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Algebra

Integrable clusters ☆

Les amas intégrables

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ABSTRACT

The goal of this note is to study quantum clusters in which cluster variables (not coefficients) commute which each other. It turns out that this property is preserved by mutations if one starts with a principal quantum seed. Remarkably, this is equivalent to the celebrated sign coherence conjecture recently proved by M. Gross, P. Hacking, S. Keel, and M. Kontsevich.

RÉSUMÉ

Le but de cet article est d'étudier les amas quantiques dont les variables d'amas (mais pas les coefficients) commutent entre elles. Cette propriété est préservée par les mutations si l'on commence par une graine quantique principale. Remarquablement, elle est équivalente à la conjecture notoire sur la cohérence de signes qui a été récemment démontrée par M. Gross, P. Hacking, S. Keel et M. Kontsevich.

Version française abrégée

Soient $\tilde{B}$ une matrice entière de taille $m \times n$, $n \le m$, et $\Lambda = (\lambda_{ij})_{1 \le i,j \le m}$ une matrice rationnelle antisymétrique compatible avec $\tilde{B}$ dans le sens de [2] (voir (1)). On associe à $\tilde{B}$ une graine $\Sigma = (\mathbf{x}, \tilde{B})$, $\mathbf{x} = (x_1, \dots, x_m)$ et son algèbre amassée supérieure $\mathcal{U} = \mathcal{U}(\Sigma) \subset \mathscr{L}_m = \mathbb{Q}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$, tandis que Λ définit un crochet de Poisson $\{\cdot,\cdot\}_\Lambda$ sur $\mathscr{L}_m$. Alors $\mathcal{U}$ est une sous-algèbre de Poisson de $\mathscr{L}_m$ et l'on obtient un schéma poissonien $\mathscr{X}$, où $\mathscr{X}_\mathbb{F} = \operatorname{Spec}(\mathcal{U} \otimes_\mathbb{Q} \mathbb{F})$ pour toute extension $\mathbb{F}$ de $\mathbb{Q}$.

La graine $\Sigma = (\mathbf{x}, \tilde{B})$ est appelée *intégrable* si, pour une Λ compatible avec $\tilde{B}$, $\{x_i, x_j\}_\Lambda = 0$ pour tous $1 \le i, j \le n$. Cette définition est justifiée par l'observation que, pour $m = 2n$, Σ intégrable et Λ inversible, le triplet $(\mathcal{U}, \{\cdot,\cdot\}_\Lambda, H)$, où $H \in \mathbb{Q}[x_1, \dots, x_n] \setminus \{0\}$, est un système intégrable avec l'hamiltonien H ; l'application naturelle $\pi_\Sigma : \mathscr{X} \to \mathbb{A}^n$ est donc un fibré lagrangien (cf. [1]).

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D'après le Lemme 2.4, une graine *principale* (voir Section 2) est toujours intégrable. On dit que Σ est *complétement intégrable* si Σ et toutes ses mutations sont intégrables.

Théorème 1. *Chaque graine principale est complétement intégrable.*

Corollaire 2. *Soit $\Sigma' = (\mathbf{y}, \tilde{B}')$ une graine équivalente par mutation à une graine principale Σ. Alors le plongement naturel $\Bbbk[y_1, \ldots, y_n] \subset \mathcal{U}(\Sigma') = \mathcal{U}$ définit une fibré lagrangien $\pi_{\Sigma'} : \mathscr{X} \to \mathbb{A}^n$.*

Problème 3. Soient Σ, Σ' des graines intégrables équivalentes par mutation. Décrire l'intersection $\pi_\Sigma^{-1}(c) \cap \pi_{\Sigma'}^{-1}(c')$ pour des points génériques $c, c' \in \mathbb{A}^n$ et déterminer pour quelles Σ, Σ' elle est transversale.

Semblablement, on dit qu'une graine quantique $\Sigma = (\mathbf{X}, \tilde{B})$, où $\mathbf{X} = (X_1, \ldots, X_m)$ (voir Section 2), est *intégrable* si $X_i X_j = X_j X_i$ pour tous $1 \le i, j \le n$ et *complétement intégrable* si Σ et toutes ses mutations sont intégrables.

Théorème 4. *Une graine quantique principale intégrable est complètement intégrable.*

En effet, on peut convertir toute graine (quantique où classique) en graine principale intégrable en dupliquant le tore (quantique) ambiant (voir [3, Section 3] et Lemme 2.6).

Un *système intégrable quantique généralisé* (cf. [8]) est un couple $(\mathcal{U}, \iota)$ où $\mathcal{U}$ est une $\Bbbk$-algèbre de dimension de Gelfand–Kirillov $2n$ et $\iota : \Bbbk[x_1, \ldots, x_n] \hookrightarrow \mathcal{U}$ est un plongement d'algèbres tel que $\iota(\Bbbk[x_1, \ldots, x_n])$ est une sous-algèbre commutative maximale de $\mathcal{U}$. Cela nous permet d'introduire un analogue quantique d'un fibré lagrangien dont la fibre lagrangienne quantique sur un idéal maximal $\mathfrak{m}$ de $\Bbbk[x_1, \ldots, x_n]$ est le $\mathcal{U}$-module à gauche $\mathcal{U}_{\iota,\mathfrak{m}} := \mathcal{U}/\mathcal{U} \cdot \iota(\mathfrak{m})$. Alors, l'intersection (quantique) de $\mathcal{U}_{\iota,\mathfrak{m}}$ et $\mathcal{U}_{\iota',\mathfrak{m}'}$ est le $\mathcal{U}$-module à gauche $\mathcal{U}/\mathcal{U} \cdot (\iota(\mathfrak{m}) + \iota'(\mathfrak{m}'))$.

Soient $\Bbbk$ un corps commutatif contenant $\mathbb{Q}(q^{\frac{1}{2}})$ et $\Sigma = (\mathbf{X}, \tilde{B})$ une graine quantique. D'après [2], on note $\mathcal{U} = \mathcal{U}(\Sigma)$ *l'algèbre amassée quantique supérieure* correspondante (voir Section 2).

Corollaire 5. *Soit $\Sigma' = (\mathbf{Y}, \tilde{B}')$ une graine quantique équivalente par mutation à une graine quantique principale intégrable Σ. Alors $(\mathcal{U}, \iota)$, où $\iota : \Bbbk[Y_1, \ldots, Y_n] \hookrightarrow \mathcal{U}(\Sigma') = \mathcal{U}$ est le plongement naturel, est un système intégrable quantique généralisé.*

1. Main results

Let $\tilde{B}$ be an integer $m \times n$ matrix, $n \le m$, and let $\Lambda = (\lambda_{ij})_{1 \le i, j \le m}$ be a rational skew-symmetric $m \times m$-matrix compatible with $\tilde{B}$ in the sense of [2, Definition 3.1], that is

$$\tilde{B}^T \Lambda = \begin{pmatrix} D & \mathbf{0} \end{pmatrix} \tag{1}$$

where D is a rational invertible diagonal $n \times n$-matrix. This defines, on the one hand, a seed $\Sigma = (\mathbf{x}, \tilde{B})$, $\mathbf{x} = (x_1, \ldots, x_m)$ and the upper cluster algebra $\mathcal{U} = \mathcal{U}(\Sigma) \subset \mathscr{L}_m = \mathbb{Q}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ and, on the other hand, the Poisson algebra structure on $\mathscr{L}_m$ via $\{x_i, x_j\}_\Lambda = \lambda_{ij} x_i x_j$, $1 \le i, j \le m$ so that $\mathcal{U}$ is a Poisson subalgebra of $\mathscr{L}_m$. This in turn defines a Poisson scheme $\mathscr{X}$ such that, for any field extension $\mathbb{F}$ of $\mathbb{Q}$, $\mathscr{X}_\mathbb{F} = \mathrm{Spec}(\mathcal{U} \otimes_\mathbb{Q} \mathbb{F})$.

We say that the seed $\Sigma = (\mathbf{x}, \tilde{B})$ is *integrable* if $\{x_i, x_j\}_\Lambda = 0$ for all $1 \le i, j \le n$ for some Λ compatible with $\tilde{B}$. It is easy to show (Lemma 2.4) that if Σ is principal (see Section 2) then it is integrable and the matrix Λ is uniquely determined by $\tilde{B}$ and D. We say that Σ is *completely integrable* if Σ and all its mutations are integrable. This definition is justified by the following observation. Let $m = 2n$ and assume that Σ is integrable and that Λ is invertible, i.e. $\{\cdot, \cdot\}_\Lambda$ is a symplectic bracket. Then for any $H \in \mathbb{Q}[x_1, \ldots, x_n] \setminus \{0\}$, the triple $(\mathcal{U}, \{\cdot, \cdot\}_\Lambda, H)$ is an integrable system with Hamiltonian H; the natural map $\pi_\Sigma : \mathscr{X} \to \mathbb{A}^n$ is a Lagrangian fibration (see [1]).

Theorem 1.1. *Every principal seed is completely integrable.*

Corollary 1.2. *For any seed $\Sigma' = (\mathbf{y}, \tilde{B}')$, $\mathbf{y} = (y_1, \ldots, y_m)$, mutation equivalent to a principal integrable seed Σ, the natural inclusion $\Bbbk[y_1, \ldots, y_n] \subset \mathcal{U}(\Sigma') = \mathcal{U}$ defines a Lagrangian fibration $\pi_{\Sigma'} : \mathscr{X} \to \mathbb{A}^n$.*

This gives rise to the following natural problem.

Problem 1.3. Given mutation equivalent integrable seeds Σ, Σ', describe the intersection $\pi_\Sigma^{-1}(c) \cap \pi_{\Sigma'}^{-1}(c')$ for generic points $c, c' \in \mathbb{A}^n$ and determine for which Σ, Σ' this intersection is transversal.

It turns out that classical results carry over verbatim to the quantum case. We say that a quantum seed $\Sigma = (\mathbf{X}, \tilde{B})$, where $\mathbf{X} = (X_1, \ldots, X_m)$ (see Section 2), is *integrable* if $X_i X_j = X_j X_i$ for all $1 \le i, j \le n$. Then, similarly to the classical case, we say that a quantum seed Σ is *completely integrable* if Σ and all its mutations are integrable.

Theorem 1.4. *Every integrable principal quantum seed is completely integrable.*

By Lemma 2.6 any (quantum or classical) seed can be converted into a principal integrable one merely by duplicating the ambient (quantum) torus as in [3, Section 3].

A *generalized quantum integrable system* (cf. [8]) is a pair $(\mathcal{U}, \iota)$ where $\mathcal{U}$ is a $\Bbbk$-algebra of Gelfand–Kirillov dimension $2n$ and $\iota : \Bbbk[x_1, \dots, x_n] \hookrightarrow \mathcal{U}$ is an embedding of algebras such that $\iota(\Bbbk[x_1, \dots, x_n])$ is a maximal commutative subalgebra of $\mathcal{U}$. This defines a quantum analogue of Lagrangian fibration, whose *quantum Lagrange fiber* over a maximal ideal $\mathfrak{m}$ of $\Bbbk[x_1, \dots, x_n]$ is the left $\mathcal{U}$-module $\mathcal{U}_{\iota,\mathfrak{m}} := \mathcal{U}/\mathcal{U} \cdot \iota(\mathfrak{m})$. Then the (quantum) intersection of $\mathcal{U}_{\iota,\mathfrak{m}}$ and $\mathcal{U}_{\iota',\mathfrak{m}'}$ is the left $\mathcal{U}$-module $\mathcal{U}/\mathcal{U} \cdot (\iota(\mathfrak{m}) + \iota'(\mathfrak{m}'))$.

Following [2], given a field $\Bbbk$ containing $\mathbb{Q}(q^{\frac{1}{2}})$ and a *quantum seed* $\Sigma = (\mathbf{X}, \tilde{B})$, we denote by $\mathcal{U} = \mathcal{U}(\Sigma)$ its *upper quantum cluster algebra* (see Section 2 for the details).

Corollary 1.5. *For any quantum seed $\Sigma' = (\mathbf{Y}, \tilde{B}')$ mutation equivalent to a given principal integrable quantum seed Σ, the natural inclusion $\iota : \Bbbk[Y_1, \dots, Y_n] \hookrightarrow \mathcal{U}(\Sigma') = \mathcal{U}$ defines a generalized quantum integrable system $(\mathcal{U}, \iota)$.*

2. Notation and proofs

We will only prove quantum results, since their classical counterparts follow by specializing q to 1.

Let $\Lambda \in \mathrm{Mat}_{m\times m}(\mathbb{Z})$ be compatible with $\tilde{B}$ in the sense of (1) where $D \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ and has positive diagonal entries. Following [2], we associate with the pair $(\Lambda, \tilde{B})$ a *quantum seed* $\Sigma = (\mathbf{X}, \tilde{B})$, $\mathbf{X} = (X_1, \dots, X_m) \in (\mathcal{F}^\times)^m$ where $\mathcal{F}$ is a skew field, such that the subalgebra $\mathscr{L}_{\mathbf{X}}$ of $\mathcal{F}$ generated by $\mathbf{X}$ over some central subfield $\Bbbk$ of $\mathcal{F}$ containing $\mathbb{Q}(q^{\frac{1}{2}})$ has presentation

$$X_i X_j = q^{\lambda_{ij}} X_j X_i, \qquad 1 \le i < j \le m.$$

Given $1 \le j \le n$, we define $\mu_j(\Sigma) = (\mathbf{X}', \mu_j(\tilde{B}))$ where $\mu_j(\tilde{B})$ is the Fomin–Zelevinsky mutation of $\tilde{B}$ from [4] and $\mathbf{X}'$ is obtained from $\mathbf{X}$ by replacing X_j with

$$X'_j = X^{[b_j]_+ - e_j} + X^{[-b_j]_+ - e_j}, \tag{2}$$

where b_j is the jth column of $\tilde{B}$, for each $a = (a_1, \dots, a_m) \in \mathbb{Z}^m$ we set

$$[a]_+ = (\max(0, a_1), \dots, \max(0, a_m)), \qquad X^a = q^{\frac{1}{2}\sum_{1\le i<j\le m} \lambda_{ji} a_i a_j} X_1^{a_1} \cdots X_m^{a_m}$$

and $\{e_i\}_{1\le i\le m}$ is the standard basis of $\mathbb{Z}^m$. After [2, Section 2], $\mu_j(\Sigma)$ is also a quantum seed and we refer to it as the jth mutation of Σ. A quantum seed Σ' is *mutation equivalent* to Σ if it can be obtained from Σ by a sequence of mutations.

Following [7], we say that an $m \times n$-matrix $\tilde{B}$ is *sign-coherent* if for every $1 \le j \le n$ there exists $\epsilon_j \in \{-1, 1\}$ such that $\epsilon_j b_{ij} \ge 0$ for all $n+1 \le i \le m$. We say that $\tilde{B}$ is *totally sign-coherent* if all matrices mutation equivalent to $\tilde{B}$ are sign-coherent.

Proposition 2.1. *Suppose $\Lambda \in \mathrm{Mat}_{m\times m}(\mathbb{Z})$ and $\tilde{B} \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ are compatible, $\tilde{B}$ is totally sign-coherent and the corresponding quantum seed Σ is integrable. Then Σ is completely integrable.*

Proof. We need the following lemma.

Lemma 2.2. *Suppose that $\Sigma = (\mathbf{X}, \tilde{B})$ is integrable and $\tilde{B}$ is sign-coherent. Then $\mu_j(\Sigma)$, $1 \le j \le n$, is integrable.*

Proof. Since $\tilde{B}$ is sign-coherent, either $X_j X^{[b_j]_+ - e_j}$ or $X_j X^{[-b_j]_+ - e_j}$ is contained in $\Bbbk[X_1, \dots, X_n]$. Since for every $1 \le i \ne j \le n$, we have $X_i X^{[b_j]_+ - e_j} = q_{ij} X^{[b_j]_+ - e_j} X_i$ and $X_i X^{[-b_j]_+ - e_j} = q_{ij} X^{[-b_j]_+ - e_j} X_i$ for some $q_{ij} \in \Bbbk^\times$, it follows that $q_{ij} = 1$. Then by (2) we have $X_i X'_j = X'_j X_i$ for all $1 \le i \ne j \le n$. □

We complete the proof by induction on the number of mutations applied to the initial seed Σ. If $\Sigma' = \mu_{j_1} \cdots \mu_{j_k}(\Sigma) = \mu_{j_1}(\Sigma'')$ where $\Sigma'' = (\mathbf{X}'', \tilde{B}'') = \mu_{j_2} \cdots \mu_{j_k}(\Sigma)$ is integrable by the induction hypothesis and $\tilde{B}''$ is sign-coherent by assumption on $\tilde{B}$. It remains to apply the lemma. □

Proof of Theorem 1.4. Recall from [5, Remark 3.2] that $\tilde{B}$ is called principal if $\tilde{B} = \binom{B}{I_n}$ where $B \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ and DB is skew symmetric for some $D \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ diagonal with positive diagonal entries. The following result was initially conjectured in [5, Conjecture 5.4 and Proposition 5.6(iii)] (see also [7, (1.8)]).

Lemma 2.3. *(See [6, Corollary 5.5]) Any principal $\tilde{B}$ is totally sign-coherent.*

Theorem 1.4 is immediate from Proposition 2.1 and the above lemma. □

Proof of Corollary 1.5. We need the following obvious classification of (quantum) integrable seeds with $m = 2n$.

Lemma 2.4. *Let*

$$\Lambda = \begin{pmatrix} \mathbf{0} & \Lambda_1 \\ -\Lambda_1^T & \Lambda_2 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} B \\ C \end{pmatrix} \tag{3}$$

with $B, C, \Lambda_1, \Lambda_2 \in \mathrm{Mat}_{n\times n}(\mathbb{Q})$ *and* $\Lambda_2^T = -\Lambda_2$. *Then* (1) *holds for some invertible diagonal* $D \in \mathrm{Mat}_{n\times n}(\mathbb{Q})$ *if and only if*

$$\det C \neq 0, \quad (DB)^T = -DB, \quad \Lambda_1 = -DC^{-1}, \quad \Lambda_2 = -(C^{-1})^T DBC^{-1}.$$

Furthermore, following [2], with each quantum seed Σ one associates the *quantum upper cluster algebra* $\mathcal{U}(\Sigma) = \bigcap_{1\le i\le n} \mathcal{U}_i$ where $\mathcal{U}_i$ is the subalgebra of $\mathscr{L}_{\mathbf{X}}$ generated by $\mathbf{X}$ and X_i'. As shown in [2, Theorem 5.1], $\mathcal{U}(\Sigma) = \mathcal{U}(\mu_j(\Sigma))$ for all $1 \le j \le n$. We need the following lemma.

Lemma 2.5. *Let* $\Sigma = (\mathbf{X}, \tilde{B})$ *be a quantum integrable seed with* $m = 2n$. *Then*

(a) $\Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ *is a maximal commutative subalgebra of* $\mathscr{L}_{\mathbf{X}}$*;*
(b) $\Bbbk[X_1, \dots, X_n]$ *is a maximal commutative subalgebra of* $\mathcal{U}(\Sigma)$.

Proof. To prove (a), note that for each $1 \le i \le n$ the centralizer $\mathscr{C}_i$ of X_i in $\mathscr{L}_{\mathbf{X}}$ is the $\Bbbk$-linear span of $\{X^a : \sum_{1\le j\le m} \lambda_{ij} a_j = 0\}$. Since Λ is as in (3) with $\det \Lambda_1 \neq 0$, $\bigcap_{1\le i\le n} \mathscr{C}_i$ is spanned by all monomials X^a with $a_j = 0$, $j > n$. This implies that $\Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ is a maximal commutative subalgebra of $\mathscr{L}_{\mathbf{X}}$.

To prove (b), denote $\mathcal{U}_i' = \mathcal{U}_i \cap \Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. It is easy to see that

$$\mathcal{U}_i' = \Bbbk[X_1^{\pm 1}, \dots, X_i, \dots, X_n^{\pm 1}].$$

Since $\mathcal{U} \cap \Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}] = \bigcap_{1\le i\le n} \mathcal{U}_i' = \Bbbk[X_1, \dots, X_n]$, part (b) is now immediate from part (a). □

The Corollary 1.5 is now immediate because each quantum seed mutation equivalent to a given principal integrable one is automatically integrable by Theorem 1.4. □

We conclude by showing that every (quantum) seed can be converted into a principal completely integrable one. Recall that in [3, Section 3], with every seed $\Sigma = (\mathbf{X}, \tilde{B})$ in $\mathscr{L}_{\mathbf{X}}$ one associates a seed $\Sigma^\bullet = (\mathbf{X}^\bullet, \tilde{B}^\bullet)$ in the duplicated quantum torus $\mathscr{L}_{\mathbf{X}}^{(2)} = \bigoplus_{e,e'\in\mathbb{Z}^m} \Bbbk X^{(e,e')}$ with $\Lambda^{(2)} = \left(\begin{smallmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & -\Lambda \end{smallmatrix}\right)$, as follows

$$X_i^\bullet = X^{(e_i, e_i)}, \qquad X_{i+n}^\bullet = X^{(b_i^{>n}, -b_i^{\le n})}, \qquad 1 \le i \le n, \qquad \tilde{B}^\bullet = \begin{pmatrix} B \\ I_n \end{pmatrix}$$

where we abbreviate $a^{>n} = \sum_{n<i\le m} a e_i$, $a^{\le n} = \sum_{1\le i\le n} a_i e_i$ for $a = (a_1, \dots, a_m) \in \mathbb{Z}^m$. By definition, $\mathscr{L}_{\mathbf{X}}$ identifies with a subalgebra of $\mathscr{L}_{\mathbf{X}}^{(2)}$ via $X^e \mapsto X^{(e,0)}$, $e \in \mathbb{Z}^m$. Then $\mathscr{L}_{\mathbf{X}}^{(2)} = \mathscr{L}_{\mathbf{X}} \cdot \mathscr{C}_{\mathbf{X}}$ and hence $\mathcal{U}(\Sigma) \cdot \mathscr{C}_{\mathbf{X}} = \mathcal{U}(\Sigma^\bullet) \cdot \mathscr{C}_{\mathbf{X}}$ as subalgebras of $\mathscr{L}_{\mathbf{X}}^{(2)}$, where $\mathscr{C}_{\mathbf{X}}$ is the subalgebra of $\mathscr{L}_{\mathbf{X}}^{(2)}$ generated by the $X^{(e_i,0)}$, $n < i \le m$ and by the $X^{(0,e_j)}$, $1 \le j \le m$.

The following is immediate from [3, (3.19)–(3.22) and Lemma 3.4].

Lemma 2.6. *The seed* $\Sigma^\bullet$ *is principal and integrable, hence completely integrable.*

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