}_n)$ with $\mathbf{D}(\hat{S}_n) = \mathbf{D}(S_n)$. So do Theorems 1.16, 1.17, 1.22, 1.24, 1.28, and 1.37 for $\hat{\mathbf{H}}(\hat{W})$ with $\hat{\mathcal{S}} = \{\hat{w}\hat{s}_i\hat{w}^{-1} \,|\, \hat{w} \in \hat{W}, i \in I\}$, $\hat{\mathbf{D}}(\hat{W}) = \hat{\mathbf{D}}(W)$, $\mathbf{K}(\hat{W}) = \mathbf{K}(W)$, and $\underline{\mathbf{K}}(\hat{W}) = \underline{\mathbf{K}}(W)$. By the very construction, the canonical homomorphism $\hat{W} \twoheadrightarrow W$ defines surjective homomorphisms of Hopf algebras $\hat{\mathbf{H}}(\hat{W}) \twoheadrightarrow \hat{\mathbf{H}}(W)$, $\underline{\mathbf{H}}(\hat{W}) \twoheadrightarrow \underline{\mathbf{H}}(W)$, and $\mathbf{H}(\hat{W}) \twoheadrightarrow \mathbf{H}(W)$.

Finally, to establish analogues of Theorems 1.20 and 1.33 for a given extended Coxeter group $\hat{W}$, in the notation of Definition 1.19, define a *(generalized) Hecke algebra* $H_{\mathbf{q}}(\hat{W})$ of $\hat{W}$, to be generated over a commutative ring $\Bbbk$ by T_i, z_i, $i \in I$ subject to relations:

- The assignments $z_i \mapsto \hat{s}_i^2$ define an injective homomorphism of algebras $\Bbbk[z_i, i \in I] \hookrightarrow \Bbbk\hat{W}$, where $\Bbbk[z_i, i \in I]$ denotes the subalgebra generated by z_i, $i \in I$.
- quadratic relations: $T_i^2 = (1 - q_i)T_i + q_iz_i$ for $i \in I$.
- braid relations: $\underbrace{T_iT_j\cdots}_{m_{ij}} = \underbrace{T_jT_i\cdots}_{m_{ij}}$ for all distinct $i, j \in I$.

Then Theorems 1.20, 1.33 and Corollary 1.21 hold verbatim for $H_{\mathbf{q}}(\hat{W})$ and $\underline{\mathbf{H}}(\hat{W})$, $\mathbf{H}(\hat{W})$.

2. New solutions of QYBE

We retain the notation of Section 1. The following is immediate.

Lemma 2.1. *Let $n \geq 3$. Then for a $\Bbbk$-module V and a $\Bbbk$-linear map $\Psi : V \otimes V \to V \otimes V$ the following are equivalent.*

(i) *the assignments $T_i \mapsto \Psi_i = \underbrace{Id_V \otimes \cdots Id_V}_{i-1} \otimes \Psi \otimes \underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-i-1}$ for $i = 1, \ldots, n-1$ define a structure of an $H_q(S_n)$-module on $V^{\otimes n}$;*

(ii) *Ψ satisfies the braid equation on $V^{\otimes 3}$ and the quadratic equation on $V^{\otimes 2}$:*

$$\Psi_1 \Psi_2 \Psi_1 = \Psi_2 \Psi_1 \Psi_2,\ \Psi^2 = (1-q)\Psi + q \cdot Id_{V \otimes V} \tag{2.1}$$

(where $\Psi_1 := \Psi \otimes Id_V$, $\Psi_2 := Id_V \otimes \Psi$).

We refer to any Ψ satisfying (2.1) as a *quadratic braiding* on V.

In a similar fashion, we obtain the following immediate result for $\mathbf{H}(S_n)$-modules.

Lemma 2.2. *Let $n \geq 3$. Then for any $\mathbb{Z}$-module U and any pair of $\mathbb{Z}$-linear maps $s, D : U \otimes U \to U \otimes U$ the following are equivalent:*

(a) *the assignments: $s_i \mapsto \underbrace{Id_U \otimes \cdots Id_U}_{i-1} \otimes s \otimes \underbrace{Id_U \otimes \cdots \otimes Id_U}_{n-i-1}$, $D_i \mapsto \underbrace{Id_U \otimes \cdots Id_U}_{i-1} \otimes D \otimes \underbrace{Id_U \otimes \cdots \otimes Id_U}_{n-i-1}$ for $i = 1, \ldots, n-1$ define a structure of an $\mathbf{H}(S_n)$-module on $U^{\otimes n}$;*

(b) *the assignments $s_1 \mapsto s \otimes Id_U$, $s_2 \mapsto Id_U \otimes s$, $D_1 \mapsto D \otimes Id_U$, $D_2 \mapsto Id_U \otimes D$ define a structure of an $\mathbf{H}(S_3)$-module on $U^{\otimes 3}$.*

We refer to any pair of $\mathbb{Z}$-linear maps $s, D : U \otimes U \to U \otimes U$ satisfying Lemma 2.2(b) as an $\mathbf{H}(S_3)$*-structure* on U.

Furthermore, given an $\mathbf{H}(S_3)$-structure $s, D : U \otimes U \to U \otimes U$ on any $\mathbb{Z}$-module U, any $\Bbbk$-module V and any $\Bbbk$-linear map $\Psi : V \otimes V \to V \otimes V$ define a $\Bbbk$-linear endomorphism Ψ_U of $(U \otimes V)^{\otimes 2}$ by:

$$\Psi_U = \tau_{23}^{-1} \circ (s \otimes \Psi + (1-q) D \otimes Id_{V \otimes V}) \circ \tau_{23}\ , \tag{2.2}$$

where $\tau_{23} : (U \otimes V) \otimes (U \otimes V) \widetilde{\to} (U \otimes U) \otimes (V \otimes V)$ is the permutation of two middle factors.

The following result was the starting point of the entire project.

Theorem 2.3. *Let $s, D : U \otimes U \to U \otimes U$ be any $\mathbf{H}(S_3)$-structure on a $\mathbb{Z}$-module U and let $\Psi : V \otimes V \to V \otimes V$ be a quadratic braiding on a $\Bbbk$-module V. Then*

(a) *The linear endomorphism Ψ_U of $(U \otimes V)^{\otimes 2}$ is also a quadratic braiding.*
(b) *The functor $F_{U^{\otimes n}}$ from Corollary 1.11 satisfies: $F_{U^{\otimes n}}(V^{\otimes n}) \cong (U \otimes V)^{\otimes n}$, where $V^{\otimes n}$ is naturally an $H_q(S_n)$-module by Lemma 2.1 via the quadratic braiding Ψ and $(U \otimes V)^{\otimes n}$ is naturally an $\mathbf{H}(S_n)$-module via the quadratic braiding Ψ_U.*

Proof. Let $\Psi : V \otimes V \to V \otimes V$ be a quadratic braiding. By Lemma 2.1, the assignment $T_i \mapsto \Psi_i$, $i = 1, \dots, n-1$ defines a $\Bbbk$-algebra homomorphism $H_q(S_n) \to End_{\Bbbk}(V^{\otimes n})$.

Furthermore, let U be a $\mathbb{Z}$-module with an $\mathbf{H}(S_3)$-structure $s, D : U \otimes U \to U \otimes U$. Then, clearly, $U^{\otimes n}$ is an $\mathbf{H}(S_n)$-module by Lemma 2.2. Tensoring these homomorphisms, we obtain an algebra homomorphism $\mathbf{H}(S_n) \otimes H_q(S_n) \to End_{\mathbb{Z}}(U^{\otimes n}) \otimes End_{\Bbbk}(V^{\otimes n}) \subset End_{\Bbbk}(U^{\otimes n} \otimes V^{\otimes n})$. Composing it with the coaction (1.1) and naturally identifying $U^{\otimes n} \otimes V^{\otimes n}$ with $(U \otimes V)^{\otimes n}$ we obtain a $\Bbbk$-algebra homomorphism $H_q(S_n) \to End_{\Bbbk}((U \otimes V)^{\otimes n})$ given by $T_i \mapsto (\Psi_U)_i$ for $i = 1, \dots, n-1$. In view of Lemma 2.1, Ψ_U is a quadratic braiding. This proves (a). Part (b) also follows.

The theorem is proved. $\square$

Remark 2.4. We found a particular case of Theorem 2.3 in [9, Formula (4.8)] and [10, Formula (32)], but the general case seems to be unavailable in the literature.

The following immediate corollary of Proposition 1.13 provides an example of an $\mathbf{H}(S_3)$-structure.

Corollary 2.5. *Let $U = \mathbb{Z}[x]$. Then the permutation of factors $s : U \otimes U \to U \otimes U$ and the Demazure operator $D = \frac{1}{1 - x_1 x_2^{-1}}(1 - s)$ on $U \otimes U = \mathbb{Z}[x_1, x_2]$ comprise an $\mathbf{H}(S_3)$-structure on U.*

3. Generalization to other groups

In this section we generalize the construction of Hecke-Hopf algebras to all groups. Indeed, let W be a group and $\mathcal{S}$ be a conjugation-invariant subset of $W \setminus \{1\}$, and let R be an integral domain.

For any functions $\chi, \sigma : W \times \mathcal{S} \to R$ let $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ be an R-algebra generated by W, as a group, and by D_s, $s \in \mathcal{S}$ subject to relations:

$$wD_s w^{-1} = \chi_{w,s} \cdot D_{wsw^{-1}} + \sigma_{w,s} \cdot (1 - wsw^{-1}) \tag{3.1}$$

for all $s \in \mathcal{S}$, $w \in W$;

$$\begin{bmatrix}|s|\\k\end{bmatrix}_{a_s} D_s(a_sD_s+b_s)(a_s^2D_s+b_s(1+a_s))\cdots(a_s^{k-1}D_s+b_s(1+a_s+\cdots+a_s^{k-2}))=0 \quad (3.2)$$

for all $s\in\mathcal{S}$ of finite order $|s|$ and $k=1,\ldots,|s|$, where we abbreviated $a_s:=\chi_{s,s}$, $b_s:=\sigma_{s,s}$ and $\begin{bmatrix}n\\k\end{bmatrix}_q=\prod_{i=1}^{k}\frac{q^{n+1-i}-1}{q^i-1}\in\mathbb{Z}_{\ge 0}[q]$ is the q-binomial coefficient.

Remark 3.1. If $a_s=\chi_{s,s}$ is an primitive $|s|$-th root of unity in $R^\times$, then the relations (3.2) simplify:

$$D_s(a_sD_s+b_s)(a_s^2D_s+b_s(1+a_s))\cdots(a_s^{|s|-1}D_s+b_s(1+a_s+\cdots+a_s^{|s|-2}))=0\ . \quad (3.3)$$

Otherwise, if a_s is not an $|s|$-th root of unity and R is a field, then $D_s=0$.

The following result generalizes Theorem 1.16.

Theorem 3.2. *For any group W, a conjugation-invariant set $\mathcal{S}\subset W\setminus\{1\}$, and any maps $\chi,\sigma:W\times\mathcal{S}\to R$, $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ is a Hopf algebra with the coproduct Δ, the counit ε, and the antipode anti-automorphism given respectively by (for $s\in\mathcal{S}$):*

$$\begin{aligned}&\Delta(s)=s\otimes s,\ \Delta(D_s)=D_s\otimes 1+s\otimes D_s,\ \varepsilon(s)=1,\ \varepsilon(D_s)=0,\\&S(s)=s^{-1},\ S(D_s)=-s^{-1}D_s\ .\end{aligned} \quad (3.4)$$

We prove Theorem 3.2 in Section 7.1.

For any $\chi,\sigma:W\times\mathcal{S}\to R$ denote by $\hat{\mathbf{D}}_{\chi,\sigma}(W)$ the R-algebra generated by all D_s, $s\in\mathcal{S}$ subject to all relations (3.2). By definition, one has an algebra homomorphism $\hat{\mathbf{D}}_{\chi,\sigma}(W)\to\hat{\mathbf{H}}_{\chi,\sigma}(W)$. This homomorphism is sometimes injective and implies a factorization of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$.

Theorem 3.3. *In the notation of Theorem 3.2, suppose that:*

- *χ and σ satisfy*

$$\chi_{w_1w_2,s}=\chi_{w_2,s}\cdot\chi_{w_1,w_2sw_2^{-1}}\in R^\times,\ \sigma_{w_1w_2,s}=\sigma_{w_2,s}+\chi_{w_2,s}\sigma_{w_1,w_2sw_2^{-1}} \quad (3.5)$$

  for all $w_1,w_2\in W$, $s\in\mathcal{S}$.
- *For any $s\in\mathcal{S}$ of finite order and $w\in W$: $\chi_{w,s}^{|s|}=1$ and there exists $\kappa_{w,s}\in\mathbb{Z}_{\ge 0}$ such that*

$$\sigma_{w,s}=\sigma_{s,s}(1+\chi_{s,s}+\cdots+\chi_{s,s}^{\kappa_{w,s}-1})\ . \quad (3.6)$$

Then $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ factors as $\hat{\mathbf{H}}_{\chi,\sigma}(W)=\hat{\mathbf{D}}_{\chi,\sigma}(W)\cdot RW$ over R (i.e., the multiplication map defines an isomorphism of R-modules $\hat{\mathbf{D}}_{\chi,\sigma}(W)\otimes RW\widetilde{\longrightarrow}\hat{\mathbf{H}}_{\chi,\sigma}(W)$) and is a free R-module.

We prove Theorem 3.3 in Section 7.2.

Remark 3.4. Any pair (χ, σ) satisfying (3.5) defines:

- A W-action on $V = \oplus_{s\in\mathcal{S}} R \cdot D_s$ via $w(D_s) = \sigma_{w,s} + \chi_{w,s} D_{wsw^{-1}}$ for $w \in W$, $s \in \mathcal{S}$ (see also Theorem 3.9(a) below).
- A function $\gamma \in Hom_R(RW \otimes V, RW)$ given by $\gamma(w \otimes D_s) = \sigma_{w,s} wsw^{-1}$ which is a Hochschild 2-cocycle, i.e., $w_1\gamma(w_2 \otimes v)w_1^{-1} - \gamma(w_1w_2 \otimes v) + \gamma(w_1 \otimes w_2(v)) = 0$ for all $w_1, w_2 \in W$, $v \in V$ (see also Proposition 4.11 with generalization to Hopf algebras).

In particular, for any function $c : W \to R$, the map $\sigma^c : W \times \mathcal{S} \to R$ given by

$$\sigma^c_{w,s} = \sigma_{w,s} + c_s - \chi_{w,s} c_{wsw^{-1}}$$

also satisfies the second condition (3.5) and thus σ^c is cohomological to σ.

Denote by $\tilde{\mathbf{D}}_{\chi,\sigma}(W)$ the subalgebra of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ generated by all D_s, $s \in \mathcal{S}$ (by definition, this is a homomorphic image of $\hat{\mathbf{D}}_{\chi,\sigma}(W)$ in $\hat{\mathbf{H}}_{\chi,\sigma}(W)$) and let

$$\mathbf{K}_{\chi,\sigma}(W) := \bigcap_{w\in W} w\tilde{\mathbf{D}}_{\chi,\sigma}(W)w^{-1} \ . \tag{3.7}$$

Denote by $\mathbf{H}_{\chi,\sigma}(W)$ the quotient algebra of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ by the ideal generated by $\mathbf{K}_{\chi,\sigma}(W) \cap Ker\ \varepsilon$.

Theorem 3.5. *In the notation of Theorem 3.2, suppose that $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ is a free R-module (e.g., R is a field). Then $\mathbf{H}_{\chi,\sigma}(W)$ is naturally a Hopf algebra.*

We prove Theorem 3.5 in Section 7.3. We will refer to $\mathbf{H}_{\chi,\sigma}(W)$ as a *Hopf envelope of* (W, χ, σ) (provided that $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ is a free R-module).

Furthermore, in the notation of Theorem 3.2 denote by $\mathbf{D}_{\chi,\sigma}(W)$ the quotient of $\tilde{\mathbf{D}}_{\chi,\sigma}(W)$ by the ideal generated by $\mathbf{K}_{\chi,\sigma}(W)$. By definition, one has an algebra homomorphism $\mathbf{D}_{\chi,\sigma}(W) \to \mathbf{H}_{\chi,\sigma}(W)$. Similarly to Theorem 3.3, this homomorphism is sometimes injective and implies a factorization of $\mathbf{H}_{\chi,\sigma}(W)$.

Theorem 3.6. *In the assumptions of Theorem 3.2, suppose that*

$$\chi_{s,s} \text{ is a primitive } |s|\text{-th root of unity } \forall\ s \in \mathcal{S} \text{ of finite order } |s| \ . \tag{3.8}$$

Then $\mathbf{H}_{\chi,\sigma}(W)$ is a Hopf algebra and it factors as $\mathbf{H}_{\chi,\sigma}(W) = \mathbf{D}_{\chi,\sigma}(W) \cdot RW$.

We prove Theorem 3.6 in Section 7.3. In fact, the lower Hecke-Hopf algebra $\underline{\mathbf{H}}(W)$ from Theorem 1.17 equals $\mathbf{H}_{\chi,\sigma}(W)$ for a special choice of χ, σ (see Proposition 7.4) which automatically satisfy (3.5), (3.6), and (3.8). For some groups W, say, complex reflection ones, we may expect an analogue of the Hecke-Hopf algebra $\mathbf{H}(W)$ as well.

Remark 3.7. We believe that classification problem of quadruples $(W, \mathcal{S}, \chi, \sigma)$ with any $s \in \mathcal{S}$ of finite order satisfying (3.5), (3.6), and (3.8), is of interest.

Similarly to Theorem 1.37, we can establish some symmetries of $\mathbf{H}_{\chi,\sigma}(W)$ in general.

Theorem 3.8. *In the notation of Theorem 3.2, suppose that* $\bar{\cdot}$ *is an involution on R such that $\overline{\chi}_{w,s} = \chi_{w,s^{-1}}$, $\overline{\sigma}_{w,s} = \sigma_{w,s^{-1}}$ for all $w \in W$, $s \in \mathcal{S}$. Then the assignments $\overline{w} = w^{-1}$, $\overline{D}_s = D_{s^{-1}}$ for $w \in W$, $s \in \mathcal{S}$ extends to a unique R-linear anti-involution of $\mathbf{H}_{\chi,\sigma}(W)$.*

The following is a generalization of parts (a) and (b) of Theorem 1.37.

Theorem 3.9. *In the assumptions of Theorem 3.3 suppose also that*

$$\sigma_{wsw^{-1},wsw^{-1}} = \sigma_{s,s} \tag{3.9}$$

for all $w \in W$, $s \in \mathcal{S}$ of finite order. Then:

(a) *Suppose that*

$$\sigma_{w,s_1}\sigma_{ws_1,s_2} \cdots \sigma_{ws_1\cdots s_{k-1},s_k} = 0 \tag{3.10}$$

for any $w \in W$, $k \geq 2$, and any $s_1, \ldots, s_k \in \mathcal{S}$ such that $s_1 \cdots s_k = 1$. Then the algebra $\mathbf{D}_{\chi,\sigma}(W)$ admits the W-action by automorphisms via $w(D_s) = \sigma_{w,s} + \chi_{w,s}D_{wsw^{-1}}$ for $w \in W$, $s \in \mathcal{S}$.

(b) *Suppose that for a given $s \in \mathcal{S}$ one has*

$$\sigma_{s^{-1},s_1}\sigma_{s^{-1}s_1,s_2} \cdots \sigma_{s^{-1}s_1\cdots s_{k-1},s_k} = 0 \tag{3.11}$$

for any $k \geq 2$ and any $s_1, \ldots, s_k \in \mathcal{S}$ such that $s_1 \cdots s_k = s$. Then $\mathbf{D}_{\chi,\sigma}(W)$ admits an s^{-1}-derivation ∂_s (i.e., $\partial_s(xy) = \partial_s(x)y + s^{-1}(x)\partial_s(y)$) such that $\partial_s(D_{s'}) = \delta_{s,s'}\sigma_{s^{-1},s}$, $s, s' \in \mathcal{S}$.

We prove Theorem 3.9 in Section 7.6. In fact, the algebra $\hat{\mathbf{D}}_{\chi,\sigma}(W)$ has these symmetries if (3.9) holds (Proposition 7.27(c)), however, (3.10) is needed for $\mathbf{K}_{\chi,\sigma}$ to be invariant the W-action and (3.11) is needed for $\mathbf{K}_{\chi,\sigma}(W)$ to be in the kernel of each ∂_s.

Remark 3.10. If R is a field, then the condition (3.10) implies that the transitive closure of the relation $ws \prec w$ iff $\sigma_{w,s} \neq 0$ is a partial order on W, which we can think of as a "generalized Bruhat order." This is justified by Proposition 7.10(b) which implies that if W is a Coxeter group and $\mathcal{S}$ is the set of all reflections in W, then (3.10) holds and the partial order coincides with the strong Bruhat order on W. It is also easy to see that the condition (3.11) holds for each simple reflection in any Coxeter group. So we can think of all s satisfying (3.11) as "generalized simple reflections."

Conjecture 3.11. *In the assumptions of Theorem 3.3 suppose that θ is an R-linear automorphism of RW such that $\theta(w) \in R^\times \cdot w$ for $w \in W$ and $\theta(s) = \chi_{s,s} \cdot s$ for $s \in \mathcal{S}$. Then θ uniquely extends to an algebra automorphism of $\mathbf{H}_{\chi,\sigma}(W)$ such that $\theta(D_s) = \sigma_{s,s} + \chi_{s,s} D_s$ for $s \in \mathcal{S}$.*

If one replaces $\mathbf{H}_{\chi,\sigma}(W)$ with $\hat{\mathbf{H}}_{\chi,\sigma}(W)$, the assertion of the conjecture is true (Proposition 7.27(b)). However, unlike that in Theorem 1.37(c), the question whether θ preserves $\mathbf{K}_{\chi,\sigma}(W)$ is still, open, which the conjecture, in fact, asserts.

The following is a natural consequence of the above results and constructions.

In the situation of Theorem 3.3 to a subset $\mathcal{S}_0 \subset \mathcal{S}$ and a function $\mathbf{q} : \mathcal{S}_0 \to R$ $(s \mapsto q_s)$ we assign a subalgebra $H_{\mathbf{q}}(W, \mathcal{S}_0)$ of $\mathbf{H}_{\chi,\sigma}(W) \otimes \Bbbk$ generated by all $s + (1 - q_s)D_s$, $s \in \mathcal{S}_0$. By the very construction, $H_{\mathbf{q}}(W, \mathcal{S}_0)$ is a left coideal subalgebra in $\mathbf{H}_{\chi,\sigma}(W)$.

We say that $H_{\mathbf{q}}(W, \mathcal{S}_0)$ is a *generalized Hecke algebra* if it is a deformation of RW_0, where W_0 is the subgroup of W generated by $\mathcal{S}_0$, or, more precisely, the restriction of the R-linear projection $\pi : \mathbf{H}_{\chi,\sigma}(W) \to RW$ given by $\pi(xw) = w$ for $x \in \mathbf{D}_{\chi,\sigma}$, $w \in W$ to $H_{\mathbf{q}}(W, \mathcal{S}_0)$, is an isomorphism of R-modules $H_{\mathbf{q}}(W, \mathcal{S}_0) \widetilde{\to} RW_0$.

Problem 3.12. *Classify generalized Hecke algebras.*

In Section 6 we solve the problem for finite cyclic groups W via generalized Taft algebras.

It would be interesting to compare our constructions with the Broue-Malle-Rouquier Hecke algebras ([6]) attached to complex reflection groups.

4. Generalization to Hopf algebras

In this section we will extend our constructions from algebras $\mathbf{H}_{\chi,\sigma}(W)$ to Hopf algebras $\mathbf{H}$ over a commutative ring R containing a Hopf subalgebra H and a left coideal subalgebra $\mathbf{D}$.

Recall that, given a coalgebra $\mathbf{H}$ over a commutative ring R, an R-submodule $\mathbf{K}$ is called a left (resp. right) coideal if $\Delta(\mathbf{K}) \subset \mathbf{H} \otimes \mathbf{K}$ (resp. $\Delta(\mathbf{K}) \subset \mathbf{K} \otimes \mathbf{H}$).

The following properties of left (and right) coideals are, apparently, well-known.

Proposition 4.1. *For any coalgebra $\mathbf{H}$ over R, one has:*

(a) *Sum of left coideals is also a left coideal.*
(b) *If $\mathbf{H}$ is a free R-module, then the intersection of left coideals is also a left coideal.*

Proof. Part (a) is immediate.

To prove (b), we need the following obvious (and, apparently, well-known) fact.

Lemma 4.2. *Let* $\mathbf{A}$ *be a free module over a commutative ring* R *and let* $\mathbf{B}$ *be an* R*-module and* $\mathbf{B_i}$, $\mathbf{i}\in\mathbf{I}$ *be a family of* R*-submodules in* $\mathbf{B}$*. Then* $\bigcap\limits_{\mathbf{i}\in\mathbf{I}}(\mathbf{A}\otimes\mathbf{B_i})=\mathbf{A}\otimes\left(\bigcap\limits_{\mathbf{i}\in\mathbf{I}}\mathbf{B_i}\right)$.

Indeed, if $\mathbf{B_i}$, $\mathbf{i}\in\mathbf{I}$ is a family of left coideals in $\mathbf{H}$, then

$$\Delta(\bigcap_{\mathbf{i}\in\mathbf{I}}\mathbf{B_i})\subset\bigcap_{\mathbf{i}\in\mathbf{I}}\Delta(\mathbf{B_i})\subset\bigcap_{\mathbf{i}\in\mathbf{I}}\mathbf{H}\otimes\mathbf{B_i}=\mathbf{H}\otimes\left(\bigcap_{\mathbf{i}\in\mathbf{I}}\mathbf{B_i}\right) .$$

by Lemma 4.2 taken with $\mathbf{A}=\mathbf{B}=\mathbf{H}$. This proves (b).

The proposition is proved. □

Let H be a Hopf algebra over R and let $\mathbf{H}$ be an H-module algebra (we denote the action by $h\otimes x\mapsto h(x)$). For any R-subalgebra $\mathbf{D}$ of $\mathbf{H}$ define

$$\mathbf{K}(H,\mathbf{D}):=\{x\in\mathbf{D}\,|\,H(x)\subset\mathbf{D}\} . \tag{4.1}$$

Lemma 4.3. $\mathbf{K}(H,\mathbf{D})$ *is a subalgebra of* $\mathbf{H}$ *invariant under the* H*-action.*

Proof. Indeed, for $x,y\in\mathbf{K}(H,\mathbf{D})$ we have $h(xy)=h_{(1)}(x)\cdot h_{(2)}(y)\in\mathbf{D}$ for all $h\in H$. Hence $xy\in\mathbf{K}(H,\mathbf{D})$ and the first assertion is proved.

Furthermore, given $x\in\mathbf{K}(H,\mathbf{D})$, $h\in H$ we have $h'(h(x))=(h'h)(x)\in\mathbf{D}$ for all $h'\in H$, therefore, $h(x)\in\mathbf{K}(H,\mathbf{D})$ for all $x\in\mathbf{K}(H,\mathbf{D})$, $h\in H$. This proves the second assertion.

The lemma is proved. □

The following is immediate.

Lemma 4.4. *Suppose that* H *is a Hopf algebra over* R *and also a subalgebra of an* R*-algebra* $\mathbf{H}$*. Then the assignments* $h\triangleright x:=h_{(1)}\cdot x\cdot S(h_{(2)})$, $h\in H$, $x\in\mathbf{H}$, *turn* $\mathbf{H}$ *into an* H*-module algebra.*

Replacing, if necessary, an H-module algebra $\mathbf{H}$ with the cross product $\tilde{\mathbf{H}}=\mathbf{H}\rtimes H$, we see that Lemma 4.4 is applicable to $\tilde{\mathbf{H}}$.

In the following result, we will use the action from Lemma 4.4 for constructing new Hopf algebras.

Theorem 4.5. *Let* $\mathbf{H}$ *be a Hopf algebra over* R*,* H *be a Hopf subalgebra of* $\mathbf{H}$*, and* $\mathbf{D}$ *be a left coideal subalgebra of* $\mathbf{H}$*. Suppose that* $\mathbf{H}$ *is free as an* R*-module. Then the ideal* $\mathbf{J}(H,\mathbf{D})$ *of* $\mathbf{H}$ *generated by* $\mathbf{K}(H,\mathbf{D})\cap Ker\ \varepsilon$ *is a Hopf ideal, hence* $\underline{\mathbf{H}}:=\mathbf{H}/\mathbf{J}(H,\mathbf{D})$ *is naturally a Hopf algebra.*

Proof. We need the following result.

Proposition 4.6. *In the assumptions of Theorem 4.5, $\mathbf{K}(H, \mathbf{D})$ is a left coideal subalgebra of $\mathbf{H}$.*

Proof. For an R-module $\mathbf{A}$ and an H-module $\mathbf{H}$ define the action of H on $\mathbf{A} \otimes \mathbf{H}$ by $h \triangleright (x \otimes y) = x \otimes h(y)$ for $x \in \mathbf{A}, y \in \mathbf{H}$.

We need the following result.

Lemma 4.7. *Let $\mathbf{H}$ be a Hopf algebra over R and let H be a Hopf subalgebra of $\mathbf{H}$. Then $h \triangleright \Delta(x) = (S(h_{(1)}) \otimes 1) \cdot \Delta(h_{(2)} \triangleright x) \cdot (h_{(3)} \otimes 1)$ (here $\triangleright$ is the adjoint action from Lemma 4.4) for $h \in H$, $x \in \mathbf{H}$, with the Sweedler notation $(\Delta \otimes 1) \circ \Delta(h) = (1 \otimes \Delta) \circ \Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$.*

Proof. Indeed, $(S(h_{(1)}) \otimes 1) \cdot \Delta(h_{(2)} \triangleright x) \cdot (h_{(3)} \otimes 1) = (S(h_{(1)}) \otimes 1) \cdot \Delta(h_{(2)}) \Delta(x) \Delta(S(h_{(3)})) \cdot (h_{(4)} \otimes 1) = (1 \otimes h_{(1)}) \cdot \Delta(x) \cdot (1 \otimes S(h_{(2)})) = h \triangleright \Delta(x)$ because $(S(h_{(1)}) \otimes 1) \cdot \Delta(h_{(2)}) = S(h_{(1)}) \cdot h_{(2)} \otimes h_{(3)} = 1 \otimes h$ and $\Delta(S(h_{(1)})) \cdot (h_{(2)} \otimes 1) = S(h_{(2)}) \cdot h_{(3)} \otimes S(h_{(1)}) = 1 \otimes S(h)$.

The lemma is proved. □

This proves that, in the assumptions of Theorem 4.5, we have $H \triangleright \Delta(x) \subset \mathbf{H} \otimes \mathbf{D}$ for all $x \in \mathbf{K}$. To finish the proof of Proposition 4.6, we need the following result.

Lemma 4.8. *For any free R-module $\mathbf{A}$ one has in the assumptions of (4.1):*

$$\{z \in \mathbf{A} \otimes \mathbf{D} \mid H \triangleright z \subset \mathbf{A} \otimes \mathbf{D}\} = \mathbf{A} \otimes \mathbf{K}(H, \mathbf{D}) \ . \tag{4.2}$$

Proof. Indeed, let $\mathbf{B}$ be an R-basis of $\mathbf{A}$. Write each $z \in \mathbf{A} \otimes \mathbf{H}$ as

$$z = \sum_{b \in \mathbf{B}} b \otimes x_b$$

where all $x_b \in \mathbf{D}$ and all but finitely many of them are 0. Then

$$h \triangleright z = \sum_{b \in \mathbf{B}} b \otimes h(x_b) \ .$$

In particular, if $h \triangleright z \in \mathbf{H} \otimes \mathbf{D}$ for some $h \in H$, then $h(x_b) \in \mathbf{D}$ for all $b \in \mathbf{B}$. Therefore, $H \triangleright z \subset \mathbf{H} \otimes \mathbf{D}$ implies that $x_b \in \mathbf{K}(H, \mathbf{D})$ for $b \in \mathbf{B}$.

This proves the inclusion of the left hand side of (4.2) into the right hand side. The opposite inclusion is obvious.

The lemma is proved. □

Therefore, Proposition 4.6 is proved. □

We need the following (probably, well-known) general result.

Proposition 4.9. *Let* $\mathbf{H}$ *be a Hopf algebra over* R *and let* $\mathbf{K} \subset \mathbf{H}$ *be a left or right coideal. Then the ideal* $\mathbf{J}$ *generated by* $\mathbf{K}^+ := \mathbf{K} \cap Ker\ \varepsilon$ *is a Hopf ideal, i.e.,* $\Delta(\mathbf{J}) \subset \mathbf{H} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{H}$, $S(\mathbf{J}) \subset \mathbf{J}$.

Proof. We will prove the assertion when $\mathbf{K}$ is a left coideal (for the right ones the proof is identical). We need the following well-known fact.

Lemma 4.10. *For any coalgebra* $\mathbf{H}$ *one has* $\Delta(h) - h \otimes 1 \in \mathbf{H} \otimes Ker\ \varepsilon$ *for all* $h \in \mathbf{H}$.

Indeed, taking into account that $\Delta(\mathbf{K}) \subset \mathbf{H} \otimes \mathbf{K}^+ \oplus \mathbf{H} \otimes 1$ for any left coideal $\mathbf{K} \subset \mathbf{H}$, where we abbreviated $\mathbf{K}^+ := \mathbf{K} \cap Ker\ \varepsilon$, Lemma 4.10 guarantees that

$$\Delta(h) - h \otimes 1 \in \mathbf{H} \otimes \mathbf{K}^+ \tag{4.3}$$

for all $h \in \mathbf{K}^+$. Therefore, $\Delta(\mathbf{K}^+) \subset \mathbf{H} \otimes \mathbf{K}^+ + \mathbf{K}^+ \otimes 1$. In turn, this implies that:

$$\Delta(\mathbf{J}) \subset (\mathbf{H}\otimes\mathbf{H})\cdot\Delta(\mathbf{K}^+)\cdot(\mathbf{H}\otimes\mathbf{H}) \subset (\mathbf{H}\otimes\mathbf{H})\cdot(\mathbf{H}\otimes\mathbf{K}^+ + \mathbf{K}^+\otimes 1)\cdot(\mathbf{H}\otimes\mathbf{H}) \subset \mathbf{H}\otimes\mathbf{J} + \mathbf{J}\otimes\mathbf{H}\ ,$$

i.e., $\mathbf{J}$ is a bi-ideal.

Furthermore, applying $\mathbf{m} \circ (S \otimes 1)$ to (4.3) and using the property of the antipode $m \circ (S \otimes 1) \circ \Delta = \varepsilon$, we obtain $\varepsilon(h) - S(h) \in S(\mathbf{H}) \cdot \mathbf{K}^+$ for all $h \in \mathbf{K}^+$, therefore, $S(\mathbf{K}^+) \subset \mathbf{H} \cdot \mathbf{K}^+$. Hence

$$S(\mathbf{J}) = \mathbf{H} \cdot S(\mathbf{K}^+) \cdot \mathbf{H} \subset \mathbf{H} \cdot (\mathbf{H} \cdot \mathbf{K}^+) \cdot \mathbf{H} = \mathbf{J}\ .$$

The proposition is proved. □

Clearly, the assertion of Theorem 4.5 follows from Propositions 4.6 and 4.9.

Theorem 4.5 is proved. □

Let H be a Hopf algebra over R, V be an $T(H)$-module (i.e., an R-linear map $H \otimes V \to V$), for an R-bilinear map $\gamma : H \times V \to H$ satisfying:

$$\gamma(1, v) = 0 \tag{4.4}$$

for all $v \in V$, let $\mathbf{H}_\gamma$ be an algebra generated by H (viewed as an algebra) and V subject to relations

$$h_{(1)} \cdot v \cdot S(h_{(2)}) = h(v) + \gamma(h, v) \tag{4.5}$$

for all $h \in H$, $v \in V$. Using the property of the antipode in H, it is easy to see that relations (4.5) are equivalent to:

$$hv = h_{(1)}(v) \cdot h_{(2)} + \beta(h \otimes v)$$

for all $h \in H$, $v \in V$ where $\beta : H \otimes V \to H$ is given by $\beta(h \otimes v) = \gamma(h_{(1)}, v)h_{(2)}$. This implies that $\mathbf{H}_\gamma = A_\mu$ in the notation of (A.10) and of Corollary A.7, where β is as above and $\nu : H \otimes V \to V \otimes H$ is given by $\nu(h \otimes v) = h_{(1)}(v) \otimes h_{(2)}$ for all $h \in H$, $v \in V$.

If $\gamma = 0$ and $T(H)$-action on V factors through an H-action, then $H_\gamma = T(V) \rtimes H$, the cross product. Using Corollary A.7, we obtain a criterion for factorization of $\mathbf{H}_\gamma$ into $T(V)$ and H.

Proposition 4.11. *Let $\gamma : H \times V \to H$ be an R-bilinear map satisfying (4.4). Then $\mathbf{H}_\gamma$ factors as $\mathbf{H}_\gamma = T(V) \cdot H$ (i.e., the multiplication map defines an isomorphism of R-modules $T(V) \otimes H \widetilde{\longrightarrow} \mathbf{H}_\gamma$) as an R-module iff V is an H-module and γ satisfies for all $h, h' \in H$, $v \in V$:*

$$\gamma(hh', v) = \gamma(h, h'(v)) + h \triangleright \gamma(h', v) \tag{4.6}$$

where $\triangleright$ denotes the adjoint action of the Hopf algebra H on itself (as in Lemma 4.4).

Proof. Let us identify both conditions of Corollary A.7 with $B = H$ and ν and γ as above. Namely, taking into account that $\nu \circ (\mathbf{m}_H \otimes Id_V)(h \otimes h' \otimes v) = \nu(hh' \otimes v) = (hh')_{(1)}(v) \otimes (hh')_{(2)}$,

$$(Id_H \otimes \mathbf{m}_H) \circ (\nu \otimes Id_H) \circ (Id_H \otimes \nu)((h \otimes h' \otimes v)) = (hh')_{(1)}(v) \otimes (hh')_{(2)} \ ,$$

the first condition of Corollary A.7 reads

$$(hh')_{(1)}(v) \otimes (hh')_{(2)} = h_{(1)}(h'_{(1)}(v)) \otimes h_{(2)}h'_{(2)} \tag{4.7}$$

for all $h, h' \in T(H)$, $v \in V$.

Furthermore, taking into account that $\beta \circ (\mathbf{m}_H \otimes Id_V)(h \otimes h' \otimes v) = \beta(hh' \otimes v)$,

$$\begin{aligned}
\mathbf{m}_H \circ (Id_H \otimes \beta)(h \otimes h' \otimes v) &= h\beta(h' \otimes h) \ , \\
\mathbf{m}_H \circ (\beta \otimes Id_H) \circ (Id_H \otimes \nu)(h \otimes h' \otimes v) &= \mathbf{m}_H \circ (\beta \otimes Id_H)(h \otimes h'_{(1)}(v) \otimes h'_{(2)}) \\
&= \beta(h \otimes h'_{(1)}(v))h'_{(2)} \ ,
\end{aligned}$$

the second condition of Corollary A.7 reads

$$\beta(hh' \otimes v) = h\beta(h' \otimes v) + \beta(h \otimes h'_{(1)}(v))h'_{(2)} \tag{4.8}$$

for all $h, h' \in H$, $v \in V$.

Let us show that (4.7) is equivalent to

$$(hh')(v) = h(h'(v)) \tag{4.9}$$

for $h \in H$, $v \in V$.

Indeed, multiplying both sides of (4.7) by $S((hh')_{(3)}) = S(h'_{(3)})S(h_{(3)})$ on the right we obtain (4.9) after cancellations. Conversely, by acting with the first factor of $\Delta(hh') = (hh')_{(1)} \otimes (hh')_{(2)} = h_{(1)}h'_{(1)} \otimes h_{(2)}h'_{(2)}$ on v and using (4.9), we obtain (4.7). Thus, (4.9) and (4.4) assert that V is an H-module (and vice versa).

Finally, let us show that (4.8) is equivalent to (4.6).

1. (4.8) => (4.6). Since $\beta(h \otimes v) = \gamma(h_{(1)}, v)h_{(2)}$, (4.8) becomes:

$$\gamma((hh')_{(1)}, v)(hh')_{(2)} = h\gamma(h'_{(1)}, v)h'_{(2)} + \gamma(h_{(1)}, h'_{(1)}(v))h_{(2)}h'_{(2)} \ .$$

Multiplying both sides by $S((hh')_{(3)}) = S(h'_{(3)})S(h_{(3)})$ on the right, we obtain after cancellations

$$\begin{aligned}\gamma(hh', v) &= h_{(1)}\gamma(h'_{(1)}, v)\varepsilon(h_{(2)})h'_{(2)}S(h'_{(3)})S(h_{(3)}) \\ &\quad + \gamma(h_{(1)}, h'_{(1)}(v))h_{(2)}h'_{(2)}S(h'_{(3)})S(h_{(3)}) \\ &= h_{(1)}\gamma(h', v)S(h_{(2)}) + \gamma(h, h'(v)),\end{aligned}$$

which coincides with (4.6).

2. (4.6) => (4.8). Since $\gamma(h, v) = \beta(h_{(1)} \otimes v)S(h_{(2)})$, (4.6) becomes:

$$\beta((hh')_{(1)} \otimes v)S((hh')_{(2)}) = \beta(h_{(1)} \otimes h'(v))S(h_{(2)}) + h \triangleright (\beta(h'_{(1)} \otimes v)S(h'_{(2)})) \ .$$

Multiplying both sides by $(hh')_{(3)} = h_{(3)}h'_{(3)}$, we obtain after cancellations

$$\begin{aligned}\beta(hh' \otimes v) &= \beta(h_{(1)} \otimes h'(v))S(h_{(2)})h_{(3)}h'_{(3)} + h_{(1)} \cdot \beta(h'_{(1)} \otimes v)S(h'_{(2)}) \cdot S(h_{(2)})h_{(3)}h'_{(3)} \\ &= \beta(h \otimes h'(v)) + h\beta(h'_{(1)} \otimes v)S(h'_{(2)}),\end{aligned}$$

which coincides with (4.8).

The proposition is proved. $\square$

It is well-known that if $\gamma = 0$, then $\mathbf{H}_\gamma$ is a Hopf algebra. Now we provide sufficient conditions on γ (one can show that they are also necessary) for $\mathbf{H}_\gamma$ to be a Hopf algebra.

Proposition 4.12. *Let H be a Hopf algebra over R, V be a $T(H)$-module, and $\gamma : H \times V \to H$ be an R-bilinear map satisfying (4.4). Suppose that:*

- *V has an H-coaction $\delta : V \to H \otimes V$ ($\delta(v) = v^{(-1)} \otimes v^{(0)}$ in a Sweedler-like notation) such that for all $v \in V$, $h \in H$ the Yetter-Drinfeld condition (see e.g., [2, Section 1.2]) holds:*

$$\delta(h(v)) = h_{(1)}v^{(-1)}S(h_{(3)}) \otimes h_{(2)}(v^{(0)}) \ . \tag{4.10}$$

- $\Delta(\gamma(h,v)) = \gamma(h,v)\otimes 1 + h_{(1)}v^{(-1)}S(h_{(3)})\otimes\gamma(h_{(2)},v^{(0)})$ *and* $\varepsilon(\gamma(h,v)) = 0$ *for* $v\in V$, $h\in H$.

Then $\mathbf{H}_\gamma$ *is a Hopf algebra with the coproduct, counit, and the antipode extending those in* H *and determined by (for* $h\in H$, $v\in V$*):*

$$\Delta(v) = v\otimes 1 + \delta(v) = v\otimes 1 + v^{(-1)}\otimes v^{(0)},\ \varepsilon(v) = 0,\ S(v) = -S(v^{(-1)})v^{(0)}$$

Proof. We need the following general result.

Lemma 4.13. *Let* H *be a Hopf algebra over* R *and let* V *be a left comodule over* H *(i.e., one has a co-associative and co-unital linear map* $\delta : V\to H\otimes V$*). Then the free product of* R*-algebras* $\mathbf{H} := H * T(V)$ *is a Hopf algebra over* R *with the coproduct, counit, and the antipode extending those on* H *and determined by (for* $h\in H$, $v\in V$*):*

$$\Delta(v) = v\otimes 1 + \delta(v) = v\otimes 1 + v^{(-1)}\otimes v^{(0)},\ \varepsilon(v) = 0,\ S(v) = -S(v^{(-1)})v^{(0)}\ .$$

Proof. Indeed, each element $x\in\mathbf{H}$ can be written as sum of elements of the form:

$$x = h_0v_1h_1\cdots v_kh_k\ ,$$

where $h_0,h_1,\ldots,h_k\in H$, $v_1,\ldots,v_k\in V$, $k\ge 0$ (with the convention $x = h_0$ if $k = 0$). By setting

$$x\mapsto\Delta(x) = \Delta(h_0)(v_1\otimes 1 + \delta(v_1))\Delta(h_1)\cdots(v_k\otimes 1 + \delta(v_k))\Delta(h_k),$$
$$x\mapsto\varepsilon(x) = \begin{cases}\varepsilon(h_0) & \text{if } k = 0\\ 0 & \text{if } k > 0\end{cases},$$
$$x\mapsto S(x) = S(h_k)(-S(v_k^{(-1)})v_k^{(0)})S(h_1)\cdots(-S(v_1^{(-1)})v_1^{(0)})S(h_k)$$

one has well-defined R-linear maps $\Delta : \mathbf{H}\to\mathbf{H}\otimes\mathbf{H}$, $\varepsilon : \mathbf{H}\to R$, and $S : \mathbf{H}\to\mathbf{H}$, respectively.

Clearly, Δ is a homomorphism of algebras. Therefore, it suffices to verify the remaining compatibility conditions only on generators $v\in V$. Indeed:

$$(m\circ(\varepsilon\otimes 1)\circ\Delta)(v) = (m\circ(\varepsilon\otimes 1))(v\otimes 1 + \delta(v)) = (m\circ(\varepsilon\otimes 1)\circ\delta)(v) = v\ ,$$
$$(m\circ(1\otimes\varepsilon)\circ\Delta)(v) = (m\circ(1\otimes\varepsilon))(v\otimes 1 + \delta(v)) = m(v\otimes 1) = v\ ,$$
$$(m\circ(S\otimes 1)\circ\Delta)(v) = (m\circ(S\otimes 1)(v\otimes 1 + v^{(-1)}\otimes v^{(0)}) = S(v) + S(v^{(-1)})v^{(0)} = 0 = \varepsilon(v)\ ,$$
$$\begin{aligned}(m\circ(1\otimes S)\circ\Delta)(v) &= (m\circ(1\otimes S)(v\otimes 1 + v^{(-1)}\otimes v^{(0)}) = v + v^{(-1)}S(v^{(0)})\\ &= v - v^{(-2)}S(v^{(-1)})v^{(0)} = v - \varepsilon(v^{(-1)})v^{(0)} = 0 = \varepsilon(v)\ .\end{aligned}$$

This finished the proof of the lemma. $\square$

Furthermore, let $\mathbf{K}_\gamma$ be the R-submodule of $\mathbf{H} = H * T(V)$ generated by 1 and

$$\delta_{h,v} := h_{(1)} \cdot v \cdot S(h_{(2)}) - h(v) - \gamma(h,v)$$

for all $h \in H$, $v \in V$.

Lemma 4.14. *In the assumptions of Proposition 4.12, $\Delta(\mathbf{K}_\gamma) \subset H \otimes \mathbf{K}_\gamma$, in particular, $\mathbf{K}_\gamma$ is a left coideal in $\mathbf{H} = H * T(V)$.*

Proof. First, prove that

$$\Delta(\delta_{h,v}) = \delta_{h,v} \otimes 1 + h_{(1)} v^{(-1)} S(h_{(3)}) \otimes \delta_{h_{(2)},v^{(0)}} \tag{4.11}$$

for all $h \in H$, $v \in V$. Indeed,

$$\begin{aligned}
\Delta(\delta_{h,v}) &= \Delta(h_{(1)}) \cdot \Delta(v) \cdot \Delta(S(h_{(2)})) - \Delta(h(v)) - \Delta(\gamma(h,v)) \\
&= \Delta(h_{(1)}) \cdot (v \otimes 1 + \delta(v)) \cdot \Delta(S(h_{(2)})) - h(v) \otimes 1 - \delta(h(v)) - \Delta(\gamma(h,v)) \\
&= (\delta_{h,v} + \gamma(h,v)) \otimes 1 + \Delta(h_{(1)}) \cdot \delta(v) \cdot \Delta(S(h_{(2)})) - \delta(h(v)) - \Delta(\gamma(h,v)) \\
&= \delta_{h,v} \otimes 1 + \Delta(h_{(1)}) \cdot \delta(v) \cdot \Delta(S(h_{(2)})) - \delta(h(v)) \\
&\quad - h_{(1)} v^{(-1)} S(h_{(3)}) \otimes \gamma(h_{(2)}, v^{(0)}) \\
&= \delta_{h,v} \otimes 1 + h_{(1)} v^{(-1)} S(h_{(3)}) \otimes \delta_{h_{(2)},v^{(0)}} \ ,
\end{aligned}$$

where we used sequentially:

(1) The fact that

$$\begin{aligned}
\Delta(h_{(1)}) \cdot (v \otimes 1) \cdot \Delta(S(h_{(2)})) &= h_{(1)} \cdot v \cdot S(h_{(4)}) \otimes h_{(2)} S(h_{(3)}) \\
&= h_{(1)} \cdot v \cdot S(h_{(3)}) \otimes \varepsilon(h_{(2)}) = h_{(1)} \cdot v \cdot S(h_{(2)}) \otimes 1 \\
&= (\delta_{h,v} + h(v) + \gamma(h,v)) \otimes 1 \ .
\end{aligned}$$

(2) The second assumption of Proposition 4.12.

(3) The Yetter-Drinfeld condition (4.10) in the form

$$\begin{aligned}
&\Delta(h_{(1)}) \cdot \delta(v) \cdot \Delta(S(h_{(2)})) - \delta(h(v)) \\
&\quad = h_{(1)} v^{(-1)} S(h_{(4)}) \otimes h_{(2)} \cdot v^{(-1)} \cdot S(h_{(3)}) - h_{(1)} v^{(-1)} S(h_{(3)}) \otimes h_{(2)}(v^{(0)}) \\
&\quad = h_{(1)} v^{(-1)} S(h_{(3)}) \otimes (\delta_{h_{(2)},v^{(0)}} + \gamma(h_{(2)}, v^{(0)}))
\end{aligned}$$

This proves (4.11).

The lemma is proved. $\square$

Note that $\mathbf{K}_\gamma^+ := \mathbf{K}_\gamma \cap Ker\ \varepsilon$ is the R-submodule of $\mathbf{H} = H * T(V)$ generated by $\delta_{h,v}$, $h \in H$, $v \in V$. In view of Proposition 4.9, this and Lemma 4.14 guarantee that the ideal $\mathbf{J}_\gamma$ generated by $\delta_{h,v}$, $h \in H$, $v \in V$, is a Hopf ideal in $\mathbf{H}$. Therefore, $\underline{\mathbf{H}} = \mathbf{H}/\mathbf{J}_\gamma$ is a Hopf algebra.

The proposition is proved. □

We conclude the section with some general facts which we will use frequently.

Lemma 4.15. *Let $\mathbf{H}$ be an R-algebra, and H, $\mathbf{D}$ subalgebras of $\mathbf{H}$ such that $\mathbf{H}$ factors as $\mathbf{H} = \mathbf{D} \cdot H$ over R (i.e., the multiplication map defines an isomorphism of R-modules $\mathbf{D} \otimes H \widetilde{\longrightarrow} \mathbf{H}$). Let $\mathbf{K} \subset \mathbf{D}$ be an R-submodule such that $H \cdot \mathbf{K} \subset \mathbf{K} \cdot H$. Then the ideal $\mathbf{J}_\mathbf{K}$ of $\mathbf{H}$ generated by $\mathbf{K}$ factors as $\mathbf{I}_\mathbf{K} \cdot H$, where $\mathbf{I}_\mathbf{K}$ is the ideal of $\mathbf{D}$ generated by $\mathbf{K}$ and the quotient algebra $\underline{\mathbf{H}} = \mathbf{H}/\mathbf{J}_\mathbf{K}$ factors as $\underline{\mathbf{H}} = \underline{\mathbf{D}} \cdot H$, where $\underline{\mathbf{D}} = \mathbf{D}/\mathbf{I}_\mathbf{K}$.*

Proof. Indeed, $\mathbf{J}_\mathbf{K} = \mathbf{D} \cdot H \cdot \mathbf{K} \cdot \mathbf{D} \cdot H \subset \mathbf{D} \cdot \mathbf{K} \cdot H \cdot \mathbf{D} \cdot H = \mathbf{D} \cdot \mathbf{K} \cdot \mathbf{D} \cdot H = \mathbf{I}_\mathbf{K} \cdot H$ (because $\mathbf{I}_\mathbf{K} = \mathbf{D} \cdot \mathbf{K} \cdot \mathbf{D}$). The opposite inclusion is obvious, therefore, $\mathbf{J}_\mathbf{K} = \mathbf{I}_\mathbf{K} \cdot H$.

Finally, $\underline{\mathbf{H}} = \mathbf{H}/\mathbf{J}_\mathbf{K} = (\mathbf{D} \cdot H)/(\mathbf{I}_\mathbf{K} \cdot H) = (\mathbf{D}/\mathbf{I}_\mathbf{K}) \cdot H = \underline{\mathbf{D}} \cdot H$ as an R-module.

The lemma is proved. □

In some cases, we can describe $\mathbf{K}(H, \mathbf{D})$ explicitly.

Lemma 4.16. *Let W be a group. Suppose that $\mathbf{H}$ is an R-algebra which factors as $\mathbf{H} = \mathbf{D} \cdot RW$ over R, where $\mathbf{D}$ is a subalgebra of $\mathbf{H}$. Then, in the notation of Proposition A.8, one has (where the RW-action on $\mathbf{H}$ is given by conjugation): $\mathbf{K}(RW, \mathbf{D}) = \bigcap\limits_{w,w' \in W: w \ne w'} Ker\ \partial_{w,w'}$. Furthermore, $\partial_{w,w}(x) = wxw^{-1}$ for all $x \in \mathbf{D}$.*

Proof. Indeed, writing (A.11) in the form: $wxw^{-1} = \sum_{w,w' \in W} \partial_{w,w'}(x) w' w^{-1}$ for $w \in W$, $x \in \mathbf{D}$, we see that $wxw^{-1} \in \mathbf{D}$ iff $\partial_{w,w'}(x) = 0$ for all $w' \ne w$, in which case $wxw^{-1} = \partial_{w,w}(x)$.

The lemma is proved. □

5. Generalized Nichols algebras and symmetries of Hecke-Hopf algebras

Let W be a monoid and let $\mathcal{R} \subset W \times W$ be a preorder on W such that $(h, 1) \in \mathcal{R}$ iff $h = 1$. We say that W is *$\mathcal{R}$-finite* if $W_g = \{w \in W \mid (w, g) \in \mathcal{R}\}$ is finite.

Clearly, any finite monoid is $\mathcal{R}$-finite with $\mathcal{R} = W \times (W \setminus \{1\}) \cup \{(1, 1)\}$. Also any Coxeter group W is $\mathcal{R}$-finite with $\mathcal{R}$ being a Bruhat order on W.

Given an $\mathcal{R}$-finite monoid W, define the algebra $\mathbf{B}(W, \mathcal{R})$ over $\mathbb{Z}$ to be generated by $d_{g,h}$, $g, h \in W$ subject to relations $d_{g,w} = 0$ if $(w, g) \notin \mathcal{R}$, $d_{1,1} = 1$ and:

$$d_{gh,w} = \sum_{w_1, w_2 \in W: w_1 w_2 = w} d_{g,w_1} d_{h,w_2} \tag{5.1}$$

for all $g, h \in W$, $w \in W$.

Proposition 5.1. *For any $\mathcal{R}$-finite monoid W one has:*

(a) *the algebra $\mathbf{B}(W, \mathcal{R})$ is a bialgebra with the coproduct Δ and the counit ε given respectively by (for all $g, h \in W$):*

$$\Delta(d_{g,h}) = \sum_{w \in W} d_{g,w} \otimes d_{w,h}, \ \varepsilon(d_{g,h}) = \delta_{g,h} \tag{5.2}$$

(b) *Suppose that φ is any anti-automorphism of W such that $(\varphi \times \varphi)(\mathcal{R}) = \mathcal{R}$. Then the assignments $d_{g,h} \mapsto d_{\varphi(g),\varphi(h)}$, $g, h \in W$ define an anti-automorphism φ^* of $\mathbf{B}(W, \mathcal{R})$ such that $(\varphi^* \otimes \varphi^*) \circ \Delta = \Delta \circ \varphi^*$ and $\varepsilon \circ \varphi^* = \varepsilon$.*

Proof. Prove (a). Let $U(\mathcal{R})$ be the free $\mathbb{Z}$-module with the free basis $d_{g,h}$, $(g, h) \in \mathcal{R}$. The following is immediate.

Lemma 5.2. *$U(\mathcal{R})$ is a coalgebra with the coproduct and the counit given by (5.2).*

This implies that the tensor algebra $T(U(\mathcal{R}))$ is naturally a bialgebra. Denote by $\hat{\mathbf{B}}(W, \mathcal{R})$ the quotient of $T(U(\mathcal{R}))$ by the ideal J generated by $d_{1,1}-1$. Since $\Delta(d_{1,1}-1) = d_{1,1} \otimes d_{1,1} - 1 \otimes 1 = (d_{11} - 1) \otimes d_{11} + 1 \otimes (d_{1,1} - 1)$ and $\varepsilon(d_{1,1} - 1) = 0$, J is a bi-ideal hence $\hat{\mathbf{B}}(W, \mathcal{R})$ is a bialgebra.

For each $g, h \in W$ and $w \in W$ define elements $\delta_{g,h;w} \in \hat{\mathbf{B}}(W, \mathcal{R})$ by:

$$\delta_{g,h;w} := d_{gh,w} - \sum_{w_1, w_2 \in W : w_1 w_2 = w} d_{g,w_1} d_{h,w_2} \ .$$

Denote by $\mathbf{K} = \mathbf{K}(W, \mathcal{R})$ the $\mathbb{Z}$-submodule $\sum_{g,h,w \in W} \mathbb{Z} \cdot \delta_{g,h;w}$ of $\hat{\mathbf{B}}(W, \mathcal{R})$.

Lemma 5.3. *$\mathbf{K}(W, \mathcal{R})$ is a two-sided coideal in $\hat{\mathbf{B}}(W, \mathcal{R})$.*

Proof. Indeed,

$$\begin{aligned}
\Delta(\delta_{g,h;w}) &= \sum_{w' \in W} d_{gh,w'} \otimes d_{w',w} - \sum_{w_1, w_2, w'_1, w'_2 \in W} d_{g,w'_1} d_{h,w'_2} \otimes d_{w'_1,w_1} d_{w'_2,w_2} \\
&= \sum_{w' \in W} \delta_{g,h;w'} \otimes d_{w',w} + \sum_{w'_1, w'_2 \in W} d_{g,w'_1} d_{h,w'_2} \otimes d_{w'_1 w'_2,w} \\
&\quad - \sum_{w_1, w_2, w'_1, w'_2 \in W} d_{g,w'_1} d_{h,w'_2} \otimes d_{w'_1,w_1} d_{w'_2,w_2} \\
&= \sum_{w'} \delta_{g,h;w'} \otimes d_{w',w} + \sum_{w'_1, w'_2 \in W} d_{g,w'_1} d_{h,w'_2} \otimes \delta_{w'_1,w'_2;w} \ ,
\end{aligned}$$

where we used that $d_{gh,w'} = \delta_{g,h;w'} + \sum\limits_{w'_1,w'_2\in W: w'_1w'_2=w'} d_{g,w'_1} d_{h,w'_2}$.

This proves that $\Delta(\mathbf{K}) \subset \hat{\mathbf{B}}(W,\mathcal{R}) \otimes \mathbf{K} + \mathbf{K} \otimes \hat{\mathbf{B}}(W,\mathcal{R})$. It remains to show that $\varepsilon(\mathbf{K}) = 0$. We have $\varepsilon(\delta_{g,h;w}) = \delta_{gh,w} - \sum\limits_{w_1,w_2\in W, w_1w_2=w} \delta_{g,w_1}\delta_{h,w_2} = \delta_{gh,w} - \delta_{gh,w} = 0$ for all $g,h,w \in W$.

The lemma is proved. □

Denote by $\mathbf{J}$ the ideal of $\hat{\mathbf{B}}(W,\mathcal{R})$ generated by $\mathbf{K} = \mathbf{K}(W,\mathcal{R})$. Let us show that $\mathbf{J}$ is a bi-ideal in $\hat{\mathbf{B}}(W,\mathcal{R})$. Lemma 5.3 implies that $\varepsilon(\mathbf{J}) = 0$ and:

$$\begin{aligned}\Delta(\mathbf{J}) &\subset (\hat{\mathbf{B}}(W,\mathcal{R}) \otimes \hat{\mathbf{B}}(W,\mathcal{R})) \cdot (\hat{\mathbf{B}}(W,\mathcal{R}) \otimes \mathbf{K} + \mathbf{K} \otimes \hat{\mathbf{B}}(W,\mathcal{R})) \cdot (\hat{\mathbf{B}}(W,\mathcal{R}) \otimes \hat{\mathbf{B}}(W,\mathcal{R}))\\ &\subset \hat{\mathbf{B}}(W,\mathcal{R}) \otimes \mathbf{K} + \mathbf{K} \otimes \hat{\mathbf{B}}(W,\mathcal{R})\ .\end{aligned}$$

Finally, since $\mathbf{B}(W,\mathcal{R}) = \hat{\mathbf{B}}(W,\mathcal{R})/\mathbf{J}$, this implies that $\mathbf{B}(W,\mathcal{R})$ is a bialgebra. This proves (a).

Prove (b) now. Clearly, the assignments $d_{g,h} \mapsto d_{\varphi(g),\varphi(h)}$, $g,h \in W$ define an anti-automorphism φ^* of the coalgebra $U(\mathcal{R})$ such that $(\varphi^* \otimes \varphi^*) \circ \Delta = \Delta \circ \varphi^*$ and $\varepsilon \circ \varphi^* = \varepsilon$. Therefore, passing to the tensor algebra $T(U(\mathcal{R}))$ this gives an anti-automorphism $\tilde{\varphi}^*$ of $T(U(\mathcal{R}))$ with the same properties. Furthermore, $\tilde{\varphi}^*(d_{11}-1) = d_{11}-1$, thus, $\tilde{\varphi}^*$ preserves the above bi-ideal J generated by $d_{11}-1$, thus, gives a well-defined anti-automorphism of $\hat{\varphi}^*$ of the quotient bialgebra $\hat{\mathbf{B}}(W,\mathcal{R})$. In turn, we have

$$\begin{aligned}\hat{\varphi}(\delta_{g,h;w}) &:= d_{\varphi(gh),\varphi(w)} - \sum_{w_1,w_2\in W:\varphi(w_1w_2)=\varphi(w)} d_{\varphi(h),\varphi(w_2)} d_{\varphi(g),\varphi(w_1)}\\ &= d_{\varphi(h)\varphi(g),\varphi(w)} - \sum_{w'_1,w'_2\in W: w'_1w'_2=\varphi(w)} d_{\varphi(h),w'_1} d_{\varphi(g),w'_2} = \delta_{\varphi(h),\varphi(g);\varphi(w)}\end{aligned}$$

for all $g,h,w \in W$. In particular $\hat{\varphi}^*(\mathbf{K}(W,\mathcal{R}) = \mathbf{K}(W,\mathcal{R})$ hence the bi-ideal $\mathbf{J}$ generated by $\mathbf{K}(W,\mathcal{R})$ is $\hat{\varphi}^*$-invariant hence one has a natural anti-automorphism φ^* on the quotient bialgebra $\mathbf{B}(W,\mathcal{R}) = \hat{\mathbf{B}}(W,\mathcal{R})/\mathbf{J}$. This proves (b).

The proposition is proved. □

The following is an immediate corollary of Propositions 5.1 and A.8.

Corollary 5.4. *Let W be a monoid and $\mathcal{R}$ be preorder on W an so that W is $\mathcal{R}$-finite. Suppose that $\mathbf{H}$ is an R-algebra which factors as $\mathbf{H} = \mathbf{D} \cdot RW$ over R where $\mathbf{D}$ is a subalgebra. Suppose that $g \cdot \mathbf{D} \subset \mathbf{D} \cdot W_g$ for all $g \in W$. Then $\mathbf{D}$ is a module algebra over $\mathbf{B}(W,\mathcal{R}) \otimes R$ via $d_{g,h} \mapsto \partial_{g,h}$.*

Remark 5.5. The "universally acting" bialgebra $\mathbf{B}(W,\mathcal{R})$ is a particular case of the bialgebras emerging in the forthcoming joint paper of Yury Bazlov with the first author [3].

For any $\mathcal{R}$-finite monoid W let $\underline{\mathbf{B}}(W,\mathcal{R})$ be the quotient algebra of $\mathbf{B}(W,\mathcal{R})$ by the ideal generated by all $d_{gh,gh} - d_{g,g}d_{h,h}$.

Proposition 5.6. *In the assumptions of Proposition 5.1, suppose that $\mathcal{R}$ is a poset. Then $\underline{\mathbf{B}}(W,\mathcal{R})$ is naturally a bialgebra.*

Proof. For $g,h \in W$ let $\delta_{g;h} \in \mathbf{B}(W,\mathcal{R})$ be given by $\delta_{g;h} := d_{gh,gh} - d_{g,g}d_{h,h}$ and let $\underline{\mathbf{K}}(W,\mathcal{R})$ be the $\mathbb{Z}$-submodule $\sum\limits_{g,h\in W} \mathbb{Z}\cdot\delta_{g;h}$ of $\mathbf{B}(W,\mathcal{R})$.

Lemma 5.7. *$\underline{\mathbf{K}}(W,\mathcal{R})$ is a two-sided coideal in $\hat{\mathbf{B}}(W,\mathcal{R})$.*

Proof. Since $\mathcal{R}$ is a partial order, then $\Delta(d_{g,g}) = d_{g,g}\otimes d_{g,g}$ for $g \in W$. Therefore,

$$\Delta(\delta_{g;h}) = d_{gh,gh}\otimes d_{gh,gh} - d_{g,g}d_{h,h}\otimes d_{g,g}d_{h,h} = d_{gh,gh}\otimes \delta_{g;h} + \delta_{g;h}\otimes d_{g,g}d_{h,h},$$
$$\varepsilon(\delta_{g;h}) = 1 - 1\cdot 1 = 0$$

for all $g,h \in W$. Finally, $\varepsilon(\delta_{g;h}) = 1 - 1\cdot 1 = 0$ for all $g,h\in W$.

The lemma is proved. $\square$

Denote $\underline{\mathbf{J}} := \mathbf{B}(W,\mathcal{R})\cdot\mathbf{K}\cdot\mathbf{B}(W,\mathcal{R})$. Similarly to the proof of Proposition 5.1, one shows that this is the bi-ideal of the bialgebra $\mathbf{B}(W,\mathcal{R})$.

Finally, since $\underline{\mathbf{B}}(W,\mathcal{R}) = \mathbf{B}(W,\mathcal{R})/\underline{\mathbf{J}}$, this implies that $\underline{\mathbf{B}}(W,\mathcal{R})$ is a bialgebra.

The proposition is proved. $\square$

Remark 5.8. If W is a group, then one can ask whether $\underline{\mathbf{B}}(W,\mathcal{R})$ is a Hopf algebra. In that case, the antipode is given by:

$$S(d_{g,h}) = \sum (-1)^{k-1} d^{-1}_{w_1,w_1} d_{w_1,w_2} d^{-1}_{w_2,w_2}\cdots d^{-1}_{w_{k-1},w_{k-1}} d_{w_{k-1},w_k} d^{-1}_{w_k,w_k},$$

where the summation is over all $k \geq 1$ and distinct $w_1,\ldots,w_k \in W$ such that $w_1 = g$, $w_k = h$.

This computation is based on the following well-known fact: any lower triangular $n\times n$ matrix $A = (a_{ij})$ over an associative unital ring $\mathcal{A}$ such that all a_{ii} are invertible in $\mathcal{A}$, is invertible over $\mathcal{A}$ and

$$(A^{-1})_{ij} = \sum_{i=i_1>i_2>\cdots>i_k=j,k\geq 1} (-1)^{k-1} a^{-1}_{i_1,i_1} a_{i_1,i_2} a^{-1}_{i_2,i_2}\cdots a^{-1}_{i_{k-1},i_{k-1}} a_{i_{k-1},i_k} a^{-1}_{i_k,i_k}$$

for $1 \leq j \leq i \leq n$.

Furthermore, for $g,h,w \in W$ define elements $v^w_{g,h} \in \underline{\mathbf{B}}(W,\mathcal{R})$ by $v^w_{g,h} := d_{w,w}d_{g,h}d^{-1}_{h,h}d^{-1}_{w,w}$ and let $\mathcal{B}(W,\mathcal{R})$ be the subalgebra of $\underline{\mathbf{B}}(W,\mathcal{R})$ generated by all $v^w_{g,h}$.

We refer to $\mathcal{B}(W,\mathcal{R})$ as the *generalized Nichols algebra of* $(W,\mathcal{R})$ due to the following result.

Theorem 5.9. *Let W be a group and $\mathcal{R}$ be a partial order on W so that W is $\mathcal{R}$-finite. Then:*

(a) *$\mathcal{B}(W,\mathcal{R})$ is an algebra over $\mathbb{Z}$ generated by $v_{g,h}^w$, $g,h,w\in W$, subject to relations $v_{g,h}^w=0$ if $(h,g)\notin\mathcal{R}$ and (for $w,w',g,h\in W$):*

$$v_{w,w}^{w'}=1,\ v_{gh,w}^{w'}=\sum_{w_1,w_2\in W:w_1w_2=w} v_{g,w_1}^{w'}v_{h,w_2}^{w'w_1} \tag{5.3}$$

(b) *$\mathcal{B}(W,\mathcal{R})$ is a module algebra over $\mathbb{Z}W$ with respect to the action given by*

$$w(v_{g,h}^{w'})=v_{g,h}^{ww'} \tag{5.4}$$

for all $w,w',g,h\in W$.

(c) *The algebra $\underline{\mathbf{B}}(W,\mathcal{R})$ is isomorphic to the cross product $\mathcal{B}(W,\mathcal{R})\rtimes\mathbb{Z}W$.*

(d) *$\mathcal{B}(W,\mathcal{R})$ is a bialgebra in the (braided monoidal) category ${}^W_W\mathcal{YD}$ of Yetter-Drinfeld modules over $\mathbb{Z}W$ (see e.g., [2, Section 1.2]) with:*
- *W-grading given by $|v_{g,h}^w|=wg(wh)^{-1}$ for all $w,g,h\in W$.*
- *The (braided) coproduct given by $\underline{\Delta}(v_{g,h}^w)=\sum\limits_{w'\in W} v_{g,w'}^w\otimes v_{w',h}^w$ for all $g,h,w\in W$.*
- *The (braided) counit given by $\underline{\varepsilon}(v_{g,h}^w)=\delta_{g,h}$ for all $g,h,w\in W$.*

Proof. Let $\mathcal{B}'(W,\mathcal{R})$ be the algebra generated by all $v_{g,h}^w$, $g,h,w\in W$, subject to the relations $v_{g,h}^w=0$ if $(h,g)\notin\mathcal{R}$ and (5.3). We need the following immediate fact.

Lemma 5.10. (5.4) *defines a W-action on $\mathcal{B}'(W,\mathcal{R})$ by algebra automorphisms.*

Therefore, $\mathcal{B}'(W,\mathcal{R})$ is a $\mathbb{Z}W$-module algebra, which, in particular, proves (b).

Prove (a) and (c) now. Denote $\underline{\mathbf{B}}'(W,\mathcal{R}):=\mathcal{B}'(W,\mathcal{R})\rtimes\mathbb{Z}W$.

Proposition 5.11. *The assignments $d_{g,h}\mapsto v_{g,h}^1\cdot h$ for $g,h\in W$ define an isomorphism of algebras $\mathbf{f}_W:\underline{\mathbf{B}}(W,\mathcal{R})\widetilde{\to}\underline{\mathbf{B}}'(W,\mathcal{R})$ such that $\mathbf{f}_W(\mathcal{B}(W,\mathcal{R}))=\mathcal{B}'(W,\mathcal{R})$.*

Proof. Since $d_{g,h}=d_{w,w}^{-1}v_{g,h}^w d_{w,w}d_{h,h}$ in $\underline{\mathbf{B}}(W,\mathcal{R})$ for all $g,h,w\in W$, $(g,h)\in\mathcal{R}$, substituting this to (5.1) gives the relations

$$\begin{aligned}v_{gh,w}^1 d_{w,w}&=\sum_{w_1,w_2\in W:w_1w_2=w} v_{g,w_1}^1 d_{w_1,w_1}d_{w_1,w_1}^{-1}v_{h,w_2}^{w_1}d_{w_1,w_1}d_{w_2,w_2}\\&=\sum_{w_1,w_2\in W:w_1w_2=w} v_{g,w_1}^1 v_{h,w_2}^{w_1}d_{w,w}\end{aligned}$$

for all $g,h,w\in W$, which gives the second relation in (5.3) with $w'=1$. Finally, using

$$d_{w_1,w_1}v_{h,w_2}^{w_3}d_{w_1,w_1}^{-1}=v_{h,w_2}^{w_1w_3} \tag{5.5}$$

in $\underline{\mathbf{B}}(W,\mathcal{R})$ for any $h,w_1,w_2,w_3\in W$ we obtain the second relation (5.3) for with any $w'\in W$ (the relations $v_{w,w}^{w'}=1$ are obvious).

Let $\underline{V}_W := \oplus_{g,h\in W:(h,g)\in\mathcal{R}}\mathbb{Z}\cdot d_{g,h}$ Clearly, the assignments $d_{g,h}\mapsto d_{g,h}$ define a canonical surjective homomorphism $T(\underline{V}_W)\twoheadrightarrow \mathbf{B}(W,\mathcal{R})$, whose kernel is the ideal of $T(\underline{V}_W)$ generated by

$$d_{1,1}-1,\ d_{gh,gh}-d_{g,g}d_{h,h},\ d_{gh,w}-\sum_{w_1,w_2\in W:w_1w_2=w} d_{g,w_1}d_{h,w_2} \tag{5.6}$$

for $g,h,w\in W$.

Then the homomorphism of algebras $\hat{\mathbf{f}}_W:\underline{\mathbf{B}}(W,\mathcal{R})\widetilde{\to}\mathbf{B}'(W,\mathcal{R})$ by $\hat{\mathbf{f}}_W(d_{g,h})=v_{g,h}^1\cdot h$ for $g,h\in W$. Clearly, the image of (5.6) under $\hat{\mathbf{f}}_W$ is

$$v_{1,1}^1-1=0,\ v_{gh,gh}^1gh-v_{g,g}^1gv_{h,h}^1h=0,\ v_{gh,w}^1w-\sum_{w_1,w_2\in W:w_1w_2=w} v_{g,w_1}^1w_1v_{h,w_2}^1w_2=0$$

because relations (5.3) hold in $\mathcal{B}(W,\mathcal{R})$ and

$$w_1v_{h,w_2}^{w_3}w_1^{-1}=v_{h,w_2}^{w_1w_3} \tag{5.7}$$

in $\underline{\mathbf{B}}'(W,\mathcal{R})$ for any $h,w_1,w_2,w_3\in W$.

This proves that $\mathbf{f}_W$ is a well-defined homomorphism of algebras $\underline{\mathbf{B}}(W,\mathcal{R})\to\underline{\mathbf{B}}'(W,\mathcal{R})$. It is clearly surjective due to (5.5). Injectivity of $\mathbf{f}_W$ follows from that the defining relations (5.3) and (5.7) of $\mathbf{B}'(W,\mathcal{R})$ (together with $v_{g,h}^w=0$ if $(h,g)\notin\mathcal{R}$) already hold in $\mathbf{B}(W,\mathcal{R})$ (since $\mathbf{f}_W(d_{g,g})=g$ the relations (5.5) match (5.7)).

The proposition is proved. □

This finishes the proof of (a) and (c).

Prove (d) now. We need the following result.

Lemma 5.12. *In the assumptions of Theorem 5.9, one has:*

(a) *The $\mathbb{Z}$-module $Y_W := \oplus_{g,h,w\in W:(h,g)\in\mathcal{R}}\mathbb{Z}\cdot v_{g,h}^w$ (convention: $v_{g,h}^w=0$ if $(h,g)\notin\mathcal{R}$) is a Yetter-Drinfeld module over W with the W-action and W-grading as in Theorem 5.9(d).*

(b) *The maps $\underline{\Delta}:Y_W\to Y_W\otimes Y_W$ and $\underline{\varepsilon}:Y_W\to\mathbb{Z}$ given by Theorem 5.9(d) turn Y_W into a coalgebra in the (braided monoidal) category ${}^W_W\mathcal{YD}$ of Yetter-Drinfeld modules over W.*

Proof. Indeed, $|w(v_{g,h}^{w'})|=|v_{g,h}^{ww'}|=ww'gh^{-1}(ww')^{-1}=w|v_{g,h}^{w'}|w^{-1}$ for all $g,h,w,w'\in W$. This proves (a).

Prove (b). Clearly, both $\underline{\Delta}$ and $\underline{\varepsilon}$ commute with W-action. Also using the standard grading on $Y_W\otimes Y_W$ via $|x\otimes y|=|x|\cdot|y|$ for homogeneous $x,y\in Y_W$, we obtain $|v_{g,h}^w|=wg(wh)^{-1}$ and

$$|v^w_{g,w'} \otimes v^w_{w',h}| = |v^w_{g,w'}| \cdot |v^w_{w',h}| = wg(ww')^{-1}ww'(wh)^{-1} = wg(wh)^{-1} = |v^w_{g,h}|$$

for all $g, h, w, w' \in W$, therefore, $|\underline{\Delta}(v^w_{g,h})| = |v^w_{g,h}|$ for all $g, h, w \in W$. Similarly, $\underline{\varepsilon}(v^w_{g,}) = |\delta_{g,h}| = \delta_{g,h} \cdot 1$ for $g, h, w \in W$. This proves that both $\underline{\Delta}$ and $\underline{\varepsilon}$ are morphisms in ${}^W_W\mathcal{Y}D$. Coassociativity of $\underline{\Delta}$ and the counit axiom follow. This proves (b).

The lemma is proved. $\square$

Lemma 5.12(b) implies that $\underline{\Delta}$ viewed as a morphism from Y_W to the algebra $T(Y_W)\otimes T(Y_W)$ extends to a homomorphism $\underline{\Delta} : T(Y_W) \to T(Y_W) \otimes T(Y_W)$ of algebras in the braided monoidal category ${}^W_W\mathcal{Y}D$. Similarly, $\underline{\varepsilon}$ extends to a homomorphism of algebras $T(Y_W) \to \mathbb{Z}$, the latter viewed as the unit object in ${}^W_W\mathcal{Y}D$. Thus, $T(Y_W)$ is a bialgebra in the braided monoidal category ${}^W_W\mathcal{Y}D$.

For $g, h, w, w' \in W$ define elements $\underline{\delta}^{w'}_w$, $\underline{\delta}_{g,h;w'} \in T(Y_W)$ by

$$\underline{\delta}^{w'}_w = v^{w'}_{w,w} - 1,\ \underline{\delta}_{g,h,w;w'} = v^{w'}_{gh,w} - \sum_{w_1,w_2\in W:w_1w_2=w} v^{w'}_{g,w_1} v^{w'w_1}_{h,w_2}$$

and denote by $\underline{\mathcal{K}}(W, \mathcal{R})$ the $\mathbb{Z}$-submodule of $T(Y_W)$ generated by all $\underline{\delta}^{w'}_w$ and $\underline{\delta}_{g,h;w'}$.

Clearly, these elements are homogeneous, more precisely, $|\underline{\delta}^{w'}_w| = 1$, $|\underline{\delta}_{g,h,w;w'}| = w'gh(w'w)^{-1}$ for all $w, w', g, h \in W$. Moreover $w''(\underline{\delta}^{w'}_w) = \underline{\delta}^{w''w'}_w$, $w''(\underline{\delta}_{g,h,w;w'}) = \underline{\delta}_{g,h,w;w''w'}$ for all $w'' \in W$, in particular, $\underline{\mathcal{K}}(W, \mathcal{R})$ is a Yetter-Drinfeld submodule of $T(Y_W)$.

The following is a braided version of Lemma 5.7.

Lemma 5.13. *$\underline{\mathcal{K}}(W, \mathcal{R})$ is a two-sided coideal in $T(Y_W)$ in ${}^W_W\mathcal{Y}D$ and $\underline{\varepsilon}(\mathcal{K}(W, \mathcal{R})) = \{0\}$.*

Proof. Indeed, $\underline{\Delta}(\underline{\delta}^{w'}_w) = v^{w'}_{w,w} \otimes v^{w'}_{w,w} - 1 \otimes 1 = \underline{\delta}^{w'}_w \otimes v^{w'}_{w,w} + 1 \otimes \underline{\delta}^{w'}_w$.

Furthermore,

$$\begin{aligned}
\underline{\Delta}(\underline{\delta}_{g,h,w;w'}) &= \underline{\Delta}(v^{w'}_{gh,w}) - \sum_{w_1,w_2\in W:w_1w_2=w} \underline{\Delta}(v^{w'}_{g,w_1})\underline{\Delta}(v^{w'w_1}_{h,w_2}) \\
&= \sum_{w''\in W} v^{w'}_{gh,w''} \otimes v^{w'}_{w'',w} \\
&\quad - \sum_{w_1,w_2,w''_1,w''_2\in W:w_1w_2=w} (v^{w'}_{g,w''_1} \otimes v^{w'}_{w''_1,w_1})(v^{w'w_1}_{h,w''_2} \otimes v^{w'w_1}_{w''_2,w_2}) \\
&= \sum_{w''\in W} v^{w'}_{gh,w''} \otimes v^{w'}_{w'',w} \\
&\quad - \sum_{w_1,w_2,w''_1,w''_2\in W:w_1w_2=w} v^{w'}_{g,w''_1} v^{w'w''_1}_{h,w''_2} \otimes v^{w'}_{w''_1,w_1} v^{w'w_1}_{w''_2,w_2},
\end{aligned}$$

where we used the fact that $(v^{w'}_{g,w''_1} \otimes v^{w'}_{w''_1,w_1})(v^{w'w_1}_{h,w''_2} \otimes v^{w'w_1}_{w''_2,w_2}) = v^{w'}_{g,w''_1} v^{w'w''_1}_{h,w''_2} \otimes v^{w'}_{w''_1,w_1} v^{w'w_1}_{w''_2,w_2}$ because $(x\otimes y)(z\otimes t) = x\cdot(|y|(z))\otimes yt$ for any $x, y, z, t \in T(Y_W)$, where y is homogeneous of degree $|y|$, and $|v^{w'}_{w''_1,w_1}|(v^{w'w_1}_{h,w''_2}) = (w'w''_1(w'w_1)^{-1})(v^{w'w_1}_{h,w''_2}) = v^{w'w''_1}_{h,w''_2}$. Finally, taking into account that

$$\sum_{w''\in W} v^{w'}_{gh,w''} \otimes v^{w'}_{w'',w} = \sum_{w''} \underline{\delta}^{w'}_{g,h,w''} \otimes v^{w'}_{w'',w} + \sum_{w''_1,w''_2\in W} v^{w'}_{g,w''_1} v^{w'w''_1}_{h,w''_2} \otimes v^{w'}_{w''_1w''_2,w} ,$$

we obtain $\underline{\Delta}(\underline{\delta}_{g,h,w;w'}) = \sum\limits_{w''\in W} \underline{\delta}^{w'}_{g,h,w''} \otimes v^{w'}_{w'',w} + \sum\limits_{w''_1,w''_2\in W} v^{w'}_{g,w''_1} v^{w'w''_1}_{h,w''_2} \otimes \underline{\delta}_{w''_1w''_2,w;w'}$.

This proves that $\underline{\Delta}(\underline{\mathcal{K}}(W,\mathcal{R})) \subset \underline{\mathcal{K}}(W,\mathcal{R}) \otimes T(Y_W) + T(Y_W) \otimes \underline{\mathcal{K}}(W,\mathcal{R})$. It remains to show that $\underline{\varepsilon}(\mathcal{K}(W,\mathcal{R})) = \{0\}$.

Indeed, $\underline{\varepsilon}(\underline{\delta}^{w'}_w) = 1-1 = 0$, $\underline{\varepsilon}(\underline{\delta}_{g,h,w;w'}) = \delta_{gh,w} - \sum\limits_{w_1,w_2\in W:w_1w_2=w} \delta_{g,w_1}\delta_{h,w_2} = \delta_{gh,w} - \delta_{gh,w} = 0$.

The lemma is proved. □

Similarly to the conclusion of the proof of Proposition 5.1, denote $\underline{\mathbf{J}}' := T(Y_W) \cdot \underline{\mathcal{K}}(W,\mathcal{R}) \cdot T(Y_W)$. This is the ideal of $T(Y_W)$ generated by $\underline{\mathcal{K}}(W,\mathcal{R})$. Let us show that $\underline{\mathbf{J}}'$ is a bi-ideal in $\hat{\mathbf{B}}(W,\mathcal{R})$. Clearly, $\varepsilon(\underline{\mathbf{J}}') = 0$ by Lemma 5.13. Furthermore, Lemma 5.13 implies that

$$\begin{aligned}\Delta(\underline{\mathbf{J}}') &\subset (T(Y_W) \otimes T(Y_W)) \cdot (T(Y_W) \otimes \mathcal{K}(W,\mathcal{R}) + \mathcal{K}(W,\mathcal{R}) \otimes T(Y_W)) \\ &\quad \cdot (T(Y_W) \otimes T(Y_W)) \\ &\subset T(Y_W) \otimes \mathcal{K}(W,\mathcal{R}) + \mathcal{K}(W,\mathcal{R}) \otimes T(Y_W)\end{aligned}$$

because

$$\begin{aligned}&(T(Y_W) \otimes T(Y_W)) \cdot (T(Y_W) \otimes Y + Y \otimes T(Y_W)) \cdot (T(Y_W) \otimes T(Y_W)) \\ &\subset T(Y_W)YT(Y_W) \otimes T(Y_W) + T(Y_W) \otimes T(Y_W)YT(Y_W)\end{aligned}$$

for any Yetter-Drinfeld submodule Y of $T(Y_W)$.

Thus, $\underline{\mathbf{J}}'$ is a bi-ideal and $\mathcal{B}(W,\mathcal{R}) = T(Y_W)/\underline{\mathbf{J}}'$ is a bialgebra in ${}^W_W\mathcal{YD}$. This proves (d).

The theorem is proved. □

It turns out that for Coxeter groups these Hopf algebras are closely related to the graded versions of Hecke-Hopf algebra.

Definition 5.14. For any Coxeter group W let $\hat{\mathbf{H}}_0(W)$ be the algebra generated by s_i, d_i, $i \in I$ subject to relations:

(i) Rank 1 relations: $s_i^2 = 1$, $d_i^2 = 0$, $s_id_i + d_is_i = 0$ for $i \in I$.

(ii) Coxeter relations: $(s_is_j)^{m_{ij}} = 1$ and linear braid relations: $\underbrace{d_is_js_i\cdots s_{j'}}_{m_{ij}} = \underbrace{s_j\cdots s_{i'}s_{j'}d_{i'}}_{m_{ij}}$ for all distinct $i,j\in I$ with $m_{ij}\neq 0$, where $i' = \begin{cases} i & \text{if } m_{ij} \text{ is even} \\ j & \text{if } m_{ij} \text{ is odd}\end{cases}$ and $\{i',j'\}=\{i,j\}$.

That is, $\hat{\mathbf{H}}_0(W)$ is given by "homogenizing" Definition 1.14.

Similarly to Section 1, for any $s\in\mathcal{S}$ there is a unique element $d_s\in\hat{\mathbf{H}}_0(W)$ such that $d_{s_i}=d_i$ for $i\in I$ and $d_{s_iss_i}=s_iD_ss_i$ for any $i\in I$, $s\in\mathcal{S}\setminus\{s_i\}$. It is easy to see that $wd_sw^{-1}=\chi_{w,s}d_{wsw^{-1}}$, where $\chi_{w,s}$ is defined in (7.5) (cf. [15, Section 5]).

Denote by $\hat{\mathbf{D}}_0(W)$ the subalgebra of $\hat{\mathbf{H}}_0(W)$ generated by d_s, $s\in\mathcal{S}$.

The following is an immediate homogeneous analogue of Theorem 1.22.

Lemma 5.15. *For any Coxeter group W, one has:*

(a) *the algebra $\hat{\mathbf{D}}_0(W)$ is generated by all d_s, $s\in\mathcal{S}$ subject to relations $d_s^2=0$, $s\in\mathcal{S}$.*
(b) *$\hat{\mathbf{H}}_0(W)$ is naturally isomorphic to the cross product $\hat{\mathbf{D}}_0(W)\rtimes\mathbb{Z}W$ with respect to the action of W on $\hat{\mathbf{D}}_0(W)$ given by $w(d_s)=\chi_{w,s}d_s$ for $w\in W$, $s\in\mathcal{S}$, $\chi_{w,s}$ is defined in (7.5).*
(c) *$\hat{\mathbf{D}}_0(W)$ is graded by W via $|d_s|=s$ for $s\in\mathcal{S}$ and is a Hopf algebra in the category ${}^W_W\mathcal{YD}$ with the braided coproduct, counit, and the antipode given respectively by (for $s\in\mathcal{S}$):*

$$\underline{\Delta}(d_s)=d_s\otimes 1+1\otimes d_s,\ \underline{\varepsilon}(d_s)=0,\ \underline{S}(d_s)=-d_s\ .$$

Remark 5.16. In fact, $\hat{\mathbf{D}}_0(W)$ is a *pre-Nichols algebra* of the braided vector space $\oplus_{s\in\mathcal{S}}\mathbb{Z}\cdot d_s$ in terminology of [14].

We can "approximate" the braided bialgebra $\mathcal{B}(W,\mathcal{R}_W)$, where $\mathcal{R}_W$ is the strong Bruhat order on W (see e.g., [4, Section 2]), by the pre-Nichols algebra $\hat{\mathbf{D}}_0(W)$.

Theorem 5.17. *Let W be a Coxeter group and $\mathcal{R}_W$ be the strong Bruhat order on W. Then*

(a) *The assignments $s_i\mapsto d_{s_i,s_i}$, $d_i\mapsto -d_{s_i,1}$, $i\in I$ define a surjective homomorphism of bialgebras*

$$\hat{\varphi}_W:\hat{\mathbf{H}}_0(W)\twoheadrightarrow\underline{\mathbf{B}}(W,\mathcal{R}_W)\ . \tag{5.8}$$

whose restriction to $\mathbb{Z}W$ is injective.
(b) *In the notation of Theorem 5.9(d), the restriction of $\hat{\varphi}_W$ to $\hat{\mathbf{D}}_0(W)$ is a surjective homomorphism of bialgebras in ${}^W_W\mathcal{YD}$*

$$\underline{\hat{\varphi}}_W : \hat{\mathbf{D}}_0(W) \twoheadrightarrow \mathcal{B}(W, \mathcal{R}_W) \ . \tag{5.9}$$

whose restriction to $\underline{Y}_W := \oplus_{s\in\mathcal{S}}\mathbb{Z}\cdot d_s$ *is injective.*

Proof. We need the following result.

Proposition 5.18. *Let* W *be a Coxeter group. Then* $gx = g(x)\cdot g + \sum\limits_{h\in W\setminus\{g\}:(h,g)\in\mathcal{R}_W} \partial_{g,h}(x)h$ *for all* $g \in W$, $x\in \hat{\mathbf{D}}(W)$, *in the notation of Proposition A.8, where* $(g,x)\mapsto g(x)$ *is the* W*-action on* $\hat{\mathbf{D}}(W)$ *given by Theorem 1.37(b)(i). In particular,* $\hat{\mathbf{D}}(W)$ *is a module algebra over* $\underline{\mathbf{B}}(W,\mathcal{R}_W)$.

Proof. Let us prove the implication

$$\partial_{g,h} \neq 0 \ => \ (h,g)\in\mathcal{R}_W \tag{5.10}$$

for all $g, h \in W$.

We need the following result.

Lemma 5.19. *Let* W *be a Coxeter group. Suppose that* $w, w' \in W$ *such that* $(w', w) \in \mathcal{R}_W$ *and let* $i \in I$ *be such that* $\ell(s_i w) = \ell(w) + 1$ *and* $\ell(s_i w') = \ell(w') + 1$. *Then* $(s_i w', s_i w) \in \mathcal{R}_W$.

Proof. Indeed, it is well-known (see e.g., [4, Theorem 2.2.2]) that $(w', w) \in \mathcal{R}_W$ iff

$$w = w_1 s_{i_1} w_2 s_{i_2} \cdots w_k s_{i_k} w_{k+1},\ \ w' = w_1\cdots w_{k+1} \tag{5.11}$$

for some $i_1,\ldots,i_k \in I$, and $k \geq 0$ such that $\ell(w) = k+\sum\limits_{r=1}^{k+1}\ell(w_r)$ and $\ell(w') = \sum\limits_{r=1}^{k+1}\ell(w_r) = \ell(w) - k$.

Then, by the assumption of the lemma, the pair $(s_i w', s_i w)$ satisfies (5.11) because $\ell(s_i w_1) = \ell(w_1) + 1$, hence $(s_i w', s_i w) \in \mathcal{R}_W$.

The lemma is proved. □

Furthermore, we prove (5.10) by induction in $\ell(g)$. If $\ell(g) = 0$, i.e., $g = 1$, then $\partial_{1,h} = \delta_{1,h}$ and we have nothing to prove. Suppose that $\ell(g) \geq 1$, i.e., $\ell(s_i g) = \ell(g) - 1$ for some $i \in I$.

We need the following result.

Lemma 5.20. *For each Coxeter group* W *one has the following symmetries of* $\hat{\mathbf{D}}(W)$*:*

(a) $\hat{\mathbf{D}}(W)$ *is a* $\mathbb{Z}W$*-module algebra via* $w(D_s) = \begin{cases} D_{wsw^{-1}} & \text{if } \ell(ws) > \ell(w) \\ 1 - D_{wsw^{-1}} & \text{if } \ell(ws) < \ell(w)\end{cases}$ *for* $w \in W$, $s\in\mathcal{S}$.

(b) *The $\mathbb{Z}$-linear transformation d_i given by $d_i(x) := s_i(x) \cdot s_i - s_i x$ for $x \in \hat{\mathbf{D}}(W)$ and $i \in I$, is an s_i-derivation d_i of $\hat{\mathbf{D}}(W)$ determined by $d_i(D_s) = \delta_{s,s_i}$.*

Proof. Prove (a). Theorem 1.22 and the fact that $(w(D_s))^2 = w(D_s)$ for $w \in W$, $s \in \mathcal{S}$ imply that the assignment $x \mapsto w(x)$ for $x \in \hat{\mathbf{D}}(W)$ is an algebra automorphism for any $w \in W$. It suffices to show that $w_1(w_2(x)) = (w_1 w_2)(x)$ for all $x \in \hat{\mathbf{D}}(W)$, $w, w' \in W$. Since the involved maps are automorphisms, it suffices to do so only on generators $x = D_s$, $s \in \mathcal{S}$. Indeed, $w(D_s) = \sigma_{w,s} + \chi_{w,s} D_{wsw^{-1}}$ for $w \in W$, $s \in \mathcal{S}$ and χ, σ given by (7.5).

Then

$$\begin{aligned} w_1(w_2(D_s)) &= w_1(\sigma_{w_2,s} + \chi_{w_2,s} D_{w_2 s w_2^{-1}}) = \sigma_{w_2,s} + \chi_{w_2,s} w_1(D_{w_2 s w_2^{-1}}) \\ &= \sigma_{w_2,s} + \chi_{w_2,s} \chi_{w_1, w_2 s w_2^{-1}} D_{w_1 w_2 s w_2^{-1} w_1^{-1}} + \chi_{w_2,s} \sigma_{w_1, w_2 s w_2^{-1}} \\ &= \sigma_{w_1 w_2, s} + \chi_{w_1 w_2, s} D_{w_1 w_2 s w_2^{-1} w_1^{-1}} \end{aligned}$$

for all $w \in W$, $s \in \mathcal{S}$, by (3.5). This proves (a).

Prove (b). Indeed,

$$d_i(xy) = s_i(xy)s_i - s_i xy = (s_i(x)s_i - s_i x)y + s_i(x)(s_i(y)s_i - s_i y) = d_i(x)y + s_i(x)d_i(y)$$

for all $x, y \in \hat{\mathbf{D}}(W)$, $i \in I$. Also, $d_i(D_s) = s_i(D_s)s_i - s_i D_s = \delta_{s,s_i}$ for all $s \in \mathcal{S}$, $i \in I$ because

$$s_i D_s = \begin{cases} D_{s_i s s_i} s_i & \text{if } s \neq s_i \\ s_i - 1 - D_i s_i & \text{if } s = s_i \end{cases}, \qquad s_i(D_s) = \begin{cases} D_{s_i s s_i} & \text{if } s \neq s_i \\ 1 - D_i & \text{if } s = s_i \end{cases}.$$

This proves (b).

The lemma is proved. □

Furthermore, if $g = s_i$ for $i \in I$, then Lemma 5.20 guarantees that $\partial_{s_i,h} = 0$ iff $h \notin \{1, s_i\}$ and $\partial_{s_i,1} = d_i$, ∂_{s_i,s_i} is the action of s_i. Together with Proposition A.8 these imply that

$$\partial_{g,h}(x) = \sum_{h_1,h_2 \in W : h_1 h_2 = h} \partial_{s_i,h_1}(\partial_{s_i g, h_2}(x)) = s_i(\partial_{s_i g, s_i h}(x)) + d_i(\partial_{s_i g, h}(x)) \tag{5.12}$$

for all $x \in \hat{\mathbf{D}}(W)$, $g, h \in W$, $i \in I$ such that $(s_i g, g) \in \mathcal{R}_W$.

In particular, for a given $i \in I$, $g, h \in W$ such that $\ell(s_i g) = \ell(g) - 1$, i.e., $(s_i g, g) \in \mathcal{R}_W$, the equation (5.12) guarantees that $\partial_{g,h} \neq 0$ implies that either $\partial_{s_i g, h} \neq 0$ or $\partial_{s_i g, s_i h} \neq 0$. Using the inductive hypothesis, we obtain the implication:

$$\partial_{g,h} \neq 0 \implies \text{either } (s_i h, s_i g) \in \mathcal{R}_W \text{ or } (h, s_i g) \in \mathcal{R}_W \tag{5.13}$$

Clearly, if $(h, s_i g) \in \mathcal{R}_W$ in (5.13), then (5.10) holds by transitivity. If $(s_i h, s_i g) \in \mathcal{R}_W$ in (5.13) and $\ell(s_i h) = \ell(h) - 1$, then $(h, g) \in \mathcal{R}_W$ by Lemma 5.19.

It remains to consider the case $(s_i h, s_i g) \in \mathcal{R}_W$, $\ell(s_i h) = \ell(h) + 1$. Indeed, $(h, s_i h), (s_i g, g) \in \mathcal{R}_W$ hence $(h, g) \in \mathcal{R}_W$ by transitivity The implication (5.10) is proved.

Finally, let us prove the claim that $\partial_{g,g}(x) = g(x)$ for all $g \in W$, $x \in \hat{\mathbf{D}}(W)$. Once again, we proceed by induction in $\ell(g)$. If $\ell(g) = 0$, i.e., $g = 1$, then we have nothing to prove. Suppose that $\ell(g) \geq 1$, i.e., $\ell(s_i g) = \ell(g) - 1$ for some $i \in I$. Taking into account that $(g, s_i g) \notin \mathcal{R}_W$ hence $\partial_{s_i g, g} = 0$ by (5.10), (5.12) implies that $\partial_{g,g}(x) = s_i(\partial_{s_i g, s_i g}(x))$ for all $x \in \hat{\mathbf{D}}(W)$. Using the inductive hypothesis in the form $\partial_{s_i g, s_i g}(x) = s_i g(x)$ for $x \in \hat{\mathbf{D}}(W)$, we obtain: $\partial_{g,g}(x) = s_i(\partial_{s_i g, s_i g}(x)) = s_i(s_i g(x)) = g(x)$, which proves the claim.

This finishes the proof of Proposition 5.18. $\square$

Prove (a) now. Indeed, $(w, s_i) \in \mathcal{R}_W$ iff $W \in \{1, s_i\}$. Therefore, the defining relations (5.1) read

$$d_{s_i g, h} = d_{s_i,1} d_{g,h} + d_{s_i,s_i} d_{g, s_i h}, \ d_{g s_i, h} = d_{g,h} d_{s_i,1} + d_{g, h s_i} d_{s_i, s_i} \tag{5.14}$$

for all $i \in I$, $g, h \in W$.

In particular, if $g = s_i, h = s_i$, we obtain:

$$d_{s_i,1} d_{s_i,s_i} + d_{s_i,s_i} d_{s_i,1} = 0 \tag{5.15}$$

because $d_{1,s_i} = 0$.

Taking $g = h$, such that $s_i g = g s_{i'}$ and $\ell(s_i g) > \ell(g)$ for some $i, i' \in I$, (5.14) implies that $d_{s_i,1} d_{g,g} = d_{g,g} d_{s_{i'},1}$. In particular, taking $g = \underbrace{s_j s_i \cdots s_{j'}}_{m_{ij}-1}$ whenever $m_{ij} \geq 2$ in the notation of Definition 5.14, we have $s_i g = g s_{i'}$ and we obtain:

$$\underbrace{d_{s_i,1} d_{s_j,s_j} d_{s_i,s_i} \cdots d_{s_{j'} s_{j'}}}_{m_{ij}} = \underbrace{d_{s_j,s_j} \cdots d_{s_{i'} s_{i'}} d_{s_{j'} s_{j'}} d_{s_{i'},1}}_{m_{ij}} \tag{5.16}$$

The relations (5.15) and (5.16) guarantee that (5.8) defines a homomorphism $\hat{\varphi}_W$ of algebras. The relations (5.14) guarantee that $\underline{\mathbf{B}}(W, \mathcal{R}_W)$ is generated by $d_{s_i,1}$ and d_{s_i,s_i}, $i \in I$ hence the homomorphism (5.8) is surjective.

Finally, taking into account that $\Delta(d_{s_i,s_i}) = d_{s_i,s_i} \otimes d_{s_i,s_i}$ and $\Delta_{d_{s_i,1}} = d_{s_i,1} \otimes d_{1,1} + d_{s_i,s_i} \otimes d_{s_i,1}$ for $i \in I$ and $d_{1,1} = 1$, we see that $\hat{\varphi}_W$ is a homomorphism of Hopf algebras.

Let us prove the second assertion of (a). First, show that $d_{w,w} \neq 1$ in $\mathcal{B}(W, \mathcal{R}_W)$ for each $w \in W \setminus \{1\}$. By the construction, $\hat{\varphi}_W(w) = d_{w,w}$ for all $w \in W$.

We need the following fact.

Lemma 5.21. *For each $s \in \mathcal{S}$ there is a unique nonzero element $d'_s \in \mathcal{B}(W, \mathcal{R}_W)$ such that $d'_s = \hat{\varphi}_W(d_s)$ for $s \in \mathcal{S}$ and $d'_{ws_iw^{-1}} = \chi_{w,s_i} d_{w,w} d'_{s_i} d^{-1}_{w,w}$ for any $w \in W$, $i \in I$.*

Proof. The uniqueness follows from the fact that d_s is determined uniquely by same property and $\hat{\varphi}_W$ is a homomorphism of algebras. The fact that $d'_s \neq 0$ follows from that $d'_{s_i} = -d_{s_1,1} \neq 0$ for all $i \in I$, which, in turn, follows from Corollary 5.4 and Proposition 5.18 since $\partial_{s_i,1}(D_i) = -1$ for $i \in I$.

The lemma is proved. $\square$

Suppose that $d_{w,w} = 1$ for some w. Lemma 5.21 implies that $d'_{ws_iw^{-1}} = -d_{w,w} d'_{s_i} d^{-1}_{w,w}$ in $\underline{\mathbf{B}}(W, \mathcal{R}_W)$ for $i \in I$ such that $\ell(ws_i) = \ell(w) - 1$ hence $ws_iw^{-1} = s_i$ and $w = 1$.

This proves that $d_{w,w} \neq 1$ for each $w \in W$, $w \neq 1$ hence $d_{w,w} \neq d_{w',w'}$ if $w \neq w'$.

Finally, since the $\mathbb{Z}$-linear span of all $d_{w,w}$ is a sub-bialgebra of $\underline{\mathbf{B}}(W, \mathcal{R})$ and each $d_{w,w}$ is grouplike, then the set $\{d_{w,w} \,|\, w \in W\}$ is $\mathbb{Z}$-linearly independent. This proves the second assertion and finishes the proof of (a).

Prove Theorem 5.17(b). The presentation (5.3) implies (by induction in length) that $\mathcal{B}(W, \mathcal{R}_W)$ is generated by all $v^g_{s_i,1}$, i.e., by all d'_s, i.e., by $\hat{\varphi}(Y_W)$. This implies that $\underline{\hat{\varphi}}$ is surjective. Also $\underline{\hat{\varphi}}$ commutes with W-action and preserves W-grading, therefore, it is a homomorphism of algebras in ${}^W_W\mathcal{YD}$. In turn, this implies that $\underline{\hat{\varphi}} \otimes \underline{\hat{\varphi}}$ is a well-defined surjective homomorphism of algebras $\hat{\mathbf{H}}_0(W) \otimes \hat{\mathbf{H}}_0(W) \twoheadrightarrow \mathcal{B}(W, \mathcal{R}_W) \otimes \mathcal{B}(W, \mathcal{R}_W)$. Note also that $\underline{\Delta}(v^g_{s_i,1}) = v^g_{s_i,1} \otimes 1 + 1 \otimes v^g_{s_i,1}$ for all $g \in W$, $i \in I$ by (5.9), that is, $\underline{\Delta}(d'_s) = d'_s \otimes 1 + 1 \otimes d'_s$ for any $s \in \mathcal{S}$. This and the above imply that

$$\underline{\Delta} \circ \underline{\hat{\varphi}} = (\underline{\hat{\varphi}} \otimes \underline{\hat{\varphi}}) \circ \underline{\Delta} \,.$$

It is also immediate that $\underline{\varepsilon}(d_{s'}) = 0$ for all $s \in \mathcal{S}$ hence $\underline{\varepsilon} \circ \underline{\hat{\varphi}} = \underline{\varepsilon}$.

Finally, note that since $d'_s \neq 0$ by Lemma 5.21 and $|d'_s| = s$ for all $s \in \mathcal{S}$, the set $\{d'_s \,|\, s \in \mathcal{S}\}$ is $\mathbb{Z}$-linearly independent. This finishes the proof of (b).

Theorem 5.17 is proved. $\square$

Remark 5.22. Theorem 5.17(b) asserts that $\mathcal{B}(W, \mathcal{R}_W)$ is essentially a pre-Nichols algebra, however, we are not yet aware of existence of the braided antipode in $\mathcal{B}(W, \mathcal{R}_W)$.

Definition 5.23. Let W be a simply-laced Coxeter group. Denote by $\mathbf{H}_0(W)$ the $\mathbb{Z}$-algebra generated by s_i, d_i, $i = 1, \ldots, n-1$ subject to relations:

- $s_i d_i + d_i s_i = 0$, $d_i^2 = 0$ for $i \in I$.
- $s_i s_j = s_j s_i$ $d_j s_i = s_i d_j$, $d_j d_i = d_i d_j$ for all $i, j \in I$ with $m_{ij} = 2$.
- $s_j s_i s_j = s_i s_j s_i$, $s_j d_i s_j = s_i d_j s_i$, $d_j s_i d_j = s_i d_j d_i + d_i d_j s_i$ for all $i, j \in I$ with $m_{ij} = 3$.

That is, the simply-laced $\mathbf{H}_0(W)$ is obtained by "homogenizing" Theorem 1.25 and is naturally a Hopf algebra. In particular, the canonical surjective algebra homomorphism $\hat{\mathbf{H}}_0(W) \twoheadrightarrow \mathbf{H}_0(W)$ is that of Hopf algebras.

The following is an immediate graded version of Proposition 1.31.

Lemma 5.24. *For any simply-laced Coxeter group W one has:*

(a) *the algebra $\mathbf{H}_0(W)$ is isomorphic to the cross product $\mathbf{D}_0(W) \rtimes \mathbb{Z}W$, where $\mathbf{D}_0(W)$ is the $\mathbb{Z}$-algebra generated by d_s, $s \in \mathcal{S}$, subject to relations (in the notation of Proposition 1.31):*
- *$d_s^2 = 0$ for all $s \in \mathcal{S}$.*
- *$d_s d_{s'} = d_{s'} d_s$ for all compatible pairs $(s, s') \in \mathcal{S} \times \mathcal{S}$ with $m_{s,s'} = 2$.*
- *$d_s d_{s'} = d_{ss's} d_s + d_{s'} d_{ss's}$ for all compatible pairs $(s, s') \in \mathcal{S} \times \mathcal{S}$ with $m_{s,s'} = 3$.*

(b) *$\mathbf{D}_0(W)$ is a (braided) Hopf algebra in the category ${}^W_W\mathcal{YD}$ so that the canonical surjective homomorphism $\hat{\mathbf{D}}_0(W) \twoheadrightarrow \mathbf{D}_0(W)$ is that of braided Hopf algebras.*

Remark 5.25. In view of Remark 5.16, for any simply-laced Coxeter group W the Hopf algebra $\mathbf{D}_0(W)$ is a pre-Nichols algebra of the Yetter-Drinfeld module $\underline{Y}_W = \oplus_{s\in\mathcal{S}}\mathbb{Z} \cdot d_s$ over W so that the canonical surjective homomorphism $\hat{\mathbf{D}}_0(W) \twoheadrightarrow \mathbf{D}_0(W)$ is that of pre-Nichols algebras.

Remark 5.26. The algebra $\mathbf{D}_0(S_n)$ coincides with the Fomin-Kirillov algebra $\mathcal{E}_n$ defined in [8].

Theorem 5.27. *For any simply-laced Coxeter group W the homomorphism (5.9) factors through the following surjective homomorphism of bialgebras in ${}^W_W\mathcal{YD}$.*

$$\mathbf{D}_0(W) \twoheadrightarrow \mathbf{B}(W, \mathcal{R}_W) \ . \tag{5.17}$$

Proof. First, prove that (5.8) factors through the homomorphism of bialgebras

$$\mathbf{H}_0(W) \twoheadrightarrow \underline{\mathbf{B}}(W, \mathcal{R}_W) \ . \tag{5.18}$$

Clearly, for distinct $i, j \in I$ we have by (5.14):

$$d_{s_i s_j,1} = d_{s_i,1} d_{s_j,1}, d_{s_i s_j, s_i} = d_{s_i,s_i} d_{s_j,1},\ d_{s_i s_j, s_j} = d_{s_i,1} d_{s_j,s_j} \tag{5.19}$$

because $d_{s_j,s_i} = d_{s_j,s_i} = d_{s_j,s_i s_j} = 0$.

Let $i, j \in I$ be such that $m_{ij} = 2$, i.e., $s_i s_j = s_j s_i$. Then using (5.19) (also with i and j interchanged where necessary), we obtain

$$d_{s_i,1} d_{s_j,1} = d_{s_j,1} d_{s_i,1},\ d_{s_i,s_i} d_{s_j,1} = d_{s_j,1} d_{s_i,s_i} \tag{5.20}$$

Now let $m_{ij} = 3$ and let $s_{ij} := s_i s_j s_i = s_j s_i s_j$.

Indeed, let us compute $d_{s_{ij},1}$ in two ways using (5.14) and (5.19) (also interchanging i and j where necessary). We obtain:

$$d_{s_{ij},s_i} = d_{s_is_js_i,s_i} = d_{s_i,1}d_{s_js_i,s_i} + d_{s_i,s_i}d_{s_js_i,1} = d_{s_i,1}d_{s_j,1}d_{s_i,s_i} + d_{s_i,s_i}d_{s_j,1}d_{s_i,1} \ ,$$
$$d_{s_{ij},s_i} = d_{s_js_is_j,s_i} = d_{s_j,1}d_{s_is_j,s_i} + d_{s_j,s_j}d_{s_is_j,s_js_i} = d_{s_j,1}d_{s_is_j,s_i} = d_{s_j,1}d_{s_j,s_j}d_{s_i,1} \ .$$

Therefore, $d_{s_j,1}d_{s_j,s_j}d_{s_i,1} = d_{s_i,1}d_{s_j,1}d_{s_i,s_i}+d_{s_i,s_i}d_{s_j,1}d_{s_i,1}$. Clearly, the above relation and (5.20) ensure that (5.18) is a well-defined homomorphism of algebras. Clearly, it commutes with the coproduct, the counit and the antipode, so is a homomorphism of Hopf algebras.

Then, copying the argument of the proof of Theorem 5.17(b), we conclude that (5.17) is surjective, commutes with the W-action, preserves W-grading, braided coproduct and the braided counut.

The theorem is proved. $\square$

Remark 5.28. In [15, Section 6] the authors conjectured that $\mathbf{D}_0(S_n)$ is, in fact, a Nichols algebra. In turn, this would imply that (5.17) is an isomorphism for $W = S_n$, $n \geq 2$. So is natural to ask whether (5.17) is an isomorphism for each simply-laced Coxeter group W.

We conclude the section with a (conjectural) generalization of (5.17) to all Coxeter groups as follows. Define a filtration on $\hat{\mathbf{H}}(W)$ by assigning the filtered degree 1 to each D_i and 0 to each s_i.

The following is an immediate consequence of Theorems 1.22 and 1.25.

Lemma 5.29. *For any Coxeter group W one has:*

(a) *The assignments $d_s \mapsto D_s$, $s \in \mathcal{S}$, define a natural isomorphism of graded algebras*

$$\hat{gr}_W : \hat{\mathbf{H}}_0(W) \widetilde{\to} gr\ \hat{\mathbf{H}}(W) \ ,$$

where $gr\ \hat{\mathbf{H}}(W)$ is the associated graded of $\hat{\mathbf{H}}(W)$.

(b) *If W is simply laced, then $\hat{gr}_W$ factors through a surjective homomorphism $\mathbf{D}_0(W) \twoheadrightarrow gr\ \mathbf{D}(W)$ of W-graded algebras.*

Remark 5.30. For any Coxeter group W the composition of $\hat{gr}_W^{-1}$ with (5.8) is a surjective homomorphism $gr\ \hat{\mathbf{D}}(W) \twoheadrightarrow \mathcal{B}(W, \mathcal{R}_W)$ of bialgebras in ${}^W_W\mathcal{YD}$. We expect this homomorphism to factor through the surjective homomorphism of bialgebras in ${}^W_W\mathcal{YD}$: $gr\ \mathbf{D}(W) \twoheadrightarrow \mathcal{B}(W, \mathcal{R}_W)$.

6. Hecke-Hopf algebras of cyclic groups and generalized Taft algebras

In this section we study a variant of the generalized Hecke-Hopf algebra for cyclic groups. In fact, these Hopf algebras are bialgebras universally coacting (in the sense of [3]) on finite dimensional principal ideal domains. It turns out that the actual (generalized)

Hecke-Hopf algebra of a cyclic group is the quotient of such a universal Hopf algebra and is isomorphic to the Taft algebras.

Let R be a commutative unital ring and let $f \in R[x] \setminus \{0\}$. Denote by $\mathbf{H}_f$ the R-algebra generated by s, D subject to relations $s^{\deg f} = 1$ and the relations given by the functional equation

$$f(ts + D) = f(t) \tag{6.1}$$

over $\Bbbk[t]$ (with the convention that if $\deg f = 0$, then s is of infinite order).

In other words, if we write $f = a_0 + a_1 x + \cdots + a_n x^n$, $a_0, \ldots, a_n \in R$, $a_n \neq 0$, then $\mathbf{H}_f$ is subject to relations $\sum\limits_{r=k}^{n} a_r \{s, D\}_{k,r-k} = a_k$ for $k = 0, \ldots, n$, where $\{\mathbf{a}, \mathbf{b}\}_{k,r-k} = \{\mathbf{b}, \mathbf{a}\}_{r-k,k}$ denotes the coefficient of t^k in the expansion of the noncommutative binomial $(\mathbf{a}t + \mathbf{b})^r$ (that is, $\{\mathbf{a}, \mathbf{b}\}_{k,r-k} = \sum\limits_{\substack{\varepsilon_1,\ldots,\varepsilon_r \in \{0,1\}: \\ \varepsilon_1 + \cdots + \varepsilon_r = k}} \mathbf{a}^{\varepsilon_1} \mathbf{b}^{1-\varepsilon_1} \cdots \mathbf{a}^{\varepsilon_r} \mathbf{b}^{1-\varepsilon_r}$). Clearly, $\mathbf{H}_{cf+d} = \mathbf{H}_f$ for any $d \in R$ and $c \in R^\times$.

Example 6.1.

- $f(x) = x + a_0$. Then $\mathbf{H}_f = R$.
- $f(x) = x^2 + a_1 x + a_0$. Then $\mathbf{H}_f$ is generated by s and D subject to relations $s^2 = 1$, $D^2 = -a_1 D$, $sD + Ds = a_1(1 - s)$. In particular, if $a_1 = -1$, then $\mathbf{H}_f = \mathbf{H}(S_2)$ by Definition 1.1.
- $f(t) = x^3 + a_2 x^2 + a_1 x + a_0$. Then $\mathbf{H}_f$ is generated by s and D subject to relations $s^3 = 1$ and

$$D^3 = -a_2 D^2 - a_1 D, \ s^2 D + sDs + Ds^2 = a_2(1 - s^2),$$
$$D^2 s + DsD + sD^2 + a_2(sD + Ds) = a_1(1 - s) \ .$$

Proposition 6.2. *For each $f(x) \in R[x]$, $\mathbf{H}_f$ is a Hopf algebra over R with the coproduct, the counit, and the antipode given respectively by*

$$\Delta(D) = D \otimes 1 + s \otimes D, \ \Delta(s) = s \otimes s, \ \varepsilon(D) = 0, \varepsilon(s) = 1,$$
$$S(s) = s^{-1}, \ S(D) = -s^{-1} D \ . \tag{6.2}$$

Proof. Denote by $\mathbf{H}'$ the free product (over R) of the cyclic group algebra $R[s]/(s^n - 1)$, where $n := \deg f$, and the polynomial algebra $R[D]$. By Lemma 4.13 taken with $H = R[s]/(s^n - 1)$, $V = RD$ and $\delta(D) = s \otimes D$, $\mathbf{H}'$ is a Hopf algebra with coproduct, counit, and antipode as in (6.2).

Let $y_k \in \mathbf{H}'$, $k = 0, \ldots, n$ be the coefficients in the expansion $f(ts + D) = \sum\limits_{k=0}^{n} y_k t^k$. In fact,

$$y_k = \sum_{i=k}^{n} a_i \{s, D\}_{i,k-i} \tag{6.3}$$

for $k = 0, \ldots, n = \deg f$, where $\sum\limits_{k=0}^{n} a_k t^k = f(t)$.

Denote by $\mathbf{K}_f$ the R-submodule of $\mathbf{H}'$ generated by 1 and $y_0, \ldots, y_{n-1}$. Let $\mathbf{H}'[t] = \mathbf{H}' \otimes_R R[t]$, which, clearly, is a Hopf algebra over $R[t]$.

Lemma 6.3. *The R-module $\mathbf{K}_f$ is a right coideal in $\mathbf{H}'$.*

Proof. We have in $\mathbf{H}'[t]$:

$$\Delta(f(st+D)) = f(ts \otimes s + D \otimes 1 + s \otimes D) = f(s \otimes (st+D) + D \otimes 1) = f(s't' + D')$$

where $s' = s \otimes 1$, $t' = 1 \otimes (st+D)$, $D' = D \otimes 1$. Taking into account that the assignment $s \mapsto s'$, $D \mapsto D'$ is an algebra homomorphism $\mathbf{H}' \to \mathbf{H}' \otimes 1$, we obtain

$$\Delta(f(st+D)) = \sum_{k=0}^{n} y'_k {t'}^k = \sum_{k=0}^{n} y_k \otimes (st+D)^k \subset \mathbf{K}_f \otimes \mathbf{H}'[t]$$

where $y'_k := y_k \otimes 1$. This implies that $\delta(y_k) \in \mathbf{K}_f \otimes \mathbf{H}'$ for $k = 0, \ldots, n = \deg f$.

The lemma is proved. $\square$

Finally, note that $\mathbf{K}_f^+ = \mathbf{K}_f \cap Ker\, \varepsilon = \sum\limits_{k=0}^{n} R \cdot (y_k - a_k)$, i.e., $\mathbf{K}_f^+$ is an R-submodule of $\mathbf{H}'$ generated by all coefficients of $f(ts+D) - f(t)$. In view of Proposition 4.9, this implies that the ideal $\mathbf{J}_f$ generated by $y_k - a_k$, $k = 0, \ldots, n$, is a Hopf ideal in $\mathbf{H}'$, i.e., $\mathbf{H}_f = \mathbf{H}'/\mathbf{J}_f$ is a Hopf algebra.

Proposition 6.2 is proved. $\square$

The following is an analogue of Theorems 1.20 and 1.33.

Proposition 6.4. *For an R-algebra $\Bbbk$, $c \in \Bbbk$ and any $f \in \Bbbk[x]$ the assignment $x \mapsto cs + D$ defines a homomorphism of algebras*

$$\varphi_c : \Bbbk[x]/(f - f(c)) \to \mathbf{H}_f \otimes_R \Bbbk \tag{6.4}$$

whose image is a left coideal subalgebra in $\mathbf{H}_f$.

Proof. Indeed, defining functional relations (6.1) imply that $f(cs+D) - f(c) = 0$. This proves that φ_c is a homomorphism of algebras.

Since $\underline{x} := \varphi_c(x) = cs + D$ and $\Delta(\underline{x}) = cs \otimes s + D \otimes 1 + s \otimes D = D \otimes 1 + s \otimes \underline{x}$. Thus, $R \cdot \underline{x}$ is a left coideal in $\mathbf{H}_f$ hence the subalgebra of $\mathbf{H}_f$ generated by $\underline{x}$ is a left coideal subalgebra in $\mathbf{H}_f$.

Proposition 6.4 is proved. $\square$

Remark 6.5. We expect that (6.4) is always injective.

For $a, b \in R$ denote by $\mathbf{H}_f(a,b)$ the R-algebra generated by D, s subject to relations $s^{\deg\ f} = 1$, the functional relations (6.1), and $sDs^{-1} = aD + b(1-s)$.

Proposition 6.6. *For any nonzero $f \in R[x]$ and $a, b \in R$, the algebra $\mathbf{H}_f(a,b)$ is a Hopf algebra with the coproduct Δ, counit ε, and the antipode S given respectively by:*

$$\Delta(s) = s \otimes s,\ \Delta(D) = D \otimes 1 + s \otimes D,\ \varepsilon(s) = 1,\ \varepsilon(D) = 0,\ S(s) = s^{-1},\ S(D) = -s^{-1}D\ .$$

Proof. Let $\mathbf{K}_{a,b}$ be the R-submodule of $\mathbf{H}_f$ generated by 1 and $sDs^{-1} - aD + bs$.

We need the following result.

Lemma 6.7. *$\mathbf{K}_{a,b}$ is a left coideal in $\mathbf{H}_f$.*

Proof. Indeed, let $\delta := sDs^{-1} - aD + bs$. Then

$$\begin{aligned}\Delta(\delta) &= (s \otimes s)\Delta(D)(s^{-1} \otimes s^{-1}) - a\Delta(D) + bs \otimes s \\ &= sDs^{-1} \otimes 1 + s \otimes sDs^{-1} - aD \otimes 1 - s \otimes aD + bs \otimes s \\ &= (\delta - bs) \otimes 1 + s \otimes \delta \in \mathbf{H}_f \otimes \mathbf{K}_{a,b}\ .\end{aligned}$$

The lemma is proved. $\square$

Finally, note that $\mathbf{K}_{a,b}^+ = \mathbf{K}_{a,b} \cap Ker\ \varepsilon = R \cdot \delta_{a,b}$, where $\delta_{a,b} = sDs^{-1} - aD - b(1-s)$. In view of Proposition 4.9, this guarantees that the ideal $\mathbf{J}_f$ generated by $\delta_{a,b}$ is a Hopf ideal in $\mathbf{H}_f$. Hence the quotient $\mathbf{H}_f(a,b) = \mathbf{H}_f/\mathbf{J}_{a,b}$ is a Hopf algebra.

The proposition is proved. $\square$

We abbreviate $\mathbf{H}_n(a,b) := \mathbf{H}_{f_n^{a,b}}(a,b)$ for $a, b \in R$, where

$$f_n^{a,b} := x(x-b)(x-b(1+a))\cdots(x-b(1+a+\cdots+a^{n-2})) \tag{6.5}$$

Note that if $a \in R \setminus \{1\}$ is a root of unity, i.e., $1 + a + \cdots + a^{n-1} = 1$, then

$$f_n^{a,b}(ax+b) = f_n^{a,b}(x) \tag{6.6}$$

because the set of roots of $f_n^{a,b}$ is invariant under the linear change $x \mapsto ax + b$.

We call $\mathbf{H}_n(a,b)$ a *generalized Taft algebra*. This terminology is justified by the following result.

Proposition 6.8. *Given a commutative unital ring R and $a, b \in R$, the Hopf algebra $\mathbf{H}_n(a,b)$ has a presentation: $s^n = 1$, $sDs^{-1} = aD + b(1-s)$, and*

$$\begin{bmatrix} n \\ k \end{bmatrix}_a D(aD+b)(a^2D+b(1+a))\cdots(a^{k-1}D+b(1+a+\cdots+a^{k-2}))=0 \tag{6.7}$$

for $k=1,\dots,n$, *where* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n+1-i}-1}{q^i-1} \in \mathbb{Z}_{\geq 0}[q]$ *is the* q*-binomial coefficient.*

In particular, if a *is a primitive* n*-th root of unity in* $R^\times$, *then* $\mathbf{H}_n(a,b)$ *has a presentation:*

$$s^n=1,\ sDs^{-1}=aD+b(1-s),$$
$$D(aD+b)(a^2D+b(1+a))\cdots(a^{n-1}D+b(1+a+\cdots+a^{n-2}))=0$$

Proof. We need the following result.

Lemma 6.9. *The algebra* $\mathbf{H}_n(a,b)$ *has a presentation* $s^n=1$, $sDs^{-1}=aD+b(1-s)$, *and the functional relations*

$$(t-D)(t-aD-b)(t-a^2D-b(1+a))\cdots(t-a^{n-1}D-b(1+a+\cdots+a^{n-2}))=f_n^{a,b}(t)\ . \tag{6.8}$$

Proof. Applying the antipode to the defining functional relations (6.1) we see that $\mathbf{H}_{f_n^{a,b}}$ has a presentation: $s^n=1$ and

$$(s^{-1}t-s^{-1}D-b(1+a+\cdots+a^{n-2}))\cdots(s^{-1}t-s^{-1}D-b)(s^{-1}t-s^{-1}D)=f_n^{a,b}(t)\ .$$

Equivalently, factoring out s^{-1} to the left from each factor, we obtain $s^n=1$ and:

$$(t-s^{n-1}Ds^{1-n}-b(1+a+\cdots+a^{n-2})s)\cdots(t-sDs^{-1}-bs)(t-D)=f_n^{a,b}(t) \tag{6.9}$$

Passing to $\mathbf{H}_n(a,b)$, we obtain one more defining relation $sDs^{-1}=aD+b(1-s)$, which immediately implies $s^kDs^{-k}=a^kD+b(1+a+\cdots+a^{k-1})(1-s)$ for $k\in\mathbb{Z}_{\geq 0}$. Taking this into account, we see that the left hand side of (6.9) becomes the left hand side of (6.8). The lemma is proved. □

We need the following combinatorial fact. For $n\geq 0$ let $f_n(t,x;p,q)\in\mathbb{Z}[t,x,p,q]$ be given by

$$f_n(t,x,p,q)=\prod_{i=0}^{n-1}\left(t+q^ix+p\frac{q^i-1}{q-1}\right)$$

with the convention that $f_0(t,x;p,q)=1$.

The following is a generalization of the q-binomial formula.

Lemma 6.10. $f_n(t,x;p,q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q f_k(0,x;p,q)f_{n-k}(t,0;p,q)$ *for* $n\geq 0$.

Applying Lemma 6.10 with $q = a$, $p = -b$, $x = -D$, we see that the left hand side of (6.8) equals $f_n(t, -D; -b, a)$ and $f_n(t, 0; -b, a) = f_n^{a,b}(t)$, so (6.8) becomes: $\sum\limits_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_a f_k(0, -D; -b, a) f_{n-k}^{a,b}(t) = 0$.

Finally, using $\mathbf{H}_n(a, b)$-linear independence of $f_k^{a,b}(t)$, $k \geq 0$ in $\mathbf{H}_n(a, b)[t]$, we obtain (6.7).

The proposition is proved. $\square$

By Proposition 6.8, for a being a primitive n-th root of unity in R, $\mathbf{H}_n(a, 0)$ is the Tuft algebra with the presentation: $\underline{s}^n = 1$, $\partial^n = 0, \underline{s}\partial = a\partial\underline{s}$.

It turns out that $\mathbf{H}_n(a, b)$ is always a module algebra over the Taft algebra, and the multiplication in the former can be expressed in terms of the action.

Corollary 6.11. *In the notation of Proposition 6.8, suppose that a is a primitive n-th root of unity in R. Then*

(a) *$\mathbf{H}_n(a, b)$ is an $\mathbf{H}_n(a^{-1}, 0)$-module algebra via: $\underline{s} \triangleright s = as$, $\partial \triangleright s = 0$, $\underline{s} \triangleright D = b + aD$.*

(b) *$s^\ell p(D) s^{-\ell} = \sum\limits_{k=0}^{\ell} (-1)^k \begin{bmatrix} k+\ell \\ \ell-1 \end{bmatrix}_a b^k ((\underline{s}^\ell \partial^k) \triangleright p(D)) \cdot s^k$ for any polynomial $p \in R[x]$ and $\ell \geq 0$.*

7. Proofs of main results

7.1. *Almost free Hopf algebras and proof of Theorems 1.16 and 3.2*

Given a group W, a conjugation-invariant subset $\mathcal{S} \subset W \setminus \{1\}$, and any maps $\chi, \sigma : W \times \mathcal{S} \to R$, let $\hat{\mathbf{H}}'_{\chi,\sigma}(W)$ be an R-algebra generated by W and D_s, $s \in \mathcal{S}$ subject to relations (3.1) for all $s \in \mathcal{S}$, $w \in W$.

Proposition 7.1. *For any maps $\chi, \sigma : W \times \mathcal{S} \to R$ one has*

(a) *$\hat{\mathbf{H}}'_{\chi,\sigma}(W)$ is a Hopf algebra with the coproduct Δ, counit ε, and the antipode S given by (3.4).*

(b) *$\hat{\mathbf{H}}'_{\chi,\sigma}(W)$ factors as $\hat{\mathbf{H}}'_{\chi,\sigma}(W) = T(V) \cdot RW$ over R, where $V = \oplus_{s\in\mathcal{S}} R \cdot D_s$, iff χ and σ satisfy (3.5).*

Proof. Prove (a). Clearly, $\hat{\mathbf{H}}'_{\chi,\sigma}(W) = \mathbf{H}_\gamma$ in the notation of Proposition 4.12, where:

- $H = RW$, $V = \oplus_{s\in\mathcal{S}} R \cdot D_s$ is a $T(H)$-module via

$$w(D_s) = \chi_{w,s} D_s \tag{7.1}$$

for $w \in W$, $s \in \mathcal{S}$ and an H-comodule via $\delta(D_s) = s \otimes D_s$.

- $\gamma : RW \times V \to R$ is given by

$$\gamma(w, D_s) = \sigma_{w,s}(1 - wsw^{-1}) \tag{7.2}$$

for $w \in W$, $s \in \mathcal{S}$.

Then, clearly the Yetter-Drinfeld condition (4.10) holds because

$$\delta(w(D_s)) = \chi_{w,s}\delta(D_{wsw^{-1}}) = wsw^{-1} \otimes \chi_{w,s}D_{wsw^{-1}} = wsw^{-1} \otimes w(D_s)$$

for all $w \in W$, $s \in \mathcal{S}$.

The second condition of Proposition 4.12 also holds automatically because

$$\Delta(\gamma(w, D_s)) = \sigma_{w,s}(1 \otimes 1 - wsw^{-1} \otimes wsw^{-1}) = \gamma(w, D_s) \otimes 1 + wsw^{-1} \otimes \gamma(w, D_s)$$

and $\varepsilon(\gamma(w, D_s)) = 0$ for all $w \in W$, $s \in \mathcal{S}$.

Thus, $\hat{\mathbf{H}}'_{\chi,\sigma}(W) = \mathbf{H}_\gamma$ is a Hopf algebra by Proposition 4.12. This proves (a).

Prove (b). It suffices to translate the conditions of Proposition 4.11. Indeed, taking into account that the first condition of (3.5) implies $\chi_{1,s} = 1$ for all $s \in \mathcal{S}$, we see that the first condition of (3.5) is equivalent to that (7.1) is a RW-action on $V = \oplus_{s\in\mathcal{S}} R \cdot D_s$.

Finally, the condition (4.6) reads for this action and γ given by (7.2):

$$\begin{aligned}\sigma_{w_1w_2,s}(1 - w_1w_2sw_2^{-1}w_1^{-1}) &= \gamma(w_1w_2, D_s) = \gamma(w_1, w_2(D_s)) + w_1\gamma(w_2, D_s)w_1^{-1} \\ &= \chi_{w_2,s}\sigma_{w_1,w_2sw_2^{-1}}(1 - w_1w_2sw_2^{-1}w_1^{-1}) + \sigma_{w_2,s}(1 - w_1w_2sw_2^{-1}w_1^{-1})\end{aligned}$$

in RW for all $w_1, w_2 \in W$, $s \in \mathcal{S}$, which is, clearly, equivalent to the second condition of (3.5). This proves (b).

Proposition 7.1 is proved. $\square$

Furthermore, we say that a family $\mathbf{f} = (f_s) \in (R[x] \setminus \{0\})^{\mathcal{S}}$ of polynomials $f_s \in R[x] \setminus \{0\}$ is *adapted* to $\mathcal{S}$ if $\deg f_s = |s|$ for all $s \in \mathcal{S}$ (with the convention $|s| = 0$ if s is of infinite order, hence, f_s is a nonzero constant in that case).

For any maps $\chi, \sigma : W \times \mathcal{S} \to R$ and any family $\mathbf{f} = (f_s) \in (R[x] \setminus \{0\})^{\mathcal{S}}$ adapted to $\mathcal{S}$ let $\hat{\mathbf{H}}_{\chi,\sigma,\mathbf{f}}(W)$ be an R-algebra generated by W and D_s, $s \in \mathcal{S}$ subject to relations (3.1) for all $s \in \mathcal{S}$, $w \in W$ and the functional relations

$$f_s(ts + D_s) = f_s(t) \tag{7.3}$$

(if s is of infinite order, i.e., $|s| = 0$, then the condition (7.3) is vacuous).

By definition, one has a surjective homomorphism of R-algebras $\pi_{\mathbf{f}} : \hat{\mathbf{H}}'_{\chi,\sigma}(W) \twoheadrightarrow \hat{\mathbf{H}}_{\chi,\sigma,\mathbf{f}}(W)$.

Proposition 7.2. *For any family* $\mathbf{f}$ *adapted to* $\mathcal{S}$, $\hat{\mathbf{H}}_{\chi,\sigma,\mathbf{f}}(W)$ *is naturally a Hopf algebra (i.e.,* $\pi_{\mathbf{f}}$ *is a homomorphism of Hopf algebras).*

Proof. Using notation from Section 6 and copying (6.3), define for each finite order element $s \in \mathcal{S}$ the elements $y_0^s, \dots, y_{|s|-1}^s \in \hat{\mathbf{H}}'_{\chi,\sigma}(W)$ by $y_k^s = \sum\limits_{i=k}^{|s|} a_i^s \{s, D_s\}_{i,k-i}$, where $\sum\limits_{k=0}^{|s|} a_{k_s} t^s = f(t)$.

Denote by $\mathbf{K}_{f_s}$ the R-submodule of $\mathbf{H}'_{\chi,\sigma}(W)$ generated by 1 and y_k^s, $k = 0, \dots, |s|-1$. (with the convention that $\mathbf{K}_{f_s} = R$ if $|s| = 0$).

Lemma 7.3. *$\mathbf{K}_{f_s}$ is a right coideal in $\hat{\mathbf{H}}'_{\chi,\sigma}(W)$ for each $s \in \mathcal{S}$.*

Proof. The proof is identical to that of Lemma 6.3. $\square$

Therefore, $\mathbf{K_f} := \sum\limits_{s \in \mathcal{S}} \mathbf{K}_{f_s}$ is a right coideal in $\hat{\mathbf{H}}'_{\chi,\sigma}(W)$ by Proposition 4.1 (for right coideals) and $\mathbf{K_f}^+ := \mathbf{K_f} \cap Ker\ \varepsilon$ is the R-submodule of $\mathbf{H}'_{\chi,\sigma}(W)$ generated by $y_k^s - a_k^s$, $k = 0, \dots, |s|$, $s \in \mathcal{S}$.

By definition, the kernel of $\pi_\mathbf{f}$ is the ideal of $\mathbf{H}'_{\chi,\sigma}(W)$ generated by $\mathbf{K_f}^+$. In view of Proposition 4.9, this guarantees that the kernel of $\pi_\mathbf{f}$ is a Hopf ideal in $\mathbf{H}'_{\chi,\sigma}(W)$. Therefore, $\mathbf{H}_{\chi,\sigma,\mathbf{f}}(W) = \hat{\mathbf{H}}'_{\chi,\sigma}(W)/(Ker\ \pi_\mathbf{f})$ is a Hopf algebra and $\pi_\mathbf{f}$ is a homomorphism of Hopf algebras.

The proposition is proved. $\square$

Proof of Theorem 3.2. Let us show that $\hat{\mathbf{H}}_{\sigma,\chi}(W) = \mathbf{H}_{\chi,\sigma,\mathbf{f}}(W)$, where

$$f_s = f_{|s|}^{a_s,b_s} = x(x - b_s)(x - b_s(1 + a_s)) \cdots (x - b_s(1 + a_s + \dots + a_s^{|s|-2})) \tag{7.4}$$

in the notation (6.5), where we abbreviated $a_s := \chi_{s,s}$ and $b_s := \sigma_{s,s}$ (with the convention $f_s = 1$ if $|s| = 0$). Indeed, in view of Proposition 6.8, since each relevant $\chi_{s,s}$ is the primitive $|s|$-th root of unity, the defining functional relation (7.3) for $\hat{\mathbf{H}}_{\chi,\sigma,\mathbf{f}}(W)$ coincides with the defining relation (3.2) for $\hat{\mathbf{H}}_{\sigma,\chi}(W)$. Thus, $\hat{\mathbf{H}}_{\sigma,\chi}(W) = \mathbf{H}_{\chi,\sigma,\mathbf{f}}(W)$ is a Hopf algebra.

Theorem 3.2 is proved. $\square$

Proof of Theorem 1.16. Similarly to Definition 1.14, for any Coxeter group $W = \langle s_i | i \in I\rangle$ let $\hat{\mathbf{H}}'(W)$ the $\mathbb{Z}$-algebra generated by s_i, D_i, $i \in I$ subject to relations:

(i) Rank 1 relations: $s_i^2 = 1$, $s_i D_i + D_i s_i = s_i - 1$ for $i \in I$.

(ii) Coxeter relations: $(s_i s_j)^{m_{ij}} = 1$ and linear braid relations: $\underbrace{D_i s_j s_i \cdots s_{j'}}_{m_{ij}} = \underbrace{s_j \cdots s_{i'} s_{j'} D_{i'}}_{m_{ij}}$ for all distinct $i, j \in I$, where $i' = \begin{cases} i & \text{if } m_{ij} \text{ is even} \\ j & \text{if } m_{ij} \text{ is odd} \end{cases}$ and $\{i', j'\} = \{i, j\}$.

Proposition 7.4. *$\hat{\mathbf{H}}'(W) = \hat{\mathbf{H}}'_{\chi,\sigma}(W)$ for any Coxeter group W the notation of Proposition 7.1 with $R = \mathbb{Z}$, where $\chi : W \times \mathcal{S} \to \{-1, 1\} \subset \mathbb{Z}$ and $\sigma : W \times \mathcal{S} \to \{0, 1\} \subset \mathbb{Z}$ are given by:*

$$\chi_{w,s} = (-1)^{\ell(w)+\frac{1}{2}(\ell(wsw^{-1})-\ell(s))}, \ \sigma_{w,s} = \frac{1-\chi_{w,s}}{2} \tag{7.5}$$

for all $w \in W$, $s \in \mathcal{S}$. In particular, $\hat{\mathbf{H}}'(W)$ is a Hopf algebra with the coproduct Δ, counit ε, and the antipode S given by (3.4).

Proof. Clearly, $\hat{\mathbf{H}}'(W)$ is generated over $\mathbb{Z}$ by $V = \oplus_{s\in\mathcal{S}}\mathbb{Z}D_s$ and the group W.

We need the following result.

Lemma 7.5. *For any Coxeter group W the map (7.5) satisfies $\chi_{s_i,s} = \begin{cases} -1 & \text{if } s = s_i \\ 1 & \text{if } s \neq s_i \end{cases}$ for $s \in \mathcal{S}$, $i \in I$. In particular,*

$$s_i D_s s_i = \begin{cases} D_{s_i s s_i} & \text{if } s \neq s_i \\ -D_{s_i} + 1 - s_i & \text{if } s = s_i \end{cases} = \chi_{s_i,s} D_{s_i s s_i} + \frac{1-\chi_{s_i,s}}{2}(1 - s_i s s_i) \tag{7.6}$$

in $\hat{\mathbf{H}}'(W)$ for all $s \in \mathcal{S}$, $i \in I$.

Proof. Clearly, χ defined by (7.5) satisfies the first assertion of the lemma because $\ell(s_i) = 1$ and $\ell(s_i s s_i) - \ell(s) \in \{-2, 2\}$ for all $i \in I$, $s \in \mathcal{S} \setminus \{s_i\}$, i.e., $\ell(s_i) + \frac{1}{2}(\ell(s_i s s_i) - \ell(s)) \in \{0, 2\}$ (of course, $\chi_{s_i,s_i} = -1$). Then (7.6) follows.

The lemma is proved. $\square$

Prove that (3.1) hold in $\hat{\mathbf{H}}'(W)$ by induction on $\ell(w)$. If $w = 1$, we have nothing to prove. If $\ell(w) = 1$, i.e., $w = s_i$ for some $i \in I$, then the (7.6) which verifies (3.1).

Suppose that $\ell(w) \geq 2$, i.e., $w = w_1w_2$ for some $w_1, w_2 \in W \setminus \{1\}$ with $\ell(w_1) + \ell(w_2) = \ell(w)$. Then using the inductive hypothesis in the form: $w_2 D_s w_2^{-1} = \chi_{w_2,s} D_{s'} + \frac{1-\chi_{w_2,s}}{2}(1 - w_2 s w_2^{-1})$, where we abbreviated $s' = w_2 s w_2^{-1}$, we obtain, by conjugating both sides with w_1:

$$\begin{aligned} wD_s w^{-1} &= w_1(w_2 D_s w_2^{-1})w_1^{-1} = \chi_{w_2,s} w_1 D_{s'} w_1^{-1} + \frac{1-\chi_{w_2,s}}{2}(1 - wsw^{-1}) \\ &= \chi_{w_2,s}(\chi_{w_1,s'} D_{w_1 s' w_1^{-1}} + \frac{1-\chi_{w_1,s'}}{2}(1 - w_1 s' w_1^{-1})) + \frac{1-\chi_{w_2,s}}{2}(1 - wsw^{-1}) \\ &= \chi_{w,s} D_{wsw^{-1}} \frac{1-\chi_{w,s}}{2}(1 - wsw^{-1}) \end{aligned}$$

by the inductive hypothesis with w' and by the first condition of (3.5).

This finishes the inductive proof of (3.1). Thus, $\hat{\mathbf{H}}'(W) = \hat{\mathbf{H}}'_{\chi,\sigma}(W)$ in the notation (3.1) with χ and σ are as in (7.5). Therefore, $\hat{\mathbf{H}}'(W)$ is a Hopf algebra by Proposition 7.1(a).

The proposition is proved. □

Finally, note that for χ and σ given by (7.5) the relations (3.2) become $D_s^2 = D_s$, $s \in \mathcal{S}$. Moreover, it follows from (7.6) that these relations considered in $\hat{\mathbf{H}}(W)$ follow from the relations $D_i^2 = D_i$, $i \in I$. This proves the following

Lemma 7.6. *$\hat{\mathbf{H}}(W) = \hat{\mathbf{H}}_{\chi,\sigma}(W)$ for any Coxeter group W and χ, σ given by (7.5).*

Thus, $\hat{\mathbf{H}}(W)$ is a Hopf algebra by Theorem 3.2.

Theorem 1.16 is proved. □

7.2. Factorization of Hecke-Hopf algebras and proof of Theorems 1.22, 3.3

Prove Theorem 3.3 first. Proposition 7.1(b) together with (3.5) guarantee that $\hat{\mathbf{H}}'_{\chi,\sigma}(W)$ factors as $\hat{\mathbf{H}}'_{\chi,\sigma}(W) = T(V) \cdot RW$ over R, where $V = \oplus_{s\in\mathcal{S}} R \cdot D_s$. To establish the factorization of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ we need the following result (which is a pre-condition in Lemma 4.15).

Proposition 7.7. *In the notation of Lemma 7.3, $w \cdot \mathbf{K}_s \cdot w^{-1} = \mathbf{K}_{wsw^{-1}}$ for all $w \in W$, $s \in \mathcal{S}$.*

Proof. The following is an immediate consequence of (3.5).

Lemma 7.8. *For any $\sigma, \chi : W \times \mathcal{S} \to R$ satisfying (3.5) one has for all $w \in W$, $s \in \mathcal{S}$:*

(a) $\chi_{1,s} = 1$, $\sigma_{1,s} = 0$, $\chi_{w^{-1},wsw^{-1}} = \frac{1}{\chi_{w,s}}$, $\sigma_{w^{-1},wsw^{-1}} = -\frac{\sigma_{w,s}}{\chi_{w,s}}$.
(b) $\chi_{wsw^{-1},wsw^{-1}} = \chi_{s,s}$, $(\chi_{s,s} - 1)\sigma_{w,s} = \chi_{w,s}\sigma_{wsw^{-1},wsw^{-1}} - \sigma_{s,s}$.

Furthermore, in the notation of Section 7.1, for $s \in \mathcal{S}$ of finite order $|s|$ we abbreviate: $a_s = \chi_{s,s}$, $b_s = \sigma_{s,s}$, $f_s := f_{|s|}^{a_s,b_s} \in R[x]$ and denote $\delta_s(t) := f_s(ts + D_s) \in \hat{\mathbf{H}}'_{\chi,\sigma}(W)[t]$.

We need the following result.

Lemma 7.9. *In the assumptions of Theorem 3.3 one has (in the notation (7.4)):*

$$f_{wsw^{-1}}(x) = f_s(\chi_{w,s} \cdot x + \sigma_{w,s}) \tag{7.7}$$

for all $w \in W$, $s \in \mathcal{S}$ of finite order. In particular, $w \cdot \delta_s(t) \cdot w^{-1} = \delta_{wsw^{-1}}\left(\frac{t-\sigma_{w,s}}{\chi_{w_s}}\right)$.

Proof. For a given $w \in W$ we abbreviate $s' := wsw^{-1}$ and $n := |s| = |s'|$. Then $a_{s'}^n = 1$ and (3.6) reads: $\sigma_{w,s} = b_s \frac{1-a_s^k}{1-a_s}$, where $k = \kappa_{w,s}$. Also $a_{s'} = a_s$ and $b_{s'} = \frac{b_s}{\chi_{w,s}} a_s^k$ by Lemma 7.8. Combining, we obtain $\sigma_{w,s} = b_{s'} \chi_{w,s} \frac{a_{s'}^{-k}-1}{1-a_{s'}}$. Then:

$$\begin{aligned} f_s(\chi_{w,s} \cdot x + \sigma_{w,s}) &= \prod_{i=1}^{n} \left(\chi_{w,s} \cdot x + \sigma_{w,s} - b_s \frac{1-a_s^i}{1-a_s} \right) \\ &= \prod_{i=1}^{n} \left(x + \frac{\sigma_{w,s}}{\chi_{w,s}} - \frac{b_s}{\chi_{w,s}} \cdot \frac{1-a_{s'}^i}{1-a_{s'}} \right) \\ &= \prod_{i=1}^{n} \left(x + b_{s'} \frac{a_{s'}^{-k}-1}{1-a_{s'}} - b_{s'} \frac{a_{s'}^{-k} - a_{s'}^{i-k}}{1-a_{s'}} \right) \\ &= \prod_{i=1}^{n} \left(x - b_{s'} \frac{1-a_{s'}^{i-k}}{1-a_{s'}} \right) = \prod_{i=1}^{n} (x - b_{s'} \frac{1-a_{s'}^i}{1-a_{s'}}) = f_{s'}(x) \ . \end{aligned}$$

This proves the first assertion of the lemma. Prove the second assertion now. Indeed, using (7.7) the form $f_s(t) = f_{s'}(p)$, where $p = \frac{t-\sigma_{w,s}}{\chi_{w_s}}$, we obtain:

$$\begin{aligned} w \cdot \delta_s(t) \cdot w^{-1} &= f_s(w \cdot (ts + D_s) \cdot w^{-1}) = f_s(ts' + \chi_{w,s} D_{s'} + \sigma_{w,s}(1-s')) \\ &= f_s((t-\sigma_{w,s})s' + \chi_{w,s} D_s + \sigma_{w,s}) = f_s(\chi_{w,s}(ps' + D_s) + \sigma_{w,s}) \\ &= f_{s'}(ps' + D_{s'}) + \sigma_{w,s}) = \delta_{wsw^{-1}}(p) \ . \end{aligned}$$

The lemma is proved. □

Since $\mathbf{K}_s$ is generated by the coefficients of $\delta_s(t)$, the second assertion of Lemma 7.9, finishes the proof of Proposition 7.7. □

In particular, $\mathbf{K} = \sum_{s \in \mathcal{S}} \mathbf{K}_s$ satisfies $w \cdot \mathbf{K} = \mathbf{K} \cdot w$ for all $w \in W$. This and the factorization of $\mathbf{H} = \hat{\mathbf{H}}'_{\chi,\sigma}(W) = T(V) \cdot RW$ guarantee that Lemma 4.15 is applicable here, therefore, $\underline{\mathbf{H}} = \hat{\mathbf{H}}_{\chi,\sigma}(W)$ factors as $\underline{\mathbf{H}} = \underline{\mathbf{D}} \cdot RW$ over R, where $\underline{\mathbf{D}} = \hat{\mathbf{D}}_{\chi,\sigma}(W) = T(V)/\langle \mathbf{K} \rangle$.

Theorem 3.3 is proved. □

Proof of Theorem 1.22. We need the following result.

Proposition 7.10. *For any Coxeter group W, one has*

(a) *the maps χ and σ defined by (7.5) satisfy (3.5).*

(b) *the map χ given by (7.5) satisfies* $\chi_{w,s} = \begin{cases} 1 & \text{if } \ell(ws) > \ell(w) \\ -1 & \text{if } \ell(ws) < \ell(w) \end{cases}$ *for all $s \in \mathcal{S}$, $w \in W$.*

Proof. Prove (a). The following immediate fact gives a “default” χ satisfying (3.5).

Lemma 7.11. *Let W be a group and $\mathcal{S}$ be a conjugation-invariant subset of W, then for any ring R, a group homomorphism $\rho : W \to R^\times$, and map $\varphi : \mathcal{S} \to R^\times$, the map $\chi = \chi^{\varphi,\rho} : W \times \mathcal{S} \to R^\times$ given by $\chi_{w,s} = \rho(w) \cdot \frac{\varphi(wsw^{-1})}{\varphi(s)}$ for $w \in W$, $s \in \mathcal{S}$ satisfies the first condition of* (3.5).

We use Lemma 7.11 with $R = \mathbb{Z}$, a homomorphism $\rho : W \to \{-1, 1\} = \mathbb{Z}^\times$, and a map $\varphi : \mathcal{S} \to \{-1, 1\}$ given respectively by: $\rho(w) = (-1)^\ell(w)$, $\varphi(s) = (-1)^{\frac{1}{2}(\ell(s)-1)}$ for $w \in W$, $s \in \mathcal{S}$. Then, clearly, χ defined by (7.5) equals $\chi^{\varphi,\rho}$ in the notation of Lemma 7.11 and, thus, satisfies the first condition (3.5).

Finally, $2\sigma_{w_2,s}+2\chi_{w_2,s}\sigma_{w_1,w_2sw_2^{-1}} = 1-\chi_{w_2,s}+\chi_{w_2,s}\cdot(1-\chi_{w_1,w_2sw_2^{-1}}) = 1-\chi_{w_1w_2,s} = 2\sigma_{w_1w_2,s}$ by the first condition (3.5). This proves (a).

Prove (b) now. Note that $\chi_{s_i,s'} = 1$ iff $s' \neq s_i$ by Lemma 7.5. This, Proposition 7.10(a) and the first equation (3.5) taken with $w_1 = s_i$, $w_2 = s_iw$ imply that

$$\chi_{w,s} = (-1)^{\delta_{s_i,wsw^{-1}}}\chi_{s_iw,s} \tag{7.8}$$

for all $w \in W$, $s \in \mathcal{S}$. Furthermore, we proceed by induction in $\ell(w)$ in the form

$$\text{If } \chi_{w,s} = -1, \text{ then } \ell(w) > \ell(ws) \tag{7.9}$$

Indeed, if $w = s_i$ for some $i \in I$, then $\chi_{s_i,s} = -1$ iff $s = s_i$ and we have nothing to prove.

Suppose that $\chi_{w,s} = -1$ for some $w \in W$ with $\ell(w) \geq 2$ and some $s \in \mathcal{S}$. Now choose $i \in I$ such that $\ell(s_iw) = \ell(w) - 1$. If $s_i = wsw^{-1}$, i.e., $s_iw = ws$, then clearly, $\ell(w) > \ell(ws)$ and we have nothing to prove. If $s_i \neq wsw^{-1}$, then (7.8) guarantees that $\chi_{s_iw,s} = -1$ and the inductive hypothesis for (s_iw, s) asserts that $\ell(s_iw) > \ell(s_iws)$. Taking into account that $\ell(s_iws) \geq \ell(ws) - 1$, we obtain $\ell(w) > \ell(ws)$, which finishes the proof of (7.9). Finally, using (7.9) let us prove

$$\text{If } \chi_{w,s} = 1, \text{ then } \ell(ws) > \ell(w) \tag{7.10}$$

Indeed, taking into account that $\chi_{s,s} = -1$ for all $s \in \mathcal{S}$ by (7.5) and using Proposition 7.10(a) again, the first equation (3.5) taken with $w_1 = ws$, $w_2 = s$ implies $\chi_{w,s} = -\chi_{ws,s}$ for all $w \in W$, $s \in \mathcal{S}$. Therefore, $\chi_{w,s} = 1$ implies that $\chi_{ws,s} = -1$ hence $\ell(ws) > \ell(wss) = \ell(w)$ by (7.9). This proves (7.10). Part (b) is proved.

The proposition is proved. $\square$

Let us show that these χ and σ also satisfy (3.6). Indeed, $|s| = 2$ $\chi_{s,s} = -1$, $\sigma_{s,s} = 1$, for all $s \in \mathcal{S}$ and $\chi^2_{w,s} = 1$ for all $w \in W$, therefore (3.6) holds automatically with $\kappa_{w,s}$ taken to be the exponent in (7.5), so that $\chi_{w,s} = (-1)^{\kappa_{w,s}}$. Therefore, $\hat{\mathbf{H}}(W) = \hat{\mathbf{H}}_{\chi,\sigma}(W)$ (by Lemma 7.6) and it factors over $\mathbb{Z}$ as $\hat{\mathbf{H}}(W) = \hat{\mathbf{D}}(W) \cdot \mathbb{Z}W$ by Theorem 3.3.

Theorem 1.22 is proved. $\square$

7.3. Left coideals and proof of Theorems 1.17, 1.24, 1.28, 3.5, and 3.6

Prove Theorem 3.5 first. Indeed, by definition (3.7),

$$\mathbf{K}_{\chi,\sigma}(W) = \{x \in \tilde{\mathbf{D}}_{\chi,\sigma}(w) \,|\, wxw^{-1} \in \tilde{\mathbf{D}}_{\chi,\sigma}(W) \ \forall\ w \in W\} = \mathbf{K}(RW, \tilde{\mathbf{D}}_{\chi,\sigma}(W)) \quad (7.11)$$

in the notation (4.1). Also, by definition, $\tilde{\mathbf{D}}_{\chi,\sigma}(W)$ is a left coideal subalgebra in $\hat{\mathbf{H}}(W)$, i.e., $\Delta(\tilde{\mathbf{D}}_{\chi,\sigma}(W)) \subset \hat{\mathbf{H}}(W) \otimes \tilde{\mathbf{D}}_{\chi,\sigma}(W)$. These and R-freeness of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ guarantee that all conditions of Theorem 4.5 are satisfied, therefore $\underline{\mathbf{H}} = \mathbf{H}_{\chi,\sigma}(W)$ is naturally a Hopf algebra.

Theorem 3.5 is proved. □

Proof of Theorem 3.6. We need the following result.

Lemma 7.12. *In the notation of Theorem 3.2, one has:*

(a) *the algebra $\hat{\mathbf{D}}_{\chi,\sigma}(W)$ is the free product (over R) of algebras $\mathbf{D}_s$, $s \in \mathcal{S}$, where $\mathbf{D}_s$ is the R-algebra generated by D_s subject to relations (3.2) (e.g., $\mathbf{D}_s = R[D_s]$ if s is of infinite order).*
(b) *If the condition (3.8) holds for all $s \in \mathcal{S}$, then $\hat{\mathbf{D}}_{\chi,\sigma}(W)$ is a free R-module.*

Proof. Part (a) is immediate from the presentation (3.2) of $\hat{\mathbf{D}}_{\chi,\sigma}(W)$.

Prove (b). If s is of finite order $|s|$ and $\chi_{s,s}$ is a primitive $|s|$-th root of unity in $R^\times$, then, according to Remark 3.1, $\mathbf{D}_s$ is generated by D_s subject to the only (monic) polynomial relation, therefore, is a free R-module. Since free product of free R-modules is also a free R-module, this finishes the proof of (b).

The lemma is proved. □

Lemma 7.13. *In the assumptions of Theorem 3.6, one has:*

(a) *$\hat{\mathbf{H}}_{\chi,\sigma}(W)$ is a free R-module.*
(b) *In the notation (7.11), one has*

$$\mathbf{K}_{\chi,\sigma}(W) = \{x \in \hat{\mathbf{D}}_{\chi,\sigma}(w) \,|\, wxw^{-1} \in \hat{\mathbf{D}}_{\chi,\sigma}(W) \ \forall\ w \in W\} = \mathbf{K}(RW, \hat{\mathbf{D}}_{\chi,\sigma}(W)) \quad (7.12)$$

and $\mathbf{K}_{\chi,\sigma}(W)$ is a left coideal in $\hat{\mathbf{H}}_{\chi,\sigma}(W)$.

Proof. In the assumptions of Theorem 3.6, one has a factorization $\hat{\mathbf{H}}_{\chi,\sigma}(W) = \hat{\mathbf{D}}_{\chi,\sigma}(W) \cdot RW$ by Theorem 3.3, in particular, $\tilde{\mathbf{D}}_{\chi,\sigma}(W) = \hat{\mathbf{D}}_{\chi,\sigma}(W)$. Taking into account that RW is also a free R-module and tensor product of free modules is free, we finish the proof of part (a).

Prove (b). (7.12) is immediate. The second assertion follows from (a) and Proposition 4.6.

The lemma is proved. □

Thus, all conditions of Theorem 4.5 are satisfied for the Hopf algebra $\hat{\mathbf{H}}_{\chi,\sigma}(W)$, therefore $\underline{\mathbf{H}} = \mathbf{H}_{\chi,\sigma}(W)$ is a Hopf algebra by Theorem 4.5. Finally, the factorization $\mathbf{H}_{\chi,\sigma}(W) = \mathbf{D}_{\chi,\sigma}(W) \cdot RW$ follows from Proposition 4.9 and Theorem 3.3.

Theorem 3.6 is proved. □

Proof of Theorem 1.17. Taking into account that $\hat{\mathbf{H}}(W) = \hat{\mathbf{H}}_{\chi,\sigma}(W)$ by Lemma 7.6 for χ and σ given by (7.5), we see that $\mathbf{K}(W) = \mathbf{K}_{\chi,\sigma}(W)$, therefore, $\underline{\mathbf{H}}(W) = \mathbf{H}_{\chi,\sigma}(W) = \mathbf{H}(W)$, which is a Hopf algebra by Theorem 3.6.

Theorem 1.17 is proved. □

Proof of Theorem 1.24. Clearly, the algebra $\hat{\mathbf{H}}(W)$ is filtered by $\mathbb{Z}_{\ge 0}$ via $\deg D_s = 1$, $s \in \mathcal{S}$, $\deg w = 0$. For each $r \in \mathbb{Z}_{\ge 0}$ denote by $\hat{\mathbf{H}}(W)_{\le r}$ the filtered component of degree r. In particular, $\hat{\mathbf{H}}(W)_{\le 0} = RW$. For each subset $X \subset \hat{\mathbf{H}}(W)$ we abbreviate $X_{\le r} := X \cap \hat{\mathbf{H}}(W)_{\le r}$.

Proposition 7.14. *For each $r \ge 0$ and $J \subset I$ the $\mathbb{Z}$-module $\mathbf{K}(W_J)_{\le r}$ is a left coideal in $\hat{\mathbf{H}}(W)$.*

Proof. We need the following result.

Lemma 7.15. *For each $r \ge 0$ the $\mathbb{Z}$-module $\hat{\mathbf{D}}(W)_{\le r}$ is a left coideal in $\hat{\mathbf{H}}(W)$.*

Proof. We proceed by induction in r. Indeed, since $\hat{\mathbf{D}}(W)_{\le 0} = \mathbb{Z}$, $\hat{\mathbf{D}}(W)_{\le 1} = \mathbb{Z} + \sum\limits_{s\in\mathcal{S}} \mathbb{Z} \cdot D_s$, the assertion is immediate for $r = 0, 1$. Suppose that $r > 1$. Clearly, $\mathbf{D}(W)_{\le r} = \mathbf{D}(W)_{\le r-1} \cdot \hat{\mathbf{D}}(W)_{\le 1}$. Therefore, $\Delta(\mathbf{D}(W)_{\le r}) \subset (\mathbf{H}(W) \otimes \mathbf{D}(W)_{\le r-1}) \cdot (\mathbf{H}(W) \otimes \mathbf{D}(W)_{\le 1}) = \mathbf{H}(W) \otimes \mathbf{D}(W)_{\le r}$ by the inductive hypothesis.

The lemma is proved. □

We need the following result.

Lemma 7.16. *For any Coxeter group W and any subset $J \subset I$ one has:*

(a) *the subalgebra of $\hat{\mathbf{H}}(W)$ generated by s_j, D_j, $j \in J$ is naturally isomorphic to $\hat{\mathbf{H}}(W_J)$.*
(b) *Under the identification from (a), $\hat{\mathbf{D}}(W_J)$ is a subalgebra $\hat{\mathbf{D}}(W)$ generated by D_s, $s \in \mathcal{S} \cap W_J$.*
(c) *$\mathbf{K}(W_J) \subset \hat{\mathbf{D}}(W_J) \subset \hat{\mathbf{H}}(W)$ is a left coideal in $\hat{\mathbf{H}}(W)$.*

Proof. Indeed, we have a natural homomorphism of algebras $\varphi_J : \hat{\mathbf{H}}(W_J) \to \hat{\mathbf{H}}(W)$ determined by $\varphi_J(s_j) = s_j$, $\varphi_J(D_j) = D_j$, $j \in J$. Clearly, the restriction of φ_J to $\mathbb{Z}W_J$

is an injective homomorphism $\mathbb{Z}W_J \hookrightarrow \mathbb{Z}W$. Also, $\varphi_J(D_s) = D_s$ for $s \in \mathcal{S}_J = \mathcal{S} \cap W_J$, which follows from (7.6) and the fact that $\mathcal{S}_J$ is the set of all reflections in W_J. In view of Lemma 7.12(a) applied to $\hat{\mathbf{D}}(W) = \hat{\mathbf{D}}_{\chi,\sigma}(W)$ with χ, σ given by (7.5), the restriction of φ_J to $\hat{\mathbf{D}}(W_J)$ is an injective homomorphism $\hat{\mathbf{D}}(W_J) \hookrightarrow \hat{\mathbf{D}}(W)$. Therefore, by Theorem 1.22, which asserts factorizations $\hat{\mathbf{H}}(W) = \hat{\mathbf{D}}(W) \cdot \mathbb{Z}W$ and $\hat{\mathbf{H}}(W_J) = \hat{\mathbf{D}}(W_J) \cdot \mathbb{Z}W_J$, the map $\varphi : \hat{\mathbf{D}}(W_J) \cdot \mathbb{Z}W_J \to \hat{\mathbf{D}}(W) \cdot \mathbb{Z}W$ is also injective as the tensor product of injective $\mathbb{Z}$-linear maps.

This proves (a) and (b).

Prove (c). By Lemma 7.6, $\hat{\mathbf{H}}(W) = \hat{\mathbf{H}}_{\chi,\sigma}(W)$ (for χ, σ defined by (7.5)). Therefore, taking into account that $\hat{\mathbf{D}}(W) = \hat{D}_{\chi,\sigma}(W)$, $\mathbf{K}(W) = \mathbf{K}_{\chi,\sigma}(W)$ is a left coideal in $\mathbf{H}(W)$ by Lemma 7.13(c). Replacing W with W_J and using (a), we finish proof of (c). $\square$

Lemmas 7.15, 7.16(c), and Proposition 4.1 guarantee that $\mathbf{K}(W_J)_{\le r} = \hat{\mathbf{D}}(W)_{\le r} \cap \mathbf{K}(W_J)$ is a left coideal in $\hat{\mathbf{H}}(W)$, which is a free $\mathbb{Z}$-module by Lemmas 7.6 and 7.13.

The proposition is proved. $\square$

Furthermore, by definition, $\mathbb{Z} + \mathbf{K}_{ij}(W) = \mathbf{K}(W_{\{i,j\}})_{\le m_{ij}}$. This and Propositions 4.1(a), 7.14 imply that $\underline{\mathbf{K}} = \mathbb{Z} + \sum\limits_{i,j \in I} \mathbf{K}_{ij}(W)$ is a left coideal in $\hat{\mathbf{H}}(W)$. Proposition 4.9 guarantees that the ideal $\underline{\mathbf{J}}(W)$ of $\hat{\mathbf{H}}(W)$ generated by $\underline{\mathbf{K}}$, is a Hopf ideal, hence $\mathbf{H}(W) = \hat{\mathbf{H}}(W)/\underline{\mathbf{J}}(W)$ is a Hopf algebra.

Theorem 1.24 is proved. $\square$

Proof of Theorem 1.28. $\underline{\mathbf{H}}(W) = \mathbf{H}_{\chi,\sigma}(W)$ for χ, σ given by (7.5) by the argument from the proof of Theorem 1.17. Therefore, the first assertion of Theorem 1.28 coincides with the second assertion of Theorem 3.6.

Prove the second assertion of Theorem 1.28. We need the following result.

Proposition 7.17. *For any subset $J \subset I$, under the natural inclusion $\hat{\mathbf{H}}(W_J) \subset \hat{\mathbf{H}}(W)$ from Lemma 7.16(a), one has $\mathbf{K}(W_J) \subset \mathbf{K}(W)$.*

Proof. We need the following immediate consequence of (3.1) and Proposition 7.10(b).

Lemma 7.18. *The following relations hold in $\hat{\mathbf{H}}'(W)$*

$$wD_sw^{-1} = \begin{cases} D_{wsw^{-1}} & \text{if } \ell(ws) > \ell(w) \\ 1 - D_{wsw^{-1}} - wsw^{-1} & \text{if } \ell(ws) < \ell(w) \end{cases} \tag{7.13}$$

for all $w \in W$, $s \in \mathcal{S}$.

Given $J \subset I$, denote $W^J := \{w \in W \mid \ell(ws_j) = \ell(w)+1 \ \forall\, j \in J\}$. It is well-known (see e.g., [5]) that W has a unique factorization $W = W^J \cdot W_J$, which we write element-wise as $w = [w]^J \cdot [w]_J$ for any $w \in W$, where $[w]^J \in W^J$ and $[w]_J \in W_J$.

Lemma 7.19. *For any Coxeter group W, and any subset $J \subset I$ one has*

(a) $w\hat{\mathbf{D}}(W_J)w^{-1} \subset \hat{\mathbf{D}}(W)$ *for* $w \in W^J$.
(b) $w\mathbf{K}(W_J)w^{-1} = [w]^J\mathbf{K}(W_J)([w]^J)^{-1} \subset \mathbf{K}(W)$ *for all* $w \in W$.

Proof. It is easy to see that $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ for any $w_1 \in W^J$, $w_2 \in W_J$. This and (7.13) imply that $wD_sw^{-1} = D_{wsw^{-1}}$ in $\hat{\mathbf{D}}(W)$ for all $w \in W^J$, $s \in \mathcal{S}_J = \mathcal{S} \cap W_J$. Hence $w\hat{\mathbf{D}}(W_J)w^{-1} \subset \hat{\mathbf{D}}(W)$ for all $w \in W^J$. This proves (a).

Prove (b) now. We have, based on the proof of (a):

$$w\mathbf{K}(W_J)w^{-1} = [w]^J[w]_J\hat{\mathbf{K}}(W_J)([w]_J)^{-1}([w]^J)^{-1} = [w]^J\mathbf{K}(W_J)([w]^J)^{-1} \subset \hat{\mathbf{D}}(W)$$

for all $w \in W$ because $w_1\mathbf{K}(W_J)w_1^{-1} = \mathbf{K}(W_J)$, $w_2\mathbf{K}(W_J)w_2^{-1} \subset \hat{\mathbf{D}}(W_J)$ for all $w_1 \in W_J$, $w_2 \in W^J$.

In particular, $\mathbf{K}(W_J) \subset \mathbf{K}(W)$. Conjugating with w and using the fact that $w\mathbf{K}(W)w^{-1} = \mathbf{K}(W)$ for all $w \in W$, we finish the proof of (b).

The lemma is proved. □

Therefore, the proposition is proved. □

Let $\underline{\mathbf{K}}(W) := \sum\limits_{w\in W, i,j\in I: i\neq j} w\mathbf{K}_{ij}(W)w^{-1}$. By definition, $w \cdot \underline{\mathbf{K}}(W) = \underline{\mathbf{K}}(W) \cdot w$ for all $w \in W$ and the ideal $\underline{\mathbf{J}}(W)$ from the proof of Theorem 1.24 is generated by $\underline{\mathbf{K}}(W)$. Also, $\underline{\mathbf{K}}(W) \subset \hat{\mathbf{D}}(W)$ by Proposition 7.17. Therefore $\mathbf{H} = \hat{\mathbf{H}}(W)$, $\mathbf{D} = \hat{\mathbf{D}}(W)$, and $\mathbf{K} = \underline{\mathbf{K}}(W) \cap Ker\ \varepsilon$ satisfy the assumptions of Lemma 4.15, thus $\underline{\mathbf{H}} = \mathbf{H}(W)$ factors as $\mathbf{H}(W) = \underline{\mathbf{D}} \cdot \mathbb{Z}W$ over $\mathbb{Z}$, where $\underline{\mathbf{D}} = \mathbf{D}(W) = \hat{\mathbf{D}}(W)/\langle\underline{\mathbf{K}}(W)\rangle$.

Theorem 1.28 is proved. □

7.4. Relations in $\mathbf{D}(W)$ *and proof of Theorem 1.34*

For all distinct $i, j \in I$, $w \in W^{\{i,j\}}$, and $s \in \mathcal{S} \cap W_{\{i,j\}}$ we have $wD_sw^{-1} = D_{wsw^{-1}}$ by (7.13). Therefore, it suffices to prove the assertion only when $w = 1$ and $W = W_{\{i,j\}}$. Define $Q_{ij}^{(n,r,p)}$ and $R_{ij}^{(n,r,t)} \in \hat{D}(W_{\{i,j\}})$ for all divisors n of $m = m_{ij}$, $r \in [1, n]$, and $1 \le p < \frac{m}{2n}$, $0 \le t \le \frac{m}{n}$ by:

$$Q_{ij}^{(n,r,p)} = \sum_{0\le a<b<\frac{m}{n}: b-a=\frac{m}{n}-p} D_{r+bn}D_{r+an} - \sum_{0\le a'<b'<\frac{m}{n}: b'-a'=p} D_{r+a'n}D_{r+b'n} + \sum_{p\le c<\frac{m}{n}-p} D_{r+cn}\ ,$$

$$R_{ij}^{(n,r)} = D_rD_{r+n}\cdots D_{r+m-n} - D_{r+m-n}\cdots D_{r+n}D_r\ ,$$

$$R_{ij}^{(n,r,t)} := \prod^{\longrightarrow}_{t\le a\le\frac{m}{n}-1}(1 - D_{r+an}) \prod^{\longrightarrow}_{0\le b\le t-1} D_{r+bn} - \prod^{\longleftarrow}_{0\le b\le t-1} D_{r+bn} \prod^{\longleftarrow}_{t\le a\le\frac{m}{n}-1}(1 - D_{r+an})\ .$$

We need the following fact.

Proposition 7.20. *For any Coxeter group W and $i, j \in I$ with $m := m_{ij} \geq 2$ one has for all divisors n of $m = m_{ij}$ and $r \in [1, n]$:*

(a) $s_i Q_{ij}^{(n,r,p)} s_i = \begin{cases} Q_{ij}^{(n,r-1,p)} & \text{if } r > 1 \\ Q_{ji}^{(n,n,p)} & \text{if } r = 1 \end{cases}$, $s_j Q_{ij}^{(n,r,p)} s_j = \begin{cases} Q_{ij}^{(n,r+1,p)} & \text{if } r < n \\ Q_{ji}^{(n,1,p)} & \text{if } r = n \end{cases}$ *for* $1 \leq p < \frac{m}{2n}$.

(b) $$s_i R_{ij}^{(n,r,t)} s_i = \begin{cases} R_{ij}^{(n,r-1,t)} & \text{if } r > 1 \\ R_{ji}^{(n,n,t-1)} & \text{if } r = 1,\ t \geq 1 \\ -R_{ij}^{(n,1,1)} & \text{if } r = 1,\ t = 0 \end{cases},$$
$$s_i R_{ij}^{(n,r,t)} s_i = \begin{cases} R_{ij}^{(n,r+1,t)} & \text{if } r < n \\ R_{ji}^{(n,1,t+1)} & \text{if } r = n,\ t < \frac{m}{n} \\ -R_{ij}^{(n,1,0)} & \text{if } r = n,\ t = \frac{m}{n} \end{cases} \text{ for } 1 \leq t \leq \frac{m}{n}.$$

Proof. We need the following immediate consequence of (7.6).

Lemma 7.21. *If $m := m_{ij} \geq 2$, then one has in $\hat{\mathbf{D}}(W_{\{i,j\}})$:*

$$s_i D_k s_i = \begin{cases} 1 - D_1 - s_i & \text{if } k = 1 \\ D_{m+2-k} & \text{if } 2 \leq k \leq m \end{cases} \tag{7.14}$$

for $k = 1, \ldots, m$, where $D_k := D_k^{i,j} = D_{\underbrace{s_i s_j \cdots s_i}_{2k-1}}$ for $k = 1, \ldots, m$.

Taking into account that $D_k^{ji} = D_{m+1-k}^{i,j}$, we will repeatedly use (7.14) in the form:

$$s_i D_k s_i = \begin{cases} 1 - D_m^{ji} - s_i & \text{if } k = 1 \\ D_{k-1}^{ji} & \text{if } 2 \leq k \leq m \end{cases}, \quad s_i D_\ell^{ji} s_i = D_{m-\ell}^{ji} \tag{7.15}$$

for $k = 1, \ldots, m$, $\ell = 1, \ldots, m-1$.

Prove (a). First, suppose that $r > 1$. Then, using (7.15), we have

$$\begin{aligned} s_i Q_{ij}^{(n,r,p)} s_i &= \sum_{0 \leq a < b < \frac{m}{n} : b - a = \frac{m}{n} - p} s_i D_{r+bn} D_{r+an} s_i \\ &\quad - \sum_{0 \leq a' < b' < \frac{m}{n} : b' - a' = p} s_i D_{r+a'n} D_{r+b'n} s_i + \sum_{p \leq c < \frac{m}{n} - p} s_i D_{r+cn} s_i \\ &= \sum_{0 \leq a < b < \frac{m}{n} : b - a = \frac{m}{n} - p} D_{r-1+bn}^{ji} D_{r-1+an}^{ji} \end{aligned}$$

$$-\sum_{0\le a'<b'<\frac{m}{n}:b'-a'=p} D^{ji}_{r-1+a'n}D^{ji}_{r-1+b'n}+\sum_{p\le c<\frac{m}{n}-p} D^{ji}_{r-1+cn}=Q^{n,r-1,p}_{ji}$$

Interchanging i and j, we also obtain $s_jQ^{(n,r,p)}_{ij}s_j=Q^{(n,r+1,p)}_{ji}$ whenever $r<n$.

Finally, suppose that $r=1$. Then, using (7.15) again, we have

$$\begin{aligned}
s_iQ^{(n,1,p)}_{ij}s_i &= s_iD_{1+m-pn}D_1s_i+\sum_{0<a<b<\frac{m}{n}:b-a=\frac{m}{n}-p} s_iD_{1+bn}D_{1+an}s_i\\
&\quad - s_iD_1D_{1+pn}s_i-\sum_{0<a'<b'<\frac{m}{n}:b'-a'=p} s_iD_{1+a'n}D_{1+b'n}s_i\\
&\quad + \sum_{p\le c<\frac{m}{n}-p} s_iD_{1+cn}s_i\\
&= D^{ji}_{m-pn}(1-D^{ji}_m-s_i)+\sum_{0<a<b<\frac{m}{n}:b-a=\frac{m}{n}-p} D^{ji}_{bn}D^{ji}_{an}\\
&\quad -(1-D^{ji}_m-s_i)D^{ji}_{pn}-\sum_{0<a'<b'<\frac{m}{n}:b'-a'=p} D^{ji}_{a'n}D^{ji}_{b'n}+\sum_{p\le c<\frac{m}{n}-p} D^{ji}_{cn}\\
&= D^{ji}_mD^{ji}_{pn}-D^{ji}_{pn}+D^{ji}_{m-pn}+\sum_{0<a<b<\frac{m}{n}:b-a=\frac{m}{n}-p} D^{ji}_{bn}D^{ji}_{an}\\
&\quad -D^{ji}_{m-pn}D^{ji}_m-\sum_{0<a'<b'<\frac{m}{n}:b'-a'=p} D^{ji}_{a'n}D^{ji}_{b'n}+\sum_{p\le c<\frac{m}{n}-p} D^{ji}_{cn}\\
&= \sum_{0\le a-1<b-1<\frac{m}{n}:b-a=\frac{m}{n}-p} D^{ji}_{bn}D^{ji}_{an}-\sum_{0\le a'-1<b'-1<\frac{m}{n}:b'-a'=p} D^{ji}_{a'n}D^{ji}_{b'n}\\
&\quad +\sum_{p\le c-1<\frac{m}{n}-p} D^{ji}_{cn}=Q^{(n,n,p)}_{ji}
\end{aligned}$$

Interchanging i and j, we obtain $s_jQ^{(n,n,p)}_{ij}s_j=Q^{(n,1,p)}_{ji}$. This proves (a).

Prove (b) now. First, suppose that $r>1$. Then, using (7.15), we have

$$\begin{aligned}
s_iR^{(n,r,t)}_{ij}s_i &= \overrightarrow{\prod_{t\le a\le\frac{m}{n}-1}}(1-s_iD_{r+an}s_i)\overrightarrow{\prod_{0\le b\le t-1}} s_iD_{r+bn}s_i\\
&\quad -\overleftarrow{\prod_{0\le b\le t-1}} s_iD_{r+bn}s_i\overleftarrow{\prod_{t\le a\le\frac{m}{n}-1}}(1-s_iD_{r+an}s_i)\\
&= \overrightarrow{\prod_{0\le b\le t-1}} D^{ji}_{r-1+bn}\overrightarrow{\prod_{t\le a\le\frac{m}{n}-1}}(1-D^{ji}_{r-1+an})\\
&\quad -\overleftarrow{\prod_{0\le b\le t-1}} D^{ji}_{r-1+bn}\overleftarrow{\prod_{t\le a\le\frac{m}{n}-1}}(1-D^{ji}_{r-1+an})=R^{(n,r-1,t)}_{ji}\ .
\end{aligned}$$

Interchanging i and j, we also obtain $s_j R_{ij}^{(n,r,t)} s_j = R_{ji}^{(n,r+1,t)}$ whenever $r < n$.

Now suppose that $r = 1$, $t \geq 1$. Then, using (7.15) again, we have

$$\begin{aligned}
s_i R_{ij}^{(n,1,t)} s_i &= \overrightarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - s_i D_{1+an} s_i) \overrightarrow{\prod_{0\leq b\leq t-1}} s_i D_{1+bn} s_i \\
&\quad - \overleftarrow{\prod_{0\leq b\leq t-1}} s_i D_{1+bn} s_i \overleftarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - s_i D_{1+an} s_i) \\
&= \overrightarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - D_{an}^{ji}) \cdot (1 - D_m^{ji} - s_i) \overrightarrow{\prod_{1\leq b\leq t-1}} D_{bn}^{ji} \\
&\quad - \overleftarrow{\prod_{1\leq b\leq t-1}} D_{bn}^{ji} \cdot (1 - D_m^{ji} - s_i) \overleftarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - D_{an}^{ji}) \\
&= \overrightarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - D_{an}^{ji}) \cdot (1 - D_m^{ji}) \overrightarrow{\prod_{1\leq b\leq t-1}} D_{bn}^{ji} \\
&\quad - \overleftarrow{\prod_{1\leq b\leq t-1}} D_{bn}^{ji} \cdot (1 - D_m^{ji}) \overleftarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - D_{an}^{ji}) = R_{ji}^{(n,n,t-1)}
\end{aligned}$$

because

$$\begin{aligned}
&- \overrightarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - D_{an}^{ji}) \cdot s_i \overrightarrow{\prod_{1\leq b\leq t-1}} D_{bn}^{ji} + \overleftarrow{\prod_{1\leq b\leq t-1}} D_{bn}^{ji} \cdot s_i \overleftarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - D_{an}^{ji}) \\
&\quad = -s_i \left(\overrightarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - D_{m-an}^{ji}) \overrightarrow{\prod_{1\leq b\leq t-1}} D_{bn}^{ji} + \overleftarrow{\prod_{1\leq b\leq t-1}} D_{m-bn}^{ji} \overleftarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - D_{an}^{ji}) \right) \\
&\quad = 0
\end{aligned}$$

which is immediate if $t = 1$ or $t = \frac{m}{n}$ and follows from the relations $D_s^2 = D_s$ if $1 < t < \frac{m}{n}$ (which we use here in the form $(1 - D_n^{ji}) D_n^{ji} = D_n^{ji}(1 - D_n^{ji}) = 0$).

Interchanging i and j, we obtain $s_j R_{ij}^{(n,n,t)} s_j = R_{ji}^{(n,1,t+1)}$ whenever $1 \leq t < \frac{m}{n}$.

Finally, suppose that $r = 1$, $t = 0$. Then, using (7.14), we have

$$\begin{aligned}
s_i R_{ij}^{(n,1,0)} s_i &= \overrightarrow{\prod_{0\leq a\leq \frac{m}{n}-1}} (1 - s_i D_{1+an} s_i) - \overleftarrow{\prod_{t\leq a\leq \frac{m}{n}-1}} (1 - s_i D_{1+an} s_i) \\
&= (D_1 + s_i) \cdot \overrightarrow{\prod_{1\leq a\leq \frac{m}{n}-1}} (1 - D_{m+1-an}) - \overleftarrow{\prod_{1\leq a\leq \frac{m}{n}-1}} (1 - D_{m+1-an}) \cdot (D_1 + s_i) \\
&= D_1 \cdot \overrightarrow{\prod_{1\leq a\leq \frac{m}{n}-1}} (1 - D_{m+1-an}) - \overleftarrow{\prod_{1\leq a\leq \frac{m}{n}-1}} (1 - D_{m+1-an}) \cdot D_1
\end{aligned}$$

$$= D_1 \cdot \prod_{1\le a'\le \frac{m}{n}-1}^{\longleftarrow} (1 - D_{1+a'n}) - \prod_{1\le a'\le \frac{m}{n}-1}^{\longleftarrow} (1 - D_{1+a'n}) \cdot D_1 = -R_{ij}^{(n,1,1)}$$

because $s_i \underbrace{(1 - D_{m+1-n}) \cdots (1 - D_{n+1})}_{\frac{m}{n}-1} - \underbrace{(1 - D_{n+1}) \cdots (1 - D_{m+1-n})}_{\frac{m}{n}-1} s_i = 0$.

Interchanging i and j, we obtain $s_j R_{ij}^{(n,1,1)} s_j = -R_{ij}^{(n,1,0)}$ whenever $1 \le t < \frac{m}{n}$. This proves (b).

The proposition is proved. $\square$

Finally, Proposition 7.20 implies that all $Q_{ij}^{(n,r,p)}$, $\overline{Q_{ij}^{(n,r,p)}}$ and $R_{ij}^{(n,r,t)}$ belong to $\mathbf{K}_{ij}(W)$, where $\bar{\cdot}$ is the anti-involution of $\hat{\mathbf{D}}(W)$ given by $\overline{D_s} = D_s$ for $s \in\in \mathcal{S}$ (see also Theorem 1.37(a) and its proof). This proves Theorem 1.34. $\square$

7.5. *Braid relations and proof of Theorems 1.20 and 1.33*

For commutative ring $\Bbbk$, $i, j \in I$ with $m_{ij} \ge 2$, $c_i, c_j \in \Bbbk$ such that $c_i = c_j$ if m_{ij} is odd, define the element in $\hat{\mathbf{H}}(W) \otimes \Bbbk$ by:

$$\Delta_{ij}^{c_i,c_j} = w_{ij}(\underbrace{\cdots (s_i - c_i s_i D_{s_i})(s_j - c_j s_j D_{s_j})}_{m} - \underbrace{\cdots (s_j - c_j s_j D_{s_j})(s_i - c_i s_i D_{s_i})}_{m})$$

where $m := m_{ij}$ and $w_{ij} = \underbrace{s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i \cdots}_{m_{ij}}$ is the longest element in $W_{\{i,j\}}$. It is easy to see that

$$S^{-1}(w_{ij}\Delta_{ij}^{c_i,c_j}) = \underbrace{\cdots (s_i + c_i D_{s_i})(s_j + c_j D_{s_j})}_{m} - \underbrace{\cdots (s_j + s_j D_{s_j})(s_i + c_i D_{s_i})}_{m} \tag{7.16}$$

and

$$\Delta_{ij}^{c_i,c_j} = (1 - c_{i_1} D_1) \cdots (1 - c_{i_m} D_m) - (1 - c_{i_m} D_m) \cdots (1 - c_{i_1} D_1) \tag{7.17}$$

where $D_k = D_k^{i,j}$ are as in Lemma 7.21, $m = m_{ij}$, and $(i_1, \ldots, i_m) = \underbrace{(\ldots, i, j)}_{m}$.

In particular, $\Delta_{ij}^{c_i,c_j} \in \hat{\mathbf{D}}(W) \otimes \Bbbk$ for all i, j with $m_{ij} \ge 2$ and $c_i, c_j \in \Bbbk$.

Proposition 7.22. *In the assumptions as above, each $\Delta_{ij}^{c_i,c_j}$ belongs to $\mathbf{K}_{ij}(W) \otimes \Bbbk$.*

Proof. We need the following result.

Lemma 7.23. *For all $i, j \in I$ with $m_{ij} \ge 2$, $c_i, c_j \in \Bbbk^\times$ such that $c_i = c_j$ if m_{ij} is odd, one has:*

$$s_i\Delta_{ij}^{c_i,c_j}s_i = \frac{1}{c_i-1}D_{i,c_i}\Delta_{ij}^{c_i,c_j}D_{i,c_i},\ s_j\Delta_{ij}^{c_i,c_j}s_j = \frac{1}{c_j-1}D_{j,c_j}\Delta_{ij}^{c_i,c_j}D_{j,c_j}\ , \tag{7.18}$$

where $D_{i,c_i} = (1-c_i)(1-D_{s_i}) = (1-c_i)(1-c_iD_{s_i})^{-1}$.

Proof. Prove the first equation (7.18). Using relations (7.14) we obtain

$$\begin{aligned} s_i\Delta_{ij}^{c_i,c_j}s_i &= s_i(1-c_{i_1}D_1)\cdots(1-c_{i_m}D_m)s_i - s_i(1-c_{i_m}D_m)\cdots(1-c_{i_1}D_1)s_i \\ &= (1-c_i(1-D_{s_i}-s_i))(1-c_{i_m}D_m)\cdots(1-c_{i_2}D_2) \\ &\quad -(1-c_{i_2}D_2)\cdots(1-c_{i_m}D_m)(1-c_i(1-D_{s_i}-s_i)) \\ &= D_{i,c_i}(1-c_{i_m}D_m)\cdots(1-c_{i_2}D_2)-(1-c_{i_2}D_2)\cdots(1-c_{i_m}D_m)D_{i,c_i} \end{aligned}$$

because $s_i(1-c_{i_m}D_m)\cdots(1-c_{i_2}D_2) = (1-c_{i_2}D_2)\cdots(1-c_{i_m}D_m)s_i$.

Taking into account that $D_{i,c_i} = (1-c_i)\cdot(1-c_iD_{s_i})^{-1}$, we obtain the first equation (7.18). The second one also follows because $\Delta_{ji}^{c_j,c_i} = -\Delta_{ij}^{c_i,c_j}$.

The lemma is proved. □

Conjugating $\Delta_{ij}^{c_i,c_j}$ with $w = \underbrace{\cdots s_js_i}_{\ell} \in W_{\{i,j\}}$, $\ell \le m$, and using (7.18) repeatedly, we obtain:

$$w\Delta_{ij}^{c_i,c_j}w^{-1} = \left(\prod_{k=1}^{\ell}\frac{1}{c_{i_\ell}-1}\right)\cdot \tilde{D}_\ell\cdots\tilde{D}_1\Delta_{ij}^{c_i,c_j}\tilde{D}_1\cdots\tilde{D}_\ell$$

where we abbreviate $\tilde{D}_k = s_{i_1}\cdots s_{i_{k-1}}D_{i_k,c_{i_k}}s_{i_{k-1}}\cdots s_{i_1} = 1-c_{i_k}(1-D_k)$ for $k = 1,\ldots,\ell$ in the notation of (7.17), where $i_k = \begin{cases} i & \text{if } k \text{ is odd} \\ j & \text{if } k \text{ is even}\end{cases}$. This implies that $w\Delta_{ij}^{c_i,c_j}w^{-1} \in \mathbf{D}(W_{\{i,j\}})\otimes\Bbbk$. Similarly, taking $w = \underbrace{\cdots s_is_j}_{\ell} \in W_{\{i,j\}}$, $\ell \le m$, one shows that $w\Delta_{ij}^{c_i,c_j}w^{-1} \in \mathbf{D}(W_{\{ij\}})_{\le m}\otimes\Bbbk$. Thus, $w\Delta_{ij}^{c_i,c_j}w^{-1} \in \mathbf{D}(W_{\{i,j\}})\otimes\Bbbk$ for any $w \in W_{\{i,j\}}$ for any $\Bbbk$ and any $c_i, c_j \in \Bbbk$ such that $c_i = c_j$ if m_{ij} is odd. Suppose that $\Bbbk$ is a free $\mathbb{Z}$-module. This implies that $\Delta_{ij}^{c_i,c_j} \in \bigcap\limits_{w\in W} w\cdot\big(\mathbf{D}(W_{\{i,j\}})_{\le m}\otimes\Bbbk\big)\cdot w^{-1}$ where the intersection is in $\hat{\mathbf{H}}(W_{\{i,j\}})\otimes\Bbbk$.

Suppose that $\Bbbk$ is a free $\mathbb{Z}$-module. Then, taking into account that

$$\bigcap_{w\in W} w\cdot(\mathbf{D}(W_{\{i,j\}})_{\le m}\otimes\Bbbk)\cdot w^{-1} = \left(\bigcap_{w\in W} w\cdot\mathbf{D}(W_{\{ij\}})_{\le m}\cdot w^{-1}\right)\otimes\Bbbk$$

by Lemma 4.2 and $\bigcap\limits_{w\in W} w\cdot\mathbf{D}(W_{\{ij\}})_{\le m}\cdot w^{-1} = \left(\bigcap\limits_{w\in W} w\cdot\mathbf{D}(W_{\{ij\}})\cdot w^{-1}\right)_{\le m} = K_{ij}(W)$, we obtain $\Delta_{ij}^{c_i,c_j} \in \mathbf{K}_{ij}(W)\otimes\Bbbk$.

Finally, we can remove freeness condition for $\Bbbk$ over $\mathbb{Z}$ by first replacing $\Bbbk$ with a commutative ring $\hat{\Bbbk}$ free over $\mathbb{Z}$ and then noting that any commutative ring $\Bbbk$ is a homomorphic image of some $\hat{\Bbbk}$. Then extending the structural homomorphism $f:\hat{\Bbbk}\twoheadrightarrow\Bbbk$ to $f:\hat{\mathbf{H}}(W_{\{i,j\}})\otimes\hat{\Bbbk}\twoheadrightarrow\hat{\mathbf{H}}(W_{\{i,j\}})\otimes\Bbbk$ we see that an inclusion $\Delta_{ij}^{\hat{c}_i,\hat{c}_j}\in\mathbf{K}_{ij}(W)\otimes\hat{\Bbbk}$ implies an inclusion $\Delta_{ij}^{c_i,c_j}\in\mathbf{K}_{ij}(W)\otimes\Bbbk$, where $c_i=f(\hat{c}_i)$, $c_j=f(\hat{c}_j)$.

The proposition is proved. $\square$

Proof of Theorems 1.20 and 1.33. Indeed, the braid relations between $T_i'=s_i+c_iD_{s_i}$ and $T_j'=s_j+c_jD_{s_j}$ (where $c_i=1-q_i$, $c_j=1-q_j$) in $\mathbf{H}(W)\otimes\Bbbk$ follow from (7.16) and Proposition 7.22 because $S^{-1}(w_{ij}\Delta_{ij}^{c_i,c_j})\in\mathbf{K}_{ij}(W)\otimes\Bbbk$.

It remains to verify the quadratic relations for T_i'. Indeed, one has:

$$\begin{aligned}T_i'^2&=(s_i+(1-q_i)D_{s_i})^2=s_i^2+(1-q_i)(s_iD_{s_i}+D_{s_i}s_i)+(1-q_i)^2D_{s_i}^2\\&=1+(1-q_i)(s_i-1)+(1-q_i)^2D_{s_i}=(1-q_i)T_i'+q_i\ .\end{aligned}$$

This proves that there is a unique homomorphism of algebras $\varphi_W:H_{\mathbf{q}}(W)\to\mathbf{H}(W)\otimes\Bbbk$ such that $\varphi_W(T_i)=T_i'$ for $i\in I$, as in Theorem 1.33.

Furthermore, Proposition 7.17 with $J=\{i,j\}$ guarantees the inclusions $\mathbf{K}_{ij}(W)\subset\mathbf{K}(W_{\{i,j\}})\subset\mathbf{K}(W)$ hence we have a surjective homomorphism of Hopf algebras $\pi_W:\mathbf{H}(W)\twoheadrightarrow\underline{\mathbf{H}}(W)$ as in Theorem 1.20. Denote $\underline{\varphi}_W:=(\pi_W\otimes 1)\circ\varphi_W$, which is a homomorphism $H_{\mathbf{q}}(W)\to\underline{\mathbf{H}}(W)\otimes\Bbbk$, as in Theorem 1.20

Let us prove it injectivity of $\underline{\varphi}_W$ (the injectivity of φ_W will follow verbatim).

Recall that for each $w\in W$ there is a unique element $T_w\in H_{\mathbf{q}}(W)$ such that $T_w=T_{i_1}\cdots T_{i_m}$ for any reduced decomposition $w=s_{i_1}\cdots s_{i_m}$ in W. Clearly, the elements T_w generate $H_{\mathbf{q}}(W)$ as a $\Bbbk$-module (in fact, they form a $\Bbbk$-basis - see Corollary 1.21 below).

Thus, to prove injectivity of $\underline{\varphi}_W$, it suffices to show that the images $\underline{\varphi}_W(T_w)$ are $\Bbbk$-linearly independent in $\mathbf{H}(W)\otimes\Bbbk$.

Proposition 7.24. *For each $w\in W$ one has: $\underline{\varphi}_W(T_w)\in w+\sum_{w':w'\prec w}\Bbbk\cdot\mathbf{D}(W)\cdot w'$, where $\prec$ denotes the strong Bruhat order on W.*

Proof. For each $w\in W$ denote $W_{\prec w}:=\{w'\in W:w'\prec w\}$ and $W_{\preceq w}:=\{w\}\sqcup W_{\prec w}$.

We need the following fact.

Lemma 7.25. *$W_{\prec w}\cdot D_i\in\mathbf{D}(W)\cdot W_{\prec w}$ for any $w\in W$.*

Proof. Since $W_{\preceq\tilde{w}}\subset W_{\prec w}$ for any $\tilde{w}\prec w$, it suffices to show that

$$\tilde{w}\cdot D_i\in\mathbf{D}\cdot W_{\preceq\tilde{w}}\tag{7.19}$$

in $\mathbf{H}(W)$ for all $\tilde{w}\in W$, $i\in I$.

Indeed, by definition of generators D_s of $\mathbf{D}(W)$, which are images of their counterparts in $\hat{\mathbf{D}}(W)$, if $\ell(\tilde{w}s_i)=\ell(\tilde{w})+1$, then (7.13) implies that $\tilde{w}\cdot D_i=D_{\tilde{w}s_i\tilde{w}^{-1}}\cdot\tilde{w}\in\mathbf{D}(W)\cdot W_{\preceq\tilde{w}}$.

Otherwise, i.e., if $\tilde{w} \in W$ is such that $\ell(\tilde{w}s_i) = \ell(\tilde{w}) - 1$, then using (7.13) again, we obtain

$$\tilde{w} \cdot D_i = \tilde{w}s_i \cdot (-D_i s_i + s_i - 1) = -D_{\tilde{w}s_i\tilde{w}^{-1}} \cdot \tilde{w} + \tilde{w} - \tilde{w}s_i \in \mathbf{D}(W) \cdot W_{\preceq \tilde{w}}$$

since $\tilde{w}s_i \cdot D_i = D_{\tilde{w}s_i\tilde{w}^{-1}} \cdot \tilde{w}s_i$.

The lemma is proved. $\square$

The following finishes the proof of the proposition.

Lemma 7.26. *For all $w \in W$ one has $\underline{\varphi}_W(T_w) \in w + \Bbbk \cdot \mathbf{D}(W) \cdot W_{\prec w}$.*

Proof. We will prove the assertion by induction in length $\ell(w)$. Indeed, if $w = 1$, we have nothing to prove. Suppose $w \neq 1$, then choose $i \in I$ such that $\ell(ws_i) = \ell(w) - 1$ (or, equivalently, $ws_i \prec w$). Using the inductive hypothesis for ws_i and that $\underline{\varphi}_W(T_w) = \underline{\varphi}_W(T_{ws_i})\underline{\varphi}_W(T_i)$, we obtain:

$$\begin{aligned}\underline{\varphi}_W(T_w) &= \underline{\varphi}_W(T_{ws_i})(s_i + (1-q_i)D_i) \in (ws_i + \Bbbk \cdot \mathbf{D}(W) \cdot W_{\prec ws_i})(s_i + (1-q_i)D_i) \\ &\subset w + (1-q_i)ws_iD_i + \Bbbk \cdot \mathbf{D}(W) \cdot W_{\prec ws_i} \cdot s_i + \Bbbk \cdot \mathbf{D}(W) \cdot W_{\prec ws_i} \cdot D_i \\ &\subset w + (1-q_i)\mathbf{D}(w)ws_i + \Bbbk \cdot \mathbf{D}(W) \cdot W_{\prec w} + \Bbbk \cdot \mathbf{D}(W) \cdot W_{\prec ws_i} \\ &= \Bbbk \cdot \mathbf{D}(W) \cdot W_{\prec w}\end{aligned}$$

because $W_{\preceq ws_i} \cup (W_{\prec ws_i} \cdot s_i) = W_{\prec w}$ for any $w \in W$ and $i \in I$ such that $\ell(ws_i) = \ell(w) - 1$.

The lemma is proved. $\square$

Therefore, Proposition 7.24 is proved. $\square$

Finally, Theorem 1.28 implies that elements $w \in W$ are $\Bbbk$-linearly independent in $\mathbf{H}(W) \otimes \Bbbk$. This and Proposition 7.24 imply that the elements $\underline{\varphi}_W(T_w)$, $w \in W$ are also $\Bbbk$-linearly independent in $\mathbf{H}(W) \otimes \Bbbk$.

This proves that $\underline{\varphi}_W$ is an injective homomorphism of algebras $H_{\mathbf{q}}(W) \hookrightarrow \mathbf{H}(W)$. Injectivity of φ_W is then immediate.

Theorems 1.20 and 1.33 are proved. $\square$

7.6. Symmetries of $\mathbf{H}(W)$ and proof of Theorems 1.37, 3.8, and 3.9

We need the following

Proposition 7.27. *In the notation of Theorem 3.2 we have:*

(a) *Suppose that $\bar{\cdot}$ is an involution on R such that $\overline{\chi}_{w,s} = \chi_{w,s^{-1}}$, $\overline{\sigma}_{w,s} = \sigma_{w,s^{-1}}$ for all $w \in W$, $s \in \mathcal{S}$. Then the assignments $\overline{w} = w^{-1}$, $\overline{D}_s = D_{s^{-1}}$ for $w \in W$, $s \in \mathcal{S}$ extends to a unique R-linear anti-involution of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$.*

(b) *Suppose that RW admits an R-linear automorphism θ such that $\theta(w) \in R^\times \cdot w$ for $w \in W$ and $\theta(s) = \chi_{s,s} \cdot s$ for $s \in \mathcal{S}$. Then θ uniquely extends to an R-linear automorphism of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ such that $\theta(D_s) = \chi_{s,s}D_s + \sigma_{s,s}$ for $s \in \mathcal{S}$. Moreover, $\theta(\mathbf{K}_{\chi,\sigma}(W)) = \mathbf{K}_{\chi,\sigma}(W)$.*

(c) *In the assumptions of Theorem 3.3, suppose that $\sigma_{wsw^{-1},wsw^{-1}} = \sigma_{s,s}$ for all $w \in W$, $s \in \mathcal{S}$ of finite order. Then $\hat{\mathbf{D}}_{\chi,\sigma}(W)$ admits:*
  (i) *A W-action by automorphisms via $w(D_s) = \sigma_{w,s} + \chi_{w,s}D_{wsw^{-1}}$ for $w \in W$, $s \in \mathcal{S}$.*
  (ii) *An s-derivation d_s (i.e., $d_s(xy) = d_s(x)y + s(x)d_s(y)$) such that $d_s(D_{s'}) = \delta_{s,s'}$, $s, s' \in \mathcal{S}$.*

These actions satisfy for all $w \in W$, $s \in \mathcal{S}$:

$$d_{wsw^{-1}} = \chi_{w,s} \cdot w \circ d_s \circ w^{-1} \tag{7.20}$$

Proof. Prove (a). It suffices to verify that $\bar{\cdot}$ preserves the defining relations of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$. Indeed,

$$\begin{aligned}\overline{w^{-1}} \cdot \overline{D}_s \cdot \overline{w} &= wD_{s^{-1}}w^{-1} = \chi_{w,s^{-1}}D_{ws^{-1}w^{-1}} + \sigma_{w,s^{-1}}(1 - ws^{-1}w^{-1})\\ &= \overline{\chi}_{w,s}\overline{D}_{wsw^{-1}} + \overline{\sigma}_{w,s}(1 - ws^{-1}w^{-1})\\ &= \overline{\chi_{w,s}D_{wsw^{-1}} + \sigma_{w,s}(1 - wsw^{-1})} = \overline{wD_sw^{-1}}\end{aligned}$$

for all $w \in W$, $s \in \mathcal{S}$, i.e., (3.1) is $\bar{\cdot}$-invariant. Clearly, applying $\bar{\cdot}$ to (3.1) for D_s, we obtain (3.1) for $D_{s^{-1}} = \overline{D}_s$, because $|s^{-1}| = |s|$, which verifies that (3.2) is also -invariant. This proves (a).

Prove (b). First, show that θ is an endomorphism of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$, i.e., that θ preserves the defining relations. Indeed, for $w \in W$ one has $\theta(w) = \tau_w \cdot w$, where $\tau_w \in R^\times$ such that $\tau_w\tau_{w^{-1}} = 1$. Therefore, abbreviating $s' = wsw^{-1}$, we obtain

$$\begin{aligned}\theta(w)\theta(D_s)\theta(w^{-1}) &= w\theta(D_s)w^{-1} = w(\chi_{s,s}D_s + \sigma_{s,s})w^{-1}\\ &= \chi_{s,s}(\chi_{w,s}D_{s'} + \sigma_{w,s}(1 - s')) + \sigma_{s,s}\\ &= \chi_{s',s'}(\chi_{w,s}D_{s'} + \sigma_{w,s}(1 - s')) + \sigma_{s,s}\\ &= \chi_{w,s}(\sigma_{s',s'} + \chi_{s',s'}D_{s'}) + \sigma_{w,s}(1 - \chi_{s',s'}s')\\ &= \theta(\chi_{w,s}D_{s'} + \sigma_{w,s}(1 - s')) = \theta(wD_sw^{-1})\end{aligned}$$

for $w \in W$, $s \in \mathcal{S}$ by Lemma 7.8(b) and the assumption of part (b). Finally, let us verify that the relations (3.3) are invariant under θ. Indeed, applying θ to the defining functional relation (7.3) for f_s defined by (7.4) and using (6.6), we obtain (abbreviating $a_s = \chi_{s,s}$, $b_s = \sigma_{s,s}$):

$$\theta(f_s(ts + D_s) - f(t)) = f_s(a_sts + a_sD_s + b_s) - f(t) = f_s(ts + D_s) - f(t) = 0$$

This proves that θ is an R-linear endomorphism of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$. It is easy to see that θ is invertible and the inverse is given by $\theta^{-1}(w) = \tau_w^{-1}w$ for $w \in W$ and $\theta^{-1}(D_s) = \sigma_{s^{-1},s^{-1}} + \chi_{s^{-1},s^{-1}}D_s$ for $s \in \mathcal{S}$. This proves the first assertion of part (b). Prove the second assertion. Indeed, we obtain for all $w \in W$: $\theta(w \cdot \tilde{\mathbf{D}}_{\chi,\sigma}(W) \cdot w^{-1}) = \theta(w) \cdot \theta(\tilde{\mathbf{D}}_{\chi,\sigma}(W)) \cdot \theta(w^{-1}) \subseteq w \cdot \tilde{\mathbf{D}}_{\chi,\sigma}(W) \cdot w^{-1}$ therefore, $\theta(\mathbf{K}_{\chi,\sigma}(W)) \subset \mathbf{K}_{\chi,\sigma}(W)$. Part (b) is proved.

Prove (c)(i). We need the following fact.

Lemma 7.28. *For each χ, σ satisfying (3.5), $\tilde{V} = \oplus_{s\in\mathcal{S}} R \cdot D_s$ is a W-module via $w(1) = 1$ and $w(D_s) = \sigma_{w,s} + \chi_{w,s}D_{wsw^{-1}}$ for $w \in W$, $s \in \mathcal{S}$.*

Proof. Indeed,

$$\begin{aligned} w_1(w_2(D_s)) &= w_1(\sigma_{w_2,s} + \chi_{w_2,s}D_{w_2sw_2^{-1}}) = \sigma_{w_2,s} + \chi_{w_2,s}w_1(D_{w_2sw_2^{-1}}) \\ &= \sigma_{w_2,s} + \chi_{w_2,s}\chi_{w_1,w_2sw_2^{-1}}D_{w_1w_2sw_2^{-1}w_1^{-1}} + \chi_{w_2,s}\sigma_{w_1,w_2sw_2^{-1}} \\ &= \sigma_{w_1w_2,s} + \chi_{w_1w_2,s}D_{w_1w_2sw_2^{-1}w_1^{-1}} \end{aligned}$$

for all $w \in W$, $s \in \mathcal{S}$, by (3.5). Also $1(D_s) = D_s$ because $\chi_{1,s} = 1$, $\sigma_{1,s} = 0$.

The lemma is proved. □

That is, the W-action lifts to $T(V)$, where $V = \oplus_{s\in\mathcal{S}} R \cdot D_s$ by algebra automorphisms (because any R-linear map $V \to T(V)$ lifts to an endomorphism of the algebra $T(V)$). Thus, it remains to show that the defining relations (3.3) are preserved under the action. Indeed, since $a_s = \chi_{s,s}$ is a primitive $|s|$-th root of unity, i.e., $1 + a_s + \cdots + a_s^{|s|-k} = -a^{-k}(1 + a_s + \cdots a_s^{k-1})$ for $0 \le k \le |s|$, the relation (3.3) with $s \in \mathcal{S}$ of finite order $|s|$ reads (in the notation (7.4)):

$$f_s(D_s) = 0\ . \tag{7.21}$$

Then, applying w to the left hand side of the above relation, we obtain (in the notation (7.4)):

$$f_s(\sigma_{w,s} + \chi_{w,s}D_{s'}) = f_{s'}(D_s) \tag{7.22}$$

by Lemma 7.9, where we abbreviated $s' = wsw^{-1}$. Finally, taking into account that $\chi_{s',s'} = \chi_{s,s}$ by Lemma 7.8(b) and $\sigma_{s',s'} = \sigma_{s,s}$ by the assumption of part (b), we obtain $f_{s'}(D_s) = f_s(D_s) = 0$. Part (c)(i) is proved.

Prove (c)(ii). We start with the following obvious general result.

Lemma 7.29. *For any R-module V and R-linear maps $f, s : V \to T(V)$ there is a unique R-linear map $d = d_{f,s} : T(V) \to T(V)$ such that*

- $d(1) = 0$, $d(v) = f(v)$ *for* $v \in V$.
- $d(xy) = d(x)y + s(x)d(y)$ *for all* $x, y \in T(V)$.

For $V = \oplus_{s\in\mathcal{S}}\mathbb{Z}\cdot D_s$, $s \in \mathcal{S}$ viewed as a $\mathbb{Z}$-linear map $V \to V \subset T(V)$ and $f : V \to \mathbb{Z} \subset T(V)$ given by $f(D_{s'}) = \delta_{s,s'}$, we abbreviate $d_s := d_{f,s}$. To prove the assertion, it suffices to show that d_s preserves the defining relations (7.21) of $\hat{\mathbf{D}}_{\chi,\sigma}(W)$, i.e., the relations of the form $f_s(D_s) = 0$ for all $s \in \mathcal{S}$ of finite order $|s|$. Indeed, if $s \neq s'$, then, clearly, $d_s(f_{s'}(D_{s'})) = 0$. Suppose that $s = s'$. Then, in the notation (7.4) one has (similarly to the proof of Lemma 7.9):

$$\begin{aligned}
d_s(f_s(D_s)) &= \sum_{k=1}^{|s|} s\left(\prod_{i=k+1}^{|s|}\left(D_s - a_s\frac{1-a_s^i}{1-a_s}\right)\right)\cdot d_s\left(D_s - b_s\frac{1-a_s^k}{1-a_s}\right) \\
&\quad \cdot \prod_{j=1}^{k-1}(D_s - b_s\frac{1-a_s^j}{1-a_s}) \\
&= \sum_{k=1}^{|s|}\left(\prod_{i=k+1}^{|s|}\left(a_sD_s + b_s - b_s\frac{1-a_s^i}{1-a_s}\right)\right)\cdot a_s\cdot\prod_{j=1}^{k-1}(D_s - b_s\frac{1-a_s^j}{1-a_s}) \\
&= \sum_{k=1}^{|s|}\left(\prod_{i=k+1}^{|s|}\left(a_sD_s - b_s\frac{a_s-a_s^i}{1-a_s}\right)\right)\cdot a_s\cdot\prod_{j=1}^{k-1}(D_s - b_s\frac{1-a_s^j}{1-a_s}) \\
&= \sum_{k=1}^{|s|} a_s^{|s|-k+1}\cdot\prod_{j=1}^{|s|-1}(D_s - b_s\frac{1-a_s^j}{1-a_s}) = 0
\end{aligned}$$

because $s(D_s) = a_sD_s + b_s$ and $a_s = \chi_{s,s}$ is a primitive s-th root of unity in $R^\times$. This proves (c)(ii).

We prove the last assertion of (c) by showing that both sides of (7.20) are wsw^{-1}-derivations which agree on generators of $\hat{\mathbf{D}}_{\chi,\sigma}(W)$. Indeed, denote $d'_s = \chi_{w,s}w \circ d_s \circ w^{-1}$ and, first, substitute $x = D_{s'}$: $d'_s(D_{s'}) = \chi_{w,s}w(d_s(\chi_{w^{-1},s'}D_{w^{-1}s'w} + \sigma_{w^{-1},s'})) = \chi_{w,s}w(\chi_{w^{-1},s'}\delta_{s,w^{-1}s'w}) = \chi_{w,s}\chi_{w^{-1},wsw^{-1}}\delta_{wsw^{-1},s'} = d_{wsw^{-1}}(D_{s'})$ by (7.8). Furthermore,

$$\begin{aligned}
d'_s(xy) &= \chi_{w,s}w(d_s(w^{-1})(x)\cdot w^{-1}(y)) \\
&= \chi_{w,s}w\left(d_s(w^{-1})(x))\cdot w^{-1}(y) + s(w^{-1}(x))\cdot d_s(w^{-1}(y))\right) \\
&= \chi_{w,s}w(d_s(w^{-1})(x)))\cdot y + \chi_{w,s}\cdot(wsw^{-1})(x)\cdot w(d_s(w^{-1}(y))) \\
&= d'_s(x)\cdot y + (wsw^{-1})(x)\cdot d'_s(y)
\end{aligned}$$

for $x, y \in \hat{\mathbf{D}}_{\chi,\sigma}(W)$. This proves that $d'_s = d_s$.

Proposition 7.27 is proved. $\square$

Proof of Theorem 3.8. Using Proposition 7.27(a), we obtain for all $w \in W$:

$$\overline{w \cdot \tilde{\mathbf{D}}_{\chi,\sigma}(W) \cdot w^{-1}} = \overline{w^{-1}} \cdot \overline{\tilde{\mathbf{D}}_{\chi,\sigma}(W)} \cdot \overline{w} \subseteq w \cdot \tilde{\mathbf{D}}_{\chi,\sigma}(W) \cdot w^{-1} ,$$

therefore, $\overline{\mathbf{K}_{\chi,\sigma}(W)} \subset \mathbf{K}_{\chi,\sigma}(W)$.

Finally, we need the following fact.

Lemma 7.30. *$\varepsilon(\overline{x}) = \overline{\varepsilon(x)}$ for $x \in \hat{\mathbf{H}}_{\chi,\sigma}(W)$.*

Proof. Since both $\bar{\cdot} \circ \varepsilon$ and $\varepsilon \circ \bar{\cdot}$ are R-antilinear ring homomorphisms $\hat{\mathbf{H}}_{\chi,\sigma}(W) \to R$, it suffices to prove the assertion only on generators of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$. Indeed, $\varepsilon(\overline{D}_s) = \overline{\varepsilon(D_s)} = 0$ for all $s \in \mathcal{S}$ and $\varepsilon(\overline{w}) = \varepsilon(w^{-1}) = 1 = \overline{\varepsilon(w)}$ for $w \in W$.

The lemma is proved. □

Therefore, the ideal $\mathbf{J}_{\chi,\sigma}(W)$ generated by $\mathbf{K}_{\chi,\sigma}(W) \cap Ker\ \varepsilon$ is $\bar{\cdot}$-invariant and $\bar{\cdot}$ is well-defined on the quotient $\mathbf{H}_{\chi,\sigma}(W) = \hat{\mathbf{H}}_{\chi,\sigma}(W)/\mathbf{K}_{\chi,\sigma}(W)$.

This proves Theorem 3.8. □

The following result correlates the automorphism θ with relations in $\mathbf{H}_{\chi,\sigma}(W)$.

Proposition 7.31. *In the assumptions of Proposition 7.27(b) suppose that $\varepsilon(\theta(x)) = \varepsilon(x)$ for all $x \in \mathbf{K}_{\chi,\sigma}(W)$. Then*

(a) *$\theta(x) = S^{-2}(x)$ for all $x \in \mathbf{K}_{\chi,\sigma}(W)$.*
(b) *Suppose that $\hat{\mathbf{H}}_{\chi,\sigma}(W)$ is a free R-module. Then $\mathbf{H}_{\chi,\sigma}(W)$ admits an R-linear automorphism θ such that the structural homomorphism $\hat{\mathbf{H}}_{\chi,\sigma} \twoheadrightarrow \mathbf{H}_{\chi,\sigma}$ is θ-equivariant.*

Proof. Prove (a). We need the following result.

Lemma 7.32. *In the assumptions of Proposition 7.27(b) one has*

$$\Delta \circ \theta = (S^{-2} \otimes \theta) \circ \Delta = (\theta \otimes 1) \circ \Delta \ . \tag{7.23}$$

Proof. Since both Δ and θ are algebra homomorphisms, hence so are $\Delta \circ \theta$, $(S^{-2} \otimes \theta) \circ \Delta$, and $(\theta \otimes 1) \circ \Delta$, it suffices to prove (7.23) only on generators of $\hat{\mathbf{H}}_{\chi,\sigma}(W)$.

Indeed, for $w \in W$ one has $\theta(w) = \tau_w \cdot w$ for some $\tau_w \in R^\times$, therefore

$$\Delta(\theta(w)) = \Delta(\tau_w \cdot w) = \tau_w \cdot w \otimes w = \theta(w) \otimes w = S^{-2}(w) \otimes \theta(w) \ .$$

Furthermore, we obtain for $s \in \mathcal{S}$ (abbreviating $a_s = \chi_{s,s}$, $b_s = \sigma_{s,s}$):

$$\begin{aligned}\Delta(\theta(D_s)) &= \Delta(b_s + a_s D_s) = b_s \cdot 1 \otimes 1 + a_s D_s \otimes 1 + a_s \cdot s \otimes D_s \\ &= \theta(D_s) \otimes 1 + \theta(s) \otimes D_s = S^{-2}(D_s) \otimes 1 + S^{-2}(s) \otimes \theta(D_s)\end{aligned}$$

because $\theta(D_s) = b_s + a_s D_s$, $\theta(s) = a_s \cdot s$ and $S^2(D_s) = S(-s^{-1}D_s) = s^{-1} \cdot D_s \cdot s$, therefore, $S^{-2}(D_s) = s \cdot D_s \cdot s^{-1} = a_s D_s + b_s(1 - s)$. This proves (7.23).

The lemma is proved. □

Finally, applying $1 \otimes \varepsilon$ to (7.23), we obtain $\theta(x) = S^{-2}(x_{(1)}) \cdot \varepsilon(\theta(x_{(2)})$ for all $x \in \mathbf{K}_{\chi,\sigma}(W)$. Since $\mathbf{K}_{\chi,\sigma}(W)$ is a left coideal by the argument from the proof of Theorem 3.5, then $x_{(2)} \in \mathbf{K}_{\chi,\sigma}(W)$, i.e., $\varepsilon(\theta(x_{(2)}) = \varepsilon(x_{(2)})$ and $\theta(x) = S^{-2}(x_{(1)}) \cdot \varepsilon(x_{(2)}) = S^{-2}(x_{(1)}\varepsilon(x_{(2)})) = S^{-2}(x)$. This proves (a).

Prove (b). The assumption of the proposition and the second assertion of Proposition 7.27(b) imply that $\mathbf{K}_{\chi,\sigma}(W)^+ = \mathbf{K}_{\chi,\sigma}(W) \cap Ker\ \varepsilon$ is θ-invariant. Therefore, the ideal $\mathbf{J}_{\chi,\sigma}(W)$ generated by $\mathbf{K}_{\chi,\sigma}(W)^+$ is also θ-invariant and θ is well-defined on the quotient $\mathbf{H}_{\chi,\sigma}(W) = \hat{\mathbf{H}}_{\chi,\sigma}(W)/\mathbf{K}_{\chi,\sigma}(W)$. This proves (b).

The proposition is proved. □

Proof of Theorem 1.37. Prove (a). Indeed, χ and σ defined by (7.5) satisfy the assumptions of Proposition 7.27(a) with the identity $\bar{\cdot}$ on $\mathbb{Z}$, therefore $\bar{\cdot}$ is a well-defined involutive anti-automorphism of $\hat{\mathbf{H}}(W) = \hat{\mathbf{H}}_{\chi,\sigma}(W)$ and it satisfies $\overline{D}_s = D_s$ for $s \in \mathcal{S}$. Copying the argument from the proof of Theorem 3.8, we see that $\mathbf{K}(W)^+ = \mathbf{K}(W) \cap Ker\ \varepsilon$ is $\bar{\cdot}$-invariant. Since all filtered components $\hat{\mathbf{D}}(W)_{\le d}$ are also $\bar{\cdot}$-invariant, replacing W with $W_{\{i,j\}}$ and taking $d = m_{ij}$, we see that all $\mathbf{K}_{ij}(W)$ are $\bar{\cdot}$-invariant. Therefore, the (Hopf) ideals $\mathbf{J}(w)$ and $\underline{\mathbf{J}}(w)$ generated respectively by $\mathbf{K}(W)^+$ and $\underline{\mathbf{K}} = \sum\limits_{j \ne i} \mathbf{K}_{ij}(W)$ are also $\bar{\cdot}$-invariant. This proves Theorem 1.37(a).

Prove Theorem 1.37(b) now. We need the following result.

Proposition 7.33. *For any Coxeter group W one has:*

(a) *For $s \in \mathcal{S}$, $\hat{\mathbf{D}}(W)$ admits an s-derivation d_s such that $d_s(D_{s'}) = \delta_{s,s'}$, $s, s' \in \mathcal{S}$.*
(b) *$d_s(\mathbf{K}(W)) = \{0\}$ for all $s \in \mathcal{S}$ and $wxw^{-1} = w(x)$ for all, $w \in W$.*

Proof. Part (a) directly follows from Proposition 7.27(c)(ii).

Prove (b). Since $s_i\mathbf{K}(W)s_i = \mathbf{K}(W)$, Theorem 1.22 and Lemma 5.20 imply that for each $x \in \mathbf{K}(W)$ one has $d_{s_i}(x) = 0$ and $s_i x s_i = s_i(x)$. This, in particular, proves the first assertion of part (b) for $s = s_i$ and the second assertion – for $w = s_i$. Let us prove the second assertion for any w. Indeed, if $\ell(w) \le 1$, we have nothing to prove. Suppose that $\ell(w) \ge 2$, i.e., $w = s_i w'$ so that $\ell(w') = \ell(w) - 1$. Then, using inductive hypothesis in the form $w'xw'^{-1} = w'(x) \in \mathbf{K}(W)$ for all $x \in \mathbf{K}(W)$, we obtain

$$wxw^{-1} = s_i \cdot (w'xw'^{-1}) \cdot s_i = s_i \cdot (w'xw'^{-1}) \cdot s_i = s_i \cdot (w'(x)) \cdot s_i$$
$$= s_i(w'(x)) = (s_i w')(x) = w(x)$$

for all $x \in \mathbf{K}(W)$, which proves the second assertion. Prove the first assertion now. Let $s \in \mathcal{S}$, choose $w \in W$ and $i \in I$ such that $s = ws_iw^{-1}$. The last assertion of Proposition 7.27 guarantees that $d_s = \chi_{w,s} \cdot w \circ d_{s_i} \circ w^{-1}$. Then

$$d_s(\mathbf{K}(W)) = \chi_{w,s} \cdot (w \circ d_{s_i} \circ w^{-1})(\mathbf{K}(W)) = \chi_{w,s} \cdot (w)(d_{s_i}(\mathbf{K}(W))) = \{0\} \ .$$

This finishes the proof of (b). The proposition is proved. $\square$

Therefore, the (Hopf) ideal $\mathbf{I}(w)$ of $\hat{\mathbf{D}}(W)$ generated $\mathbf{K}(W)$ is invariant both under the W-action and under all d_s. This proves that the quotient $\underline{\mathbf{D}}(W) = \hat{\mathbf{D}}(W)/\mathbf{I}(w)$ has a natural W-action and s-derivations d_s. Similarly, let $\underline{\mathbf{K}}(W) \subset \hat{\mathbf{D}}(W)$ be as in the proof of Theorem 1.28. By definition, $\underline{\mathbf{K}}(W) \subset \mathbf{K}(W)$ is W-invariant and is annihilated by all d_s, Therefore, the ideal $\underline{\mathbf{I}}(W)$ generated by $\underline{\mathbf{K}}(W)$ is also invariant both under the W-action and under all d_s hence the quotient $\mathbf{D}(W) = \hat{\mathbf{D}}(W)/\underline{\mathbf{I}}(W)$ has a natural W-action and s-derivations d_s. This proves Theorem 1.37(b).

Prove Theorem 1.37(c) now. We need the following result.

Proposition 7.34. *Suppose that $W = \langle s_i \,|\, i \in I\rangle$ is a finite Coxeter group. Then $\varepsilon(\theta(x)) = \varepsilon(x)$ for all $x \in \mathbf{K}(W)$.*

Proof. We need the following result.

Lemma 7.35. *Suppose that $W = \langle s_i \,|\, i \in I\rangle$ is a finite Coxeter group. Then in the notation of Lemma 5.20(a), one has*

$$\theta(x) = \hat{\tau}(w_0(x)) \tag{7.24}$$

for $x \in \hat{\mathbf{D}}(W)$, where w_0 is the longest element of W and τ is an automorphism of $\hat{\mathbf{H}}(W)$ determined by $\hat{\tau}(s_i) = s_{\tau(i)}$, $\hat{\tau}(D_i) = D_{\tau(i)}$, where σ is a certain permutation of I.

Proof. Lemma 5.20(a) taken with $s = s_i$ immediately implies that

$$w(D_i) = \begin{cases} D_{ws_iw^{-1}} & \text{if } \ell(ws_i) = \ell(w) + 1 \\ 1 - D_{ws_iw^{-1}} & \text{if } \ell(ws_i) = \ell(w) - 1 \end{cases} \tag{7.25}$$

for all $i \in I$, $w \in W$.

Furthermore, clearly, $w_0 s_i w_0^{-1} = s_{\tau(i)}$ for all $i \in I$ and some permutation τ of I (which satisfies $m_{\tau(i),\tau(j)} = m_{ij}$ for all $i, j \in I$). It is also clear that the assignments $s_i \mapsto s_{\tau(i)}$, $D_i \mapsto D_{\tau(i)}$ define an automorphism $\hat{\tau}$ of $\hat{\mathbf{H}}(W)$. Since $\ell(w_0 s_i) = \ell(w_0) - 1$, we have by (7.25):

$$w_0(D_i) = 1 - D_{w_0 s_i w_0^{-1}} = 1 - D_{\tau(i)} \ .$$

Since the θ and another automorphism of $\hat{\mathbf{D}}(W)$ given by $x \mapsto \hat{\tau}(w_0(x))$ agree on generators D_i, $i \in I$, this proves (7.24). The lemma is proved. $\square$

Furthermore, Lemma 7.35 and the immediate fact that $\varepsilon \circ \hat{\tau} = \varepsilon$ imply

$$\varepsilon(\theta(x)) = \varepsilon(\hat{\tau}(w_0(x))) = \varepsilon(w_0(x))$$

for all $x \in \hat{\mathbf{D}}(W)$. If $x \in \mathbf{K}(W)$, then $w_0(x) = w_0 \cdot x \cdot w_0^{-1}$ by Proposition 7.33(b) and $\varepsilon(w_0(x)) = \varepsilon(x)$.

The proposition is proved. $\square$

Finally, we need the following result.

Lemma 7.36. *$\theta(\mathbf{K}_{ij}(W)) = \mathbf{K}_{ij}(W)$ for any Coxeter group W and any distinct $i, j \in I$.*

Proof. By definition, for any subset J of I, θ preserves the subalgebra $\hat{\mathbf{H}}(W_J) \subset \hat{\mathbf{H}}(W)$, e.g., $\theta(\hat{\mathbf{D}}(W_J)) = \hat{\mathbf{D}}(W_J)$. Also, by Proposition 7.27(b), $\theta(\mathbf{K}(W_J)) = \mathbf{K}(W_J)$. Since θ preserves each filtered component $\hat{\mathbf{D}}(W)_{\leq d}$, θ also preserves each filtered component $\mathbf{K}(W_J)_{\leq d} \subset \mathbf{K}(W)_{\leq d}$. Note that if $m_{ij} \geq 2$, then the subgroup $W_{\{i,j\}}$ of W is finite. These arguments and Proposition 7.34 guarantee that $\varepsilon(\theta(x)) = \varepsilon(x)$ for $x \in \mathbf{K}(W_{\{i,j\}})_{\leq d}$ whenever $m_{ij} \geq 2$, $d \geq 0$. Taking $d = m_{ij}$ (and taking into account that $\mathbf{K}_{ij}(W) = \{0\}$ whenever $m_{ij} = 0$), we finish the proof.

The lemma is proved. $\square$

Therefore, the Hopf ideal $\underline{\mathbf{J}}(w)$ of $\hat{\mathbf{H}}(W)$ generated $\underline{\mathbf{K}} = \sum\limits_{j \neq i} \mathbf{K}_{ij}(W)$ is θ-invariant. Hence θ is a well-defined automorphism of the quotient $\mathbf{H}(W) = \hat{\mathbf{H}}(W)/\underline{\mathbf{J}}(w)$.

This proves Theorem 1.37(c). $\square$

Proof of Theorem 3.9. We need the following result.

Proposition 7.37. *In the assumptions of Proposition 7.27(c) and notation of Proposition A.8:*

(a) *the condition (3.10) implies:*
  - (i) $\sigma_{w,s_1}\sigma_{ws_1,s_2}\cdots\sigma_{ws_1\cdots s_{k-1},s_k}\partial_{ws_1\cdots s_k,w} = 0$ *for any* $w \in W$, $s_1, \ldots, s_k \in \mathcal{S}$, $k \geq 1$.
  - (ii) $w(x) = \partial_{w,w}(x)$ *for all* $w \in W$, $x \in \hat{\mathbf{D}}_{\chi,\sigma}(W)$.

(b) *The condition (3.11) for a given $s \in \mathcal{S}$ implies that*
  - (i) $\sigma_{s^{-1},s_1}\sigma_{s^{-1}s_1,s_2}\cdots\sigma_{s^{-1}s_1\cdots s_{k-1},s_k}\partial_{s^{-1}s_1\cdots s_k,1} = \delta_{k,1}\delta_{s,s_1}\sigma_{s^{-1},s}$ *for all* $s_1, \ldots, s_k \in \mathcal{S}$, $k \geq 1$.
  - (ii) $\partial_{s^{-1},1} = -\sigma_{s^{-1},s}d_{s^{-1}}$ *in the notation of Proposition 7.27(c)(i).*

Proof. Prove (a). Denote $\partial^w_{s_1,\dots,s_k} = \sigma_{w,s_1}\sigma_{ws_1,s_2}\cdots\sigma_{ws_1\cdots s_{k-1},s_k}\partial_{ws_1\cdots s_k,w}$ for all $w \in W$, $s_1,\dots,s_k \in \mathcal{S}$, $k \ge 0$ (with $k = 0$ this is just $\partial_{w,w}$).

Lemma 7.38. *$\partial^w_{s_1,\dots,s_k}(D_{s_{k+1}}x) = ws_1\cdots s_k(D_s)\partial^w_{s_1,\dots,s_k}(x) - \partial^w_{s_1,\dots,s_{k+1}}(x)$ for each $w \in W$, $s_1,\dots,s_{k+1} \in \mathcal{S}$, $k \ge 0$, $x \in \hat{\mathbf{D}}_{\chi,\sigma}(W)$.*

Proof. Clearly, $\partial_{w,w''}(D_s) = \delta_{w,w''}w(D_s) - \delta_{w'',ws}\sigma_{w,s}$ for all $w, w'' \in W$, $s \in \mathcal{S}$. This and Proposition A.8 imply

$$\partial_{w,w'}(D_s x) = w(D_s)\partial_{w,w'}(x) - \sigma_{w,s}\partial_{ws,w'}(x) \tag{7.26}$$

for all $w, w' \in W$, $s \in \mathcal{S}$, $x \in \hat{\mathbf{D}}_{\chi,\sigma}(W)$. Furthermore, (7.26) implies that

$$\partial_{ws_1\cdots s_k,w}(D_{s_{k+1}}x) = ws_1\cdots s_k(D_s)\partial_{w,ws_1\cdots s_k}(x) - \sigma_{ws_1\cdots s_k,s_{k+1}}\partial_{ws_1\cdots s_{k+1},w}(x)\ .$$

The lemma is proved. □

Furthermore, we will show that $\partial^w_{s_1,\dots,s_k}(y) = 0$ for all $s_1,\dots,s_k \in \mathcal{S}$, $k \ge 1$ and $\partial_{w,w}(y) = w(y)$ (i.e., when $k = 0$) by induction in the filtered degree of $y \in \hat{\mathbf{D}}_{\chi,\sigma}(W)$ (the algebra is naturally filtered by $\mathbb{Z}_{\ge 0}$ via $\deg D_s = 1$ for $s \in \mathcal{S}$). Indeed, if $y \in \hat{\mathbf{D}}_{\chi,\sigma}(W)_{\le 0} = R$, then $\partial_{w,w'}(y) = \delta_{w,w'}y$, therefore, $\partial^w_{s_1,\dots,s_k}(y) = \sigma_{w,s_1}\sigma_{ws_1,s_2}\cdots\sigma_{ws_1\cdots s_{k-1},s_k}\delta_{s_1\cdots s_k,1}y = 0$ by the condition (3.10) and $\partial_{w,w}(y) = w(y) = y$. If $y \in \hat{\mathbf{D}}_{\chi,\sigma}(W)_{\le r}$, $r > 0$, then y is an R-linear combination of the elements of the form $D_s x$, where $x \in \hat{\mathbf{D}}_{\chi,\sigma}(W)_{\le r-1}$. By R-linearity it suffices to prove the assertion only for $y = D_s x$. Then, using the inductive hypothesis in the form: $\partial^w_{s_1,\dots,s_k}(x) = 0$, $\partial^w_{s_1,\dots,s_k,s}(x) = 0$, $\partial_{w,w}(x) = w(x)$ Lemma 7.38 guarantees that $\partial^w_{s_1,\dots,s_k}(y) = 0$ and same lemma taken with $k = 0$ implies that $\partial_{w,w}(D_s x) = w(D_s)\partial_{w,w}(x) - \partial^w_s(x) = w(D_s)\partial_{w,w}(x) = w(D_s)w(x) = w(y)$.

This proves (a).

Prove (b) now. Denote $\tilde{\partial}^s_{s_1,\dots,s_k} = \sigma_{s^{-1},s_1}\sigma_{s^{-1}s_1,s_2}\cdots\sigma_{s^{-1}s_1\cdots s_{k-1},s_k}\partial_{s^{-1}s_1\cdots s_k,1}$ for $s, s_1,\dots,s_k \in \mathcal{S}$, $k \ge 0$ (if $k = 0$, this is just $\partial_{s,1}$).

Lemma 7.39. *$\tilde{\partial}^s_{s_1,\dots,s_k}(D_{s_{k+1}}x) = s^{-1}s_1\cdots s_k(D_s)\tilde{\partial}^s_{s_1,\dots,s_k}(x) - \tilde{\partial}^s_{s_1,\dots,s_{k+1}}(x)$ for $s, s_1,\dots, s_{k+1} \in \mathcal{S}$, $x \in \hat{\mathbf{D}}_{\chi,\sigma}(W)$.*

Proof. Let $u := s^{-1}s_1\cdots s_k$. Then $\partial_{u,1}(D_{s_{k+1}}x) = u(D_s)\partial_{u,1}(x) - \sigma_{u,s_{k+1}}\partial_{u,1}(x)$ by (7.26).

The lemma is proved. □

Furthermore, similarly to the proof of part (a), we will show that $\tilde{\partial}^s_{s_1,\dots,s_k}(y) = 0$ for $k \ge 1$ and all $s_1,\dots,s_k \in \mathcal{S}$ and $\partial_{s^{-1},1}(y) = -\sigma_{s^{-1},s}d_{s^{-1}}(y)$ (i.e., when $k = 0$) by induction in the filtered degree of $y \in \hat{\mathbf{D}}_{\chi,\sigma}(W)$. Indeed, if $y \in \hat{\mathbf{D}}_{\chi,\sigma}(W)_{\le 0} = R$, then $\partial_{w,w'}(y) = \delta_{w,w'}y$, therefore,

$$\tilde{\partial}^s_{s_1,\ldots,s_k}(y) = \sigma_{s^{-1},s_1}\sigma_{s^{-1}s_1,s_2}\cdots\sigma_{s^{-1}s_1\cdots s_{k-1},s_k}\delta_{s_1\cdots s_k,s}y = 0$$

by the condition (3.11) and $\partial_{s^{-1},1}(y) = d_s(y) = 0$.

If $y \in \hat{\mathbf{D}}_{\chi,\sigma}(W)_{\le r}$, $r > 0$, then y is an R-linear combination of the elements of the form $D_{s'}x$, where $x \in \hat{\mathbf{D}}_{\chi,\sigma}(W)_{\le r-1}$. By R-linearity it suffices to prove the assertion only for $y = D_{s'}x$. Then, using the inductive hypothesis in the form: $\partial^w_{s_1,\ldots,s_k}(x) = \delta_{k,1}\delta_{s,s_1}\sigma_{s^{-1},s}\cdot x$, $\partial^w_{s_1,\ldots,s_k,s'}(x) = 0$, $\partial_{s^{-1},1}(x) = \sigma_{s^{-1},s}d_{s^{-1}}(x)$ Lemma 7.39 guarantees that

$$\tilde{\partial}^s_{s_1,\ldots,s_k}(y) = \delta_{k,1}\delta_{s,s_1}\sigma_{s^{-1},s}s^{-1}s_1\cdots s_k(D_{s'})x = \delta_{k,1}\delta_{s,s_1}\sigma_{s^{-1},s}D_{s'}x = \delta_{k,1}\delta_{s,s_1}\sigma_{s^{-1},s}\cdot y$$

for all $k \ge 1$ and same lemma taken with $k = 0$ implies that

$$\begin{aligned}\partial_{s^{-1},1}(y) &= s^{-1}(D_{s'})\partial_{s^{-1},1}(x) - \tilde{\partial}^s_{s'}(x) = -s^{-1}(D_{s'})\sigma_{s^{-1},s}d_{s^{-1}}(x) - \delta_{s,s'}\sigma_{s^{-1},s}x\\ &= -\sigma_{s^{-1},s}d_{s^{-1}}(y)\ .\end{aligned}$$

This proves (b). The proposition is proved. □

Finally, Lemma 4.16 and Proposition 7.37(a) imply that $wxw^{-1} = w(x)$, $\partial_{s^{-1},1}(x) = 0$ for all $x \in \mathbf{K}_{\chi,\sigma}(W)$, $w \in W$, $s \in \mathcal{S}$. Therefore, the ideal $\mathbf{I}$ of $\hat{\mathbf{D}}_{\chi,\sigma}(W)$ generated by $\mathbf{K}_{\chi,\sigma}(W) \cap Ker\ \varepsilon$ is invariant under both W-action and the s^{-1}-derivation $\partial_s := -\partial_{s^{-1},1} = \sigma_{s^{-1},s}d_s$. Hence $\mathbf{D}_{\chi,\sigma}(W) = \hat{\mathbf{D}}_{\chi,\sigma}(W)/\mathbf{I}$ is also invariant under these symmetries.

Theorem 3.9 is proved. □

7.7. Simply-laced Hecke-Hopf algebras and proof of Theorems 1.3, 1.9, 1.25 and Propositions 1.6, 1.7, 1.31

We need the following result.

Proposition 7.40. *For any Coxeter group W and $i, j \in I$ one has:*

(a) *If $m_{ij} = 2$, then $\mathbf{K}_{ij}(W) = \mathbb{Z}\cdot K_{ij}$, where $K_{ij} = D_iD_j - D_jD_i$.*
(b) *If $m_{ij} = 3$, then*

$$\begin{aligned}\mathbf{K}_{ij}(W) &= \mathbb{Z}\cdot K_{ij} + \mathbb{Z}\cdot K_{ji} + \mathbb{Z}\cdot(K_{ij}D_i - D_iK_{ji}) + \mathbb{Z}\cdot(K_{ji}D_i - D_iK_{ji})\\ &\quad + \mathbb{Z}\cdot(D_jK_{ji} - K_{ij}D_j)\end{aligned}$$

Proof. Indeed, by Proposition 7.33(b),

$$\mathbb{Z} + \mathbf{K}_{ij}(W) \subseteq \mathbf{K}'_{ij}(W) := \{x \in \hat{\mathbf{D}}(W_{\{i,j\}})_{\le m_{ij}}\} \mid d_s(x) = 0, s \in \mathcal{S}\cap W_{\{i,j\}}\}\ . \quad (7.27)$$

Prove (a) now. Clearly, if $m_{ij} = 2$, then each $x \in \mathbf{K}_{ij}(W)$ is of the form $x = a + bD_iD_j + cD_jD_i$ for some $a, b, c \in \mathbb{Z}$. Since $s_i(D_j) = D_j$, then, clearly, $d_i(x) = bD_j + cD_j$, $d_j(x) = bD_i + cD_i$, $\varepsilon(x) = a$. Thus, $d_i(x) = d_j(x) = \varepsilon(x) = 0$ iff $a = 0$, $b + c = 0$. This proves (a).

Prove (b) now. Fix $i, j \in I$ with $m_{ij} = 3$. Then, according to Theorem 1.22, $\hat{\mathbf{D}}(W_{\{i,j\}})$ is an algebra generated by $D_1 := D_i$, $D_2 := D_{ij}$, $D_3 := D_j$ subject to relations $D_k^2 = D_k$ for $k = 1, 2, 3$. Denote also $d_1 := d_i$, $d_2 = s_id_js_i = s_jd_is_j$, $d_3 = d_j$ so that $d_k(xy) = d_k(x)y + s_k(x)d_k(y)$ for all $x, y \in \hat{\mathbf{D}}(W_{\{i,j\}})$, where $s_1 := s_i$, $s_2 := s_is_js_i = s_js_is_j$, $s_3 := s_j$.

In particular, $K_{ij} = D_1D_3 - D_2D_1 - D_3D_2 + D_2$, $K_{ji} = D_3D_1 - D_1D_2 - D_2D_3 + D_2$.

Lemma 7.41. *In the assumptions of Proposition 7.40(b), one has:*

(a) $\mathbf{K}'_{ij}(W) = \mathbf{K}'_{ij}(W)_{\le 2} + \mathbb{Z} \cdot (K_{ij}D_1 - D_1K_{ji}) + \mathbb{Z} \cdot (K_{ji}D_1 - D_1K_{ji}) + \mathbb{Z} \cdot (D_3K_{ji} - K_{ij}D_3)$.

(b) $\mathbf{K}'_{ij}(W)_{\le 2} = \mathbb{Z} + \mathbb{Z} \cdot K_{ij} + \mathbb{Z} \cdot K_{ji}$.

Proof. Since $\hat{\mathbf{D}}(W_{\{ij\}})$ is the free product of three copies of $\hat{\mathbf{D}}(W_{\{i\}})$, it is a free $\mathbb{Z}$-module (this also follows from by Lemmas 7.6 and 7.13). In particular, $\hat{\mathbf{D}}(W_{\{i,j\}})_{\le 3}$ is a free $\mathbb{Z}$-module with a basis 1, D_1, D_2, D_3, D_aD_b, $D_aD_bD_{6-a-b}$, $D_aD_bD_a$ for all distinct $a, b \in \{1, 2, 3\}$, that is, each $x \in \hat{\mathbf{D}}(W_{\{i,j\}})_{\le 3}$ can be uniquely written as

$$x = a_0 + \sum_{\ell=1}^{3} a_\ell D_\ell + \sum_{a,b\in\{1,2,3\},a\ne b} (f_{a,b}D_aD_b + g_{a,b}D_aD_bD_{6-a-b} + h_{a,b}D_aD_bD_a) , \quad (7.28)$$

where all a_k, $f_{a,b}$, $g_{a,b}$, $h_{a,b}$ are integers.

Let us show first that $d_k(x) = 0$ for some x in (7.28) and some $k \in \{1, 2, 3\}$ implies that then $h_{k,b} = 0$ for $b \in \{1, 2, 3\} \setminus \{k\}$. Indeed, $d_k(D_aD_b) = \delta_{k,a}D_b + \delta_{k,b}s_k(D_a)$, $d_k(D_aD_bD_c) = d_k(D_aD_b)D_c + \delta_{k,c}s_k(D_aD_b) = \delta_{k,a}D_bD_c + \delta_{k,b}s_k(D_a)D_c + \delta_{k,c}s_k(D_aD_b)$. Therefore,

$$\begin{aligned} d_k(x) = a_k &+ \sum_{a\ne b}(f_{a,b}(\delta_{k,a}D_b + \delta_{k,b}s_k(D_a)) \\ &+ \sum_{a\ne b} g_{a,b}(\delta_{k,a}D_bD_{6-k-b} + \delta_{k,b}s_k(D_a)D_{6-a-k} + \delta_{k,6-a-b}s_k(D_aD_b)) \\ &+ \sum_{a\ne b} h_{a,b}(\delta_{k,a}D_bD_k + s_k(D_kD_b) + \delta_{k,b}s_k(D_a)D_a) \end{aligned}$$

Taking into account that

$$s_k(D_a) = \begin{cases} 1 - D_k & \text{if } a = k \\ D_{6-a-k} & \text{if } k \in \{1,3\},\ a \neq k \\ 1 - D_{6-a-k} & \text{if } k = 2,\ a \neq k \end{cases} \tag{7.29}$$

we see that $d_k(x) = \sum\limits_{b \neq k} h_{k,b}(D_b D_k + D_k D_{6-k-b}) + \cdots$, where $\cdots$ stand for the linear combination of monomials not containing D_k. Thus, $d_k = 0$ implies $h_{k,b} = 0$ for $b \in \{1,2,3\} \setminus \{k\}$. In particular, $d_k(x) = 0$ for $k = 1,2,3$ implies that $h_{a,b} = 0$ for all distinct $a, b \in \{1,2,3\}$.

Based on the above computations, using (7.29) again we obtain for $k \in \{1,2,3\}$:

$$\begin{aligned} d_k(x) = a_k &+ \sum_{a \neq b} (f_{a,b}(\delta_{k,a} D_b + \delta_{k,b} s_k(D_a)) \\ &+ \sum_{a \neq b} g_{a,b}(\delta_{k,a} D_b D_{6-k-b} + \delta_{k,b} s_k(D_a) D_{6-a-k} + \delta_{k,6-a-b} s_k(D_a D_b)) \\ &= z + \sum_{b \neq k} g_{k,b} D_b D_{6-k-b} + \sum_{a \neq k} g_{a,k} s_k(D_a) D_{6-a-k} + \sum_{a,b:6-a-b=k} g_{a,b} s_k(D_a) s_k(D_b) \\ &= z' + \sum_{b \neq k} g_{k,b} D_b D_{6-k-b} + \sum_{a \neq k} g_{a,k}(1 - \delta_{k,2}) D_{6-a-k} D_{6-a-k} \\ &+ \sum_{a,b:6-a-b=k} g_{a,b} D_{6-a-k} D_{6-b-k} \\ &= z'' + \sum_{b \neq k} g_{k,b} D_b D_{6-k-b} + \sum_{a,b:6-a-b=k} g_{a,b} D_b D_a \end{aligned}$$

for some $z, z', z'' \in \hat{\mathbf{D}}(W_{\{i,j\}})_{\le 1}$. Thus, fixing k', k'' such that $\{k, k', k''\} = \{1,2,3\}$, we obtain:

$$d_k(x) = g_{k,k'} D_{k'} D_{k''} + g_{k,k''} D_{k''} D_{k'} + g_{k',k''} D_{k''} D_{k'} + g_{k'',k'} D_{k'} D_{k''} + z'' \ .$$

Since $z'' \in \hat{\mathbf{D}}(W_{\{i,j\}})_{\le 1}$, the equations $d_k(x) = 0$ for $k = 1,2,3$ imply that $g_{k,k'} + g_{k'',k'} = 0$ for each permutation (k, k', k'') of $\{1,2,3\}$.

Thus, (7.28) with $d_k(x) = 0$ for $k = 1,2,3$ becomes

$$\begin{aligned} x = y &+ g_{1,2}(D_1 D_2 D_3 - D_3 D_2 D_1) + g_{1,3}(D_1 D_3 D_2 - D_2 D_3 D_1) \\ &+ g_{2,1}(D_2 D_1 D_3 - D_3 D_1 D_2) \end{aligned} \tag{7.30}$$

for some $y \in \hat{\mathbf{D}}(W_{\{i,j\}})_{\le 2}$. Note that: $D_1 K_{ji} = D_1 D_3 D_1 - D_1 D_2 D_3$, $K_{ij} D_1 = D_1 D_3 D_1 - D_3 D_2 D_1$,

$$D_1K_{ij} = D_1D_3 - D_1D_2D_1 - D_1D_3D_2 + D_1D_2,\ D_3K_{ji}$$
$$= D_3D_1 - D_3D_1D_2 - D_3D_2D_3 + D_3D_2\ ,$$
$$K_{ji}D_1 = D_3D_1 - D_1D_2D_1 - D_2D_3D_1 + D_2D_1,\ K_{ij}D_3$$
$$= D_1D_3 - D_2D_1D_3 - D_3D_2D_3 + D_2D_3\ ,$$

which, in particular, imply that $D_1D_2D_3 - D_3D_2D_1 = K_{ij}D_1 - D_1K_{ji} = -(K_{ji}D_3 - D_3K_{ij}$,

$$D_1D_3D_2 - D_2D_3D_1 = K_{ji}D_1 - D_1K_{ji} + D_1D_2 - D_2D_1 + D_1D_3 - D_3D_1\ ,$$
$$D_2D_1D_3 - D_3D_1D_2 = D_3K_{ji} - K_{ij}D_3 + D_2D_3 - D_3D_2 + D_1D_3 - D_3D_1\ .$$

Thus, (7.30) becomes:

$$x \in \hat{\mathbf{D}}(W_{\{i,j\}})_{\le 2} + \mathbb{Z}\cdot(K_{ij}D_1 - D_1K_{ji}) + \mathbb{Z}\cdot(K_{ji}D_1 - D_1K_{ji}) + \mathbb{Z}\cdot(D_3K_{ji} - K_{ij}D_3)$$

Note that $K_{ij}D_1 - D_1K_{ji}$, $K_{ji}D_1 - D_1K_{ji}$ and $D_3K_{ji} - K_{ij}D_3$ belong to $\mathbf{K}'_{ij}(W)$. This proves (a).

Prove (b). Repeating the argument from the proof of (a), we see that $\hat{\mathbf{D}}(W_{\{i,j\}})_{\le 2}$ is a free $\mathbb{Z}$-module with a basis 1, D_1, D_2, D_3, D_aD_b, for all distinct $a, b \in \{1,2,3\}$, that is, each $x \in \hat{\mathbf{D}}(W_{\{i,j\}})_{\le 2}$ can be uniquely written as

$$x = a_0 + \sum_{\ell=1}^{3} a_\ell D_\ell + \sum_{a,b\in\{1,2,3\},a\ne b} f_{a,b}D_aD_b\ , \tag{7.31}$$

where all a_k, $f_{a,b}$ are integers.

Using the argument from the proof of (a) and (7.29) we obtain for $k \in \{1,2,3\}$:

$$d_k(x) = a_k + \sum_{a\ne b}(f_{a,b}(\delta_{k,a}D_b + \delta_{k,b}s_k(D_a)) = a_k + \sum_{b\ne k} f_{k,b}D_b + \sum_{a\ne k} f_{a,k}s_k(D_a)$$

Thus, fixing k', k'' such that $\{k,k',k''\} = \{1,2,3\}$, we obtain

$$d_k(x) = a_k + \begin{cases} f_{k,k'}D_{k'} + f_{k,k''}D_{k''} + f_{k',k}D_{k''} + f_{k'',k}D_{k'} & \text{if } k \in \{1,3\} \\ f_{k,k'}D_{k'} + f_{k,k''}D_{k''} + f_{k',k}(1 - D_{k''}) + f_{k'',k}(1 - D_{k'}) & \text{if } k = 2 \end{cases}.$$

Thus, $d_k(x) = 0$ for $k = 1,2,3$ imply that $a_1 = a_3 = 0$, $f_{12} + f_{31} = 0$, $f_{13} + f_{21} = 0$, $f_{12} + f_{31} = 0$, $f_{31} + f_{23} = 0$, $f_{32} + f_{13} = 0$, $a_2 + f_{12} + f_{32} = 0$.

Therefore, $x = a_0 + f_{13}K_{ij} + f_{31}K_{ji}$. This proves (b).

The lemma is proved. $\square$

To finish the proof of Proposition 7.40(b), it suffices to show that $\mathbf{K}'_{ij}(W) = \mathbb{Z} + \mathbf{K}_{ij}(W)$ for $m_{ij} = 3$. We already have the inclusion $\mathbb{Z} + \mathbf{K}_{ij}(W) \subset \mathbf{K}'_{ij}(W)$ by (7.27). To show the opposite inclusion note that Lemma 7.41 implies that $\mathbf{K}'_{ij}(W)$ is a $\mathbb{Z}$-submodule of $\hat{\mathbf{D}}(W_{\{ij\}})$ generated by 1, K_{ij}, K_{ji}, $\tilde{K}_{ij} = K_{ij}D_i - D_iK_{ji} = -(K_{ji}D_j - D_jK_{ij}) = -\tilde{K}_{ji}$, $\tilde{K}'_{ij} = K_{ji}D_i - D_iK_{ji}$, and $\tilde{K}'_{ji} = K_{ij}D_j - D_jK_{ij}$.

Thus, to prove the inclusion $\mathbf{K}'_{ij}(W) \subset \mathbb{Z} + \mathbf{K}_{ij}(W)$ it suffices to show that

$$\{wK_{ij}w^{-1}, w\tilde{K}_{ij}w^{-1}, w\tilde{K}_{ij}w^{-1}\} \subset \hat{\mathbf{D}}(W_{\{ij\}}) \tag{7.32}$$

for each $w \in W_{\{ij\}}$.

Note that $Q_{ij}^{(1,1,1)} = K_{ij}$, $Q_{ji}^{(1,1,1)} = K_{ji}$ in the notation of Proposition 7.20(a). Thus,

$$s_iK_{ij}s_i = s_jK_{ij}s_j = K_{ji},\ s_iK_{ji}s_i = s_jK_{ji}s_j = K_{ij}$$

by Proposition 7.20(a), which implies that $K_{ij}, K_{ji} \in \mathbf{K}_{ij}(W)$. Also

$$\begin{aligned} s_i\tilde{K}_{ij}s_i &= K_{ji}(1 - s_i - D_i) - (1 - s_i - D_i)K_{ij} = K_{ji}(1 - D_i) - (1 - D_i)K_{ij} \\ &= K_{ji} - K_{ij} - \tilde{K}'_{ij} \in \hat{\mathbf{D}}(W_{\{ij\}}) \end{aligned}$$

because $K_{ji}s_i - s_iK_{ij} = 0$. In particular, $s_i\tilde{K}'_{ij}s_i = K_{ij} - K_{ji} - \tilde{K}_{ij} \in \hat{\mathbf{D}}(W_{\{ij\}})$.

Furthermore,

$$\begin{aligned} s_j\tilde{K}_{ij}s_j &= K_{ji}s_jD_is_j - s_jD_is_jK_{ij} = \tilde{K}''_{ij} \in \hat{\mathbf{D}}(W_{\{ij\}}) \ , \\ s_j\tilde{K}'_{ij}s_j &= K_{ij}s_jD_is_j - s_jD_is_jK_{ji} = \tilde{K}''_{ji} \in \hat{\mathbf{D}}(W_{\{ij\}}) \end{aligned}$$

where we abbreviated $\tilde{K}''_{ij} = K_{ji}D_{ij} - D_{ij}K_{ij}$. Finally,

$$\begin{aligned} s_i\tilde{K}''_{ij}s_i &= K_{ij}s_iD_{ij}s_i - s_iD_{ij}s_iK_{ji} = K_{ij}D_j - D_jK_{ji} = \tilde{K}'_{ij} \in \hat{\mathbf{D}}(W_{\{ij\}}) \ , \\ s_j\tilde{K}''_{ij}s_j &= K_{ij}s_jD_{ij}s_j - s_jD_{ij}s_jK_{ji} = K_{ij}D_i - D_iK_{ji} = \tilde{K}_{ij} \in \hat{\mathbf{D}}(W_{\{ij\}}) \ . \end{aligned}$$

This proves the inclusions (7.32). Thus, $\mathbf{K}'_{ij}(W) = \mathbb{Z} + \mathbf{K}_{ij}(W)$. Together with Lemma 7.41 this finishes the proof of Proposition 7.40(b).

Proposition 7.40 is proved. □

Proof of Theorem 1.25. In the assumptions of Theorem 1.25, suppose that $m_{ij} = 3$ and let $K'_{ij} \in \hat{\mathbf{H}}(W)$ denote $K'_{ij} = -D_js_iD_j + s_iD_jD_i + D_iD_js_i + s_iD_js_i$. Clearly,

$$K'_{ij}s_i = -D_jD_{ij} + D_{ij}(1 - D_i - s_i) + D_iD_j + s_iD_j = D_iD_j - D_jD_{ij} + D_iD_{ij} + D_{ij} = K_{ij}$$

in the notation of Proposition 7.40. We also abbreviate $K'_{ij} := K_{ij} = D_iD_j - D_jD_i$ if $m_{ij} = 2$.

Thus, $\mathbf{H}(W)$ is the quotient of $\hat{\mathbf{H}}(W)$ by the ideal generated by K'_{ij} for all distinct $i, j \in I$ Theorem 1.25 is proved. □

Therefore, Theorem 1.3 is proved. □

Proof of Proposition 1.6 and Theorem 1.9. Since S_n is simply-laced, $\mathbf{H}(S_n)$ is covered by Theorem 1.25. Then Theorem 1.22 guarantees the factorization of $\mathbf{H}(S_n)$. Also the first assertion of Theorem 1.33 for $W = S_n$, $\Bbbk = \mathbb{Z}[q, q^{-1}]$ coincides with the assertion of Theorem 1.9.

Proposition 1.6 and Theorem 1.9 are proved. □

Proof of Proposition 1.31. In the proof of Proposition 7.17, we established that $\mathbf{D}(W) = \hat{\mathbf{D}}(W)/\langle\underline{\mathbf{K}}(W)\rangle$ for any Coxeter group W, where

$$\underline{\mathbf{K}}(W) = \sum_{w\in W, i,j\in I: i\neq j} w\mathbf{K}_{ij}(W)w^{-1} = \sum_{i,j\in I: i\neq j, w\in W^{\{i,j\}}} w\mathbf{K}_{ij}(W)w^{-1} \tag{7.33}$$

by Lemma 7.19, where $W^{\{i,j\}} = \{w \in W \mid \ell(ws_i) = \ell(w) + 1, \ell(ws_j) = \ell(w) + 1\}$.

Now suppose that W is simply-laced, i.e., $m_{ij} \in \{0, 2, 3\}$. Then, in view Proposition 7.40, the equation (7.33) reads

$$\underline{\mathbf{K}}(W) = \sum_{i,j\in I: i\neq j, w\in W^{\{i,j\}}} wK_{ij}(W)w^{-1} \tag{7.34}$$

For each compatible pair $(s, s') \in \mathcal{S} \times \mathcal{S}$ define an element $K_{s,s'} \in \hat{\mathbf{H}}(W)$ by

$$K_{s,s'} := \begin{cases} 0 & \text{if } m_{s,s'} = 0 \\ D_sD_{s'} - D_{s'}D_s & \text{if } m_{s,s'} = 2 \\ D_sD_{s'} - D_{ss's}D_s - D_{s'}D_{ss's} + D_{ss's} & \text{if } m_{s,s'} = 3 \end{cases}$$

in the notation of Proposition 1.31. Since $wD_iw^{-1} = D_{ws_iw^{-1}}$ whenever $\ell(ws_i) = \ell(w)+1$ by (7.13), in each of these cases, one has, in the notation of Proposition 7.40, $K_{s,s'} = wK_{ij}w^{-1}$ for some distinct $i, j \in I$, $w \in W^{\{i,j\}}$.

Therefore, for each simply-laced Coxeter group W, (7.34) reads: $\underline{\mathbf{K}}(W) = \sum_{\substack{i,j\in I:\\ i\neq j, w\in W^{\{i,j\}}}} wK_{s,s'}(W)$, where the summation is over all compatible pairs $(s, s') \in \mathcal{S}\times\mathcal{S}$.

The proposition is proved. □

Proof of Proposition 1.7. Let $W = S_n$ and let $s = (i, j)$, $s' = (k, \ell)$, $1 \le i < j \le n$, $1 \le k < \ell \le n$ be distinct transpositions in S_n.

- Clearly, $m_{s,s'} = 2$, i.e., $s's = s's$ iff $\{i, j\} \cap \{k, \ell\} = \emptyset$; then (s, s') is compatible.
- Clearly, $m_{s,s'} = 3$ iff either $i = k$ or $j = \ell$ or $j = k$ or $i = \ell$; then (s, s') is compatible precisely in the last two cases.

Finally, this characterization of compatible pairs in S_n and Proposition 1.31 finish the proof. □

7.8. Action on Laurent polynomials and verification of Conjecture 1.40

Let $\mathcal{Q}_I$ be the field of fractions of the Laurent polynomial ring $\mathcal{L}_I = \mathbb{Z}[t_i^{\pm 1}, i \in I]$. So $\mathcal{Q}_I$ is a purely transcendental field generated by t_i, $i \in I$. Since $\mathcal{L}_I$ is a group ring of $\mathbb{Z}^I = \oplus_{i\in I}\mathbb{Z}\alpha_i$, then the natural reflection action of W on $\mathbb{Z}^I$ ($s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$) extends to a W-action on $\mathcal{Q}_I$ by automorphisms.

Proposition 7.42. *For any Coxeter group W the assignments $D_i \mapsto \frac{1}{1-t_i}(1-s_i)$, $s_i \mapsto s_i$, $i \in I$, define a homomorphism of algebras $\hat{p}_W : \hat{\mathbf{H}}(W) \to \mathcal{Q}_I \rtimes \mathbb{Z}W$. Under this homomorphism, $\hat{p}_W(\mathbf{K}_{ij}(W)) = \{0\}$ whenever $m_{ij} \in \{0, 2, 3\}$.*

Proof. It suffices to verify only relations involving D_i's. Indeed, let us abbreviate $\tau_i = \frac{1}{1-t_i}$ and $\underline{D}_i := \tau_i(1-s_i) \in \mathcal{Q}_I \rtimes \mathbb{Z}W$. Taking into account that $s_i\tau_i s_i = \frac{1}{1-t_i^{-1}} = 1-\tau_i$, we obtain

$$\begin{aligned}\underline{D}_i^2 &= \tau_i(1-s_i)\tau_i(1-s_i) = (\tau_i^2 - \tau_i s_i \tau_i)(1-s_i) = (\tau_i^2 - \tau_i(1-\tau_i)\cdot s_i)(1-s_i)\\ &= (\tau_i^2 + \tau_i(1-\tau_i))(1-s_i) = \underline{D}_i\end{aligned}$$

for $i \in I$. Furthermore, let us verify linear braid relations in $\hat{\mathbf{H}}(W)$, which we write in the form $wD_iw^{-1} = D_{i'}$ whenever $ws_iw^{-1} = s_{i'}$ and $\ell(ws_i) = \ell(w)+1$. Indeed, for such i, i' and w one has $wt_iw^{-1} = w(t_i) = t_{i'}$ therefore, $w\underline{D}_iw^{-1} = w\tau_i(1-s_i)w^{-1} = \tau_{i'}(1-s_{i'}) = \underline{D}_{i'}$.

This proves the first assertion of the proposition.

Let us prove the second assertion.

Indeed, if $m_{ij} = 0$, then $\mathbf{K}_{ij}(W) = \{0\}$ and we have nothing to prove.

If $m_{ij} = 2$, then, according to Proposition 7.40(a), $\mathbf{K}_{ij}(W) = \mathbb{Z}\cdot K_{ij}$, where $K_{ij} = D_iD_j - D_jD_i$. Clearly, in this case, $s_is_j = s_js_i$, $s_it_j = t_js_i$ hence $s_i\tau_j = \tau_js_i$, therefore, $\underline{D}_i\underline{D}_j = \tau_i(1-s_i)\tau_j(1-s_j) = \tau_j(1-s_j)\tau_i(1-s_i) = \underline{D}_j\underline{D}_i$, i.e., $\hat{p}_W(\mathbf{K}_{ij}(W)) = 0$.

Let now $m_{ij} = 3$. Then, according to Proposition 7.40(b), $\mathbf{K}_{ij}(W) = \mathbb{Z}\cdot K_{ij} + \mathbb{Z}\cdot K_{ji}$, where $K_{ij} = D_iD_j - D_jD_{ij} - D_{ij}D_i + D_{ij}$, and $D_{ij} = s_iD_js_i = s_jD_is_j$. Thus, it suffices to show that $\hat{p}_W(K_{ij}) = 0$. Indeed, $\hat{p}_W(K_{ij}) = \underline{D}_i\underline{D}_j - \underline{D}_j\underline{D}_{ij} - \underline{D}_{ij}\underline{D}_i + \underline{D}_{ij}$, where $\underline{D}_{ij} = \tau_{ij}(1-s_{ij})$, and $\tau_{ij} = s_i\tau_js_i = s_j\tau_is_j = \frac{1}{1-t_it_j}$, $s_{ij} = s_is_js_i = s_j\tau_is_j$. Let us compute:

$$\underline{D}_i\underline{D}_j = \tau_i(1-s_i)\tau_j(1-s_j) = \tau_i(\tau_j - \tau_{ij}s_i)(1-s_j) = \tau_i\tau_j(1-s_j) - \tau_i\tau_{ij}(s_i - s_is_j)\ ,$$

$$\underline{D}_j\underline{D}_{ij} = \tau_j(1-s_j)\tau_{ij}(1-s_{ij}) = \tau_j(\tau_{ij} - \tau_is_j)(1-s_{ij}) = \tau_j\tau_{ij}(1-s_{ij}) - \tau_i\tau_j(s_j - s_is_j)\ ,$$

$$\begin{aligned}\underline{D}_{ij}\underline{D}_i &= \tau_{ij}(1-s_{ij})\tau_i(1-s_i) = \tau_{ij}(\tau_i - (1-\tau_j)s_{ij})(1-s_i)\\ &= \tau_{ij}\tau_i(1-s_i) - \tau_{ij}(1-\tau_j)(s_{ij} - s_is_j)\ .\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{p}_W(K_{ij}) &= \tau_i(\tau_j - \tau_{ij}s_i)(1-s_j) = \tau_i\tau_j(1-s_j) - \tau_i\tau_{ij}(s_i - s_is_j) - \tau_j\tau_{ij}(1-s_{ij})\\ &\quad + \tau_i\tau_j(s_j - s_is_j) - \tau_{ij}\tau_i(1-s_i) + \tau_{ij}(1-\tau_j)(s_{ij} - s_is_j) + \tau_{ij}(1-s_{ij})\\ &= k_{ij} - \tau_i\tau_js_j - \tau_i\tau_{ij}(s_i - s_is_j) + \tau_j\tau_{ij}s_{ij} + \tau_i\tau_j(s_j - s_is_j)\\ &\quad + \tau_{ij}\tau_is_i + \tau_{ij}(1-\tau_j)(s_{ij} - s_is_j) - \tau_{ij}s_{ij}\\ &= k_{ij} - k_{ij}s_is_j\ ,\end{aligned}$$

where $k_{ij} = \tau_i\tau_j - \tau_j\tau_{ij} - \tau_{ij}\tau_i + \tau_{ij}$. Thus, $\hat{p}_W(K_{ij}) = 0$ because $k_{ij} = 0$. This finishes the proof of the second assertion of the proposition.

The proposition is proved. $\square$

Verification of Conjecture 1.40 in the simply-laced case. The following is an immediate corollary of Proposition 7.42.

Corollary 7.43. *Suppose that W is a simply-laced Coxeter group, i.e., $m_{ij} \in \{0,2,3\}$ for $i,j \in I$. Then, in the notation of Proposition 7.42, the assignments $D_i \mapsto \frac{1}{1-t_i}(1-s_i)$, $s_i \mapsto s_i$, $i \in I$, define a homomorphism of algebras $p_W : \mathbf{H}(W) \to \mathcal{Q}_I \rtimes \mathbb{Z}W$.*

Note that $\mathcal{Q}_I \rtimes \mathbb{Z}W$ naturally acts on $\mathcal{Q}_I$ via $(tw)(t') = t \cdot w(t')$ for $t, t' \in \mathcal{Q}_I$, $w \in W$. Composing this with p_W gives an action of $\mathbf{H}(W)$ on $\mathcal{Q}_I$, under which $\mathcal{L}_I$ is invariant, thus both $\mathcal{Q}_I$ and $\mathcal{L}_I$ are module algebras over $\mathbf{H}(W)$. For W simply-laced this, taken together with Corollary 7.43 turns $\mathcal{Q}_I$ into a module algebra over $\mathbf{H}(W)$, so that $\mathcal{L}_I$ is a module subalgebra. Since the above action of $\mathbf{H}(W)$ on $\mathcal{L}_I$ coincides with (1.4), this verifies Conjecture 1.40 for all simply-laced W. $\square$

Proof of Proposition 1.13. Let $W = S_n$, so that $I = \{1, \ldots, n-1\}$ and $\mathcal{L}_I = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}]$. Also denote $\mathcal{P}_n := \mathbb{Z}[x_1, \ldots, x_n]$ and let $\mathcal{Q}_n$ be the field of fractions of $\mathcal{P}_n$. We identify $\mathcal{Q}_I$ with the subfield of $\mathcal{Q}_n$ generated by $t_i = \frac{x_i}{x_{i+1}}$, $i = 1, \ldots, n-1$. We have a natural S_n-action on $\mathcal{Q}_n$ by permutations so that its restriction to $\mathcal{Q}_I$ coincides with the natural S_n-action on $\mathcal{Q}_I$. In particular, this defines a natural action of $\mathcal{Q}_I \rtimes \mathbb{Z}S_n$ on $\mathcal{Q}_n$ via $(tw)(x) = t \cdot w(x)$ for $t \in \mathcal{Q}_I$, $x \in \mathcal{Q}_n$, $w \in S_n$.

For $W = S_n$ this, taken together with Corollary 7.43 defines a structure of a module algebra over $\mathbf{H}(S_n)$ on $\mathcal{Q}_n$, so that $\mathcal{P}_n$ is a module subalgebra. This proves Proposition 1.13 because the above action coincides with the one given by (1.4). $\square$

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Appendix A. Deformed semidirect products

For readers' convenience, in this section we state relevant results about deformations of cross products, see also [16] and the forthcoming joint paper of Yury Bazlov with the first author [3].

Throughout this section, we fix a commutative ring R. Let A and B be unital R-algebras and let $\Psi : B \otimes A \to A \otimes B$ be an R-linear map (all tensor products are over R).

Define a (possibly non-associative) multiplication on $A \otimes B$ by: $(a' \otimes b)(a \otimes b') = a'\Psi(b \otimes a)b'$ for all $a, a' \in A$, $b, b' \in B$ and denote the resulting algebra by $A \otimes_\Psi B$.

Note that $1 \otimes 1$ is a unit of $A \otimes_\Psi B$ iff

$$\Psi(1 \otimes a) = a \otimes 1, \;\; \Psi(b \otimes 1) = 1 \otimes b \tag{A.1}$$

for all $a \in A$, $b \in B$ (in that case, $A \otimes 1$ and $1 \otimes B$ are subalgebras of $A \otimes_\Psi B$).

We need the following result from [13] (due to its importance, we provide a proof).

Proposition A.1 *([13, Proposition 21.4]). Let A and B be associative unital R-algebras and $\Psi : B \otimes A \to A \otimes B$ be an R-linear map satisfying (A.1). Then the R-algebra $A \otimes_\Psi B$ is associative iff the following diagrams are commutative:*

$$\begin{array}{ccc} B \otimes A \otimes A & \xrightarrow{(1\otimes\Psi)\circ(\Psi\otimes 1)} & A \otimes A \otimes B \\ \Big\downarrow {\scriptstyle 1\otimes \mathbf{m}_A} & & {\scriptstyle \mathbf{m}_A \otimes 1} \Big\downarrow \\ B \otimes A & \xrightarrow{\Psi} & A \otimes B \end{array} \;, \quad \begin{array}{ccc} B \otimes B \otimes A & \xrightarrow{(\Psi\otimes 1)\circ(1\otimes\Psi)} & A \otimes B \otimes B \\ \Big\downarrow {\scriptstyle \mathbf{m}_B\otimes 1} & & {\scriptstyle 1 \otimes \mathbf{m}_B} \Big\downarrow \\ B \otimes A & \xrightarrow{\Psi} & A \otimes B \end{array} \;. \tag{A.2}$$

where $\mathbf{m}_A$ (resp. $\mathbf{m}_B$) is the multiplication map $A \otimes A \to A$ (resp. $B \otimes B \to B$).

Proof. Indeed, suppose that $A \otimes_\Psi B$ is an associative algebra. Clearly, the associativity equations

$$(1 \otimes b)(aa' \otimes 1) = ((1 \otimes b)(a \otimes 1))(a' \otimes 1), \;\; (1 \otimes b'b)(a \otimes 1) = (1 \otimes b')(1 \otimes b)(a \otimes 1) \tag{A.3}$$

for all $a, a' \in A$, $b, b' \in B$ are respectively equivalent to the commutativity of the diagrams (A.2).

Conversely, suppose that the diagrams (A.2) are commutative, that is, (A.3) hold. These, taken together with obvious relations $(a \otimes b) = (a \otimes 1)(1 \otimes b)$ for $a \in A$, $b \in B$ and:

$$(a'' \otimes 1)(zz') = ((a'' \otimes 1)z)z', \ (zz')(1 \otimes b'') = z(z'(1 \otimes b'')) \tag{A.4}$$

for any $a'' \in A$, $b'' \in B$, $z, z' \in A \otimes B$, imply

$$\begin{aligned}(a'' \otimes b)(aa' \otimes b') &= ((a'' \otimes b)(a \otimes 1))(a' \otimes b'),\\ (a' \otimes b'b)(a \otimes b'') &= (a' \otimes b')((1 \otimes b)(a \otimes b''))\end{aligned} \tag{A.5}$$

for all $a, a', a'' \in A$, $b, b', b'' \in B$. In view of the obvious relations $aa' \otimes b' = (a \otimes 1)(a' \otimes b')$ and $a' \otimes b'b = (a' \otimes b')(1 \otimes b)$, the relations (A.5) are equivalent to

$$z((a \otimes 1)z') = (z(a \otimes 1))z', \ z((1 \otimes b)z') = (z(1 \otimes b))z' \tag{A.6}$$

for all $a, a' \in A$, $b, b' \in B$, $z \in A \otimes B$.

Finally, using by (A.6) repeatedly, we obtain

$$\begin{aligned}(a_1 \otimes b_1)((a_2 \otimes b_2)(a_3 \otimes b_3)) &= (a_1 \otimes b_1)(a_2\Psi(b_2 \otimes a_3)b_3)\\ &= (a_1 \otimes b_1)((a_2 \otimes 1)(\Psi(b_2 \otimes a_3)b_3)\\ &= ((a_1 \otimes b_1)(a_2 \otimes 1))(\Psi(b_2 \otimes a_3)b_3)\\ &= (a_1\Psi(b_1 \otimes a_2))((1 \otimes b_2)(a_3 \otimes b_3))\\ &= (a_1\Psi(b_1 \otimes a_2))(1 \otimes b_2))(a_3 \otimes b_3)\\ &= (a_1\Psi(b_1 \otimes a_2)b_2)(a_3 \otimes b_3)\\ &= ((a_1 \otimes b_1)(a_2 \otimes a_2))(a_3 \otimes b_3)\ .\end{aligned}$$

This proves associativity of $A \otimes_\Psi B$.

The proposition is proved. $\square$

We say that $A \otimes_\Psi B$ is *left* associative (resp. *right* associative) if the first (resp. the second) diagram (A.2) is commutative. According to Proposition A.1, $A \otimes_\Psi B$ is an associative R-algebra iff it is both left and right associative. In particular, taking $B = RW$, where W is a monoid acting on A by R-linear endomorphisms and $\Psi : RW \otimes A \to A \otimes RW$ given by $\Psi(w \otimes a) = w(a) \otimes w$, $w \in W$, $a \in A$, we recover the following well-known result.

Corollary A.2. *(semidirect product) Let W be a monoid and A be an RW-module algebra (i.e., W acts on A by R-linear algebra endomorphisms). Then the space $A \otimes RW$ is an associative R-algebra with the product given by $(a' \otimes w)(a \otimes w') = a' \cdot w(a) \otimes ww'$ for all $a, a' \in A$, $w, w' \in W$.*

For an R-module V denote by $T(V)$ its tensor algebra $\oplus_{n \geq 0} V^{\otimes n}$ of V.

Proposition A.3. *In the assumptions of Proposition A.1 suppose that $A = T(V)$ for some R-module V. Then for any R-linear map $\mu : B \otimes V \to T(V) \otimes B$ satisfying*

$$\mu(1 \otimes x) = x \otimes 1 \tag{A.7}$$

for all $x \in T(V)$, there exists a unique $\Psi^\mu : B \otimes T(V) \to T(V) \otimes B$ such that $T(V) \otimes_{\Psi^\mu} B$ is left associative with unit $1 \otimes 1$ and $\Psi^\mu|_{B\otimes V} = \mu$.

Proof. Define $\Psi^\mu := \oplus_{n\geq 0} \Psi^{(n)}$, where $\Psi^{(n)}$ is an R-linear map $B \otimes V^{\otimes n} \to T(V) \otimes B$ given by

- $\Psi^{(0)}(b \otimes 1) = 1 \otimes b$ for all $b \in B$.
- $\Psi^{(n)} = \mu_n \circ \cdots \circ \mu_1$ for $n \geq 1$, where $\mu_i : T(V)^{\otimes i-1} \otimes B \otimes V^{\otimes n+1-i} \to T(V)^{\otimes i} \otimes B \otimes V^{\otimes n-i}$ is given by $\mu_i = 1 \otimes \cdots \otimes 1 \otimes \mu \otimes 1 \otimes \cdots \otimes 1$.

Taking into account that $V^{\otimes m} \otimes V^{\otimes n} = V^{\otimes m+n}$ for $m, n \geq 0$, we immediately obtain

$$\Psi^{(m+n)} = (\mu_{m+n} \circ \cdots \circ \mu_{m+1}) \circ (\mu_m \circ \cdots \circ \mu_1) = (1 \otimes \Psi^{(n)}) \circ (\Psi^{(m)} \otimes 1)$$

which implies that $\Psi^\mu = (\mathbf{m}_{T(V)} \otimes 1) \circ (1 \otimes \Psi^\mu) \circ (\Psi^\mu \otimes 1)$ i.e., the following diagram is commutative.

$$\begin{array}{ccc} B \otimes T(V) \otimes T(V) & \xrightarrow{(1\otimes\Psi^\mu)\circ(\Psi^\mu\otimes 1)} & T(V) \otimes T(V) \otimes B \\ \downarrow{\scriptstyle 1\otimes \mathbf{m}_{T(V)}} & & {\scriptstyle \mathbf{m}_{T(V)}\otimes 1}\downarrow \\ B \otimes T(V) & \xrightarrow{\Psi^\mu} & T(V) \otimes B \end{array} .$$

The above diagram is the first diagram (A.2) for $A = T(V)$, hence, $T(V) \otimes_{\Psi^\mu} B$ is left associative. By the construction, Ψ^μ satisfies (A.1).

Clearly, Ψ^μ is uniquely determined by the assumptions of the proposition.

The proposition is proved. □

Proposition A.4. *Let V be an R-module, B be an R-algebra, and $\mu : B \otimes V \to T(V) \otimes B$ be an R-linear map satisfying (A.7). Then $T(V) \otimes_{\Psi^\mu} B$ is an associative R-algebra iff the following diagram is commutative:*

$$\begin{array}{ccc} B \otimes B \otimes V & \xrightarrow{(\Psi^\mu\otimes 1)\circ(1\otimes\mu)} & T(V) \otimes B \otimes B \\ \downarrow{\scriptstyle \mathbf{m}_B\otimes 1} & & {\scriptstyle 1\otimes \mathbf{m}_B}\downarrow \\ B \otimes T(V) & \xrightarrow{\Psi^\mu} & T(V) \otimes B \end{array} . \tag{A.8}$$

Proof. We need the following result.

Lemma A.5. *In the assumptions of Proposition A.4, commutativity of (A.8) implies that the following diagram is commutative for all $n \geq 0$.*

$$\begin{tikzcd} B\otimes B\otimes V^{\otimes n} \arrow[r, "(\Psi^\mu\otimes 1)\circ(1\otimes\Psi^\mu)"] \arrow[d, "\mathbf{m}_B\otimes 1"] & T(V)\otimes B\otimes B \arrow[d, "1\otimes\mathbf{m}_B"'] \\ B\otimes V^{\otimes n} \arrow[r, "\Psi^\mu"] & T(V)\otimes B \end{tikzcd} . \tag{A.9}$$

Proof. We proceed by induction in n. If $n = 0, 1$, the assertion is obvious. Suppose that $n \geq 2$. Tensoring the commutative diagram (A.9) for $V^{\otimes n-1}$ with V from the right and then horizontally composing with the commutative diagram (A.8) (which is tensored with $T(V)$ from the left), followed by the multiplications $T(V) \otimes T(V) \to T(V)$ and $B \otimes B \to B$, we obtain a commutative diagram:

$$\begin{tikzcd} B\otimes B\otimes V^{\otimes n} \arrow[r] \arrow[d, "\mathbf{m}_B\otimes 1"] & T(V)\otimes B\otimes B\otimes V \arrow[r] \arrow[d, "1\otimes\mathbf{m}_B\otimes 1"'] & T(V)\otimes T(V)\otimes B\otimes B \arrow[r, "\mathbf{m}_{T(V)}"] \arrow[d, "1\otimes 1\otimes\mathbf{m}_B"] & T(V)\otimes B\otimes B \arrow[d, "1\otimes\mathbf{m}_B"] \\ B\otimes V^{\otimes n} \arrow[r] & T(V)\otimes B\otimes V \arrow[r] & T(V)\otimes T(V)\otimes B \arrow[r, "\mathbf{m}_{T(V)}"] & T(V)\otimes B \end{tikzcd} .$$

Finally, left associativity of $T(V) \otimes_{\Psi^\mu} B$, i.e., commutativity of the first diagram (A.2) established in Proposition A.3 for $A = T(V)$ implies that the composition of top (resp. bottom) horizontal arrows in the above diagram is $(\Psi^\mu \otimes 1) \circ (1 \otimes \Psi^\mu)$ (resp. Ψ^μ). This finishes the proof of the lemma. □

Clearly, commutativity of (A.9) for all $n \geq 0$ is equivalent to commutativity of the second diagram (A.2) with $A = T(V)$, i.e., to the right associativity of $T(V) \otimes_{\Psi^\mu} B$.

The proposition is proved. □

For each R-linear map $\mu : B \otimes V \to T(V) \otimes B$ consider the category $\mathcal{C}_\mu$ whose objects are associative R-algebras A generated by B and V such that:

- $b \cdot v = \mathbf{m}_A \circ \mu(v \otimes b)$ for all $b \in B$, $v \in V$;
- The assignment $b \mapsto 1 \cdot b$ is a (not necessarily injective) algebra homomorphism $\iota_A : B \to A$;

morphisms are surjective algebra homomorphisms $f : A \twoheadrightarrow A'$ such that $\iota_{A'} = f \circ \iota_A$, $\iota_{A',V} = f \circ \iota_{A,V}$, where $\iota_{A'',V}$ stands for the natural (not necessarily injective) R-linear map $V \to A''$.

Clearly, $\mathcal{C}_\mu$ is a partially ordered set with a unique maximal element A_μ, i.e., for any $A \in \mathcal{C}_\mu$ one has a surjective algebra homomorphism $A_\mu \twoheadrightarrow A$. It is also clear that A_μ is the quotient of the free product $T(V) * B$ by the ideal I_μ generated by all elements of the form

$$b * v - \mathbf{j}(\mu(b \otimes v)) \tag{A.10}$$

for all $b \in B$, $v \in V$, where $\mathbf{j} : V \otimes B \hookrightarrow T(V) * B$ is a natural embedding given by $\mathbf{j}(v' \otimes b') = v' * b'$ for all $b' \in B$, $v' \in V$.

For any (associative or not) ring A denote by J_A the left ideal generated by all elements of the form $r_{a,b,c} = a(bc) - (ab)c$, $a, b, c \in A$. The identity $a \cdot r_{b,c,d} + r_{a,b,c} \cdot d = r_{ab,c,d} - r_{a,bc,d} + r_{a,b,cd}$ for $a, b, c, d \in R$ implies that J_A is also a right ideal. Then denote $\underline{A} := A/J_A$. Clearly, $\underline{A}$ is associative and is universal in the sense that for any surjective homomorphism $A \twoheadrightarrow A'$ where A' is an associative ring there is a surjective homomorphism $\underline{A} \twoheadrightarrow A'$.

Theorem A.6. *For any (unital associative) R-algebra B, an R-module V and an R-linear map $\mu : B \otimes V \to T(V) \otimes B$ satisfying (A.7) one has $\underline{T(V) \otimes_{\Psi^\mu} B} = A_\mu$. In particular, $A_\mu = T(V) \otimes_{\Psi^\mu} B$ iff $T(V) \otimes_{\Psi^\mu} B$ is associative, i.e., iff the diagram (A.8) is commutative.*

Proof. Denote $A'_\mu = \underline{T(V) \otimes_{\Psi^\mu} B}$ and by π_μ the structural homomorphism $T(V) \otimes_{\Psi^\mu} B \twoheadrightarrow A'_\mu$. Clearly, A'_μ is an R-algebra and:

- A'_μ is generated by V and B.
- $b \cdot v = \mathbf{m}_{A'_\mu} \circ \mu(v \otimes b)$ for all $b \in B$, $v \in V$.
- The assignment $b \mapsto \pi_\mu(1 \otimes b)$ is an algebra homomorphism $B \to A'_\mu$.

Therefore, A'_μ is an object of the category $\mathcal{C}_\mu$ and thus one has a canonical surjective algebra homomorphism $\pi'_\mu : A_\mu \twoheadrightarrow A'_\mu$. On the other hand, universality of A'_μ implies that there is a canonical surjective R-algebra homomorphism $A'_\mu \twoheadrightarrow A_\mu$. Thus, π'_μ is an isomorphism, hence it is the identity, i.e., $A'_\mu = A_\mu$.

The theorem is proved. $\square$

In some cases conditions (A.7) and (A.8) can be simplified. The following is immediate consequence of Proposition A.4.

Corollary A.7. *Let V be an R-module, B be an R-algebra, and $\mu : B \otimes V \to T(V) \otimes B$ be given by $\mu = \nu + \beta$, where $\nu : B \otimes V \to V \otimes B$ and $\beta : B \otimes V \to B$ are R-linear maps such that $\nu(1 \otimes v) = v \otimes 1$ for all $v \in V$. Then $A_\mu = T(V) \otimes B$ as an R-module iff the following conditions hold.*

- $\nu \circ (\mathbf{m}_B \otimes Id_V) = (Id_B \otimes \mathbf{m}_B) \circ (\nu \otimes Id_B) \circ (Id_B \otimes \nu)$ *in* $Hom_R(B \otimes B \otimes V, V \otimes B)$.
- $\beta \circ (\mathbf{m}_B \otimes Id_V) = \mathbf{m}_B \circ (Id_B \otimes \beta) + \mathbf{m}_B \circ (\beta \otimes Id_B) \circ (Id_B \otimes \nu)$ *in* $Hom_R(B \otimes B \otimes V, B)$.

We conclude with the discussion of factorizable (in the sense of Proposition A.1) algebras with $B = RW$, the *linearization* of a monoid W, so that RW is naturally an algebra over R.

Proposition A.8. *Given an R-algebra $\mathbf{H}$, suppose that it factors as $\mathbf{H} = \mathbf{D} \cdot RW$ over R, where W is a monoid (i.e., the multiplication map defines an isomorphism of R-modules $\mathbf{D} \otimes RW \widetilde{\longrightarrow} \mathbf{H}$) and both $\mathbf{D}$ and RW are subalgebras of $\mathbf{H}$. Then for any $g, h \in W$ there exists a unique R-linear map $\partial_{g,h} : \mathbf{D} \to \mathbf{D}$ such that:*

$$gx = \sum_{w \in W} \partial_{g,w}(x) w \tag{A.11}$$

for all $g \in W$, $x \in \mathbf{D}$.

Moreover, the family $\{\partial_{g,h}\}$ satisfies: $\partial_{g,h}(xy) = \sum_{w \in W} \partial_{g,w}(x)\partial_{w,h}(y)$ for all $g, h \in W$, $x, y \in \mathbf{D}$ and $\partial_{gh,w}(x) = \sum_{w_1, w_2 \in W : w_1 w_2 = w} \partial_{g,w_1}(\partial_{h,w_2}(x))$ for all $g, h, w \in W$, $x \in \mathbf{D}$.

Proof. Indeed, the existence and uniqueness of follows from the factorization of $\mathbf{H}$, i.e., that $\mathbf{H}$ is a free left $\mathbf{D}$-module with the basis W. To prove the second assertion, note that

$$gxy = \sum_{w \in W} \partial_{g,w}(x) wy = \sum_{w \in W} \partial_{g,w}(x) \left(\sum_{h \in W} \partial_{w,h}(y) h \right) = \sum_{h \in W} \left(\sum_{w \in W} \partial_{g,w}(x) \partial_{w,h}(y) \right) h$$

for $g \in W$, $x, y \in \mathbf{D}$ and

$$\begin{aligned} ghx &= \sum_{w_2 \in W} g \partial_{h,w_2}(x) w_2 = \sum_{w_2 \in W} \left(\sum_{w_1 \in W} \partial_{g,w_1}(\partial_{h,w_2}(x)) w_1 \right) w_2 \\ &= \sum_{w_1, w_2 \in W} \partial_{g,w_1}(\partial_{h,w_2}(x)) w_1 w_2 \end{aligned}$$

for $g, h \in W$, $x \in \mathbf{D}$. The proposition is proved. □

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