A short proof of Kontsevich’s cluster conjecture

Une courte démonstration d’une conjoncture de Kontsevich

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We give an elementary proof of the Kontsevich conjecture that asserts that the iterations of the noncommutative rational map

\[ K_r : (x, y) \mapsto (xyx^{-1}, (1 + yr)x^{-1}) \]

are given by noncommutative Laurent polynomials.

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The aim of this Note is to give an elementary proof of the following Kontsevich conjecture.

Recall that the Kontsevich map \( K_r, r \in \mathbb{Z}_{>0} \), is the following (birational) automorphism of a noncommutative plane (i.e., of the free skew-field with generators \( x \) and \( y \)):

\[ K_r : (x, y) \mapsto (xyx^{-1}, (1 + y^r)x^{-1}) \]

**Conjecture 1 (M. Kontsevich).** For any \( r_1, r_2 \in \mathbb{Z}_{>0} \) all iterations \( \cdots K_{r_1}K_{r_2}(x, y), k \geq 1 \), are given by noncommutative Laurent polynomials in \( x \) and \( y \).

The conjecture extends the rank 2 Laurent phenomenon established previously for cluster algebras and their quantum versions (see [4,6,5,2,1]) to the “fully noncommutative” setting.

The Kontsevich conjecture was first proved for \( r_1 = r_2 = 2 \) by A. Usnich in [8] and was later settled by A. Usnich in [9] in greater generality when \( r_1 = r_2 = r \) (with \( 1 + y^r \) replaced by any monic palindromic polynomial \( H(y) \)) by means of derived categories. Independently, Conjecture 1 was verified for \((r_1, r_2) \in \{(2, 2), (4, 1), (1, 4)\} \) in [3] along with the positivity conjecture: for \((r_1, r_2) \in \{(2, 2), (4, 1), (1, 4)\} \) all noncommutative Laurent polynomials in question have nonnegative integer coefficients.

Our goal is to give a short proof of Conjecture 1 by adjusting the ideas of [4,6,5,2,1] to the noncommutative rank 2 setting. Note, however, that there is currently no theory of “fully noncommutative” cluster algebras, as well as a higher rank
Kontsevich conjecture. We are planning to construct examples of such higher rank noncommutative clusters elsewhere and thus provide some higher rank analogues of the Kontsevich conjecture.

To present our proof of Conjecture 1, we need some notation. Denote
\[(x_k, y_k) := \cdots \cdot K_{r_1} K_{r_2} K_{r_1} (x, y)\]
and denote \(z := [x, y] = xy^{-1}y^{-1}.\) Then it is easy to see by induction that \([x_k, y_k] = [x, y] = z\) for all \(k.\) This taken together with the recursion \(x_{k+1} = x_k y_k z_k^{-1}\) and \(y_{k+1} = (1 + y_k^2) x_k^{-1},\)
\[
r_k = \begin{cases} r_1 & \text{if } k \text{ is odd,} \\ r_2 & \text{if } k \text{ is even.} \end{cases}
\]
gives the following three recursions (they first appeared in [3, Section 2.2]):
\[x_{k+1} = z y_k,\]
\[y_{k+1} z y_{k-1} = 1 + y_k^2,\]
\[y_{k+1}^2 z y_k = y_k y_{k+1}.\]

Let \(\mathcal{F}_2 = \mathbb{Q}(y_1^{\pm 1}, y_2^{\pm 1})\) be the group algebra of the free group in 2 generators. It was proved by A.I. Malcev (see, e.g., [7, Section 8.7]) that \(\mathcal{F}_2\) is a divisible algebra, i.e., it embeds in a division ring (we denote the smallest one by \(\text{Frac}(\mathcal{F}_2)\)).

Define elements \(y_k \in \text{Frac}(\mathcal{F}_2), k \in \mathbb{Z} \setminus \{1, 2\},\) recursively by Eq. (2) where \(z := [y_2^{-1}, y_1] = y_2^{-1} y_1 y_2 y_1^{-1}.\)

Note that \(y_0, y_3 \in \mathcal{F}\) and let \(\mathcal{A} = \mathcal{A}(r_1, r_2)\) be the subalgebra of \(\mathcal{F}\) generated by \(y_0, y_1, y_2, y_3, z, z^{-1}.\) We will refer to \(\mathcal{A}\) as a (purely) noncommutative cluster algebra of type \((r_1, r_2).\)

**Lemma 2.** The elements \(y_k \in \text{Frac}(\mathcal{F}_2)\) satisfy for all \(k \in \mathbb{Z}\) Eq. (3).

**Proof.** Indeed, (3) is obvious for \(k = 1.\) Let us prove it for \(k \geq 1\) by induction. We will use the inductive hypothesis in the form \(y_k y_{k+1} z_k^{-1} = y_{k-1} y_k.\) Indeed, since \(y_{k+1} z_k y_{k-1} = (1 + y_k^2) y_k y_{k-1}^{-1} z_k,\) we obtain
\[
y_{k+1} y_k y_{k+1} - y_k y_{k+1} z_k^{-1} = (1 + y_k^2) y_k y_{k-1}^{-1} z_k^{-1} - y_k (1 + y_k^2) y_k y_{k-1}^{-1} z_k^{-1} = (1 + y_k^2) y_k y_{k-1}^{-1} z_k^{-1} - y_k (1 + y_k^2) y_k y_{k-1}^{-1} z_k^{-1} = (1 + y_k^2) y_k y_{k-1}^{-1} z_k^{-1} = 0\]
by the inductive hypothesis. The relation (3) for \(k \leq 0\) also follows. □

Thus, based on the above discussion, Conjecture 1 directly follows from our main result.

**Main Theorem 3.** Each \(y_k\) belongs to \(\mathcal{A},\) e.g., \(y_k\) is a noncommutative Laurent polynomial in \(y_1, y_2.\)

**Proof.** Denote by \(\mathcal{A}_k = \mathcal{A}(r_1, r_2)\) the subalgebra of \(\mathcal{F}_2\) generated by \(y_k, y_{k+1}, y_{k+2}, y_{k+3}, z^{\pm 1}.\) It suffices to prove the following result (which is a noncommutative version of [2, Formula (4.12)] and [1, Lemma 5.8]).

**Theorem 4.** \(\mathcal{A}_k = \mathcal{A} \text{ for all } k \in \mathbb{Z}.\)

**Proof.** Since \(\mathcal{A} = \mathcal{A}_0,\) it suffices to prove that \(\mathcal{A}_k = \mathcal{A}_{k+1}\) for \(k \in \mathbb{Z},\) i.e., that for all \(k \in \mathbb{Z}\) one has
\[y_{k+4} \in \mathcal{A}_k, \quad y_k \in \mathcal{A}_{k+1}.\]

**Proposition 5.** For each \(n \in \mathbb{Z}\) one has: \(y_{k+4} z = z y_k (y_{k+3} z)^{r_1} - \sum_{j=0}^{r_1} (y_{k+1} z y_{k+2} z)^{r_1-1} (y_{k+3} z)^{r_1},\)

**Proof.** For simplicity (and without loss of generality) we assume that \(k = 0.\) We start with the following technical result.

**Lemma 6.** For each \(m \geq 0\) we have: \(y_1^m (y_2 z)^m = 1 + \sum_{k=0}^{m-1} y_1^k (y_2 z)^k (y_3 z)^k.\)

**Proof.** We proceed by induction on \(m.\) For \(m = 0\) the assertion is clear. Assume that \(m > 0\) and it holds for \(m - 1.\) Let us prove it for \(m.\) Note that (2) and (3) imply that
\[y_{k+1} z = 1 + (y_k z)^k.\]
Indeed, using (5), we obtain
\[ y_1^m(y_3z)^n = y_1^{m-1}(y_1y_3z)(y_3z)^{m-1} = y_1^{m-1}(1 + (y_2z)^2)(y_3z)^{m-1} = y_1^{m-1}(y_2z)^2(y_3z)^{m-1} + y_1^{m-1}(y_3z)^{m-1} \]
\[ = y_1^{m-1}(y_2z)^2(y_3z)^{m-1} + 1 + \sum_{k=0}^{m-2} y_1^k(y_2z)^2(y_3z)^k = 1 + \sum_{k=0}^{m-1} y_1^k(y_2z)^2(y_3z)^k. \]

The lemma is proved. \(\Box\)

Furthermore, compute:
\[ y_4z = y_2^{-2}(y_3z)^{r_1} + 1 = y_2^{-2}(y_3z)^{r_1} + y_2^{-1} = (zy_0 - y_2^{-1}(y_1))z(y_3z)^{r_1} + y_2^{-1} \]
\[ = zy_0(y_3z)^{r_1} - y_2^{-1}(y_1y_3z)(y_3z)^{r_1-1} - 1 = zy_0(y_3z)^{r_1} - y_2^{-1}(y_1^{r_1-1}(1 + (y_2z)^2)(y_3z)^{r_1-1} - 1). \]

We have:
\[ y_1^{r_1-1}(1 + (y_2z)^2)(y_3z)^{r_1-1} - 1 = y_1^{r_1-1}(y_2z)^2(y_3z)^{r_1-1} + y_1^{r_1-1}(y_3z)^{r_1-1} - 1. \]

Using Lemma 6 and taking into account that \( y_1^m y_2 = y_2(y_1)^m \) for \( m > 0 \), we obtain:
\[ y_1^{r_1-1}(1 + (y_2z)^2)(y_3z)^{r_1-1} - 1 = y_1^{r_1-1}(y_2z)^2(y_3z)^{r_1-1} + \sum_{k=0}^{r_1-2} y_1^k(y_2z)^2(y_3z)^k = \sum_{k=0}^{r_1-1} y_1^k(y_2z)^2(y_3z)^k \]
\[ = y_2 \sum_{k=0}^{r_1-1} (zy_1)^kz(y_2z)^{r_1-1} - (y_3z)^k. \]

Therefore, \( y_4z = zy_0(y_3z)^{r_1} - \sum_{k=0}^{r_1-1} (zy_1)^kz(y_2z)^{r_1-1} - (y_3z)^k \). This proves Proposition 5. \(\Box\)

Proposition 5 gives us the first inclusion (4). Prove second inclusion (4) now. We need the following obvious fact. Let \( \sigma \) be the anti-automorphism of \( \mathcal{J}_2 \) given by: \( \sigma(y_1) = y_2, \sigma(y_1) = y_2 \) (so that \( \sigma(z) = z \).

**Lemma 7.** \( \sigma(y_k) = y_3-k \) for \( k \in \mathbb{Z} \), in particular, \( \sigma(A_k(r_1, r_2)) = A_{-k}(r_2, r_1) \) for \( k \in \mathbb{Z} \).

This immediately implies the second inclusion (4): \( y_{1-k} \in A_{-k}, k \in \mathbb{Z}, \) and Theorem 4 is proved. \(\Box\)

Therefore, Theorem 3 is proved. \(\Box\)

And, ultimately, Conjecture 1 is proved.

**Example 8.** Let \( r_1 = r_2 = 2 \). We have: \( y_{k+1}zy_{k-1} = y_k^2 + 1, \ y_{k-1}y_{k+1}z = y_kzy_kz + 1 \) for all \( k \in \mathbb{Z} \). This implies:
\[ y_4z = y_2^{-1}(y_3zy_3z + 1) = (zy_0 - y_2^{-1}y_1^2)y_3(y_3z) + y_2^{-1} \]
\[ = zy_0y_3zy_3z - y_2^{-1}(y_1(y_1y_3z)y_3z - 1). \]

Note that \( y_1(y_1y_3z)y_3z - 1 = y_1(y_2zy_2 + 1)y_3z - 1 = y_1y_2zy_2z + y_1y_3z - 1 = y_2zy_1zy_2z + y_3z(y_3z)^2 \). Therefore,
\[ y_4z = zy_0(y_3z)^2 - (zy_1zy_2z + zy_2z). \]

The noncommutative cluster algebra \( A = A(r_1, r_2) \) has a number symmetries in addition to the anti-involution \( \sigma : A(r_1, r_2) \to A(r_2, r_1) \) from Lemma 7: the translation \( y_k \mapsto y_{k+1}, \) \( k \in \mathbb{Z} \), defines an isomorphism \( \tau : A(r_1, r_2) \cong A(r_2, r_1) \), which is an automorphism when \( r_1 = r_2 \).

We conclude with a brief discussion of the presentation of \( A \).

**Proposition 9.** The generators \( y_0, y_1, y_2, y_3, z^\pm \) of \( A \) satisfy for \( i = 0, 1, 2, j = 1, 2 \):
\[ y_1y_{i+1} = y_{i+1}zy_1, \quad y_{j+1}zy_{j-1} = y_0^{y_j} + 1, \]
\[ y_{j-1}y_{j+1}z = (y_2z)^j + 1, \quad y_3zy_0 - zy_0y_3z = y_2^{y_j-1}(y_2z)^{j+1} - z(y_1z)^{j-1}(y_2z)^{j+1}. \]
Proof. Only the last relation needs to be proved (the first three relations are (3), (2), and (5) respectively). Indeed, using the available relations in $F_2$, we obtain:

$$y_0 y_3 z = ((1 + (y_1 z)^r_1) z^{-1} y_2^{-1})(y_1^{-1} (1 + (y_2 z)^r_2))$$
$$= (1 + (y_1 z)^r_1) z^{-1} y_1^{-1} z^{-1} y_2^{-1} (1 + (y_2 z)^r_2) = h_{r_1}(y_1 z) h_{r_2}(y_2 z),$$

where $h_r(y) = y^{-1} + y^{-r-1}$. Similarly,

$$y_3 y_0 z = ((1 + y_2^2) y_1^{-1})(z^{-1} y_2^{-1} (1 + y_1^r)) = (1 + y_2^2) y_2^{-1} y_1^{-1} (1 + y_1^r) = h_{r_2}(y_2) h_{r_1}(y_1).$$

Taking into account that $y_1 y_2 y_1^{-1} = y_2 z$ and $y_2^{-1} y_1 y_2 = z y_1$, we obtain:

$$y_3 y_0 = y_2^{-1} y_1^{-1} + h_{r_2}(y_2) y_4^{-1} + y_2^{-1} y_4^{-1} + y_2^{-1} y_1^{-1}$$
$$= y_2^{-1} y_1^{-1} + (zy_1)^{r_1-1} y_2^{-1} + y_1^{-1} h_{r_2}(y_2 z) + y_1^{-1} y_2^{-1} y_2^{-1}$$
$$= y_2^{-1} y_1^{-1} + z(y_1 z)^{r_1-1} (y_2 z)^{-1} + z(y_1 z)^{-1} h_{r_2}(y_2 z) + z(y_1 z)^{-1} (y_2 z)^{-1}$$
$$= y_2^{-1} y_1^{-1} + z(y_1 z)^{r_1-1} (y_2 z)^{2-1}.$$

The proposition is proved.

We expect that the relations in Proposition 9 are defining.

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References