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Quantum Folding

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In the present paper, we introduce a quantum analog of the classical folding of a simply laced Lie algebra $\mathfrak{g}$ to the nonsimply laced algebra $\mathfrak{g}^\sigma$ along a Dynkin diagram automorphism σ of $\mathfrak{g}$. For each quantum folding, we replace $\mathfrak{g}^\sigma$ by its Langlands dual $\mathfrak{g}^{\sigma\vee}$ and construct a nilpotent Lie algebra $\mathfrak{n}$ which interpolates between the nilpotnent parts of $\mathfrak{g}$ and $\mathfrak{g}^{\sigma\vee}$, together with its quantized enveloping algebra $U_q(\mathfrak{n})$ and a Poisson structure on $S(\mathfrak{n})$. Remarkably, for the pair $(\mathfrak{g}, \mathfrak{g}^{\sigma\vee}) = (\mathfrak{so}_{2n+2}, \mathfrak{sp}_{2n})$, the algebra $U_q(\mathfrak{n})$ admits an action of the Artin braid group Br_n and contains a new algebra of quantum $n \times n$ matrices with an adjoint action of $U_q(\mathfrak{sl}_n)$, which generalizes the algebras constructed by Goodearl and Yakimov in ["Poisson structures on affine spaces and flag varieties. II." *Transactions of the American Mathematical Society* 361, no. 11 (2009): 5753–780.]. The hardest case of quantum folding is, quite expectably, the pair $(\mathfrak{so}_8, G_2)$ for which the PBW presentation of $U_q(\mathfrak{n})$ and the corresponding Poisson bracket on $S(\mathfrak{n})$ contains more than 700 terms each.

1 Introduction and Main Results

This work is motivated by the classical "folding" result for a simply laced semisimple Lie algebra $\mathfrak{g}$ and an *admissible* diagram automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ (in the sense of

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[17, §12.1.1], see Section 2.1)

$$\text{The fixed Lie algebra } \mathfrak{g}^\sigma = \{x \in \mathfrak{g} : \sigma(x) = x\} \text{ is also semisimple.} \tag{1.1}$$

Our goal is to find a quantum version of this result. Note, however, that the embedding of associative algebras $U(\mathfrak{g}^\sigma) \hookrightarrow U(\mathfrak{g})^\sigma \subset U(\mathfrak{g})$ induced by the inclusion $\mathfrak{g}^\sigma \hookrightarrow \mathfrak{g}$ does not admit a naive quantum deformation (see Appendix A). On the other hand, there exists a "crystal" version of the desired homomorphism. Namely, let $B_\infty(\mathfrak{g})$ be the famous Kashiwara crystal introduced in [15]. The following result was proved by Lusztig in [17, Section 14.4].

Proposition 1.1. Let σ be an admissible diagram automorphism of $\mathfrak{g}$. Then σ acts on $B_\infty(\mathfrak{g})$ and the fixed point set $B_\infty(\mathfrak{g})^\sigma$ is naturally isomorphic to $B_\infty(\mathfrak{g}^{\sigma\vee})$, where $\mathfrak{g}^{\sigma\vee}$ is the Langlands dual Lie algebra of $\mathfrak{g}^\sigma$. □

Note that one can identify (in many ways) the $\mathbb{C}(q)$-linear span of $B_\infty(\mathfrak{g})$ with the quantized enveloping algebra $U_q^+(\mathfrak{g})$ of $\mathfrak{g}_+$, where $\mathfrak{g}_+$ stands for the "upper triangular" Lie subalgebra of $\mathfrak{g}$. This leads us to the following definition.

Definition 1.2. A *quantum folding* of $\mathfrak{g}$ is a $\mathbb{C}(q)$-linear embedding (not necessarily an algebra homomorphism!)

$$\iota : U_q^+(\mathfrak{g}^{\sigma\vee}) \hookrightarrow U_q^+(\mathfrak{g})^\sigma \subset U_q^+(\mathfrak{g}) \tag{1.2}$$

(here $U_q^+(\mathfrak{g}^{\sigma\vee})$ comes with powers of q depending on σ; see Section 2.2 for the details). □

We construct all relevant quantum foldings below (Proposition 1.19) and now focus on a rich algebraic structure that can be attached to each quantum folding.

Definition 1.3. We say that a $\Bbbk$-algebra A generated by a totally ordered set X_A is *Poincare–Birkhoff–Witt* (*PBW*) if the set $M(X_A)$ of all ordered monomials in X_A is a basis of A. More generally, we say that A is *sub-PBW* if $M(X_A)$ spans A as a $\Bbbk$-vector space (but $M(X_A)$ is not necessarily linearly independent).

For a given sub-PBW algebra A we say that an algebra $U = U(A, X_A)$ is a *uberalgebra* of A if

(a) U is generated by X_A and is a PBW algebra with these generators;

(b) The identity map $X_A \to X_A$ extends to a surjective algebra homomorphism $U \twoheadrightarrow A$. □

In general, it is not clear whether a given sub-PBW algebra A admits a uberalgebra. A criterion for uniqueness is based on the following notion of tameness of (A, X_A). We need some notation. First, consider the natural filtration $\Bbbk = A_0 \subset A_1 \subset \cdots$ given by $A_k = \mathrm{Span}\{1, X_A, X_A \cdot X_A, \ldots, (X_A)^k\}$, $k \in \mathbb{Z}_{\geq 0}$. Next, for each $X, X' \in X_A$ with $X < X'$ let $d(X, X')$ be the smallest number d such that $X'X \in A_d$. We also denote by $d_0 = d_0(A, X_A)$ the maximum of all the $d(X, X')$.

Definition 1.4. We say that a sub-PBW algebra (A, X_A) is *tame* if the set $M(X_A) \cap A_{d_0}$ is linearly independent. □

Lemma 1.5. A tame sub-PBW algebra (A, X_A) admits at most one (up to isomorphism) uberalgebra $U(A, X_A)$. □

In what follows, we will construct uberalgebras $U(\iota)$ for several quantum foldings ι as in (1.2), and these uberalgebras will depend on $\mathfrak{g}$ and σ rather than on a particular choice of ι and most algebras generated by the image of ι will be tame. We need more notation.

Definition 1.6. Let A and B be PBW-algebras and let $\iota : A \hookrightarrow B$ be an injective linear map (not necessarily an algebra homomorphism). We say that ι is *liftable* if:

(i) For any ordered monomial $X = X_1^{m_1} \cdots X_N^{m_N} \in M(X_A)$ one has

$$\iota(X) = \iota(X_1)^{m_1} \cdots \iota(X_N)^{m_N}.$$

(ii) There exists a finite subset $Z_0 \subset \langle A \rangle_\iota$, where $\langle A \rangle_\iota$ is the subalgebra of B generated by $\iota(A)$, such that $\langle A \rangle_\iota$ is sub-PBW with respect to $\iota(X_A) \cup Z_0$ (with some ordering of $\iota(X_A) \cup Z_0$ compatible with the ordering of X).

(iii) There exists a uberalgebra $U(\iota) := U(\langle A \rangle_\iota, \iota(X_A) \cup Z_0)$ for $\langle A \rangle_\iota$ and a surjective homomorphism $\mu := \mu_\iota : U(\iota) \to A$ such that for all $x \in X_A$, $\mu(\iota(x)) = x$ and $\mu(Z_0) = 0$. □

If ι is liftable and $(\langle A \rangle_\iota, \iota(X_A) \cup Z_0)$ is tame (hence $U(\iota)$ is unique), in what follows we refer to an ι as a *tame* liftable quantum folding.

For each liftable ι we have a diagram:

$$\begin{tikzcd} U(\iota) \arrow[r, "\hat{\iota}"] \arrow[d, shift left, "\mu_\iota"] & B \\ A \arrow[u, shift left, "\tilde{\iota}"] \arrow[ur, "\iota"'] & \end{tikzcd} \tag{1.3}$$

satisfying $\mu_\iota \circ \tilde{\iota} = \mathrm{id}_A$ and $\hat{\iota} \circ \tilde{\iota} = \iota$ where

- $\tilde{\iota} : A \hookrightarrow U(\iota)$ is the canonical splitting of μ given by $\tilde{\iota}(X_1^{m_1} \cdots X_N^{m_N}) = \iota(X_1)^{m_1} \cdots \iota(X_N)^{m_N}$ in the notation of Definition 1.6(i).
- $\hat{\iota} : U(\iota) \to B$ is the structural algebra homomorphism given by $\hat{\iota}(X) = X$ for $X \in \iota(X_A) \cup Z_0$ (e.g., the image of $\hat{\iota}$ is $\langle A \rangle_\iota$).

Note the following easy lemma.

Lemma 1.7. In the notation of Definition 1.6, write $X_A = \{X_1, \ldots, X_N\}$ as an ordered set. Fix $1 \le k \le N$ and $f \in \sum_{M \in M(X_A \setminus \{X_k\})} \Bbbk M$. If the uberalgebra $U(\iota)$ is optimal PBW (in the sense of Definition 2.17) then

(i) (A, X'_A), where $X'_A = \{X_1, \ldots, X'_k := X_k + f, X_{k+1}, \ldots, X_N\}$, is a PBW algebra.

(ii) The injective linear map $\iota' : A \to B$ given by the formula

$$\iota'(X_1^{m_1} \cdots (X'_k)^{m_k} X_{k+1}^{m_{k+1}} \cdots X_N^{m_N}) = \iota(X_1)^{m_1} \cdots \iota(X'_k)^{m_k} \iota(X_{k+1}^{m_{k+1}}) \cdots \iota(X_N^{m_N})$$

is liftable with $\langle A \rangle_{\iota'} = \langle A \rangle_\iota$, $U(\iota') = U(\iota)$ and $\mu_\iota = \mu_{\iota'}$. □

The following is our first main result (see Section 3 for greater details).

Theorem 1.8. For the pair $(\mathfrak{g}, \mathfrak{g}^{\sigma\vee}) = (\mathfrak{so}_{2n+2}, \mathfrak{sp}_{2n})$, $n \ge 3$ there exists a tame liftable quantum folding $\iota : U_q^+(\mathfrak{sp}_{2n}) \hookrightarrow U_q^+(\mathfrak{so}_{2n+2})$. The corresponding uberalgebra $U(\iota)$ is isomorphic to $S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n)$, where V is the standard n-dimensional $U_q(\mathfrak{sl}_n)$-module, and $S_q(V \otimes V)$ is a quadratic PBW-algebra in the category of $U_q(\mathfrak{sl}_n)$-modules. More precisely:

(i) The algebra $S_q(V \otimes V)$ is isomorphic to $T(V \otimes V)/\langle (\Psi - 1)(V^{\otimes 4}) \rangle$, where $\Psi : V^{\otimes 4} \to V^{\otimes 4}$ is a $\mathbb{C}(q)$-linear map given by:

$$\Psi = \Psi_2 \Psi_1 \Psi_3 \Psi_2 + (q - q^{-1})(\Psi_1 \Psi_2 \Psi_1 + \Psi_1 \Psi_3 \Psi_2) + \left(q - q^{-1}\right)^2 \Psi_1 \Psi_2 \tag{1.4}$$

where $\Psi_i : V^{\otimes 4} \to V^{\otimes 4}$, $1 \le i \le 3$ is, up to a power of q, the braiding operator in the category of $U_q(\mathfrak{sl}_n)$-modules that acts in the ith and $(i+1)$st factors and satisfies the normalized Hecke equation $(\Psi_i - q^{-1})(\Psi_i + q) = 0$.

(ii) The covariant $U_q(\mathfrak{sl}_n)$-action on the algebra $S_q(V \otimes V)$ is determined by the natural action of the Hopf algebra $U_q(\mathfrak{sl}_n)$ on $V \otimes V$.

(iii) The algebra $S_q(V \otimes V)$ is PBW with respect to any ordered basis of $V \otimes V$. □

Remark 1.9. Strictly speaking, the cross product $S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n)$ is "braided" in the sense of Majid [18] because $U_q(\mathfrak{sl}_n)$ is a braided Hopf algebra (see [17] and Section 2.2). □

We prove Theorem 1.8 in Section 3. In particular, the key ingredient in our proof of part (iii) is the following surprising result.

Proposition 1.10. The map Ψ satisfies:

(i) The braid equation in $(V \otimes V)^{\otimes 3}$:

$$(\Psi \otimes 1)(1 \otimes \Psi)(\Psi \otimes 1) = (1 \otimes \Psi)(\Psi \otimes 1)(1 \otimes \Psi). \tag{1.5}$$

(ii) The cubic version of the Hecke equation:

$$(\Psi - 1)(\Psi + q^2)(\Psi + q^{-2}) = 0.$$

In particular, Ψ is invertible and

$$\Psi^{-1} = \Psi_2\Psi_1\Psi_3\Psi_2 + (q - q^{-1})(\Psi_2\Psi_3\Psi_2 + \Psi_1\Psi_3\Psi_2) + (q - q^{-1})^2\Psi_3\Psi_2.$$

(iii) $\dim(\Psi - 1)(V \otimes V) = \dim \Lambda^2 V$. □

This and the following general fact that we failed to find in the literature, although numerous special cases are well known (cf. for example [8, 11, 20]), settle Theorem 1.8(iii) (see Section 3 for details).

Theorem 1.11. Let Y be a finite-dimensional $\mathbb{C}(q)$-vector space and let Ψ be an invertible $\mathbb{C}(q)$-linear map $Y \otimes Y \to Y \otimes Y$ satisfying the braid equation. Assume that:

(i) the specialization $\Psi|_{q=1}$ of Ψ is the permutation of factors $\tau : Y \otimes Y \to Y \otimes Y$,

(ii) $\dim(\Psi - 1)(Y \otimes Y) = \dim \Lambda^2 Y$.

Then the algebra $S_\Psi(Y) = T(Y)/\langle(\Psi - 1)(Y \otimes Y)\rangle$ is a flat deformation of the symmetric algebra $S(Y)$ (hence $S_\Psi(Y)$ is PBW for any ordered basis of Y). □

We prove Theorem 1.11 in Section 2.6.

An explicit PBW presentation of both $S_q(V \otimes V)$ and $U(\iota)$ is more cumbersome, so we postpone it until Proposition 3.6. Below we provide a presentation of $U(\iota)$ by a minimal set of Chevalley-like generators satisfying Serre-like relations.

Theorem 1.12. The uberalgebra $U(\iota) = S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n)$, $n \geq 2$ is generated by $u_1, \ldots, u_{n-1}$, w, and z subject to the following relations (for all relevant i, j):

$$[u_i, [u_i, u_j]_q]_{q^{-1}} = 0 \quad \text{if } |i-j| = 1, \quad u_i u_j = u_j u_i \quad \text{if } |i-j| > 1,$$

$$u_i w = w u_i \quad \text{if } i \neq 1, \quad u_i z = z u_i \quad \text{if } i \neq 2, \quad zw = wz,$$

$$[u_1, [u_1, [u_1, w]]_{q^2}]_{q^{-2}} = 0, \quad [u_2, [u_2, z]_q]_{q^{-1}} = 0,$$

$$[w, [w, u_1]_{q^2}]_{q^{-2}} = -hwz, \quad [z, [u_2, [u_1, w]_{q^2}]_q]_q = [w, [u_1, [u_2, z]_q]_q]_{q^2},$$

$$2[z, [z, u_2]_q]_{q^{-1}} = h(z[u_1, u_2]_q w + w[u_2, u_1]_{q^{-1}} z + w u_1 [z, u_2]_q + [u_2, z]_{q^{-1}} u_1 w),$$

where $h = q - q^{-1}$ and we abbreviate $[a, b]_v = ab - vba$ and $[a, b] = [a, b]_1 = ab - ba$ (with the convention that $u_2 = 0$ if $n = 2$). □

Remark 1.13. Under the decomposition $V \otimes V = S_q^2 V \oplus \Lambda_q^2 V$ in the category of $U_q(\mathfrak{sl}_n)$-modules the generator w (respectively, z) of $U(\iota)$ is a lowest weight vector in the simple $U_q(\mathfrak{sl}_n)$-module $S_q^2 V$ (respectively, $\Lambda_q^2 V$), with the convention that u_i equals the $(n-i)$th standard Chevalley generator E_{n-i} of $U_q^+(\mathfrak{sl}_n)$.

It is easy to show that $S_q(V \otimes V)/\langle \Lambda_q^2 V\rangle \cong S_q(S^2 V)$, $S_q(V \otimes V)/\langle S_q^2 V\rangle \cong S_q(\Lambda^2 V)$, where $S_q(S^2 V)$ and $S_q(\Lambda^2 V)$ are respectively the algebras of quantum symmetric and quantum exterior matrices studied in [9, 14, 19, 22]. Due to this and the canonical identification $S(V \otimes V) = S(\Lambda^2 V \oplus S^2 V) = S(\Lambda^2 V) \otimes S(S^2 V)$, we can view $S_q(V \otimes V)$ as a deformation of the braided (in the category of $U_q(\mathfrak{sl}_n)$-modules) tensor product $S_q(\Lambda^2 V) \otimes S_q(S^2 V)$ (see also Remark 1.16 for the Poisson version of this discussion). This point of view is supported by the observation that our braiding operator Ψ given by (1.4) is a deformation of the braiding $\Psi' := \Psi_2 \Psi_1 \Psi_3 \Psi_2$ of $V \otimes V$ with itself in the category of $U_q(\mathfrak{sl}_n)$-modules. Note, however, that the latter braiding Ψ' does not satisfy the condition (ii) of Theorem 1.11, therefore, the quadratic algebra $S_{\Psi'}(V \otimes V)$ (as defined in Theorem 1.11) is not a flat deformation of $S(V \otimes V)$. □

Remark 1.14. In all quantum foldings we constructed so far the image of ι is contained in $U_q(\mathfrak{g})^{\mathrm{gr}\,\sigma}$, where $(\cdot)^{\mathrm{gr}\,\sigma}$ is the graded fixed point algebra defined for any graded algebra $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ and any automorphism σ of A by: $A^{\mathrm{gr}\,\sigma} = \bigoplus_{\gamma\in\Gamma}\{a \in A_\gamma : \sigma(a) = a\}$ (in our case, Γ is the root lattice of $\mathfrak{g}$). One can show that the subalgebra of $U_q^+(\mathfrak{so}_{2n+2})$ generated by the image of $\iota : U_q^+(\mathfrak{sp}_{2n}) \hookrightarrow U_q^+(\mathfrak{so}_{2n+2})$ is isomorphic to $U_q^+(\mathfrak{so}_{2n+2})^{\mathrm{gr}\,\sigma}$, but we do not expect this to happen in general (e.g., it fails for the pair $(\mathfrak{g}, \mathfrak{g}^{\sigma\vee}) = (\mathfrak{so}_8, G_2)$). We will discuss the relationship between quantum foldings and graded fixed points of diagram auromorphisms in a separate publication. □

Theorem 1.8 implies that the "classical limit" $S(V \otimes V)$ of $S_q(V \otimes V)$ has a quadratic Poisson bracket which we present in the following

Corollary 1.15. In the notation of Theorem 1.8, let $\{X_i\}$, $i = 1, \ldots, n$ be the standard basis of V. Then the formulae (for all $1 \le i \le j \le k \le l \le n$, where we abbreviated $X_{ij} = X_i \otimes X_j$ for $1 \le i, j \le n$):

$$\{X_{ij}, X_{kl}\} = (\delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl})X_{ij}X_{kl} - 2(X_{il}X_{kj} + X_{ki}X_{lj}),$$

$$\{X_{ij}, X_{lk}\} = (\delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl})X_{ij}X_{lk} - 2(X_{kj}X_{li} + X_{ik}X_{lj}),$$

$$\{X_{ji}, X_{kl}\} = (\delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl})X_{ji}X_{kl} - 2(X_{jl}X_{ki} + X_{kj}X_{li}),$$

$$\{X_{ji}, X_{lk}\} = (\delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl})X_{ji}X_{lk} - 2(X_{jk}X_{li} + X_{ki}X_{lj}),$$

$$\{X_{ik}, X_{jl}\} = (\delta_{ij} + \delta_{il} - \delta_{jk} + \delta_{kl})X_{ik}X_{jl} - 2(X_{il}X_{jk} - X_{ij}X_{kl} + X_{ji}X_{lk}),$$

$$\{X_{ik}, X_{lj}\} = (\delta_{ij} + \delta_{il} - \delta_{jk} + \delta_{kl})X_{ik}X_{lj} - 2X_{jk}X_{li},$$

$$\{X_{ki}, X_{jl}\} = (\delta_{ij} + \delta_{il} - \delta_{jk} + \delta_{kl})X_{jl}X_{ki} - 2X_{jk}X_{li},$$

$$\{X_{ki}, X_{lj}\} = (\delta_{ij} + \delta_{il} - \delta_{jk} + \delta_{kl})X_{ki}X_{lj} - 2X_{kj}X_{li},$$

$$\{X_{il}, X_{jk}\} = (\delta_{ij} + \delta_{ik} - \delta_{jl} - \delta_{kl})X_{il}X_{jk} + 2(X_{ij}X_{lk} - X_{ji}X_{kl}),$$

$$\{X_{il}, X_{kj}\} = (\delta_{ij} + \delta_{ik} - \delta_{jl} - \delta_{kl})X_{il}X_{kj} + 2(X_{ik}X_{lj} - X_{jl}X_{ki}),$$

$$\{X_{li}, X_{kj}\} = (\delta_{ij} + \delta_{ik} - \delta_{jl} - \delta_{kl})X_{kj}X_{li},$$

$$\{X_{li}, X_{jk}\} = (\delta_{ij} + \delta_{ik} - \delta_{jl} - \delta_{kl})X_{jk}X_{li}$$

define a Poisson bracket on $S(V \otimes V)$. □

Remark 1.16. Goodearl and Yakimov [10] constructed quadratic Poisson brackets on $S(\Lambda^2 V)$ and $S(S^2 V)$. In parallel with Remark 1.13, one can show that the ideal of $S(V \otimes V)$ generated by $\Lambda^2 V = \mathrm{Span}\{X_{ij} - X_{ji}\}$ (respectively, by $S^2 V = \mathrm{Span}\{X_{ij} + X_{ji}\}$) is Poisson; hence, the quotient of $S(V \otimes V)$ by this ideal is the Poisson algebra $S(S^2 V)$ (respectively, $S(\Lambda^2 V)$) from [10]. Therefore, we can view the bracket given by Corollary 1.15 as a certain deformation of the Poisson bracket on $S(V \otimes V)$ obtained by lifting the brackets on $S(\Lambda^2 V)$ and $S(S^2 V)$. □

We construct more liftable quantum foldings when σ is an involution.

Theorem 1.17. If $(\mathfrak{g}, \mathfrak{g}^{\sigma\vee}) = (\mathfrak{sl}_n \times \mathfrak{sl}_n, \mathfrak{sl}_n)$, $n = 3, 4$, then there exists a tame liftable quantum folding $\iota : U_q^+(\mathfrak{g}^{\sigma\vee}) \hookrightarrow U_q^+(\mathfrak{g})$ such that $U(\iota)$ is a q-deformation of the universal enveloping algebra $U(V_n \rtimes (\mathfrak{sl}_n)_+)$, $n = 3, 4$, where V_n is a finite-dimensional module (regarded as an abelian Lie algebra) over $(\mathfrak{sl}_n)_+$. More precisely,

(i) For $(\mathfrak{sl}_3 \times \mathfrak{sl}_3, \mathfrak{sl}_3)$, $V_3 = \mathbf{1}$ is the trivial one-dimensional $(\mathfrak{sl}_3)_+$-module and the uberalgebra $U(\iota)$ is generated by u_1, u_2, and z subject to the following relations

- z is central;
- $u_i^2 u_j - (q^2 + q^{-2}) u_i u_j u_i + u_j u_i^2 = (q - q^{-1}) u_i z$ for $\{i, j\} = \{1, 2\}$.

(ii) For $(\mathfrak{sl}_4 \times \mathfrak{sl}_4, \mathfrak{sl}_4)$, the $(\mathfrak{sl}_4)_+$-module V_4 has a basis $z_{12}, z_{13}, z_{23}, z_{1,23}, z_{12,3}$, and $z_{12,23}$, the action of Chevalley generators e_1, e_2, and e_3 of $(\mathfrak{sl}_4)_+$ on V_4 is given by the following diagram:

$$\begin{tikzcd}
z_{12} \arrow[d, "e_3"'] & z_{13} & z_{23} \arrow[d, "e_1"] \\
z_{12,3} \arrow[dr, "e_2"'] & & z_{1,23} \arrow[dl, "e_2"] \\
& z_{12,23} &
\end{tikzcd}$$

where an arrow from z to z' labeled by e_i means that $e_i(z) = z'$, while $e_j(z) = 0$ for all $j \neq i$. The uberalgebra $U(\iota)$ is a quantized enveloping algebra of the Lie algebra $V_4 \rtimes (\mathfrak{sl}_4)_+$ and it is generated by u_1, u_2, u_3, $z_{12} = z_{21}$, $z_{23} = z_{32}$, and z_{13} subject to the relations:

- $u_i z_{ij} = z_{ij} u_i$, $i < j$, $u_1 u_3 = u_3 u_1$,

- $[u_i, [u_i, u_j]_{q^2}]_{q^{-2}} = hu_i z_{ij}$, $|i-j| = 1$,
- $[u_i, [u_i, z_{j2}]_{q^2}]_{q^{-2}} = hu_i z_{13}$ for $\{i, j\} = \{1, 3\}$,
- $(q + q^{-1})[z_{12}, z_{23}] = [u_2, [z_{12}, u_3]_{q^{-2}}]_{q^2} - [u_2, [z_{23}, u_1]_{q^{-2}}]_{q^2}$,
- $[u_2, z_{13}] + [z_{i2}, z_{j2}] = h(z_{j2}u_i u_2 - u_2 u_i z_{j2})$ for $\{i, j\} = \{1, 3\}$,
- $2[z_{i2}, [z_{i2}, u_j]_{q^{-2}}]_{q^2} + [z_{i2}, [u_i, z_{2j}]_{q^{-2}}]_{q^2} + [u_i, [z_{i2}, z_{2j}]_{q^{-2}}]_{q^2} =$
  $= h\{z_{i2}, z_{13} + (q^2 + 1 + q^{-2})u_i u_2 u_j - u_j u_2 u_i - u_i u_j u_2 - u_2 u_i u_j\}$ for $\{i, j\} = \{1, 3\}$,
- $[z_{i2}, z_{13}] - [u_i, [u_2, [u_j, z_{i2}]]] = h(u_i z_{j2} z_{i2} - z_{i2} z_{j2} u_i) + h^2(u_i u_2 u_j z_{i2} - z_{i2} u_j u_2 u_i)$

for $\{i, j\} = \{1, 3\}$, where we abbreviated $[a, b]_v = ab - vba$, $[a, b] = ab - ba$, $\{a, b\} = ab + ba$, and $h = q - q^{-1}$. □

Remark 1.18. In case of $(\mathfrak{sl}_3 \times \mathfrak{sl}_3, \mathfrak{sl}_3)$ the uberalgebra $U(\iota)$ is a PBW on the ordered set $\{u_1, u_2, u_{21} = u_1 u_2 - q^{-2} u_2 u_1 - z, z\}$ subject to the following relations:

- the element z is central,
- $u_1 u_{21} = q^2 u_{21} u_1$, $u_2 u_{21} = q^{-2} u_{21} u_2$,
- $u_1 u_2 = q^{-2} u_2 u_1 + u_{21} + z$.

In particular, $S(\mathbf{1} \rtimes (\mathfrak{sl}_3)_+)$ is generated by $\tilde{u}_1, \tilde{u}_2, \tilde{u}_{12}$, and $\tilde{z}$ and the following:

- $\{\tilde{u}_1, \tilde{z}\} = \{\tilde{u}_2, \tilde{z}\} = \{\tilde{u}_{21}, \tilde{z}\} = 0$,
- $\{\tilde{u}_1, \tilde{u}_2\} = -2\tilde{u}_2\tilde{u}_1 + 4\tilde{u}_{21} + 2\tilde{z}$,
- $\{\tilde{u}_1, \tilde{u}_{21}\} = 2\tilde{u}_{21}\tilde{u}_1$, $\{\tilde{u}_2, \tilde{u}_{21}\} = -2\tilde{u}_{21}\tilde{u}_2$

defines a Poisson bracket on $S(\mathbf{1} \rtimes (\mathfrak{sl}_3)_+)$. □

The PBW-presentation of the uberalgebra $U(\iota)$ for the folding $(\mathfrak{sl}_4 \times \mathfrak{sl}_4, \mathfrak{sl}_4)$ is more cumbersome (see Theorem 4.7). Similarly to the previous discussion, the PBW property of $U(\iota)$ defines a Poisson bracket on $S(V_4 \rtimes (\mathfrak{sl}_4)_+)$ which, unlike that on $S(V_3 \rtimes (\mathfrak{sl}_3)_+)$, includes cubic terms (Theorem 4.8). It would be interesting to construct both the uberalgebra and the corresponding Poisson bracket for the folding $(\mathfrak{sl}_n \times \mathfrak{sl}_n, \mathfrak{sl}_n)$, $n \geq 4$.

Now we will explicitly construct all tame liftable quantum foldings ι used in Theorems 1.8 and 1.17 along with their (yet conjectural) generalizations to all semisimple Lie algebras. We need some notation.

Given a semisimple simply laced Lie algebra $\mathfrak{g}$ with an admissible diagram automorphism σ, let I be the set of vertices of the Dynkin diagram of $\mathfrak{g}$ and we denote by the

same letter σ the induced bijection $\sigma : I \to I$. Denote by s_i, $i \in I$ (respectively, by s'_r, $r \in I/\sigma$) the simple reflections of the root lattice of $\mathfrak{g}$ (respectively, of $\mathfrak{g}^{\sigma\vee}$). Let $W(\mathfrak{g}) = \langle s_i : i \in I\rangle$ (respectively, $W(\mathfrak{g}^{\sigma\vee}) = \langle s'_r : r \in I/\sigma\rangle$) be the corresponding Weyl group.

Denote by $\hat{w}_\circ$ (respectively, $w_\circ$) the longest element of $W(\mathfrak{g})$ (respectively, of $W(\mathfrak{g}^{\sigma\vee})$). Furthermore, denote by $R(w_\circ)$ the set of all reduced decompositions of $w_\circ$, that is, of all sequences $\mathbf{i} = (i_1, \dots, i_m) \in (I/\sigma)^m$ where $m = \ell(w_\circ)$ is the Coxeter length of $w_\circ$ such that $s_{i_1} \cdots s_{i_m} = w_\circ$. Similarly, one defines the set $R(\hat{w}_\circ)$ of all reduced decompositions of $\hat{w}_\circ$.

Note that each admissible diagram automorphism σ defines an automorphism of $W(\mathfrak{g})$ via $s_i \mapsto s_{\sigma(i)}$ and its fixed subgroup $W(\mathfrak{g})^\sigma$ is isomorphic to $W(\mathfrak{g}^\sigma) = W(\mathfrak{g}^{\sigma\vee})$ via $s_r \mapsto \hat{s}_r = \prod_{i\in\mathcal{O}_r} s_i$ where $\mathcal{O}_r \subset I$ is the rth σ-orbit in I (see Proposition 2.4). We denote this natural isomorphism $W(\mathfrak{g}^{\sigma\vee}) \xrightarrow{\sim} W(\mathfrak{g})^\sigma$ by $w \mapsto \hat{w}$.

Thus, one can assign to each $\mathbf{i} = (i_1, \dots, i_m) \in R(w_\circ)$ its *lifting* $\hat{\mathbf{i}} \in R(\hat{w}_\circ)$ via:

$$\hat{\mathbf{i}} = (\mathcal{O}_{i_1}, \dots, \mathcal{O}_{i_m})$$

(in fact, $\hat{\mathbf{i}}$ is unique up to reordering of each set $\mathcal{O}_{r_k}$).

Following Lusztig [17, §40.2], for each $\hat{\mathbf{i}} \in R(\hat{w}_\circ)$ (respectively, $\mathbf{i} \in R(w_\circ)$ one defines a *modified* PBW-basis $M(X_{\hat{\mathbf{i}}})$ of $U_q^+(\mathfrak{g})$ (respectively, $M(X_{\mathbf{i}})$ of $U_q^+(\mathfrak{g}^{\sigma\vee})$), see Section 2.4 for details (this modification will ensure the commutativity of the triangle in (1.3).

One can show (see Lemma 2.11) that for any $\hat{\mathbf{i}} \in R(\hat{w}_\circ)$ the PBW basis $M(X_{\hat{\mathbf{i}}})$ does not depend on the choice of a lifting $\hat{\mathbf{i}} \in R(\hat{w}_\circ)$ of $\mathbf{i} \in R(w_\circ)$. Moreover, the action of σ on $U_q^+(\mathfrak{g})$ preserves $M(X_{\hat{\mathbf{i}}})$ for each such lifting $\hat{\mathbf{i}}$.

The following result serves as a definition of quantum folding for all $\mathfrak{g}$ and σ (see Lemma 2.11 for details).

Proposition 1.19. Given an admissible diagram automorphism σ of $\mathfrak{g}$, for each $\mathbf{i} \in R(w_\circ)$ there is a natural injective $\mathbb{C}(q)$-linear map

$$\iota_{\mathbf{i}} : U_q^+(\mathfrak{g}^{\sigma\vee}) \hookrightarrow U_q^+(\mathfrak{g})^\sigma \subset U_q^+(\mathfrak{g}) \tag{1.6}$$

which maps the modified PBW-basis $M(X_{\mathbf{i}})$ bijectively onto the fixed point set $M(X_{\hat{\mathbf{i}}})^\sigma$ of $M(X_{\hat{\mathbf{i}}})$. □

In fact, the tame liftable foldings ι used in Theorems 1.8 and 1.17 were of the form $\iota_{\mathbf{i}}$, $\mathbf{i} \in R(w_\circ)$.

Theorem 1.20. Let $\mathfrak{g}$ be a simply laced semisimple Lie algebra and let σ be its admissible diagram automorphism of order 2. Then for any reduced decompositions $\mathbf{i}$ and $\mathbf{i}'$ of $w_\circ$ the subalgebras of $U_q^+(\mathfrak{g})$ generated by the images of $\iota_{\mathbf{i}}$ and $\iota_{\mathbf{i}'}$ are isomorphic. □

This theorem is proved in Section 2.4.

However, if the order of σ is at least 3, it frequently happens that the image of ι generates a nonsub-PBW algebra hence the uberalgebra $U(\iota_{\mathbf{i}})$ does not always exists (see Section 4.3). In order to restore the (sub-)PBW behavior of the algebras in question, we propose the modification, which we refer to as the *enhanced* uberalgebra $\hat{U}(\iota)$.

Indeed, in the assumptions of Definition 1.3 let us relax the assumption that $Z_0 \subset \langle A \rangle_\iota$ in Definition 1.6. Suppose that B is PBW domain. Then we take Z_0 to be a finite subset of $\mathcal{F}\mathrm{rac}(\iota(A)) \cap B$, where $\mathcal{F}\mathrm{rac}(\iota(A)) \subset \mathcal{F}\mathrm{rac}(B)$ is the skew-subfield of the skew-filed $\mathcal{F}\mathrm{rac}(B)$ generated by $\iota(A)$ (B is an Ore domain so its skew-field of fractions $\mathcal{F}\mathrm{rac}(B)$ is well defined, see [3, Appendix A] for details).

We will refer to a map ι satisfying Definition 1.6 "relaxed" in such a way as *enhanced* liftable and to its uberalgebra (which we denote by $\hat{U}(\iota)$) as an *enhanced* uberalgebra of ι. (A *tame* enhanced liftable ι is introduced accordingly). By construction, $\hat{U}(\iota)$ satisfies the diagram (1.3), however, it need not be generated by A (unlike all known $U(\iota)$ for liftable ι).

Theorem 1.21. Let $n \geq 3$ and let $(\mathfrak{g}, \mathfrak{g}^{\sigma\vee}) = (\mathfrak{sl}_3^{\times n}, \mathfrak{sl}_3)$ where $\mathfrak{sl}_3^{\times n} = \mathfrak{sl}_3 \times \cdots \times \mathfrak{sl}_3$ and σ is a cyclic permutation of factors. Then for both reduced decompositions $\mathbf{i}_1 = (121)$ and $\mathbf{i}_2 = (212)$ of $w_\circ \in W(\mathfrak{g}^{\sigma\vee})$ the quantum folding $\iota_{\mathbf{i}_r}$, $r = 1, 2$ is enhanced liftable and the enhanced uberalgebras $\hat{U}(\iota_{\mathbf{i}_1})$ and $\hat{U}(\iota_{\mathbf{i}_2})$ are isomorphic. More precisely,

(i) $\hat{U}(\iota_{\mathbf{i}_1})$ is generated by Chevalley-like generators u_1, u_2, and $z_1, \ldots, z_{n-1}$, subject to Serre-like relations

- $z_k z_l = z_l z_k$ for $k, l = 1, \ldots, n-1$,
- $u_i z_{k,i} = q^{n-2k} z_{k,i} u_i$ for $i = 1, 2$, $k = 1, \ldots, n-1$,
- $u_i^2 u_j - (q^n + q^{-n}) u_i u_j u_i + u_j u_i^2 = (q^{-1} - q) u_i \sum_{k=1}^{n-1} q^k z_{k,i}$ for $\{i, j\} = \{1, 2\}$,

where we abbreviated $z_{k,1} = z_{n-k,2} = z_k$.

(ii) The enhanced uberalgebra $\hat{U}(\iota_{i_1})$ is a PBW algebra in the totally ordered set of generators $\{u_2, u_{21}, u_1, z_1, \ldots, z_{n-1}\}$, where $u_{21} = u_1 u_2 - q^{-n} u_2 u_1 - \sum_{k=1}^{n-1} \frac{q-q^{-1}}{q^k - q^{-k}} z_k$, subject to the commutation relations:

- $z_k z_l = z_l z_k$ for all $1 \le k, l \le n-1$,
- $u_1 u_{21} = q^n u_{21} u_1$, $u_2 u_{21} = q^{-n} u_{21} u_2$,
- $u_1 z_k = q^{n-2k} z_k u_1$, $u_2 z_k = q^{2k-n} z_k u_2$, $u_{21} z_k = z_k u_{21}$ for all $1 \le k \le n-1$,
- $u_1 u_2 = q^{-n} u_2 u_1 + u_{21} + \sum_{k=1}^{n-1} \frac{q-q^{-1}}{q^k - q^{-k}} z_k$.

(iii) $\hat{U}(\iota_{\mathbf{i}_1})$ a quantum deformation of the enveloping algebra $U(\mathbf{1}^{n-1} \rtimes (\mathfrak{sl}_3)_+)$, where $\mathbf{1}$ is the trivial one-dimensional $(\mathfrak{sl}_3)_+$-module. □

We prove Theorem 1.21 in Section 4.1.

Remark 1.22. It follows from Theorem 1.21 that the following defines a Poisson bracket on $S(\mathbf{1}^{n-1} \rtimes (\mathfrak{sl}_3)_+)$

- $\{\tilde{u}_1, \tilde{u}_{21}\} = n\tilde{u}_1 \tilde{u}_{21}$, $\{u_2, \tilde{u}_{21}\} = -n\tilde{u}_1 \tilde{u}_{21}$,
- $\{\tilde{u}_1, \tilde{z}_k\} = (n-2k)\tilde{u}_1 \tilde{z}_k$, $\{\tilde{u}_2, \tilde{z}_k\} = (k-2n)\tilde{u}_2 \tilde{z}_k$ for $k = 1, \dots, n-1$,
- $\{\tilde{u}_1, \tilde{u}_2\} = n(2\tilde{u}_{21} - \tilde{u}_1 \tilde{u}_2) + 2\sum_{k=1}^{n-1} \tilde{z}_k$,

where $\tilde{u}_1$,$\tilde{u}_{21}$, and $\tilde{u}_2$, and $\tilde{z}_k$, $k = 1, \dots, n-1$ are PBW generators of $S(\mathbf{1}^{n-1} \rtimes (\mathfrak{sl}_3)_+)$ obtained by certain specialization at $q = 1$ from generators of $U(\iota_{\mathbf{i}_1})$. Note that the quotient by the Poisson ideal generated by $\tilde{z}_1, \dots, \tilde{z}_{n-1}$ is the Poisson algebra $S(\mathfrak{sl}_3^+)$ with the standard Poisson bracket multiplied by n. □

Theorem 1.23. Let $(\mathfrak{g}, \mathfrak{g}^{\sigma\vee}) = (\mathfrak{so}_8, G_2)$ where σ is a cyclic permutation of three vertices of Dynkin diagram of type D_4. Then for both reduced decompositions $\mathbf{i}_1 = (121212)$ and $\mathbf{i}_2 = (212121)$ of $w_\circ \in W(\mathfrak{g}^{\sigma\vee})$ the quantum folding $\iota_{\mathbf{i}_k}$, $k = 1, 2$ is enhanced liftable and the enhanced uberalgebras $\hat{U}(\iota_{\mathbf{i}_1})$ and $\hat{U}(\iota_{\mathbf{i}_2})$ are isomorphic to each other and to a quantum deformation of the universal enveloping algebra $U(\mathfrak{n}_{G_2} \rtimes (\mathfrak{sl}_2)_+)$, where $\mathfrak{n}_{G_2}$ is a certain non-Abelian nilpotent 13-dimensional Lie algebra with the covariant $(\mathfrak{sl}_2)_+$-action. More precisely,

(i) $\mathfrak{n}_{G_2} \rtimes (\mathfrak{sl}_2)_+$ is generated by $u = e_1$, w, z_1, and z_2 subject to the following relations:

- $[u, [u, [u, [u, w]]]] = [w, [w, u]] = 0$;
- $[u, [u, z_i]] = [z_i, [z_i, [z_i, u]]] = [w, z_i] = 0$, $[w, [z_i, u]] = [z_1, z_2]$ for $i = 1, 2$,
- $[z_i, [u, z_i]] = [z_1, [u, z_2]] + [z_2, [u, z_1]]$ for $i = 1, 2$.

(ii) $\mathfrak{n}_{G_2}$ is the Lie ideal in $\mathfrak{n}_{G_2} \rtimes (\mathfrak{sl}_2)_+$ with the basis w_i, $1 \le i \le 5$ and z_i, $1 \le i \le 8$ and the multiplication table (only nonzero Lie brackets are shown)

- $[w_1, w_4] = -3w_5$, $[w_2, w_3] = w_5$,
- $[w_1, z_3] = [w_1, z_4] = 3z_5$, $[w_2, z_1] = [w_2, z_2] = [z_2, z_1] = -z_5$,
- $[w_2, z_3] = [w_2, z_4] = [z_1, w_3] = [z_2, w_3] = 2z_6$,
- $[w_3, z_3] = [w_3, z_4] = [z_3, z_4] = z_7$, $[w_4, z_1] = [w_4, z_2] = -3z_7$,
- $[z_1, z_3] = [z_2, z_4] = 2z_8$,
- $[z_1, z_4] = z_6 + z_8$, $[z_2, z_3] = -z_6 + z_8$.

(iii) $\hat{U}(\iota_{\mathbf{i}_1})$ is generated by Chevalley-like generators u and w, and z_1 and z_2 and satisfies the following Serre-like relations (the list is incomplete):

- $[u, [u, [u, [u, w]_{q^{-3}}]_{q^{-1}}]_q]_{q^3} = 0$,
- $[w, [w, u]_{q^{-3}}]_{q^3} = [w, z_1]_{q^3} = [z_2, w]_{q^3} = q([z_1, w]_q + [w, z_2]_q)$,
- $[z_1, z_2] = (q + q^{-1})[z_1, [w, u]_{q^{-3}}] - [z_2, [w, u]_{q^{-3}}]$,
- $[z_1, [u, w]_q] = [[w, u]_q, z_2]$, $[z_1, [z_1, u]_{q^{-1}}]_q = [z_2, [z_2, u]_{q^{-1}}]_q$,
- $[z_1, [z_1, u]_{q^{-1}}]_q = q([z_1, [z_2, u]_{q^{-1}}]_q + [z_2, [z_1, u]_{q^{-1}}]_q) + (q - q^{-1})(q[z_1, u]_{q^{-1}} z_2 - z_1[z_2, u]_{q^{-1}})$. □

We prove Theorem 1.23 in Section 5.

Remark 1.24. The nontameness of the quantum folding assigned to $(\mathfrak{so}_4, G_2)$ causes serious computational problems for the corresponding uberalgebra and the Poisson bracket on $S(\mathfrak{n}_{G_2})$. At the moment the Poisson bracket involves around 700 terms and the PBW presentation of $U(\iota_{\mathbf{i}_1})$ is even more complicated (they can be found at http://ishare.ucr.edu/jacobg/G2.pdf.

This is one of the reasons why Theorem 1.23(iii) contains only a partial Serre-like presentation of $U(\iota_{\mathbf{i}_1})$ in Chevalley-like generators u, w, z_1, and z_2. We dropped here the most notorious relations involving more than 30 terms each (see the above mentioned webpage). □

Taking into account Theorems 1.8, 1.20, 1.21, and 1.23, we propose the following conjecture.

Conjecture 1.25. Let σ be any admissible diagram automorphism of $\mathfrak{g}$ such that $\mathfrak{g}^{\sigma\vee}$ has no Lie ideals of type G_2. Then there exists a (unique) $\mathfrak{g}_+$-module $V_{\mathfrak{g}}$ such that:

(i) for any $\mathbf{i} \in R(w_\circ)$, the folding $\iota_{\mathbf{i}}$ is tame enhanced liftable,

(ii) the corresponding enhanced uberalgebra $\hat{U}(\iota_{\mathbf{i}})$ is a flat deformation of both the universal enveloping algebra $U(\mathfrak{n} \rtimes \mathfrak{g}_+)$ and the symmetric algebra $S(V_{\mathfrak{g}} \rtimes \mathfrak{g}_+)$,

(iii) The skew field of fractions $\mathcal{F}\mathrm{rac}(U(\iota_{\mathfrak{i}}))$ is generated by $\tilde{\iota}_{\mathfrak{i}}(E_r)$, $r \in I/\sigma$, where E_r are Chevalley generators of $U_q^+(\mathfrak{g}^{\sigma^\vee})$ (and $\tilde{\iota}_{\mathfrak{i}} : U_q^+(\mathfrak{g}^{\sigma^\vee}) \hookrightarrow U(\iota_{\mathfrak{i}})$ is the lifting of $\iota_{\mathfrak{i}}$ given by (1.3). □

If σ is an involution, we drop "enhanced" in Conjecture 1.25 because we expect that $\hat{U}(\iota_{\mathfrak{i}}) = U(\iota_{\mathfrak{i}})$.

In particular, the conjecture implies that one can canonically assign to each simply laced Lie algebra $\mathfrak{g}$ a finite-dimensional $\mathfrak{g}_+$-module $V_{\mathfrak{g}}^{(k)}$ for each $k \geq 2$ (by taking $\mathfrak{g}^{\times k}$ and its natural diagram automorphism σ, the cyclic permutation of factors so that $(\mathfrak{g}^{\times k})^{\sigma^\vee} = \mathfrak{g}$)). Theorem 1.17 implies that such a $V_{\mathfrak{g}}^{(k)}$ will be rather nontrivial even for $\mathfrak{g} = \mathfrak{sl}_n$. It would also be interesting to explicitly compute the Poisson bracket on $S(V_{\mathfrak{g}}^{(k)} \rtimes \mathfrak{g}_+)$ predicted by Conjecture 1.25. It should be noted that if $\mathfrak{g}$ has a diagram automorphism σ', then the corresponding uberalgebra also admits an automorphism extending σ'. For example, in the notation of Theorems 1.21 and 1.17, the uberalgebra for the folding $(\mathfrak{g}, \mathfrak{g}^{\sigma^\vee}) = (\mathfrak{sl}_3^{\times k}, \mathfrak{sl}_3)$ has an automorphism σ' defined by $u_1 \mapsto u_2$, $u_2 \mapsto u_1$, $z_i \mapsto z_{k-i}$, $1 \leq i \leq k-1$, while the uberalgebra for the folding $(\mathfrak{sl}_4 \times \mathfrak{sl}_4, \mathfrak{sl}_4)$ has an automorphism defined by $e_1 \mapsto e_3$, $z_{12} \mapsto z_{32}$, $z_{32} \mapsto z_{12}$ and e_2 and z_{13} are fixed.

Note also that the part (iii) of Conjecture 1.25 holds for all cases we considered so far, in particular, for the folding $(\mathfrak{g}, \mathfrak{g}^{\sigma^\vee}) = (\mathfrak{sl}_3^{\times n}, \mathfrak{sl}_3)$, the skew-field $\mathcal{F}\mathrm{rac}(U(\iota))$ is generated by u_1 and u_2 (one can show that each z_k, $1 \leq k \leq n-1$ in Theorem 1.21(i) is a rational "function" of u_1 and u_2; see Lemma 4.4) and for $(\mathfrak{g}, \mathfrak{g}^{\sigma^\vee}) = (\mathfrak{so}_8, G_2)$ the skew-field $\mathcal{F}\mathrm{rac}(U(\iota))$ is generated by u and w (both z_1 and z_2 in Theorem 1.23(iii) are rational "functions" of u and w).

2 General Properties of Quantum Foldings and PBW Algebras

2.1 Folding of semisimple Lie algebras

Recall that each semisimple Lie algebra $\mathfrak{g} = \langle e_i, f_i : i \in I \rangle$ is determined by its Cartan matrix $A = (a_{ij})_{i,j \in I}$ (see, e.g. [21]) via:

$$(\mathrm{ad}\ e_i)^{1-a_{ij}} e_j = 0 = (\mathrm{ad}\ f_i)^{1-a_{ij}} f_j, \quad [e_i, f_j] = 0, \quad i \neq j$$

and

$$[[e_i, f_i], e_j] = a_{ij} e_j, \quad [[e_i, f_i], f_j] = -a_{ij} f_j, \quad i, j \in I.$$

Denote by $\mathfrak{g}_+$ the Lie subalgebra of $\mathfrak{g}$ generated by the e_i, $i \in I$.

We say that a bijection $\sigma : I \to I$ is a diagram automorphism of $\mathfrak{g}$ if $a_{\sigma(i),\sigma(j)} = a_{ij}$ for all $i, j \in I$. It is well known that such σ defines a unique automorphism, which we also denote by σ, of the Lie algebra $\mathfrak{g}$ via

$$\sigma(e_i) = e_{\sigma(i)}, \ \sigma(f_i) = f_{\sigma(i)}, \quad i \in I.$$

After [17, §12.1.1], a diagram automorphism σ is said to be *admissible* if for all $i \in I$, $k \in \mathbb{Z}$, $a_{i,\sigma^k(i)} = 0$, whenever $\sigma^k(i) \neq i$.

In what follows we denote by I/σ the quotient set of I by the equivalence relation which consists of all pairs $(i, \sigma^k(i))$. In other words, we use I/σ as the indexing set for orbits of the cyclic group $\langle\sigma\rangle = \{1, \sigma, \sigma^2, \ldots\}$ action on I.

The following result is well known (cf. for example [13, Proposition 7.9])

Theorem 2.1. Let σ be an admissible diagram automorphism of $\mathfrak{g}$. Then the fixed Lie subalgebra $\mathfrak{g}^\sigma = \{x \in \mathfrak{g} : \sigma(x) = x\}$ of $\mathfrak{g}$ is semi-simple, with:

- the Chevalley generators e'_r, f'_r, $r \in I/\sigma$ given by

$$e'_r = \sum_{i \in \mathcal{O}_r} e_i, \quad f'_r = \sum_{i \in \mathcal{O}_r} f_i,$$

  where $\mathcal{O}_r$ is the rth orbit of the $\langle\sigma\rangle$-action on I.
- the Cartan matrix $A' = (a'_{r,s})$, $r, s \in I/\sigma$ given by

$$a'_{r,s} = \sum_{i \in \mathcal{O}_r} a_{i,j} \tag{2.1}$$

  for all $j \in \mathcal{O}_s$, $r, s \in I/\sigma$. □

2.2 Quantized enveloping algebras and Langlands dual folding

For any indeterminate v and for any $m \leq n \in \mathbb{Z}_{\geq 0}$, set

$$[n]_v := \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]_v! := \prod_{j=1}^{n} [j]_v, \quad \begin{bmatrix} n \\ m \end{bmatrix}_v := \frac{[n]_v!}{[m]_v![n-m]_v!}.$$

For each semisimple Lie algebra $\mathfrak{g}$ we fix symmetrizers $d_i \in \mathbb{N}$ such that $d_i a_{ij} = a_{ij} d_j$ for all $i, j \in I$. Then denote by $C = (d_i a_{ij})$ the *symmetrized Cartan matrix* of $\mathfrak{g}$ (it depends on the choice of symmetrizers) and let $q_i := q^{d_i}$.

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$ which is a $\mathbb{C}(q)$-algebra generated by the elements $E_i, F_i, K_i^{\pm 1}$, $i \in I$ subject to the relations

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,$$

as well as *quantum Serre relations*

$$\sum_{b=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ b \end{bmatrix}_{q_i} E_i^r E_j E_i^{1-a_{ij}-b} = 0 = \sum_{b=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ b \end{bmatrix}_{q_i} F_i^r F_j F_i^{1-a_{ij}-b} \tag{2.2}$$

for all $i \neq j$.

We denote by $U_q^+(\mathfrak{g})$ (resp. by $U_q^{\leq 0}(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by the E_i, $i \in I$. (resp. by the $F_i, K_i^{\pm 1}$, $i \in I$). Note that $U_q(\mathfrak{g})$ and $U_q^+(\mathfrak{g})$ are completely determined by the symmetrized Cartan matrix C.

We now define the folding of symmetrized Cartan matrices for a given admissible diagram automorphism σ. For each $I \times I$ symmetric matrix C and a bijection $\sigma : I \to I$ denote by $C^\sigma = (c^\sigma_{r,s})$ the $I/\sigma \times I/\sigma$ symmetric matrix with the entries:

$$c^\sigma_{r,s} = \sum_{i \in \mathcal{O}_r, j \in \mathcal{O}_s} c_{i,j} \tag{2.3}$$

for all $j \in \mathcal{O}_s$, $r, s \in I/\sigma$.

Lemma 2.2. Let $C = A$ be the Cartan matrix of a simply laced semisimple Lie algebra $\mathfrak{g}$ with an admissible diagram automorphism σ. Then C^σ is a symmetrized Cartan matrix of $\mathfrak{g}^{\sigma\vee}$, where $\mathfrak{g}^{\sigma\vee}$ is the Langlands dual Lie algebra of the semisimple Lie algebra $\mathfrak{g}^\sigma$. More precisely, $C^\sigma = D^\sigma (A')^{\mathrm{T}}$ where A' is the Cartan matrix of $\mathfrak{g}^\sigma$ (given by (2.1)) and D^σ is the diagonal matrix $\mathrm{diag}(|\mathcal{O}_r|, r \in I/\sigma)$. □

Proof. By (2.1), (2.3) and the symmetry of A we have for all $r, s \in I/\sigma$

$$c^\sigma_{r,s} = \sum_{i \in \mathcal{O}_r, j \in \mathcal{O}_s} a_{i,j} = \sum_{i \in \mathcal{O}_r} \sum_{j \in \mathcal{O}_s} a_{j,i} = \sum_{i \in \mathcal{O}_r} a'_{s,r} = |\mathcal{O}_r| a'_{s,r}. \qquad \blacksquare$$

This motivates the following notation. For each $\mathfrak{g}$ and σ as above denote by $U_q(\mathfrak{g}^{\sigma\vee})$ the quantized enveloping algebra determined by the matrix C^σ from Lemma 2.2.

2.3 Braid groups and their folding

Given a semisimple Lie algebra $\mathfrak{g}$ with the Cartan matrix $A=(a_{ij})_{i,j\in I}$, let $Q=\bigoplus_{i\in I}\mathbb{Z}\alpha_i$ be the root lattice of $\mathfrak{g}$. Recall that the *Weyl group* $W(\mathfrak{g})$ is generated by the simple reflections $s_i : Q\to Q$ given by

$$s_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i$$

for $i, j\in I$. It is well known that $W(\mathfrak{g})$ is a Coxeter group with the presentation

$$(s_is_j)^{m_{ij}}=1,\quad \text{where } m_{ij}=\begin{cases}1 & \text{if } i=j,\\ 2 & \text{if } a_{ij}=0,\\ 3 & \text{if } a_{ij}=a_{ji}=-1,\\ 4 & \text{if } a_{ij}a_{ji}=2,\\ 6 & \text{if } a_{ij}a_{ji}=3.\end{cases} \tag{2.4}$$

For each $w\in W(\mathfrak{g})$ denote by $R(w)$ the set of all reduced decompositions $\mathbf{i}=(i_1,\dots,i_m)\in I^m$ such that

$$w=s_{i_1}\cdots s_{i_m}$$

and m is minimal (this m is the Coxeter length $\ell(w)$). We denote by $w_\circ$ the longest element of $W(\mathfrak{g})$.

The Artin braid group $Br_{\mathfrak{g}}$ is generated by the T_i, $i\in I$ subject to the relations (for all $i, j\in I$):

$$\underbrace{T_iT_j\cdots}_{m_{ij}}=\underbrace{T_jT_i\cdots}_{m_{ij}}. \tag{2.5}$$

To each $w\in W(\mathfrak{g})$ one associates the element $T_w\in Br_{\mathfrak{g}}$ such that

$$T_w=T_{i_1}\cdots T_{i_m} \tag{2.6}$$

for each $\mathbf{i}=(i_1,\dots,i_m)\in R(w)$ (it follows from relations (2.5) that T_w is well defined).

Lemma 2.3 ([7, Theorem 4.21] and [5, Lemma 5.2]). For each $w\in W$ we have

$$T_{w_\circ}T_wT_{w_\circ}^{-1}=T_{w_\circ ww_\circ}.$$

In particular, the element

$$C_{\mathfrak{g}} = \begin{cases} T_{w_\circ} & \text{if } w_\circ \text{ is in the center of } W, \\ T_{w_\circ}^2 & \text{if } w_\circ \text{ is not in the center of } W \end{cases}$$

is in the center of $Br_{\mathfrak{g}}$. Moreover, the center of $Br_{\mathfrak{g}}$ is generated by all $C_{\mathfrak{g}'}$, where $\mathfrak{g}'$ runs over the simple Lie ideals of $\mathfrak{g}$. □

Now we return to the folding situation. Let $\mathfrak{g}$ be a semisimple Lie algebra and let σ be its admissible diagram automorphism.

Note that σ defines an automorphism of $W(\mathfrak{g})$ (respectively, of $Br_{\mathfrak{g}}$) via $\sigma(s_i) = s_{\sigma(i)}$ (respectively, $\sigma(T_i) = T_{\sigma(i)}$) for $i \in I$. Denote by $\hat{w}_\circ$ (respectively, $w_\circ$) the longest element of $W(\mathfrak{g})$ (respectively, of $W(\mathfrak{g}^{\sigma\vee})$). Since σ preserves the Coxeter length, it follows that $\sigma(\hat{w}_\circ) = \hat{w}_\circ$. The following result provides a "folding" isomorphism of the corresponding Weyl and braid groups.

Proposition 2.4. For each semisimple simply laced Lie algebra $\mathfrak{g}$ and its admissible diagram automorphism σ we have:

(i) The assignment

$$s_r \mapsto \hat{s}_r = \prod_{i \in \mathcal{O}_r} s_i, \quad r \in I/\sigma \tag{2.7}$$

extends to an isomorphism of groups $\hat{\cdot} : W(\mathfrak{g}^{\sigma\vee}) \xrightarrow{\sim} W(\mathfrak{g})^\sigma \subset W(\mathfrak{g})$.

(ii) The assignment

$$T_r \mapsto \hat{T}_r = \prod_{i \in \mathcal{O}_r} T_i, \quad r \in I/\sigma \tag{2.8}$$

extends to an isomorphism of groups $Br_{\mathfrak{g}^{\sigma\vee}} \xrightarrow{\sim} (Br_{\mathfrak{g}})^\sigma \subset Br_{\mathfrak{g}}$. Under this isomorphism the element T_w of $Br_{\mathfrak{g}^{\sigma\vee}}$ is mapped to the element $T_{\hat{w}}$ of $Br_{\mathfrak{g}}$. □

Proof. It is easy to see that (2.7) defines a group homomorphism because it respects the Coxeter relations (2.4). The injectivity also follows. Let us prove surjectivity, that is, that each element $w \in W(\mathfrak{g})^\sigma$ factors into a product of the $\hat{s}_r$, $r \in I/\sigma$. We proceed by induction on the Coxeter length of w, the induction base being trivial. We need the following well-known result.

Lemma 2.5. Let $w \in W(\mathfrak{g})$ and $i \neq j$ be such that $\ell(s_i w) = \ell(s_j w) = \ell(w) - 1$. Then there exists w' such that $w = \underbrace{s_i s_j \cdots}_{m_{ij}} w'$ and $\ell(w') = \ell(w) - m_{ij}$. In particular, if $I_0 \subset I$ satisfies

- $\ell(s_i w) = \ell(w) - 1$ for each $i \in I_0$,
- $s_i s_{i'} = s_{i'} s_i$ for all $i, i' \in I_0$,

then there exists w'' such that $w = (\prod_{i \in I_0} s_i) \cdot w''$ and $\ell(w'') = \ell(w) - |I_0|$. □

Indeed, let $w \in W(\mathfrak{g})^\sigma$. Then there exists $i \in I$ such that $w = s_i w'$ for some w' with $\ell(w') = \ell(w) - 1$. Applying σ^k, we obtain: $s_i w' = s_{\sigma^k(i)} \sigma^k(w')$, hence $\ell(s_{\sigma^k(i)} w) = \ell(w) - 1$. Thus the set $I_0 = \{i, \sigma(i), \sigma^2(i) \ldots\} = \mathcal{O}_r$ satisfies the conditions of Lemma 2.5. Therefore, $w = \hat{s}_r w''$ with $\ell(w'') = \ell(w) - |\mathcal{O}_r|$. In particular, $w'' \in W(\mathfrak{g})^\sigma$ and $\ell(w'') < \ell(w)$ so we finish the proof by induction. This proves (i).

To prove (ii) note that (2.8) defines a group homomorphism because it respects the Coxeter relations (2.5). The injectivity also follows. Let us prove surjectivity, that is, that each element $g \in (Br_{\mathfrak{g}})^\sigma$ factors into a product of the $\hat{T}_r^{\pm 1}$, $r \in I/\sigma$.

Following [5, 7], denote by $Br_{\mathfrak{g}}^+$ the *positive braid monoid*, that is, the monoid generated by the T_i, $i \in I$ subject to (2.5).

Lemma 2.6 ([5, Proposition 5.5; 7, Proposition 4.17])**.**

(i) The assignment $T_i \mapsto T_i$ defines an injective homomorphism of monoids $Br_{\mathfrak{g}}^+ \hookrightarrow Br_{\mathfrak{g}}$. In other words, $Br_{\mathfrak{g}}^+$ is naturally a submonoid of $Br_{\mathfrak{g}}$.

(ii) For each $g \in Br_{\mathfrak{g}}$ there exists an element $g^+ \in Br_{\mathfrak{g}}^+$ such that $g = C g^+$ for some central element C of $Br_{\mathfrak{g}}$. □

Note that the central element $C_{\mathfrak{g}} = T_{\hat{w}_\circ}^2$ from Lemma 2.3 is the product of all generators $C_{\mathfrak{g}'}$ of the center of $Br_{\mathfrak{g}}^+$, where $\mathfrak{g}'$ runs over simple Lie ideals of $\mathfrak{g}$. This and Lemma 2.6 imply that for each $g \in Br_{\mathfrak{g}}$ there exists $N \geq 0$ such that $C_{\mathfrak{g}}^N \cdot g \in Br_{\mathfrak{g}}^+$. Taking into account that $T_{\hat{w}_\circ}$ and hence $C_{\mathfrak{g}}$ is fixed under σ, it suffices to prove that any element $g^+ \in (Br_{\mathfrak{g}}^+)^\sigma$ factors into a product of the $\hat{T}_r$, $r \in I/\sigma$.

We need the following result which is parallel to Lemma 2.5.

Lemma 2.7 ([5, Lemma 2.1])**.** Let $g^+ \in Br_{\mathfrak{g}}^+$ and $i \neq j$ be such that $g^+ = T_i \cdot g_i^+ = T_j \cdot g_j^+$ for some $g_i^+, g_j^+ \in Br_{\mathfrak{g}}^+$. Then there exists $h^+ \in Br_{\mathfrak{g}}^+$ such that $g^+ = \underbrace{T_i T_j \cdots}_{m_{ij}} h^+$. In particular, if $I_0 \subset I$ satisfies

- $g^+ \in T_i \cdot Br_{\mathfrak{g}}^+$ for each $i \in I_0$,
- $T_i T_{i'} = T_{i'} T_i$ for all $i, i' \in I_0$,

then there exists $h^+ \in Br_{\mathfrak{g}}^+$ such that $g^+ = (\prod_{i \in I_0} T_i) \cdot h^+$. □

We proceed by induction on length of elements in $Br_{\mathfrak{g}}^+$. Indeed, let $g^+ \in (Br_{\mathfrak{g}}^+)^\sigma$. Then there exists $i \in I$ such that $g^+ = T_i \cdot g_i^+$ for some $g_i^+ \in Br_{\mathfrak{g}}^+$. Applying σ^k, we obtain: $T_i \cdot g_i^+ = T_{\sigma^k(i)} \cdot \sigma^k(g^+)$, where $\sigma^k(g^+) \in Br_{\mathfrak{g}}^+$. Thus the set $I_0 = \{i, \sigma(i), \sigma^2(i) \ldots\} = \mathcal{O}_r$ satisfies the conditions of Lemma 2.7. Therefore, $g^+ = \hat{T}_r \cdot h^+$ for some $h^+ \in Br_{\mathfrak{g}}^+$. In particular, $h^+ \in (Br_{\mathfrak{g}}^+)^\sigma$ and is shorter than g^+, so we finish the proof by induction. This proves (ii). ■

2.4 PBW bases and quantum folding

Lusztig proved in [17] that $Br_{\mathfrak{g}}$ acts on $U_q(\mathfrak{g})$ by algebra automorphisms via:

$$T_i(K_j^{\pm 1}) = K_j^{\pm 1} K_i^{\mp a_{ij}}, \quad T_i(E_i) = -K_i^{-1} F_i, \quad T_i(F_i) = -E_i K_i, \quad i, j \in I,$$
$$T_i(E_j) = \sum_{s+r=-a_{ij}} (-1)^r q_i^{-r} E_i^{(r)} E_j E_i^{(s)}, \quad T_i(F_j) = \sum_{s+r=-a_{ij}} (-1)^r q_i^r F_i^{(s)} F_j F_i^{(r)}, \quad i \neq j, \tag{2.9}$$

where $Y_i^{(k)} = \frac{1}{[k]_{q_i}!} Y_i^k$ (these automorphisms T_i are denoted $T'_{i,-1}$ in [17]). Lusztig also proved in [17] that $T_w(E_i) \in U_q^+(\mathfrak{g})$ if and only if $\ell(ws_i) = \ell(w) + 1$. Using this, for each $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_\circ)$, define the ordered set $X_{\mathbf{i}} = \{X_1, X_2, \ldots, X_m\} \subset U_q^+(\mathfrak{g})$ by:

$$X_k = X_{\mathbf{i},k} = c_k^{-1} T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}),$$

where $c_k = \gamma(s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) - \alpha_{i_k})$ and $\gamma : Q \to \mathbb{C}(q)^\times$ is the unique group homomorphism defined by $\gamma(\alpha_i) = q_i - q_i^{-1}$. It should be noted that for any $w, w' \in W(\mathfrak{g})$, $i, i' \in I$ such that $w\alpha_i = w'\alpha'_i$, $\gamma(w\alpha_i - \alpha_i) = \gamma(w'\alpha_{i'} - \alpha_{i'})$.

We will need the following useful Lemma which is, most likely, well known.

Lemma 2.8. Suppose that $i, j \in I$ and $w \in W(\mathfrak{g})$ satisfy $w\alpha_i = \alpha_j$. Then $T_w(E_i) = E_j$. □

Proof. We use induction on $\ell(w)$, the induction base being trivial. Since $w\alpha_i = \alpha_j$, we have $\ell(ws_i) = \ell(w) + 1$ and $ws_i w^{-1} = s_j$. Then by [2, Lemma 9.9], there exist $k \in I$ and a $\mathbf{i} \in R(w)$ such that $\mathbf{i}$ terminates with $(\ldots, i, k) \in R(w_\circ(i,k)s_i)$, where $w_\circ(i,k)$ denotes the

longest element of the subgroup of $W(\mathfrak{g})$ generated by s_i, s_k. Since by [17, §§39.2.2–4]

$$T_{w_\circ(i,k)s_i}(E_i) = \begin{cases} E_k, & a_{ik} = a_{ki} = -1, \\ E_i & \text{otherwise}, \end{cases} \tag{2.10}$$

we conclude that either $T_w(E_i) = T_{w'}(E_i)$ with $w'\alpha_i = \alpha_j$ or $T_w(E_i) = T_{w'}(E_k)$ with $w'\alpha_k = \alpha_j$ and in both cases $\ell(w') < \ell(w)$. ■

For each $a = (a_1, \dots, a_m) \in \mathbb{Z}_{\geq 0}^m$ define the monomial $X_{\mathbf{i}}^a \in U_q^+(\mathfrak{g})$ by

$$X_{\mathbf{i}}^a = X_1^{a_1} \cdots X_m^{a_m},$$

The following result is well known.

Proposition 2.9 ([17, Corollary 40.2.2]). The set $M(X_{\mathbf{i}})$ of all monomials $X_{\mathbf{i}}^a$ is a PBW-basis of $U_q^+(\mathfrak{g})$. □

Remark 2.10. The basis $M(X_{\mathbf{i}})$ differs from Lusztig's PBW basis from [17] in that we do not divide the monomials by q-factorials but rather by some factors which vanish at $q = 1$. □

Let σ be an admissible diagram automorphism of a semisimple simply laced Lie algebra $\mathfrak{g}$. For each $r \in I/\sigma$ define the element $\hat{E}_r := \prod_{i \in \mathcal{O}_r} E_i \in U_q^+(\mathfrak{g})$ and the set $\hat{E}_r^\bullet \subset U_q^+(\mathfrak{g})$ of all monomials $\prod_{i \in \mathcal{O}_r} E_i^{a_i}$, $a_i \in \mathbb{Z}_{\geq 0}$. Note that $\hat{E}_r$ is fixed under the action of σ on $U_q^+(\mathfrak{g})$ and the set $\hat{E}_r^\bullet$ is σ-invariant. Moreover, $(\hat{E}_r^\bullet)^\sigma = \{\hat{E}_r^k \mid k \in \mathbb{Z}_{\geq 0}\}$. The following result is obvious.

Lemma 2.11. Assume that σ is an admissible diagram automorphism of $\mathfrak{g}$ and let $\mathbf{i} = (r_1, \dots, r_m) \in R(w_\circ)$. Let $\hat{\mathbf{i}} \in R(\hat{w}_\circ)$ be any lifting of $\mathbf{i}$ (as defined in Section 1). Then

(i) $M(X_{\hat{\mathbf{i}}}) = \hat{X}_1^\bullet \cdots \hat{X}_m^\bullet$ up to multiplication by nonzero scalars, where $\hat{X}_k^\bullet = \hat{T}_{r_1} \cdots \hat{T}_{r_{k-1}}(\hat{E}_{r_k}^\bullet)$, $1 \leq k \leq m$.

(ii) The basis $M(X_{\hat{\mathbf{i}}})$ is invariant under the action of σ on $U_q^+(\mathfrak{g})$ and the fixed point set $M(X_{\hat{\mathbf{i}}})^\sigma$ coincides, up to scalars, with the set

$$\hat{X}_{\mathbf{i}}^a = \hat{X}_1^{a_1} \cdots \hat{X}_m^{a_m},$$

where

$$\hat{X}_k = \hat{c}_k^{-1}\hat{T}_{r_1}\hat{T}_{r_2}\cdots\hat{T}_{r_{k-1}}(\hat{E}_{r_k})$$

and $\hat{c}_k = \prod_{i\in\mathcal{O}_{r_k}} \gamma(\hat{s}_{r_1}\cdots\hat{s}_{r_{k-1}}(\alpha_i) - \alpha_i)$. □

This motivates the following definition.

Definition 2.12 (ith quantum folding). For each $\mathbf{i} \in R(w_\circ)$ define an injective linear map $\iota_{\mathbf{i}} : U_q^+(\mathfrak{g}^{\sigma\vee}) \hookrightarrow U_q^+(\mathfrak{g})^\sigma \subset U_q^+(\mathfrak{g})$ by the formula

$$\iota_{\mathbf{i}}(X_{\mathbf{i}}^a) = \hat{X}_{\mathbf{i}}^a$$

for $a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m$. □

Remark 2.13. Combinatorially, $\iota_{\mathbf{i}}$ is a bijection $M(X_{\mathbf{i}}) \xrightarrow{\sim} M(X_{\hat{\mathbf{i}}})^\sigma$, which can be interpreted as a certain bijection of Kashiwara crystals. More precisely, we can view $\iota_{\mathbf{i}}$ as the composition of the canonical isomorphism $B_\infty(\mathfrak{g}^{\sigma\vee}) \cong B_\infty(\mathfrak{g}^\vee)$ with the Lusztig's σ-equivariant identification $\hat{L}_{\mathbf{i}} : B_\infty(\mathfrak{g}) \xrightarrow{\sim} M(X_{\hat{\mathbf{i}}})$ so that the following diagram commutes, in view of Proposition 1.1 and Lemma 2.11:

$$\begin{array}{ccc} B_\infty(\mathfrak{g}) & \xrightarrow{\hat{L}_{\mathbf{i}}} & M(X_{\hat{\mathbf{i}}}) \\ \uparrow & & \uparrow \\ B_\infty(\mathfrak{g}^{\sigma\vee}) & \xrightarrow{\hat{L}_{\mathbf{i}}^\sigma} & M(X_{\hat{\mathbf{i}}}) \end{array}$$

where the vertical arrows are natural inclusions of σ-fixed point subsets. □

Remark 2.14. The Definition 2.12 makes sense for any $\mathbf{i} \in R(w)$ where w is any element of the Weyl group. In that case we replace $U_q^+(\mathfrak{g})$ by the algebra $U_q(w)$ introduced in [6, 17] and extensively studied in [24, 25]. One can also define quantum foldings for some pairs $(\mathfrak{g}, \mathfrak{g}')$ not related by a diagram automorphism (e.g., for $(\mathfrak{sp}_6, G_2)$). Namely, following [16], one can embed $B_\infty(\mathfrak{g}')$ into $B_\infty(\mathfrak{g})$ and then extend the embedding linearly to $\iota : U_q(\mathfrak{g}') \hookrightarrow U_q(\mathfrak{g})$ using Lusztig's identifications $B_\infty(\mathfrak{g}') \cong M(X_{\mathbf{i}})$ and $B_\infty(\mathfrak{g}) \cong M(X_{\hat{\mathbf{i}}})$. □

We conclude this section with the proof of Theorem 1.20.

Proof of Theorem 1.20. It is sufficient to prove the statement for two reduced decompositions $\mathbf{i}, \mathbf{i}' \in R(w_\circ)$ which differ by one braid relation involving $r, s \in I/\sigma$ and thus it

suffices to consider the rank 2 case. We have the following two possibilities:

1°. $|\mathcal{O}_r| = 2$, $|\mathcal{O}_s| = 1$ and $\mathbf{i} = (r, s, r, s)$, $\mathbf{i}' = (s, r, s, r)$. Let $\mathcal{O}_r = \{i, j\}$, $\mathcal{O}_s = \{k\}$, with $a_{i,k} = a_{j,k} = -1$. Then $\hat{\mathbf{i}} = (i, j, k, i, j, k)$, $\hat{\mathbf{i}}' = (k, i, j, k, i, j)$ and the elements $\hat{X}_s$ for these two decompositions are, respectively,

$$\hat{X}_1 = E_i E_j, \quad \hat{X}_2 = \frac{[E_i, [E_j, E_k]_{q^{-1}}]_{q^{-1}}}{(q-q^{-1})^2}, \quad \hat{X}_3 = \frac{[E_i, E_k]_{q^{-1}}[E_j, E_k]_{q^{-1}}}{(q-q^{-1})^2}, \quad \hat{X}_4 = E_k,$$

while $\hat{X}'_1 = \hat{X}_4$, $\hat{X}'_2 = \hat{X}^*_3$, $\hat{X}'_3 = \hat{X}^*_2$, and $\hat{X}'_4 = \hat{X}^*_1$, where $*$ is the unique anti-automorphism of $U_q^+(\mathfrak{g})$ such that $E_i^* = E_i$ for $i \in I$. A straightforward computation shows that

$$\hat{X}'_2 = q^{-2}(\hat{X}_3 + (q-q^{-1})^{-1}[\hat{X}_4, \hat{X}_1]\hat{X}_4), \quad \hat{X}'_3 = \hat{X}_2 + q^{-1}(q-q^{-1})^{-1}[\hat{X}_4, \hat{X}_1].$$

Therefore, the subalgebra of $U_q^+(\mathfrak{g})$ generated by $\hat{X}_r$ contains all elements $\hat{X}'_r$. Applying $*$ we obtain the opposite inclusion.

2°. $|\mathcal{O}_r| = |\mathcal{O}_s| = 2$ and $\mathbf{i} = (r, s, r)$, $\mathbf{i}' = (s, r, s)$. Let $\mathcal{O}_r = \{i, j\}$, $\mathcal{O}_s = \{k, l\}$ with $a_{i,k} = -1 = a_{j,l} = -1$ and $a_{i,l} = a_{j,k} = 0$. Then, we have $\hat{X}_1 = E_i E_j = \hat{X}'_3$, $\hat{X}_3 = E_k E_l = \hat{X}'_1$ and

$$\hat{X}_2 = \frac{[E_i, E_k]_{q^{-1}}[E_j, E_l]_{q^{-1}}}{(q-q^{-1})^2}, \quad \hat{X}'_2 = \frac{[E_k, E_i]_{q^{-1}}[E_l, E_j]_{q^{-1}}}{(q-q^{-1})^2}.$$

It is easy to check that

$$\hat{X}'_2 = \hat{X}_2 + q^{-1}(q-q^{-1})^{-1}[\hat{X}_3, \hat{X}_1],$$

hence $\hat{X}'_2$ is contained in the subalgebra of $U_q^+(\mathfrak{g})$ generated by $\hat{X}_k$, $1 \le k \le 3$. Interchanging the role of r and s completes the proof. ■

2.5 Diamond Lemma and specializations of PBW algebras

We will use the following version of Bergman's Diamond Lemma ([4]). Let A be an associative $\Bbbk$-algebra and suppose that A is generated by a totally ordered set X_A. Let $M(X_A)$ be the set of ordered monomials on the X_A.

Proposition 2.15. Assume that the defining relations for (A, X_A) are

$$X'X = \sum_{M \in M(X_A)} c^M_{X,X'} M, \quad X < X',$$

where $c_{X,X'}^{XX'} \in \Bbbk^\times$ and for any $M \in M(X_A)$, $c_{X,X'}^{M} \neq 0$ implies that $M < X'X$ in the lexicographic order. If for all $X < X' < X''$ there exists a unique $S \subset M(X_A)$ and a unique $\{a_M : M \in S\} \subset \Bbbk^\times$ such that

$$X''X'X = \sum_{M \in S} a_M M,$$

then (A, X_A) is a PBW algebra □

Note that, unlike [20], we do not require (A, X_A) to be quadratic. In fact, in most cases where we will need to apply the Diamond Lemma, this will not be the case.

We will now list some elementary properties of specializations which will be needed later. The simplest instance of specialization is given by the following definition. Throughout this section, let $\Bbbk = \mathbb{C}(t)$ (later on, we set $t = q - 1$) and denote by $\Bbbk_0$ the set of all $f = f(t) \in \Bbbk$ such that $f(0)$ is defined. Clearly, $\Bbbk_0$ is a (local) subalgebra of $\Bbbk$ and for each nonzero $f \in \Bbbk$ either $f \in \Bbbk_0$ or $f^{-1} \in \Bbbk_0$.

Definition 2.16. Let U and V be $\Bbbk$-vector spaces with bases B_U and B_V respectively. Let $F : U \to V$ be a $\Bbbk$-linear map that all matrix coefficients $c_{b,b'} = c_{b,b'}(t) \in \Bbbk$ defined by

$$F(b) = \sum_{b' \in B_V} c_{b',b} b'$$

for $b \in B_U$, belong to $\Bbbk_0$. Then the *specialization* F_0 of F is a $\Bbbk$-linear map $U \to V$ given by

$$F_0(b) = \sum_{b' \in B_V} c_{b',b}(0) b'$$

for $b \in B_U$ (here, unlike in the literature on deformation theory, we preserve the ground field $\Bbbk = \mathbb{C}(t)$ after the specialization because it is more convenient to view both F and F_0 as $\Bbbk$-linear maps $U \to V$). □

Similarly, let (A, X_A) be a PBW algebra. Then, in the notation of Definition 1.3, it has a unique presentation:

$$X'X = \sum_{M \in M(X_A)} c_{X,X'}^{M} M$$

for all $X, X' \in X_A$ such that $X < X'$, where all $c_{X,X'}^{M} \in \Bbbk$.

Definition 2.17. We say that the PBW algebra (A, X_A) is *specializable* if all the $c_{X,X'}^{M}$ belong to $\Bbbk_0$ and in that case define the specialization (A_0, X_A) to be the associative

$\Bbbk$-algebra with the unique presentation:

$$X'X = \sum_{M \in M(X_A)} c^M_{X,X'}(0)M$$

for all $X, X' \in X_A$ with $X < X'$, where all $c^M_{X,X'} \in \Bbbk$.

We say that a specializable PBW algebra (A, X_A) is *optimal* if (A_0, X_A) is just the polynomial algebra $\Bbbk[X_A]$, that is, if $c^M_{X,X'}(0) = \delta_{M,XX'}$ for all relevant X, X', and M (i.e., the defining relations in (A_0, X_A) are $X'X = XX'$). In that case we define a bi-differential bracket $\{\cdot,\cdot\}$ on $\Bbbk[X_A]$ by

- (Leibniz rule) $\{xy, z\} = x\{y, z\} + y\{x, z\}$, $\{x, yz\} = \{x, y\}z + y\{x, z\}$ for all $x, y, z \in \Bbbk[X_A]$,
- (skew symmetry) $\{x, y\} = -\{y, x\}$ for all $x, y \in \Bbbk[X_A]$,
- for all $X, X' \in X_A$ with $X < X'$

$$\{X', X\} = \sum_{M \in M(X_A)} \left.\frac{\partial c^M_{X,X'}}{\partial t}\right|_{t=0} M. \tag{2.11}$$

□

Proposition 2.18. Let (A, X_A) be a specializable PBW-algebra. Then:

(i) Its specialization (A_0, X_A) is also PBW.

(ii) If, additionally, (A, X_A) is optimal, then the bracket $\{\cdot,\cdot\}$ on $A_0 = \Bbbk[X_A]$ is Poisson. □

Proof. Denote $A'_0 = \sum_{M \in M(X_A)} \Bbbk_0 M$. Clearly, this is a $\Bbbk_0$-subalgebra of A and a free $\Bbbk_0$-module. Taking into account that (t) is a (unique) maximal ideal in $\Bbbk_0$, we see that the specialization A_0 of A, is canonically isomorphic to $\Bbbk \otimes_{\Bbbk_0} A'_0/tA'_0$.

To prove (i), suppose that in A_0 we have $\sum_{M \in M(X_A)} c_M M = 0$, where all $c_M \in \mathbb{C}$. This implies that $\sum_{M \in M(A_X)} c_M M \in tA'_0$, hence $\sum_{M \in M(X_A)} c_M M = \sum_{M \in M(X_A)} tc'_M M$ for some $c'_M \in \Bbbk_0$. Since A'_0 is a free $\Bbbk_0$-module, this implies that $c_M = c'_M = 0$ for all M and completes the proof of (i).

Now we prove (ii). Optimality of (A, X_A) implies that the commutator $[a, b]$ of any $a, b \in A'_0$ belongs to the ideal tA_0. For any $a_0, b_0 \in A'_0/tA'_0$ denote

$$\{a_0, b_0\} = \pi\left(\frac{[a, b]}{t}\right),$$

where $\pi : A_0' \twoheadrightarrow A_0'/tA_0'$ is the canonical projection and $a, b \in A_0'$ are any elements such that $\pi(a) = a_0$, $\pi(b) = b_0$. Clearly, the bracket $\{a_0, b_0\}$ is well defined (i.e., it does not depend on the choice of representatives a and b). This bracket is Poisson because the original commutator bracket was skew-symmetric, and satisfied both the Liebniz rule and Jacobi identities.

It remains to verify (2.11). Indeed, let $X, X' \in X_A$ with $X < X'$. We have

$$\{X', X\} = \pi\left(\frac{X'X - XX'}{t}\right) = \pi\left(\frac{c_{X,X'}^{XX'} - 1}{t}\right) XX' + \sum_{M \neq XX'} \pi\left(\frac{c_{X,X'}^{M}}{t}\right) M.$$

This gives (2.11) because $\pi(f) = f(0)$ and $\pi(\frac{f - f(0)}{t}) = \frac{\partial f}{\partial t}\Big|_{t=0}$ for any $f \in \Bbbk_0$. The proposition is proved. ■

2.6 Nichols algebras and proof of Theorem 1.11

We will now prove Theorem 1.11, which allows us to establish the PBW property when an algebra is quadratic and is defined in terms of a braiding. Retain the notation of Section 2.5.

Proof. Let Y be a $\Bbbk$-vector space and $\Psi : Y \otimes Y \to Y \otimes Y$ be linear map. For each $k \geq 2$ we define the linear maps $\Psi_i : Y^{\otimes k} \to Y^{\otimes k}$, $i = 1, \ldots, k-1$ by the formula:

$$\Psi_i = 1^{\otimes i-1} \otimes \Psi \otimes 1^{\otimes k-i-1}.$$

If Ψ is invertible and satisfies the braid equation (1.5), then for each $k \geq 2$ one obtains the representation of the braid group $Br_{\mathfrak{sl}_k}$ on $Y^{\otimes k}$ (in the notation of Section 2.4) via $T_i \mapsto \Psi_i$ for $i = 1, \ldots, k-1$.

Therefore, one can define the braided factorial $[k]!_\Psi : Y^{\otimes k} \to Y^{\otimes k}$ by the formula:

$$[k]!_\Psi = \sum_{w \in S_k} \Psi_w,$$

where Ψ_w is the image of T_w (given by (2.6)) in $End_{\Bbbk}(Y^{\otimes k})$.

It is well known (see, e.g. [23]) that $I_\Psi := \bigoplus_{k \geq 2} \ker[k]!_\Psi$ is a two-sided ideal in the tensor algebra $T(Y)$. The quotient algebra $\mathcal{B}_\Psi(Y) := T(Y)/I_\Psi$ is called the *Nichols–Woronowicz algebra* of the braided vector space (Y, Ψ).

For each linear map $\Psi : Y \otimes Y \to Y \otimes Y$ denote $A_\Psi(Y) := T(Y)/\langle \ker(\Psi + 1)\rangle$.

We need the following result.

Proposition 2.19. Let Y be a $\Bbbk$-vector space and let $\Psi : Y \otimes Y \to Y \otimes Y$ be an invertible $\Bbbk$-linear map satisfying the braid equation. Assume that:

(i) The specialization $\Psi_0 = \Psi|_{t=0}$ of Ψ is a well defined (with respect to a basis of Y) invertible linear map $Y \otimes Y \to Y \otimes Y$ satisfying the braid equation.

(ii) The Nichols–Woronowicz algebra $\mathcal{B}_{\Psi_0}(Y)$ is isomorphic to $A_{\Psi_0}(Y)$ as a graded vector space.

(iii) $\dim\ker(\Psi + 1) = \dim\ker(\Psi_0 + 1)$.

Then $\mathcal{B}_\Psi(Y) \cong \mathcal{B}_{\Psi_0}(Y)$ as a graded vector space and one has an isomorphism of algebras

$$\mathcal{B}_\Psi(Y) \xrightarrow{\sim} A_\Psi(Y). \qquad \square$$

Proof. We will need two technical results.

Lemma 2.20. Let U and V be finite-dimensional $\mathbb{C}(t)$ vector spaces and $F : U \to V$ be a linear map such that its specialization F_0 at $t = 0$ is a well-defined map $U \to V$ (with respect to some bases of B_U and B_V). Then

(a) $\dim F_0(U) \le \dim F(U)$ and $\dim\ker F \le \dim\ker F_0$.

(b) Assume that $\dim\ker F = \dim\ker F_0$. Then there is a linear map $G : V \to V$ such that:

  (i) $G(V) = \ker F$,

  (ii) the specialization G_0 of G at $t = 0$ is well defined,

  (iii) $G_0(V) = \ker F_0$. $\square$

Proof. Fix bases B_U and B_V and identify U with $\Bbbk^n$, V with $\Bbbk^m$, and $F : \Bbbk^n \to \Bbbk^n$ with its $m \times n$ matrix.

It is well known (and easy to show) that for each nonzero $F \in \mathrm{Mat}_{m\times n}(\Bbbk)$ there exist $g_t \in GL_m(\Bbbk_0)$ and $h_t \in GL_n(\Bbbk_0)$ such that

$$F = g_t P h_t, \tag{2.12}$$

where P is an $m \times n$-matrix such that $P_{ij} = 0$ unless $(i, j) \in \{(1, 1), \dots, (r, r)\}$, and $P_{ii} = t^{\lambda_i}$ for $i = 1, \dots, r$, where $r = \mathrm{rank}(F)$ and $\lambda_i \in \mathbb{Z}$. Therefore, F_0 is well defined if and only if all $\lambda_i \ge 0$.

In particular, $F_0 = g_0 P_0 h_0$ and $\operatorname{rank}(F_0) \le k$, where P_0 is the specialization of P at $t = 0$. That is,

$$\dim F_0(U) = \operatorname{rank}(M_0) \le \operatorname{rank}(M_t) = \dim F(U).$$

This in conjunction with the equality $\dim \ker F + \operatorname{rank}(F) = \dim V$ proves (a)

Now we prove (b). Clearly, the condition $\dim \ker F = \dim \ker F_0$ is equivalent to $\operatorname{rank}(P_0) = r$, that is, in the decomposition (2.12) one has $\lambda_1 = \cdots = \lambda_r = 0$, that is, $P = P_0$ is the matrix (not depending on t) of the standard projection $\Bbbk^n \to \Bbbk^r \subset \Bbbk^m$.

Let $P^\perp \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be the standard projection $\Bbbk^n \to \operatorname{Span}\{e_{r+1}, \dots, e_n\} \subset \Bbbk^n$ (e.g., $PP^\perp = 0$). Denote $G := h_t^{-1} P^\perp$ so that the specialization G_0 of G at $t = 0$ is well defined and given by $G_0 = h_0^{-1} P^\perp$.

Clearly,

$$G(\Bbbk^m) = h_t^{-1}(\operatorname{Span}\{e_{r+1}, \dots, e_n\}) = \ker P h_t = \ker F.$$

Similarly, $G_0(\Bbbk^m) = \ker F$. This proves (b). ■

Lemma 2.21. Let $\mathcal{F}$ be a free $\Bbbk$-algebra on y_i, $i \in I$ where I is a finite set. Fix a grading on $\mathcal{F}$ with $\deg y_i \in \mathbb{Z}_{>0}$. Fix any finite subset B_t of specializable (with respect to the natural monomial basis of $\mathcal{F}$) homogeneous elements in $\mathcal{F}$. Then $\dim\langle B_t\rangle_n \ge \dim\langle B_0\rangle_n$, where $\langle B_t\rangle_n$ (respectively, $\langle B_0\rangle_n$) is the nth homogeneous component of the ideal in $\mathcal{F}$ generated by B_t (respectively, by the specialization $B_{t=0}$ of B_t). □

Proof. Clearly

$$\langle B_t\rangle_n = \bigoplus_{i+j+k=n} \sum_{b\in B_t:\deg b=j} \mathcal{F}_i b \mathcal{F}_k.$$

Define $\widehat{\langle B_t\rangle_n} = \bigoplus_{i+j+k=n} \bigoplus_{b\in B_t:\deg b=j} \mathcal{F}_i b \mathcal{F}_k$ and let $F : \widehat{\langle B_t\rangle_n} \to \mathcal{F}_n$ be the natural map which is the identity on each summand. Clearly the specialization of F_0 at $t = 0$ with respect to the natural monomial basis in both spaces is well defined and the image of F (respectively, of F_0) is $\langle B_t\rangle_n$ (respectively, $\langle B_0\rangle_n$). Then the assertion follows from Lemma 2.20(a). ■

The algebra $\mathcal{B}_\Psi(Y)$ is graded and $\mathcal{B}_\Psi(Y)_k$ is isomorphic to $[k]_\Psi!(Y^{\otimes k})$ as a vector space. Therefore,

$$\dim(\mathcal{B}_{\Psi_0})_k \le \dim(\mathcal{B}_\Psi)_k \tag{2.13}$$

by Lemma 2.20(a) with $V = Y^{\otimes k}$ and $F = [k]!_\Psi$.

On the other hand, note that $\ker(\Psi+1)=\ker[2]!_\Psi$ and so we have a structural homomorphism of graded algebras $A_\Psi(Y)\twoheadrightarrow \mathcal{B}_\Psi(Y)$. In particular, we obtain a surjective homomorphism of vector spaces $Y^{\otimes k}/\langle B_t\rangle_k \twoheadrightarrow \mathcal{B}_\Psi(Y)_k$ where B is the image of natural basis in $Y^{\otimes k}$ under the map G from Lemma 2.20(b). It follows from Lemma 2.21 that $\dim A_{\Psi_0}(Y)_k \geq \dim A_\Psi(Y)_k$. Combining this with (2.13) and the obvious inequality $\dim A_\Psi(Y)_k \geq \dim \mathcal{B}_\Psi(Y)_k$ we obtain $\dim A_\Psi(Y)_k \geq \dim A_{\Psi_0}(Y)_k$ which implies $\dim A_\Psi(Y)_k = \dim \mathcal{B}_\Psi(Y)_k = \dim \mathcal{B}_{\Psi_0}(Y)_k$ for all k. This completes the proof of Proposition 2.19. ■

Now we can complete the proof of Theorem 1.11. First, in the notation of Proposition we use Proposition 2.19 with Y^* and $-\Psi^*: Y^*\otimes Y^* \to Y^*\otimes Y^*$.

Taking $t=q-1$ and $\Psi_0^*=\tau$ in Proposition 2.19, we see that $\mathcal{B}_{-\Psi_0^*}(Y^*)=\Lambda(Y^*)=A_{-\Psi_0^*}(Y^*)$ hence $A_{-\Psi^*}(Y^*)\cong \mathcal{B}_{-\Psi^*}(Y^*)$ is a flat deformation of the exterior algebra $\Lambda(Y^*)$. Taking into account that $(\ker(-\Psi^*+1))^\perp=(\Psi-1)(Y\otimes Y)$ we see that the quadratic dual $A_{-\Psi^*}(Y^*)^!=S_\Psi(Y)$ is a flat deformation of $S(Y)$.

Theorem 1.11 is proved. ■

2.7 Module algebras and semi-direct products

It is well known that $U_q(\mathfrak{g})$ is a Hopf algebra with:

- The coproduct $\Delta: U_q(\mathfrak{g})\to U_q(\mathfrak{g})\otimes U_q(\mathfrak{g})$ given by

$$\Delta(E_i)=E_i\otimes 1+K_i\otimes E_i,\quad \Delta(F_i)=F_i\otimes K_i^{-1}+1\otimes F_i.$$

- The counit $\varepsilon: U_q(\mathfrak{g})\to \mathbb{C}(q)$ given by

$$\varepsilon(E_i)=\varepsilon(F_i)=0,\quad \varepsilon(K_i^{\pm 1})=1.$$

- The antipode $S: U_q(\mathfrak{g})\to U_q(\mathfrak{g})$ given by

$$S(F_i)=-F_iK_i,\quad S(E_i)=-K_i^{-1}E_i,\quad S(K_i)=K_i^{-1}.$$

In particular, $U_q(\mathfrak{g})$ admits the (left) adjoint action on itself, which we denote by $(u,v)\mapsto (\operatorname{ad} u)(v)$. The action is given by:

$$(\operatorname{ad} u)(v)=u_{(1)}vS(u_{(2)}),$$

where $\Delta(u) = u_{(1)} \otimes u_{(2)}$ in the Sweedler notation. By definition,

$$(\mathrm{ad}\ K_i)(u) = K_i u K_i^{-1}, \quad (\mathrm{ad}\ E_i)(u) = E_i u - K_i u K_i^{-1} E_i, \quad (\mathrm{ad}\ F_i)(u) = (F_i u - u F_i) K_i.$$

In particular, the quantum Serre relations can be written as

$$(\mathrm{ad}\ E_i)^{1-a_{ij}}(E_j) = 0. \tag{2.14}$$

We will also need the right action of $U_q(\mathfrak{g})$ on itself. Let $*$ be the unique anti-automorphism of $U_q(\mathfrak{g})$ defined by $E_i^* = E_i$, $F_i^* = F_i$, and $K_i^* = K_i^{-1}$. Then we define $\mathrm{ad}^* u = * \circ \mathrm{ad}\, u \circ *$. In particular, we have

$$(\mathrm{ad}^* K_i)(u) = K_i u K_i^{-1}, \quad (\mathrm{ad}^* E_i)(u) = u E_i - E_i K_i u K_i^{-1}, \quad (\mathrm{ad}^* F_i)(u) = K_i^{-1}[u, F_i]$$

and

$$T_i(E_j) = (\mathrm{ad}^* E_i)^{(-a_{ij})}(E_j). \tag{2.15}$$

It is easy to see that $\mathrm{ad}^* u$ is in fact the right adjoint action for a different co-product on $U_q(\mathfrak{g})$. Note that for all $i, j \in I$

$$(\mathrm{ad}\ E_i)(E_j) = (\mathrm{ad}^* E_j)(E_i), \tag{2.16}$$

while for all $i, j \in I$ and $w \in W$ such that $w\alpha_i = \alpha_j$ we have, by Lemma 2.8

$$T_w((\mathrm{ad}\ E_i)(u)) = (\mathrm{ad}\ E_j)(T_w(u)), \quad T_w((\mathrm{ad}^* E_i)(u)) = (\mathrm{ad}^* E_j)(T_w(u)). \tag{2.17}$$

Given a bialgebra U, refer to an algebra in the category U-mod as a *module algebra* over U. The following Lemma is obvious.

Lemma 2.22. Let $\mathcal{A}$ be a left module algebra over $U_q(\mathfrak{g})$. Then the action of Chevalley generators on $\mathcal{A}$ satisfies:

$$K_i(ab) = K_i(a)K_i(b), \quad E_i(ab) = E_i(a)b + K_i(a)E_i(b), \quad F_i(ab) = F_i(a)K_i^{-1}(b) + aF_i(b)$$

for all $a, b \in \mathcal{A}$ and $i \in I$. □

Definition 2.23. For any bialgebra B and its module algebra $\mathcal{A}$ define the cross product $\mathcal{A} \rtimes B$ to be the vector space $\mathcal{A} \otimes B$ with the associative product given by

$$(a \otimes b)(a' \otimes b') = a \cdot (b_{(1)}(a)) \otimes b_{(2)} \cdot b'$$

for all $a \in \mathcal{A}$, $b \in B$ (where $\Delta(b) = b_{(1)} \otimes b_{(2)}$ in Sweedler notation). In what follows, we suppress tensors and will write $a \cdot b$ instead of $a \otimes b$ and $b \cdot a$ instead of $(1 \otimes b)(a \otimes 1)$ in the algebra $\mathcal{A} \rtimes B$. □

Similarly, one can replace B by a *braided bialgebra*, that is, a bialgebra in a braided category $\mathcal{C}$ and $\mathcal{A}$ by a module algebra over B in $\mathcal{C}$. Our main example is when $\mathcal{C}_Q$ is the category of Q-graded vector spaces with the braiding $\Psi_{U,V} : U \otimes V \to V \otimes U$ for $U, V \in \mathrm{Ob}\mathcal{C}$ given by

$$\Psi_{U,V}(u \otimes v) = q^{(\mu,\nu)} v \otimes u, \quad u \in U_\mu,\ v \in V_\nu,$$

where $(\cdot, \cdot)$ is the inner product on Q given by

$$(\alpha_i, \alpha_j) = d_i a_{ij}, \tag{2.18}$$

where $C = (d_i a_{ij})_{i,j \in I}$ is a symmetrized Cartan matrix of a semisimple lie algebra $\mathfrak{g}$.

Let $Y = \mathbb{C}(q) \otimes_{\mathbb{Z}} Q$. As Lusztig proved [17], the Nichols–Woronowicz algebra $\mathcal{B}_{\Psi_{Y,Y}}(Y)$ (see Section 2.6) is naturally isomorphic to $U_q^+(\mathfrak{g})$. The following obvious fact is parallel to Lemma 2.22.

Lemma 2.24. Let $\mathcal{A} = \bigoplus_{\nu \in Q} A_\nu$ be a module algebra over (the braided bialgebra) $U_q^+(\mathfrak{g})$. Then

(i) For any $a \in \mathcal{A}_\nu$, $b \in \mathcal{A}$, $i \in I$ one has

$$E_i(ab) = E_i(a)b + q^{(\alpha_i,\nu)} a E_i(b).$$

(ii) The braided cross product $\mathcal{A} \rtimes U_q^+(\mathfrak{g})$ is the algebra generated by $\mathcal{A}$ and $U_q^+(\mathfrak{g})$ (and isomorphic to $\mathcal{A} \otimes U_q^+(\mathfrak{g})$ as a vector space) subject to the relations

$$E_i \cdot a = E_i(a) + q^{(\alpha_i,\nu)} a \cdot E_i$$

for all $a \in \mathcal{A}_\nu$, $i \in I$. In particular, if $\mathcal{A}$ is a PBW algebra, then so is $\mathcal{A} \rtimes U_q^+(\mathfrak{g})$. □

Remark 2.25. In fact if $\mathcal{A} = \bigoplus_\nu \mathcal{A}_\nu$ is a $U_q(\mathfrak{g})$-module algebra with $K_i|_{\mathcal{A}_\nu} = q^{(\alpha_i,\nu)}$, then the braided cross product $\mathcal{A} \rtimes U_q^+(\mathfrak{g})$ is simply the subalgebra of the ordinary cross product $\mathcal{A} \rtimes U_q(\mathfrak{g})$ generated by $\mathcal{A}$ and $U_q^+(\mathfrak{g})$. Moreover, $\mathcal{A} \rtimes U_q(\mathfrak{g}) \cong \mathcal{A} \rtimes U_q^+(\mathfrak{g}) \otimes U_q^{\leq 0}(\mathfrak{g})$ as a vector space. □

Using Lemma 2.22 and [17, §3.1.5] we obtain for any $U_q(\mathfrak{g})$-module algebra $\mathcal{A}$ and $a, b \in \mathcal{A}$

$$E_i^{(r)}(ab) = \sum_{p=0}^{r} q_i^{p(r-p)} E_i^{(r-p)}(K_i^p(a)) E_i^{(p)}(b),$$

$$F_i^{(r)}(ab) = \sum_{p=0}^{r} q_i^{-p(r-p)} F_i^{(p)}(a) K_i^{-p}(F_i^{(r-p)}(b)), \tag{2.19}$$

where $Y_i^{(r)} := ([r]_{q_i}!)^{-1} Y_i^r$. In particular, if $K_i(a) = v_i a$, then

$$E_i^{(r)}(ab) = \sum_{p=0}^{r} q_i^{p(r-p)} v_i^p E_i^{(r-p)}(a) E_i^{(p)}(b). \tag{2.20}$$

Given $\mathbf{i} = (i_1, \ldots, i_k) \in I^k$ and $m = (m_1, \ldots, m_k) \in \mathbb{Z}_{\geq 0}^k$, let $E_{\mathbf{i}}^{(m)} = E_{i_1}^{(m_1)} \cdots E_{i_k}^{(m_k)}$. Using an obvious induction, we immediately obtain from (2.20)

$$E_{\mathbf{i}}^{(m)}(ab) = \sum_{m', m'' \in \mathbb{Z}_{\geq 0}^k : m' + m'' = m} q^{\frac{1}{2}\sum_{1 \leq r,s \leq k} m_r' m_s'' (\alpha_{i_r}, \alpha_{i_s})} v^{m''} E_{\mathbf{i}}^{(m')}(a) E_{\mathbf{i}}^{(m'')}(b), \tag{2.21}$$

where $v^{m''} = \prod_{r=1}^k v_{i_r}^{m_r''}$.

Following Kashiwara and Lusztig [15, 17], define for all $i \in I$ and for all $u \in U_q^+(\mathfrak{g})$ the elements $r_i(u)$, ${}_ir(u)$ by

$$[u, F_i] = \frac{K_i \cdot {}_ir(u) - r_i(u) \cdot K_i^{-1}}{q_i - q_i^{-1}}. \tag{2.22}$$

Lemma 2.26 ([17, Lemma 1.2.15, Proposition 3.1.6]). For all $i, j \in I$, $u, v \in U_q^+(\mathfrak{g})$

(i) ${}_ir(u) = r_i(u^*)^*$ and $r_i({}_jr(u)) = {}_jr(r_i(u))$;
(ii) $r_i(E_j) = \delta_{ij}$ and $r_i(uv) = r_i(u) K_i^{-1} v K_i + u r_i(v)$;
(iii) If $u \in \ker {}_ir$ then $(\mathrm{ad} F_i)(u) = (q_i - q_i^{-1})^{-1} r_i(u)$;
(iv) ${}_ir(\mathrm{ad} E_j(u)) = q_i^{-a_{ij}} E_j \cdot {}_ir(u) - q_i^{a_{ij}} K_j \cdot {}_ir(u) \cdot K_j^{-1} \cdot E_j$;
(v) $\bigcap_{i \in I} \ker {}_ir = \bigcap_{i \in I} \ker r_i = \mathbb{C}(q)$. □

Recall that for any $J \subset I$ the parabolic subalgebra $\mathfrak{p}_J$ of $\mathfrak{g}$ is the Lie subalgebra generated by $\mathfrak{g}_+$ and f_j, $j \in J$. Let $U_q(\mathfrak{p}_J)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $U_q^+(\mathfrak{g})$ and by the F_j, $K_j^{\pm 1}$, $j \in J$. Let $U_q(\mathfrak{g}_J) = \langle E_j, F_j, K_j^{\pm 1} : j \in J \rangle$ and define

$$U_q(\mathfrak{r}_J) = \{u \in U_q^+(\mathfrak{g}) : {}_j r(u) = 0, \ \forall j \in J\}.$$

Clearly $U_q(\mathfrak{p}_J)$ is a Hopf subalgebra of $U_q(\mathfrak{g})$, $U_q(\mathfrak{g}_J)$ is a Hopf subalgebra of $U_q(\mathfrak{p}_J)$, and $U_q(\mathfrak{r}_J)$ is a subalgebra of $U_q(\mathfrak{p}_J)$. The following corollary is an immediate consequence of Lemma 2.26.

Corollary 2.27 (Quantum Levi factorization).

(i) $U_q(\mathfrak{r}_J)$ is preserved by the adjoint action of $U_q(\mathfrak{g}_J)$ on $U_q(\mathfrak{p}_J)$. In particular, $U_q(\mathfrak{r}_J)$ is a $U_q(\mathfrak{g}_J)$-module algebra.

(ii) $U_q(\mathfrak{p}_J) = U_q(\mathfrak{r}_J) \rtimes U_q(\mathfrak{g}_J)$. □

3 Folding $(\mathfrak{so}_{2n+2}, \mathfrak{sp}_{2n})$ and proof of Theorems 1.8 and 1.12

3.1 The algebras $\mathcal{U}_{q,n}^+$ and $\mathcal{U}_{q,n}$

In what follows we take $I = \{-1, 0, \dots, n-1\}$ for $\mathfrak{g} = \mathfrak{so}_{2n+2}$ so that σ interchanges -1 and 0 and fixes each $i = 1, \dots, n-1$. Accordingly, we set $I/\sigma := \{0, 1, \dots, n-1\}$ for $\mathfrak{g}^{\sigma^\vee} = \mathfrak{sp}_{2n}$.

Let $\mathcal{U}_{q,n}$ be the associative $\mathbb{C}(q)$-algebra generated by $U_q(\mathfrak{sl}_n)$ and w and z subject to the following relations

$$[F_i, w] = 0 = [F_i, z], \quad K_i w K_i^{-1} = q^{-2\delta_{i,1}} w, \quad K_i z K_i^{-1} = q^{-\delta_{i,2}} z, \tag{3.1a}$$

$$[E_i, w] = 0, \quad i \neq 1, \quad [E_i, z] = 0, \quad i \neq 2, \quad [z, w] = 0, \tag{3.1b}$$

$$[E_1, [E_1, [E_1, w]_{q^{-2}}]]_{q^2} = 0 = [E_2, [E_2, z]_q]_{q^{-1}}, \tag{3.1c}$$

$$[w, [w, E_1]_{q^2}]_{q^{-2}} = -hwz, \quad [z, [E_2, [E_1, w]_{q^2}]_q]_q = [w, [E_1, [E_2, z]_q]_q]_{q^2}, \tag{3.1d}$$

$$2[z, [z, E_2]_q]_{q^{-1}} = h(z[E_1, E_2]_q w + w[E_2, E_1]_{q^{-1}} z + w E_1 [z, E_2]_q + [E_2, z]_{q^{-1}} E_1 w), \tag{3.1e}$$

where $h = q - q^{-1}$ and we abbreviate $[a, b]_v = ab - vba$ and $[a, b] = [a, b]_1 = ab - ba$ (with the convention that $E_2 = 0$ if $n = 2$) and the E_i, F_i, and $K_i^{\pm 1}$, $1 \le i \le n-1$, are Chevalley generators of $U_q(\mathfrak{sl}_n)$. Let $\mathcal{U}_{q,n}^+$ be the associative $\mathbb{C}(q)$-algebra generated by the E_i, $1 \le i \le n-1$ and by w and z subject to the relations (3.1b)–(3.1e). Let V be the standard

$U_q(\mathfrak{sl}_n)$-module. We denote $S_q(V \otimes V) := S_\Psi(V \otimes V)$ in the notation of Theorem 1.11, where $\Psi : V^{\otimes 4} \to V^{\otimes 4}$ is the $U_q(\mathfrak{sl}_n)$-equivariant map given by (1.4).

The following theorem is the main result of the section.

Theorem 3.1.

(i) $S_q(V \otimes V)$ is a PBW algebra on any ordered basis of $V \otimes V$.

(ii) The algebra $\mathcal{U}^+_{q,n}$ is isomorphic to the braided cross product $S_q(V \otimes V) \rtimes U^+_q(\mathfrak{sl}_n)$ and in particular is PBW.

(iii) The algebra $\mathcal{U}_{q,n}$ is isomorphic to the cross product $S_q(V \otimes V) \rtimes U_q(\mathfrak{sl}_n)$ and also to the tensor product of $\mathcal{U}^+_{q,n}$ and the subalgebra of $U_q(\mathfrak{sl}_n)$ generated by the F_i, $K_i^{\pm 1}$, $i \in I$. In particular, $\mathcal{U}^+_{q,n}$ is a subalgebra of $\mathcal{U}_{q,n}$.

(iv) The assignment $w \mapsto E_0$, $z \mapsto 0$ defines a homomorphism $\mu : \mathcal{U}_{q,n} \to U_q(\mathfrak{sp}_{2n})$. Its image is the (parabolic) subalgebra of $U_q(\mathfrak{sp}_{2n})$ generated by $U^+_q(\mathfrak{sp}_{2n})$ and $K_i^{\pm 1}$, F_i, $1 \le i \le n-1$.

(v) The assignment $w \mapsto E_0 E_{-1}$ and

$$z \mapsto \frac{1}{qh}([E_{-1}, [E_1, E_0]_q]_q + [E_0, [E_1, E_{-1}]_q]_q)$$

defines an algebra homomorphism $\hat{\iota} : \mathcal{U}_{q,n} \to U_q(\mathfrak{so}_{2n+2})$. Its image is contained in the (parabolic) subalgebra of $U_q(\mathfrak{so}_{2n+2})$ generated by $U^+_q(\mathfrak{so}_{2n+2})$ and $K_i^{\pm 1}$, F_i, $1 \le i \le n-1$.

(vi) The assignments

$$T_i(w) = \begin{cases} (q+q^{-1})^{-1}[[w, E_1]_{q^{-2}}, E_1], & i = 1, \\ w, & i \neq 1, \end{cases} \qquad T_i(z) = \begin{cases} [z, E_2]_{q^{-1}}, & i = 2, \\ z, & i \neq 2 \end{cases} \tag{3.2}$$

extend Lusztig's action (2.9) of the braid group $Br_{\mathfrak{sl}_n}$ on $U_q(\mathfrak{sl}_n)$ to an action on $\mathcal{U}_{q,n}$ by algebra automorphisms. Moreover, μ and $\hat{\iota}$ are $Br_{\mathfrak{sl}_n}$-equivariant. □

This theorem is proved in the rest of Section 3.

Remark 3.2. It is interesting to observe a complement to Theorem 3.1(iv): the quotient algebra $\mathcal{U}_{q,n}/\langle w \rangle$ is isomorphic to the (parabolic) subalgebra of $U_q(\mathfrak{so}_{2n})$ generated by $U^+_q(\mathfrak{so}_{2n})$ and K_i, K_i^{-1}, F_i, $i = 1, \ldots, n-1$. □

Remark 3.3. Note that the subalgebra $S_q(V \otimes V)$ of $\mathcal{U}_{q,n}$ is not preserved by the action of $Br_{\mathfrak{sl}_n}$. For example, $T_3^2(w), T_2^2(z) \notin S_q(V \otimes V)$. □

Remark 3.4. The image of $S_q(V \otimes V)$ under μ is isomorphic to $U_q(\mathfrak{r}_J) \subset U_q(\mathfrak{sp}_{2n})$, as defined in Section 2.7, where $J = \{1, \ldots, n-1\}$. Furthermore, $\hat{\imath}(S_q(V \otimes V))$ is a quantum deformation of the coordinate ring of $\mathcal{M}_{\leq 2}$, where $\mathcal{M}_{\leq 2}$ is the variety of all matrices with the symmetric part of rank at most 2. Moreover, both homomorphisms are compatible with the cross-product structure, for example, $\hat{\imath}(\mathcal{U}_{q,n}) = \hat{\imath}(S_q(V \otimes V)) \rtimes U_q(\mathfrak{sl}_n)$. □

3.2 Structure of algebra $S_q(V \otimes V)$

Let $\{v_i\}$, $1 \leq i \leq n$ be the standard basis of the n-dimensional $U_q(\mathfrak{sl}_n)$ module V. Let $X_{i,j} = v_i \otimes v_j$ be the standard basis of $V \otimes V$. In particular, we have

$$\begin{aligned} E_i(v_j) &= \delta_{i,j-1} v_{j-1}, \quad E_i(X_{j,k}) = \delta_{i,j-1} X_{j-1,k} + \delta_{i,k-1} q^{\delta_{i,j} - \delta_{i,j-1}} X_{j,k-1}, \\ F_i(v_j) &= \delta_{i,j} v_{j+1}, \quad F_i(X_{j,k}) = \delta_{i,j} q^{\delta_{i,k-1} - \delta_{i,k}} X_{j+1,k} + \delta_{i,k} X_{j,k+1} \end{aligned} \tag{3.3}$$

for all $1 \leq i < n$ and for all $1 \leq j, k \leq n$. Let $T : V \otimes V \to V \otimes V$ be the $\mathbb{C}(q)$-linear map defined by

$$T(X_{ij}) = q^{-\delta_{ij}} X_{ji}, \quad T(X_{ji}) = q^{\delta_{ij}} X_{ij} - (q - q^{-1}) X_{ji}$$

for all $1 \leq i \leq j \leq n$. It is well known that T is an isomorphism of $U_q(\mathfrak{sl}_n)$-modules, satisfies $(T - q^{-1})(T + q) = 0$ and the braid equation on $V^{\otimes 3}$. Define $\Psi_i : V^{\otimes k} \to V^{\otimes k}$ by $\Psi_i = 1^{\otimes i-1} \otimes T \otimes 1^{\otimes k-i-1}$. Then Ψ_i are isomorphisms of $U_q(\mathfrak{gl}_n)$-modules. Let

$$\Psi = \Psi_2 \Psi_1 \Psi_3 \Psi_2 + (q - q^{-1})(\Psi_1 \Psi_2 \Psi_1 + \Psi_1 \Psi_3 \Psi_2) + (q - q^{-1})^2 \Psi_1 \Psi_2.$$

It will be convenient for us to regard Ψ as an element of the Hecke algebra $H(S_n)$. Recall that $H(S_n)$ is the quotient of the group algebra over $\mathbb{C}(q)$ of the braid group $Br_{\mathfrak{sl}_n}$ by the ideal generated by $(T_i - q^{-1})(T_i + q)$, $1 \leq i \leq n-1$. In particular, $V^{\otimes n}$ is an $H(S_n)$-module. A well-known result of Jimbo [12, Proposition 3] provides a quantum analogue of Schur–Weyl duality, namely the image of $U_q(\mathfrak{gl}_n)$ in $\mathrm{End}\, V^{\otimes n}$ is the centralizer of the image of $H(S_n)$ and vice versa. It is also well known that the Hecke algebra $H(S_n)$ is semi-simple.

Proof of Proposition 1.10. Since $H(S_n)$ is semi-simple, to prove part (i) (respectively, part (ii)) of Proposition 1.10, it is sufficient to show that these identities hold in

any simple finite dimensional representation of the Hecke algebra $H(S_6)$ (respectively, $H(S_4)$). For, we use a realization of the multiplicity free direct sum of all simple finite-dimensional $H(S_n)$-modules, known as the Gelfand model, constructed in [1, Theorem 1.2.2], which we briefly review for the reader's convenience.

Let I_n be the set of involutions in the symmetric group S_n and let $I_{n,k} \subset I_n$ be the set of all involutions containing k cycles of length 2 so each $I_{n,k}$ is an orbit for the action of S_n on I_n. Given $w \in I_{n,k}$, one defines $\hat{\ell}(w) = \min\{\ell(v) : vwv^{-1} = \prod_{i=1}^{k} s_{2i-1}\}$. Let $\mathcal{V}_n^{(k)} = \mathrm{Span}\{C_w : w \in I_{n,k}\}$ and set $\mathcal{V}_n = \bigoplus_{0 \le k \le n/2} \mathcal{V}_n^{(k)}$. Then

$$T_i(C_w) = \begin{cases} -qC_w, & s_i w s_i = w,\ \ \ell(ws_i) < \ell(w), \\ q^{-1}C_w, & s_i w s_i = w,\ \ \ell(w) < \ell(ws_i), \\ qC_{s_i w s_i} - (q - q^{-1})C_w, & s_i w s_i \neq w,\ \ \hat{\ell}(w) < \hat{\ell}(s_i w s_i), \\ q^{-1}C_{s_i w s_i}, & s_i w s_i \neq w,\ \ \hat{\ell}(s_i w s_i) < \hat{\ell}(w) \end{cases}$$

defines a representation of the Hecke algebra $H(S_n)$ on $\mathcal{V}_n$ which realizes the Gelfand model for $H(S_n)$. Clearly, $\mathcal{V}_n^{(k)}$ is an $H(S_n)$-submodule of $\mathcal{V}_n$ and $\mathcal{V}_n^{(0)}$ is the trivial $H(S_n)$-module. A straightforward computation then shows that the matrix of Ψ on $\mathcal{V}_4^{(1)}$ with respect to the basis $C_{(i,j)}$, $1 \le i < j \le 4$, is

$$\begin{pmatrix} 0 & q^3h & -q^3h & 0 & 0 & q^4 \\ 0 & -q^2 & 0 & 0 & 0 & 0 \\ -q^{-3}h & -q^{-2}h^2 & -h^2 & -q^{-2} & 0 & qh \\ 0 & 0 & -q^2 & 0 & 0 & 0 \\ -q^{-3}h & -q^{-2}h^2 & -h^2 & 0 & -q^{-2} & -q^{-1}h \\ q^{-4} & q^{-3}h & q^{-1}h & 0 & 0 & 0 \end{pmatrix},$$

while $\Psi|_{\mathcal{V}_4^{(2)}} = \mathrm{id}$. Here, we abbreviate $h = q - q^{-1}$. Part (ii) is now straightforward. Part (i) is checked similarly and we omit the details.

It remains to prove (iii). Let $\tau = \tau_{V\otimes V, V\otimes V}$ be the permutation of factors. Note that by the quantum Schur–Weyl duality, the vector subspace $(V^{\otimes m})^+$ of $U_q(\mathfrak{sl}_n)$-highest weight vectors in $V^{\otimes m}$ is isomorphic to the direct sum of simple $H(S_m)$-modules S^λ, where λ runs over the set of all partitions of m with at most $\dim V$ nonzero parts. In particular, if $\dim V \ge m$ then $(V^{\otimes m})^+ \cong \mathcal{V}_m \cong \bigoplus_\lambda S^\lambda$. To complete the argument, we need the following result, which is an immediate consequence of Schur–Weyl duality.

Lemma 3.5. Let $\Psi' \in H(S_m)$ be such that Ψ' is specializable at $q=1$ on $\mathcal{V}_m$ with respect to the basis C_w, $w \in I_m$ and suppose that $\dim \Psi'(S^\lambda) = \dim \Psi'|_{q=1}(S^\lambda)$ for all partitions λ of m. Then for any V, $\dim \Psi'(V^{\otimes m}) = \dim \Psi'|_{q=1}(V^{\otimes m})$. □

Recall that $\mathcal{V}_4^{(0)} = S^{(4)}$ and it is easy to see that $\mathcal{V}_4^{(1)} = S^{(2,1^2)} \oplus S^{(3,1)}$ while $\mathcal{V}_4^{(2)} = S^{(2,2)} \oplus S^{(1^4)}$. Therefore, $(\Psi - 1)(S^\lambda) = (\tau - 1)(S^\lambda) = 0$ for $\lambda \in \{(4), (2,2), (1^4)\}$. Finally, one can easily show that $\dim(\Psi - 1)(S^\lambda) = \dim(\tau - 1)(S^\lambda) = 2$ for $\lambda \in \{(2,1^2), (3,1)\}$. ■

We can now prove the first part of Theorem 3.1.

Proof of part (i) of Theorem 3.1. By Proposition 1.10, Ψ satisfies the braid relation and condition (ii) of Theorem 1.11. Since Ψ_i specializes to the transposition of factors with respect to the standard basis of $V^{\otimes 4}$, it follows that Ψ specializes to the permutation of factors in $(V^{\otimes 2})^{\otimes 2}$. It remains to apply Theorem 1.11. ■

Proposition 3.6. The algebra $S_q(V \otimes V)$ is generated by the elements X_{ij}, $1 \le i, j \le n$, subject to the following relations for all $1 \le i \le j \le k \le l \le n$

$$
\begin{aligned}
X_{ij}X_{kl} &= q^-_{ikj}q^-_{ilj}X_{kl}X_{ij} + hq^{-d_{ij}}(q^-_{ikj}X_{kj}X_{il} + q^-_{ikl}q^-_{ilj}X_{lj}X_{ki}) \\
&\quad + h^2(q^-_{ikl}q^-_{jil}X_{jl}X_{ki} + q^-_{ikj}q^-_{jil}X_{kj}X_{li}) + h^3q^{-d_{ik}}q^-_{jil}X_{jk}X_{li}, \\
X_{ij}X_{lk} &= q^-_{ikj}q^-_{ilj}X_{lk}X_{ij} + hq^{-d_{ij}}(q^-_{ilj}X_{lj}X_{ik} + q^-_{ikj}q^\pm_{kli}X_{kj}X_{li}) + h^2q^-_{jil}q^\pm_{lki}X_{jk}X_{li}, \\
X_{ji}X_{kl} &= q^-_{ikj}q^-_{ilj}X_{kl}X_{ji} + hq^{-d_{il}}(q^-_{ikl}X_{jl}X_{ki} + q^-_{ikj}X_{kj}X_{li}) + h^2q^-_{kil}X_{jk}X_{li}, \\
X_{ji}X_{lk} &= q^-_{ikj}q^-_{ilj}X_{lk}X_{ji} + hq^{-d_{ik}}(q^-_{ilj}X_{lj}X_{ki} + q^\pm_{kli}X_{jk}X_{li}), \\
X_{ik}X_{jl} &= q^-_{jil}q^\pm_{jkl}X_{jl}X_{ik} + hq^{-d_{ik}}(-q^-_{ilj}X_{kl}X_{ij} + q^-_{jil}q^-_{jlk}X_{lk}X_{ji} + q^\pm_{kji}X_{jk}X_{il}) \\
&\quad + h^2q^-_{jik}(q^{-d_{il}+d_{jk}}X_{jk}X_{li} - X_{kj}X_{il} + q^-_{ilj}X_{kl}X_{ji}), \\
X_{ik}X_{lj} &= q^-_{jil}q^\pm_{jkl}X_{lj}X_{ik} + hq^{-d_{ij}}q^\pm_{jki}q^\pm_{jli}X_{jk}X_{li}, \\
X_{ki}X_{jl} &= q^-_{jil}q^\pm_{jkl}X_{jl}X_{ki} + hq^{d_{jk}}q^-_{jil}X_{jk}X_{li}, \\
X_{ki}X_{lj} &= q^-_{jil}q^\pm_{jkl}X_{lj}X_{ki} + hq^{-d_{ij}}q^\pm_{jli}X_{kj}X_{li}, \\
X_{il}X_{jk} &= q^\pm_{lji}q^\pm_{lki}X_{jk}X_{il} + hq^{-d_{il}}q^-_{ikj}(q^{-d_{ij}+d_{kl}}X_{kl}X_{ji} - X_{lk}X_{ij}) \\
&\quad + h^2(q^-_{jil}q^{-d_{ik}+d_{jl}}X_{jl}X_{ki} - X_{lj}X_{ik}),
\end{aligned}
$$

$$X_{il}X_{kj} = q^{\pm}_{lji}q^{\pm}_{lki}X_{kj}X_{il} + hq^{d_{jk}}q^{-}_{jil}(q^{-d_{ik}+d_{jl}}X_{jl}X_{ki} - X_{lj}X_{ik}),$$

$$X_{li}X_{jk} = q^{\pm}_{lji}q^{\pm}_{lki}X_{jk}X_{li},$$

$$X_{li}X_{kj} = q^{\pm}_{lji}q^{\pm}_{lki}X_{kj}X_{li},$$

where $h = q - q^{-1}$, $q^{+}_{abc} = q^{\delta_{ab}+\delta_{bc}}$, $q^{-}_{abc} = (q^{+}_{abc})^{-1}$, and $q^{\pm}_{abc} = q^{\delta_{ab}-\delta_{bc}}$. □

Proof. One can show that for all $1 \le i \le j \le k \le l \le n$:

$$\Psi(X_{ij}X_{kl}) = q^{-}_{ikj}q^{-}_{ilj}X_{kl}X_{ij} + hq^{-}_{ikj}(q^{-\delta_{jl}}X_{ki}X_{lj} + q^{-\delta_{ij}}X_{kj}X_{il}) + h^2q^{-}_{ikj}X_{ki}X_{jl},$$

$$\Psi(X_{ij}X_{lk}) = q^{-}_{ikj}q^{-}_{ilj}X_{lk}X_{ij} + hq^{-}_{ilj}(q^{-\delta_{jk}}X_{li}X_{kj} + q^{-\delta_{ij}}X_{lj}X_{ik}) + h^2q^{-}_{ilj}X_{li}X_{jk},$$

$$\Psi(X_{ik}X_{jl}) = q^{-}_{ilk}q^{\pm}_{kji}X_{jl}X_{ik} + h(q^{-\delta_{kl}}q^{\pm}_{kji}X_{ji}X_{lk} + q^{-\delta_{ij}}q^{\pm}_{jki}X_{jk}X_{il} - q^{-\delta_{ik}}q^{-}_{ilj}X_{kl}X_{ij}),$$
$$+ h^2(q^{\pm}_{kji}X_{ji}X_{kl} - q^{-\delta_{ik}-\delta_{jl}}X_{ki}X_{lj} - q^{-}_{jik}X_{kj}X_{il}) - h^3q^{-\delta_{ik}}X_{ki}X_{jl},$$

$$\Psi(X_{ik}X_{lj}) = q^{-}_{ilk}q^{\pm}_{kji}X_{lj}X_{ik} + hq^{\delta_{jk}}q^{-}_{ilk}X_{li}X_{jk},$$

$$\Psi(X_{il}X_{jk}) = q^{-}_{jik}X_{jk}X_{il}q^{+}_{jlk} + h(-q^{-\delta_{il}}q^{-}_{ikj}X_{lk}X_{ij} + q^{-\delta_{ij}}q^{+}_{jlk}X_{ji}X_{kl}),$$
$$- h^2(q^{-\delta_{il}-\delta_{jk}}X_{li}X_{kj} + q^{-}_{jil}X_{lj}X_{ik}) - h^3q^{-\delta_{il}}X_{li}X_{jk},$$

$$\Psi(X_{il}X_{kj}) = q^{-}_{jik}X_{kj}X_{il}q^{+}_{jlk} - h(q^{-\delta_{il}}q^{\pm}_{kji}X_{lj}X_{ik} - q^{-\delta_{ik}}q^{+}_{jlk}X_{ki}X_{jl}) - h^2q^{-\delta_{il}+\delta_{jk}}X_{li}X_{jk},$$

$$\Psi(X_{ji}X_{kl}) = q^{-}_{ikj}q^{-}_{ilj}X_{kl}X_{ji} + hq^{-}_{ikj}(q^{\delta_{ij}}X_{ki}X_{jl} + q^{-\delta_{il}}X_{kj}X_{li}),$$

$$\Psi(X_{ji}X_{lk}) = q^{-}_{ikj}q^{-}_{ilj}X_{lk}X_{ji} + hq^{-}_{ilj}(q^{\delta_{ij}}X_{li}X_{jk} + q^{-\delta_{ik}}X_{lj}X_{ki}),$$

$$\Psi(X_{jk}X_{il}) = q^{\pm}_{ijl}q^{\pm}_{ikl}X_{il}X_{jk} + h(q^{\delta_{ij}}(q^{\pm}_{ikj}X_{ik}X_{jl} + q^{\pm}_{ikl}X_{ij}X_{lk}) - q^{-\delta_{il}}(q^{-}_{kjl}X_{kl}X_{ji} + q^{\pm}_{ikl}X_{jl}X_{ik}))$$
$$+ h^2(X_{jk}X_{il} + q^{-\delta_{il}-\delta_{jk}}X_{kj}X_{li} + q^{\pm}_{ijk}X_{ki}X_{jl} + q^{\pm}_{ikl}X_{ji}X_{lk} - q^{+}_{jik}X_{ij}X_{kl})$$
$$- h^3q^{\delta_{ik}}X_{ji}X_{kl},$$

$$\Psi(X_{jk}X_{li}) = q^{-}_{jlk}q^{+}_{jik}X_{li}X_{jk},$$

$$\Psi(X_{jl}X_{ik}) = q^{\pm}_{ijk}q^{+}_{ilk}X_{ik}X_{jl} + h(q^{-\delta_{jl}}q^{-}_{ikj}X_{lk}X_{ji} - q^{\delta_{ij}}X_{ij}X_{kl}q^{+}_{ilk} + q^{-\delta_{ik}}X_{jk}X_{il}q^{+}_{ilk})$$
$$- h^2(q^{-\delta_{ik}-\delta_{jl}}X_{lj}X_{ki} + q^{\pm}_{ijl}X_{li}X_{jk} + q^{+}_{ilk}X_{ji}X_{kl}),$$

$$\Psi(X_{jl}X_{ki}) = q^{\pm}_{ijk}q^{+}_{ilk}X_{ki}X_{jl} - hq^{-\delta_{jl}}q^{+}_{jik}X_{li}X_{jk},$$

$$\Psi(X_{ki}X_{jl}) = q^{-}_{ilk}q^{\pm}_{kji}X_{jl}X_{ki} + h(q^{-\delta_{il}}q^{\pm}_{kji}X_{jk}X_{li} - q^{-\delta_{ij}}q^{-}_{ilj}X_{kl}X_{ji} + q^{-\delta_{ij}}X_{ji}X_{kl}q^{+}_{ikj})$$
$$- h^2(X_{ki}X_{jl} + q^{-}_{jil}X_{kj}X_{li}),$$

$$\Psi(X_{ki}X_{lj}) = q^{-}_{ilk}q^{\pm}_{kji}X_{lj}X_{ki} + hq^{\delta_{ik}}q^{-}_{ilk}X_{li}X_{kj},$$

$$\begin{aligned}\Psi(X_{kj}X_{il}) = {} & q^{-}_{jlk}q^{+}_{jik}X_{il}X_{kj} - h(q^{\delta_{jk}}q^{-}_{ilk}X_{jl}X_{ki} + q^{\delta_{ij}}(q^{-}_{ilj}X_{kl}X_{ij} - q^{+}_{ikj}X_{ij}X_{kl}) - q^{-\delta_{jl}}q^{+}_{jik}X_{ik}X_{lj}) \\ & - h^2(q^{-\delta_{il}+\delta_{jk}}X_{jk}X_{li} + X_{kj}X_{il} - q^{-}_{ilj}X_{kl}X_{ji} + q^{\pm}_{ijl}X_{ki}X_{lj} + q^{+}_{ikj}X_{ji}X_{kl}) \\ & + h^3q^{-\delta_{il}}X_{kj}X_{li},\end{aligned}$$

$$\Psi(X_{kj}X_{li}) = q^{-}_{jlk}q^{+}_{jik}X_{li}X_{kj},$$

$$\begin{aligned}\Psi(X_{kl}X_{ij}) = {} & q^{+}_{ikj}q^{+}_{ilj}X_{ij}X_{kl} - h(q^{-\delta_{kl}}q^{\pm}_{kji}X_{lj}X_{ki} + q^{-\delta_{ij}}q^{+}_{ilj}X_{kj}X_{il}) \\ & - h^2(q^{\pm}_{ikl}X_{li}X_{kj} + q^{+}_{ilj}X_{ki}X_{jl}),\end{aligned}$$

$$\Psi(X_{kl}X_{ji}) = X_{ji}X_{kl}q^{+}_{ikj}q^{+}_{ilj} - h(q^{\delta_{ij}}q^{+}_{ilj}X_{ki}X_{jl} + q^{-\delta_{kl}}q^{+}_{jik}X_{li}X_{kj}),$$

$$\begin{aligned}\Psi(X_{li}X_{jk}) = {} & q^{-}_{jik}X_{jk}X_{li}q^{+}_{jlk} - h(q^{-\delta_{ij}}q^{-}_{ikj}X_{lk}X_{ji} - q^{-\delta_{ij}}q^{+}_{ilj}X_{ji}X_{lk}) \\ & - h^2(X_{li}X_{jk} + q^{-}_{jik}X_{lj}X_{ki}),\end{aligned}$$

$$\Psi(X_{li}X_{kj}) = q^{-}_{jik}X_{kj}X_{li}q^{+}_{jlk} - h(q^{-\delta_{ij}}q^{\pm}_{jki}X_{lj}X_{ki} - q^{-\delta_{ik}}q^{+}_{ilk}X_{ki}X_{lj}) - h^2X_{li}X_{kj},$$

$$\begin{aligned}\Psi(X_{lj}X_{ik}) = {} & X_{ik}X_{lj}q^{\pm}_{ijk}q^{+}_{ilk} - h(q^{\delta_{ij}}q^{-}_{ikj}X_{lk}X_{ij} - q^{\delta_{ij}}q^{+}_{ilj}X_{ij}X_{lk} + q^{-\delta_{ik}}q^{+}_{jlk}X_{jk}X_{li}) \\ & - h^2(X_{lj}X_{ik} - q^{-}_{ikj}X_{lk}X_{ji} + q^{\pm}_{ijk}X_{li}X_{kj} - q^{+}_{ilj}X_{ji}X_{lk}) + h^3q^{-\delta_{ik}}X_{lj}X_{ki},\end{aligned}$$

$$\Psi(X_{lj}X_{ki}) = q^{\pm}_{ijk}q^{+}_{ilk}X_{ki}X_{lj} - hq^{\delta_{ij}}q^{\pm}_{ikj}X_{li}X_{kj},$$

$$\begin{aligned}\Psi(X_{lk}X_{ij}) = {} & q^{+}_{ikj}q^{+}_{ilj}X_{ij}X_{lk} - h(q^{-\delta_{ij}}q^{+}_{ikj}X_{lj}X_{ik} + q^{-\delta_{ij}}q^{+}_{jlk}X_{kj}X_{li}) \\ & + h^2(q^{\pm}_{kji}X_{lj}X_{ki} - X_{li}X_{jk}q^{+}_{ikj} - X_{ki}X_{lj}q^{+}_{ilk}) + h^3q^{\delta_{ik}}X_{li}X_{kj},\end{aligned}$$

$$\Psi(X_{lk}X_{ji}) = X_{ji}X_{lk}q^{+}_{ikj}q^{+}_{ilj} - h(q^{\delta_{ij}}q^{+}_{ikj}X_{li}X_{jk} + q^{\delta_{ij}}q^{+}_{ilk}X_{ki}X_{lj}) + h^2q^{+}_{jik}X_{li}X_{kj}.$$

Since the quotient $\mathcal{S} = (V \otimes V)/(\Psi - 1)(V \otimes V)$ is a flat deformation of $S^2(V \otimes V)$, the canonical images of the $X_{ab}X_{cd}$ with $(a, b) \preccurlyeq (c, d)$ (where the order is defined by $(a, b) \prec (c, d)$, if $\min(c, d) < \min(a, b)$ or $\min(a, b) = \min(c, d)$ and $\max(c, d) < \max(a, b)$, while $(i, j) \preccurlyeq (j, i)$, for all $i \leq j$) form a basis of $\mathcal{S}$. Using this basis we obtain the formulae in the proposition from the above formulae for Ψ. ■

Remark 3.7. It is easy to check that the quotient of $S_q(V \otimes V)$ by the ideal generated by the elements $X_{ij} - qX_{ji}$, $1 \leq i < j \leq n$ (respectively, by the elements $X_{ij} + q^{-1}X_{ji}$, $1 \leq i < j \leq n$ and X_{ii}, $1 \leq i \leq n$) is isomorphic to the algebra $S_q(S^2V)$ (respectively, $S_q(\Lambda^2V)$; cf. [14, Teorem 0.2] and [22, (1.1)], respectively, and also [9, 19, 26]). □

We can now prove Corollary 1.15.

Proof. The algebra $S_q(V \otimes V)$ is clearly optimal specializable with respect to its PBW basis on the X_{ij}, $1 \le i, j \le n$ with the total order defined as in Proposition 3.6. It remains to apply Propositions 2.18(ii) and 3.6. ■

3.3 Cross product structure of $\mathcal{U}^+_{q,n}$ and $\mathcal{U}_{q,n}$

In this section, we will use the usual numbering of nodes in the Dynkin diagram of $\mathfrak{so}_{2n+2}$, that is, the simple root α_{n-1} corresponds to the triple node. Retain the notations of Section 3.2.

Proposition 3.8 (Theorem 3.1(ii,iii)).

(i) The natural homomorphism $\mathcal{U}^+_{q,n} \to \mathcal{U}_{q,n}$ is injective and as vector spaces $\mathcal{U}_{q,n} \cong \mathcal{U}^+_{q,n} \otimes U_q^{\le 0}(\mathfrak{sl}_n)$, where $U_q^{\le 0}(\mathfrak{sl}_n)$ is defined as in Section 2.2.

(ii) The assignment $w \mapsto w' := X_{n,n}$, $z \mapsto z' := q(X_{n-1,n} - qX_{n,n-1})$ defines isomorphisms of algebras $\psi : \mathcal{U}_{q,n} \to S_q(V \otimes V) \rtimes U_q(\mathfrak{sl}_n)$ and $\psi_+ : \mathcal{U}^+_{q,n} \to S_q(V \otimes V) \rtimes U_q^+(\mathfrak{sl}_n)$. □

Proof. First we prove that the elements w', z' satisfy the relations (3.1b) and (3.1c). It follows from (3.3) that w' (respectively, z') is a lowest weight vector of the $U_q(\mathfrak{sl}_n)$-submodule of $V \otimes V$ isomorphic to $V_{2\varpi_1}$ (respectively, V_{ϖ_2}), where ϖ_i is the ith fundamental weight of $\mathfrak{sl}_n$. In particular, we have

$$E_i^{2\delta_{i,n-1}+1}(w') = 0 = E_i^{\delta_{i,n-2}+1}(z'), \qquad F_i(w') = F_i(z') = 0. \tag{3.4}$$

Using Lemma 2.24(ii) we immediately conclude that w' and z' satisfy (3.1a), (3.1c) and the first two relations in (3.1b). To prove the last relation in (3.1b) note that

$$[w', z'] = [X_{n,n}, qX_{n-1,n} - q^2 X_{n,n-1}] = 0$$

since $X_{n,n-1}X_{nn} = q^2 X_{nn}X_{n,n-1}$ and $[X_{n-1,n}, X_{nn}] = q^2(q - q^{-1})X_{nn}X_{n,n-1}$ by Proposition 3.6. To prove the first relation in (3.1d), note that

$$\begin{aligned}[w', [w', E_{n-1}]_{q^{-2}}]_{q^2} &= [[E_{n-1}, w']_{q^{-2}}], w']_{q^2} = [X_{n-1,n} + q^{-1}X_{n,n-1}, X_{n,n}]_{q^2} \\ &= (1 - q^2)(X_{n,n}X_{n-1,n} - qX_{n,n}X_{n,n-1}) = (q^{-1} - q)w'z'.\end{aligned}$$

The remaining identities are checked similarly. Using Lemma 2.24(ii), we rewrite them in the form $\sum Y_i m_i$, where $m_i \in \{1, E_{n-1}, E_{n-2}, E_{n-1}E_{n-2}, E_{n-2}E_{n-1}\}$ and in particularly are linearly independent and $Y_i \in S_q(V \otimes V)$. Then we check that $Y_i = 0$ which can be done either using the presentation from Proposition 3.6 or by observing that $\mathrm{Im}(\Psi - 1) = \ker((\Psi + q^2)(\Psi + q^{-2}))$. This is a rather tedious, albeit simple, computation, which was performed on a computer.

Thus, we proved that $\psi : \mathcal{U}_{q,n} \to S_q(V \otimes V) \rtimes U_q(\mathfrak{sl}_n)$ is a surjective homomorphism of algebras. The same argument shows that we have a surjective homomorphism of algebras $\psi_+ : \mathcal{U}^+_{q,n} \to S_q(V \otimes V) \rtimes U^+_q(\mathfrak{sl}_n)$.

To complete the proof of the proposition, we prove first that ψ_+ is an isomorphism. Let $\mathcal{F}$ be the free algebra on the E_i, $1 \le i \le n-1$, w and z and define a grading on $\mathcal{F}$ by $\deg E_i = \deg w = 1$ and $\deg z = 2$. Let $\mathcal{I}_q$ be the kernel of the structural homomorphism $\mathcal{F} \twoheadrightarrow \mathcal{U}^+_{q,n}$. It is easy to see that $\mathcal{I}_q$ is homogeneous with respect to this grading. Regard $S_q(V \otimes V) \rtimes U^+_q(\mathfrak{sl}_n)$ as a graded algebra with the grading induced by the homomorphism ψ_+. By Lemma 2.21, we have $\dim(\mathcal{U}^+_{q,n})_k \le \dim(\mathcal{F}/\mathcal{I}_1)_k$ for all k where $\mathcal{I}_1$ is the specialization of $\mathcal{I}_q$ at $q = 1$. On the other hand, it is easy to see that $\mathcal{F}/\mathcal{I}_1$ is isomorphic to $U(\mathfrak{n})$ where $\mathfrak{n} = (V \otimes V) \rtimes (\mathfrak{sl}_n)_+$, which we can regard as a graded Lie algebra with the grading compatible with that on $\mathcal{U}^+_{q,n}$. Since both $U(\mathfrak{n})$ and $S_q(V \otimes V) \rtimes U^+_q(\mathfrak{sl}_n)$ are PBW algebras on the set of the same cardinality, it follows that $\dim U(\mathfrak{n})_k = \dim(S_q(V \otimes V) \rtimes U^+_q(\mathfrak{sl}_n))_k$ for all k. This and the obvious inequality $\dim(\mathcal{U}^+_{q,n})_k \ge \dim(S_q(V \otimes V) \rtimes U^+_q(\mathfrak{sl}_n))_k$ proves the second assertion in part (ii).

Let ${\mathcal{U}'_{q,n}}^+$ be the subalgebra of $\mathcal{U}_{q,n}$ generated by the E_i, $i \in I$ and by w, z. Clearly, we have a canonical surjective homomorphism $\pi : \mathcal{U}^+_{q,n} \to {\mathcal{U}'_{q,n}}^+$ and $\psi_+ = \psi \circ \pi$. Since ψ_+ is an isomorphism and both ψ and π are surjective, it follows that π is an isomorphism and proves the first assertion in part (i). To establish the remaining assertions, we need the following easy Lemma.

Lemma 3.9. Let $\psi : A \to B$ be a surjective homomorphism of algebras. Let $A^\pm$ (respectively, $B^\pm$) be subalgebras of A (respectively, B) such that the multiplication map $A^+ \otimes A^- \to A$ (respectively, $B^+ \otimes B^- \to B$) is surjective (respectively, bijective). Suppose that the restriction of ψ to $A^\pm$ is an isomorphism onto $B^\pm$. Then ψ is an isomorphism of algebras and $A \cong A^+ \otimes A^-$ as vector spaces. □

Applying this Lemma with $A = \mathcal{U}_{q,n}$, $B = S_q(V \otimes V) \rtimes U_q(\mathfrak{sl}_n)$, $A^+ = \mathcal{U}^+_{q,n}$, $A^- = B^- = U^{\le 0}_q(\mathfrak{sl}_n)$, and $B^+ = S_q(V \otimes V) \rtimes U^+_q(\mathfrak{sl}_n)$ completes the proof of the proposition. ■

3.4 Structural homomorphisms

In this section we prove parts (iv) and (v) of Theorem 3.1. We use the numeration of nodes in the Dynkin diagram of $\mathfrak{so}_{2n+2}$ and $\mathfrak{sp}_{2n}$ introduced in Section 3.1. Note first that part (iv) of Theorem 3.1 is trivial since modulo the ideal generated by z its defining relations are precisely the defining relations of $U_q(\mathfrak{sp}_{2n})$ where w corresponds to E_0.

To prove part (v) of Theorem 3.1, let $W = E_0 E_{-1}$ and

$$\begin{aligned} Z &= (q^2-1)^{-1}([E_0,[E_1,E_{-1}]_q]_q + [E_{-1},[E_1,E_0]_q]_q) \\ &= (1-q^{-2})^{-1}((\mathrm{ad}^* E_0)(\mathrm{ad}^* E_1)(E_{-1}) + (\mathrm{ad}^* E_{-1})(\mathrm{ad}^* E_1)(E_0)) \\ &= (1-q^{-2})^{-1}((\mathrm{ad}^* E_0)(\mathrm{ad}\, E_{-1})(E_1) + (\mathrm{ad}^* E_{-1})(\mathrm{ad}\, E_0)(E_1)) \end{aligned}$$

be the images of w and z in $U_q(\mathfrak{so}_{2n+2})$. Clearly,

$$ {}_i r(W) = r_i(W) = 0, \quad i>0, \quad {}_0 r(W) = E_{-1}, \quad {}_{-1} r(W) = E_0. \tag{3.5}$$

Using Lemma 2.27 we obtain

$$ {}_i r(Z) = 0, \quad i>0, \quad {}_0 r(Z) = q(\mathrm{ad}^* E_1)(E_{-1}), \quad {}_{-1} r(Z) = q(\mathrm{ad}^* E_1)(E_0). \tag{3.6}$$

It is easy to check that $Z^* = Z$, hence $r_i(Z) = 0$ for all $i > 0$. Finally, we have

$$\begin{aligned} Z &= q[W, E_1]_{q^{-2}} - q\,\frac{[2]_q}{q-q^{-1}}(\mathrm{ad}\, E_0)(\mathrm{ad}\, E_{-1})(E_1) \\ &= q[E_1, W]_{q^{-2}} - q\,\frac{[2]_q}{q-q^{-1}}(\mathrm{ad}^* E_0)(\mathrm{ad}^* E_{-1})(E_1). \end{aligned} \tag{3.7}$$

Proof of Theorem 3.1(v). We need to show that the elements W and Z satisfy the relations (3.1a)–(3.1e). The last two identities in (3.1a) are trivial, while the first follows from (2.22) and (3.5), (3.6). Furthermore, observe that

$$[E_i, W] = (\mathrm{ad}\, E_i)(W), \quad i>1, \qquad [E_1,[E_1,[E_1,W]_{q^{-2}}]]_{q^2} = (\mathrm{ad}\, E_1)^3(W),$$

while

$$[E_i, Z] = (\mathrm{ad}\, E_i)(Z), \quad i>0,\ i\neq 2, \qquad [E_2,[E_2,Z]_{q^{-1}}]_q = (\mathrm{ad}\, E_2)^2(Z).$$

The first two identities in (3.1b) are now immediate from (2.19). The first identity in (3.1c) follows from (2.20) since $(\operatorname{ad} E_1)^2(E_i) = 0$, $i \in \{-1, 0\}$ by quantum Serre's relations. The second is also a consequence of quantum Serre relations since $\operatorname{ad} E_2$ commutes with $\operatorname{ad}^* E_i$, $\operatorname{ad} E_j$, $i, j \in \{0, -1\}$. To prove the last relation in (3.1b), note that since

$$(\operatorname{ad}^* E_i)^2(\operatorname{ad}^* E_1)(E_j) = 0, \quad (\operatorname{ad} E_i)(\operatorname{ad}^* E_j)(\operatorname{ad} E_i)(E_1) = 0, \quad \{i, j\} = \{0, 1\}$$

by quantum Serre relations, it follows that

$$E_i(\operatorname{ad}^* E_i)(\operatorname{ad}^* E_1)(E_j) = q(\operatorname{ad}^* E_i)(\operatorname{ad}^* E_1)(E_j)E_i,$$
$$E_i(\operatorname{ad}^* E_j)(\operatorname{ad}^* E_1)(E_i) = q^{-1}(\operatorname{ad}^* E_j)(\operatorname{ad}^* E_1)(E_i)E_i,$$

hence

$$WZ = ZW.$$

To prove the first identity in (3.1d), notice that since $[Z, W] = 0$ we obtain from (3.7)

$$\begin{aligned}[W, [W, E_1]_{q^{-2}}]_{q^2} - (q^{-1} - q)WZ &= [W, [W, E_1]_{q^{-2}} - q^{-1}Z]_{q^2} \\ &= [2]_q(q - q^{-1})^{-1}[W, (\operatorname{ad} E_0)(\operatorname{ad} E_{-1})(E_1)]_{q^2}.\end{aligned}$$

Since for all $x \in U_q^+(\mathfrak{so}_{2n+2})$

$$[W, x]_{q^2} = E_0[E_{-1}, x]_q + q[E_0, x]_q E_{-1}$$

and $[E_i, x]_q = (\operatorname{ad} E_i)(x)$ for $x = (\operatorname{ad} E_0)(\operatorname{ad} E_{-1})(E_1)$ and $i \in \{0, -1\}$, it follows from the quantum Serre relations that

$$[W, (\operatorname{ad} E_0)(\operatorname{ad} E_{-1})(E_1)]_{q^2} = 0,$$

which together with (3.7) implies the first relation in (3.1d). To prove the remaining identities, we use Lemma 2.26(i). Note that

$${}_ir([a, b]_v) = [{}_ir(a), b]_{vq^{-(\delta,\alpha_i)}} + q^{-(\gamma,\alpha_i)}[a, {}_ir(b)]_{vq^{(\gamma,\alpha_i)}},$$

for all $a \in U_q^+(\mathfrak{g})_\gamma$, $b \in U_q^+(\mathfrak{g})_\delta$, $\gamma, \delta \in Q$ and $v \in \mathbb{C}(q)^\times$.

To prove (3.1d), note that

$$[E_2, [E_1, W]_{q^2}]_q = q^3(\mathrm{ad}^* E_2)(\mathrm{ad}^* E_1)(W),$$

while

$$[E_1, [E_2, Z]_q]_q = q^2(\mathrm{ad}^* E_1)(\mathrm{ad}^* E_2)(Z).$$

Thus, we want to show that ${}_ir(x) = 0$ for all $i \in I$ where

$$x = q[Z, (\mathrm{ad}^* E_2)(\mathrm{ad}^* E_1)(W)]_q - [W, (\mathrm{ad}^* E_1)(\mathrm{ad}^* E_2)(Z)]_{q^2}.$$

This is trivial if $i > 2$. For $i = 2$ we obtain

$$\begin{aligned}
{}_2r(x) &= (q - q^{-1})(q^2[Z, (\mathrm{ad}^* E_1)(W)] - [W, (\mathrm{ad}^* E_1)(Z)]_{q^2}) \\
&= (q - q^{-1})q^2(Z(\mathrm{ad}^* E_1)(W) + (\mathrm{ad}^* E_1)(Z)W) - ((\mathrm{ad}^* E_1)(W)Z + q^{-2}W(\mathrm{ad}^* E_1)(Z)) \\
&= (q - q^{-1})q^2((\mathrm{ad}^* E_1)(ZW) - (\mathrm{ad}^* E_1)(WZ)) = (q - q^{-1})(\mathrm{ad}^* E_1)[Z, W] = 0,
\end{aligned}$$

where we used already established (3.1b) and (2.20). Similarly,

$$\begin{aligned}
{}_1r(x) &= (q - q^{-1})(q[Z, (\mathrm{ad}^* E_2)(W)]_q - q^2[W, (\mathrm{ad}^* E_2)(Z)]) \\
&= (q - q^{-1})q^2(\mathrm{ad}^* E_2)([W, Z]) = 0.
\end{aligned}$$

The computation of ${}_ir(x)$ for $i \in \{-1, 0\}$ and the ones for the last identity, are rather tedious and where performed on a computer. ■

Remark 3.10. It can be shown that the kernel of the homomorphism $\hat{\imath} : \mathcal{U}^+_{q,n} \to U_q(\mathfrak{so}_{2n+2})$ is generated by an element of degree 3 in $S_q(V \otimes V)$ which is a lowest weight vector of a simple $U_q(\mathfrak{sl}_n)$-submodule of $(V \otimes V)^{\otimes 3}$ isomorphic to $V_{2\varpi_3}$. On the other hand, the image of $\hat{\imath}$ equals to the subalgebra of σ-invariant elements in $U^+_q(\mathfrak{so}_{2n+2})$ graded by Q^σ. □

3.5 Braid group action on $\mathcal{U}_{q,n}$

Proof of part (vi) of Theorem 3.1. Let $\tilde{\mathcal{U}}_{q,n}$ be the algebra generated by $U_q(\mathfrak{sl}_n)$ and w, z subjects to the relations (3.1a), and (3.1b), (3.1c), except the commutativity relation $[w, z] = 0$. Clearly that $\mathcal{U}_{q,n}$ is a quotient of $\tilde{\mathcal{U}}_{q,n}$. First we prove the following:

Proposition 3.11. The formulae (3.2) extend the action of $Br_{\mathfrak{sl}_n}$ on $U_q(\mathfrak{sl}_n)$ to an action on $\tilde{\mathcal{U}}_{q,n}$ by algebra automorphisms. □

Proof. We note the following useful lemma:

Lemma 3.12. In $\tilde{\mathcal{U}}_{q,n}$ we have

$$[F_i, T_1(w)] = -\delta_{i,1}[w, E_1]_{q^{-2}} K_1, \quad [F_i, T_2(z)] = -\delta_{i,2} q^{-1} z K_2,$$

$$[T_1(E_1), T_1(w)]_{q^{-2}} = [w, E_1]_{q^{-2}}, \quad [T_2(E_2), T_2(z)]_{q^{-1}} = z,$$

$$[T_2(E_1), w]_{q^{-2}} = [[E_1, w]_{q^{-2}}, E_2]_{q^{-1}}, \quad [T_1(E_2), z]_{q^{-1}} = [[E_2, z]_{q^{-1}}, E_1]_{q^{-1}}.$$ □

Clearly, $[T_i(F_j), T_i(w)] = 0$ (respectively, $[T_i(F_j), T_i(z)] = 0$) for all j and for all $i \neq 1$ and (respectively, for all $i \neq 2$). Since

$$[T_1(F_1), T_1(w)] = -[2]_q^{-1}[E_1, [E_1, [E_1, W]_{q^{-2}}]]_{q^2} K_1 = 0,$$

$$[T_2(F_2), T_2(z)] = [E_2, [E_2, z]_{q^{-1}}]_q K_2 = 0.$$

we conclude that $[T_1(F_i), T_1(w)] = 0$ unless $i = 2$, while by Lemma 3.12

$$[T_1(F_2), T_1(w)] = [[F_1, F_2]_q, T_1(w)] = [[F_1, T_1(w)], F_2]_q = -[w, E_1]_{q^{-2}}[K_1, F_2]_q = 0,$$

$$[T_2(F_j), T_2(z)] = [[F_2, F_j]_q, T_2(z)] = [[F_2, T_2(z)], F_j]_q = -q^{-1} z [K_2, F_j]_q = 0, \quad j = 1, 3.$$

The remaining identity in (3.1a) is clearly preserved. Similarly, for all i and for all $j \neq 1$ (respectively, $j \neq 2$) we obtain $[T_i(E_j), T_i(w)] = 0$ (respectively, $[T_i(E_j), T_i(z)] = 0$). The remaining identities follow from Lemma 3.12 and direct computations. For example, for $i = 1, 3$

$$\begin{aligned}[T_2(E_i), T_2(z)] &= q^{-2} E_2 E_i E_2 z - q^{-2} E_2 z E_2 E_i - q^{-1} E_i E_2 E_2 z \\ &\quad + E_i E_2 z E_2 + q^{-1} z E_2 E_2 E_i - z E_2 E_i E_2 \\ &= [z, (\mathrm{ad}\, E_2)^{(2)}(E_i)]_{q^{-2}} = 0,\end{aligned}$$

where we used (3.1c) and quantum Serre relations. It is not hard to check, using the above lemma, that the maps T_i are invertible with their inverses given on

w and z by

$$T_1^{-1}(w) = [2]_q^{-1}[E_1, [E_1, w]_{q^{-2}}], \quad T_2^{-1}(z) = [E_2, z]_{q^{-1}},$$

while $T_i^{-1}(w) = w$ if $i \neq 1$ and $T_i^{-1}(z) = z$ if $i \neq 2$. Thus, we conclude that the T_i are automorphisms of $\tilde{\mathcal{U}}_{q,n}$. Finally, the only braid relations that need to be checked are $T_1T_2T_1(w) = T_2T_1T_2(w) = T_2T_1(w)$ and $T_2T_iT_2(z) = T_iT_2T_i(z) = T_iT_2(z)$, where $i = 1, 3$. This is done by a direct computation. ■

To complete the proof of part (vi) of Theorem 3.1 it suffices to show that the kernel of the canonical map $\tilde{\mathcal{U}}_{q,n} \to \mathcal{U}_{q,n}$ is preserved by the T_i, $1 \leq i \leq n-1$. For example, consider (3.1d). Note that $[w, [w, E_1]_{q^{-2}}]_{q^2} = [[E_1, w]_{q^{-2}}, w]_{q^2}$. Using Lemma 3.12 we obtain:

$$[[T_i(E_1), T_i(w)]_{q^{-2}}, T_i(w)]_{q^2} = \begin{cases} [[E_1, w]_{q^{-2}}, w]_{q^2}, & i > 2, \\ [[w, [w, E_1]_{q^{-2}}]_{q^2}], E_2]_{q^{-1}}, & i = 2, \\ [2]_q^{-1}[[[w, [w, E_1]_{q^{-2}}]_{q^2}, E_1]_{q^{-2}}, E_1], & i = 1, \end{cases}$$

while $T_i(w)T_i(z) = wz$, $i > 2$,

$$T_1(w)T_1(z) = [2]_q^{-1}[[wz, E_1]_{q^{-2}}, E_1]$$

and

$$T_2(w)T_2(z) = [wz, E_2]_{q^{-1}}.$$

Thus, the first relation in (3.1d) is preserved. The computations for the remaining relations are rather tedious and where performed on a computer. The relations can be checked in many different ways; perhaps, the simplest is to use the isomorphism $\mathcal{U}_{q,n} \cong S_q(V \otimes V) \rtimes U_q(\mathfrak{sl}_n)$, which allows us to write any element of $\mathcal{U}_{q,n}$ as $\sum_i Y_i m_i$, where $Y_i \in S_q(V \otimes V)$ and the m_i are linearly independent elements of $U_q(\mathfrak{sl}_n)$. Writing a relation in this form, we then check that $(\Psi + q^2)(\Psi + q^{-2})(Y_i) = 0$ hence $Y_i \in \mathrm{Im}(\Psi - 1)$. ■

3.6 Liftable quantum foldings and $\mathcal{U}_{q,n}^+$ as a uberalgebra

In this section, we use the standard numbering of the nodes of all Dynkin diagrams.

Theorem 3.13. In the notation of Theorem 3.1, $\iota_{\mathbf{i}}$ for any $\mathbf{i} \in R(w_\circ)$ is a tame liftable folding with $U(\iota_{\mathbf{i}}) = \mathcal{U}_{q,n}^+$ and $\mu_{\iota_{\mathbf{i}}} = \mu$. In particular, $\tilde{\iota}_{\mathbf{i}}$ splits μ and we have a commutative

diagram

$$\begin{tikzcd} \mathcal{U}^+_{q,n} \arrow[r, "\hat{\iota}"] & U^+_q(\mathfrak{so}_{2n+2}) \\ U^+_q(\mathfrak{sp}_{2n}) \arrow[u, "\tilde{\iota}_{\mathbf{i}}"] \arrow[ur, "\iota_{\mathbf{i}}"'] & \end{tikzcd} \tag{3.8}$$

where all maps commute with the right multiplication with $U^+_q(\mathfrak{sl}_n)$. □

Proof. Let $w'_\circ$ be the longest element in $W(\mathfrak{sl}_n) = W((\mathfrak{sp}_{2n})_J)$, where $J = \{1, \dots, n-1\}$ and let $\mathbf{i}' = (n-1, n-2, n-1, \dots, 1, \dots, n-1) \in R(w'_\circ)$. Set

$$\mathbf{i}_r = (n, n-1, n, n-2, n-1, n, \dots, r, \dots, n), \quad 1 \le r \le n.$$

First, we prove the Theorem for $\mathbf{i} = \mathbf{i}_\circ$ where $\mathbf{i}_\circ$ is the concatenation $\mathbf{i}_1\mathbf{i}'$.

Given $\mathbf{j} = (r_1, \dots, r_k) \in (I/\sigma)^k$, write $w_{\mathbf{j}} = s_{r_1} \cdots s_{r_k}$ and $T_{\mathbf{j}} = T_{w_{\mathbf{j}}}$ (respectively, $\hat{T}_{\mathbf{j}} = T_{\hat{w}_{\mathbf{j}}}$). It is easy to check that

$$w_{\mathbf{i}_r}(\alpha_s) = \alpha_{r+n-s-1}, \quad r < s \le n-1, \ 1 \le r \le n. \tag{3.9}$$

In particular, $T_{\mathbf{i}_1}(E_i) = E_{n-i}$, $1 \le i \le n-1$ by Lemma 2.8, hence $T_{\mathbf{i}_1}$ acts as the diagram automorphism τ of $U^+_q(\mathfrak{sl}_n) = U^+_q((\mathfrak{sp}_{2n})_J)$. Define the elements $y^+_{ij} \in U^+_q(\mathfrak{sp}_{2n})$, $x^+_{ij} \in U_q(\mathfrak{so}_{2n+2})$, $1 \le i \le j \le n$, and $x^-_{ij} \in U^+_q(\mathfrak{sl}_n) = U^+_q((\mathfrak{so}_{2n+2})_J) = U^+_q((\mathfrak{sp}_{2n})_J)$ by:

$$\begin{aligned} X_{\mathbf{i}} &= \{y^+_{nn}, y^+_{n-1,n}, y^+_{n-1,n-1}, \dots, y^+_{1n}, \dots, y^+_{11}, x^-_{12}, \dots, x^-_{1,n}, \dots, x^-_{n-1,n}\}, \\ \hat{X}_{\mathbf{i}} &= \{x^+_{nn}, x^+_{n-1,n}, x^+_{n-1,n-1}, \dots, x^+_{1n}, \dots, x^+_{11}, x^-_{12}, \dots, x^-_{1,n}, \dots, x^-_{n-1,n}\} \end{aligned} \tag{3.10}$$

as ordered sets, in the notation of Section 2.4. In particular, for all $a^+_{ij}, a^-_{ij} \in \mathbb{Z}_{\ge 0}$, we have

$$\iota_{\mathbf{i}}\left(\overrightarrow{\prod_{1\le i\le j\le n}} (y^+_{ij})^{a^+_{ij}} \overrightarrow{\prod_{1\le i<j\le n}} (x^-_{ij})^{a^-_{ij}}\right) = \overrightarrow{\prod_{1\le i\le j\le n}} (x^+_{ij})^{a^+_{ij}} \overrightarrow{\prod_{1\le i<j\le n}} (x^-_{ij})^{a^-_{ij}},$$

where both products are taken in the same order as in (3.10).

Identify $\mathcal{U}^+_{q,n}$ with $S_q(V\otimes V)\rtimes U^+_q(\mathfrak{sl}_n)$ using the isomorphism ψ_+ from Proposition 3.8. Define $\tilde{\iota}_{\mathbf{i}}: U^+_q(\mathfrak{sp}_{2n})\to \mathcal{U}^+_{q,n}$ by

$$\tilde{\iota}_{\mathbf{i}}\left(\prod_{1\le i\le j\le n}^{\rightarrow}(y^+_{ij})^{a^+_{ij}}\prod_{1\le i<j\le n}^{\rightarrow}(x^-_{ij})^{a^-_{ij}}\right)=\prod_{1\le i\le j\le n}^{\rightarrow}\tilde{X}_{ji}^{a^+_{ij}}\prod_{1\le i<j\le n}^{\rightarrow}(x^-_{ij})^{a^-_{ij}}, \tag{3.11}$$

where $\tilde{X}_{ji}=(q-q^{-1})^{i+j-2n}X_{ji}$ and the product in the right hand side is taken in the total order defined in the proof of Proposition 3.6

We will need the following result, where we abbreviate $\tilde{E}_i:=\operatorname{ad}E_i$, $\tilde{E}^*_i:=\operatorname{ad}^*E_i$.

Proposition 3.14. The elements $x^+_{ij}\in U^+_q(\mathfrak{so}_{2n+2})$ and $y^+_{ij}\in U^+_q(\mathfrak{sp}_{2n})$ defined in (3.10) are given by the following formulae

$$x^+_{ij}=(\hat{c}^+_{ij})^{-1}\tilde{E}_j\cdots\tilde{E}_{n-1}\tilde{E}_i\cdots\tilde{E}_{n-2}\tilde{E}^*_n\tilde{E}^*_{n+1}(E_{n-1}),\quad 1\le i<j\le n, \tag{3.12}$$

$$x^+_{ii}=(\hat{c}^+_{ii})^{-1}\tilde{E}^{(2)}_i\cdots\tilde{E}^{(2)}_{n-1}(E_nE_{n+1}),\quad 1\le i\le n, \tag{3.13}$$

$$y^+_{ij}=(c^+_{ij})^{-1}\tilde{E}_i\cdots\tilde{E}_{j-1}\tilde{E}^{(2)}_j\cdots\tilde{E}^{(2)}_{n-1}(E_n),\quad 1\le i\le j\le n, \tag{3.14}$$

where $c^+_{ij}=(q+q^{-1})^{1-\delta_{ij}}(q-q^{-1})^{2n-i-j}$, $\hat{c}^+_{ij}=(q-q^{-1})^{2n-i-j+1-\delta_{ij}}$. □

Proof. We only prove (3.12) and (3.13). The argument for (3.14) is nearly identical and is omitted. By the definition of the $\hat{X}_{\mathbf{i}}$ given in Section 2.4 we have for all $1\le i\le j\le n$

$$x^+_{ij}=\gamma(\beta_{ij})^{-1}(q-q^{-1})^{1+\delta_{ij}}\hat{T}_{\mathbf{i}_{i+1}}T_i\cdots T_{i+n-j-1}(E_{i+n-j}E^{\delta_{ij}}_{n+1}),$$

where $\beta_{ij}=\deg x^+_{ij}=\hat{w}_{\mathbf{i}_{i+1}}s_i\cdots s_{i+n-j-1}(\alpha_{i+n-j}+\delta_{ij}\alpha_{n+1})$. It is easy to see, using (3.9), that $\beta_{ij}=\alpha_i+\cdots+\alpha_{j-1}+2(\alpha_j+\cdots+\alpha_{n-1})+\alpha_n+\alpha_{n+1}$, hence $\gamma(\beta_{ij})(q-q^{-1})^{-1-\delta_{ij}}=\hat{c}^+_{ij}$.

Since

$$\hat{w}_{\mathbf{i}_i}=s_{n+1}s_n\cdots s_i\hat{w}_{\mathbf{i}_{i+1}},\quad 1\le i\le n-1, \tag{3.15}$$

and $\ell(w_{\mathbf{i}_i})=\ell(w_{\mathbf{i}_{i+1}})+n-i+2$, we have for all $1\le i\le n$

$$\hat{T}_{\mathbf{i}_{i+1}}(E_i)=T_{n+1}\cdots T_{i+1}T_{\mathbf{i}_{i+2}}(E_i)=T_{n+1}\cdots T_{i+1}(E_i).$$

Then it is easy to see, using (2.15), (2.16) and the obvious observation that $\mathrm{ad}\, E_r$, $\mathrm{ad}^* E_s$ commute if $a_{rs} = 0$, that

$$\hat{c}_{in}^{+} x_{in}^{+} = \tilde{E}_{n+1}^{*} \cdots \tilde{E}_{i+1}^{*}(E_i) = \tilde{E}_i \cdots \tilde{E}_{n-2} \tilde{E}_n^{*} \tilde{E}_{n+1}^{*}(E_{n-1}). \tag{3.16}$$

To establish (3.12) for $1 \le i < j < n$, note that by (3.9) and Lemma 2.8 we have $\hat{T}_{\mathbf{i}_{i+1}}(E_{i+n-j}) = E_j$, $i < j < n$. Therefore,

$$\begin{aligned}\hat{c}_{ij}^{+} x_{ij}^{+} &= \hat{T}_{\mathbf{i}_{i+1}} T_i \cdots T_{i+n-j-1}(E_{i+n-j}) = \hat{T}_{\mathbf{i}_{i+1}} \tilde{E}_i^{*} \cdots \tilde{E}_{i+n-j-1}^{*}(E_{i+n-j}) \\ &= \hat{T}_{\mathbf{i}_{i+1}} \tilde{E}_{i+n-j} \cdots \tilde{E}_{i+1}(E_i) = \tilde{E}_j \cdots \tilde{E}_{n-1}(\hat{T}_{\mathbf{i}_{i+1}}(E_i)) = \hat{c}_{in}^{+} \tilde{E}_j \cdots \tilde{E}_{n-1} x_{in}^{+},\end{aligned}$$

where we used [2, Lemma 3.5], (3.16) and (2.17).

Finally, we prove by a downward induction on i that

$$\hat{T}_{\mathbf{i}_{i+1}} T_i \cdots T_{n-1}(E_n) = T_{\sigma^{n-i}(n)} T_{n-1} \cdots T_{i+1}(E_i). \tag{3.17}$$

If $i = n-1$, it follows from Lemma 2.8 that $T_n T_{n+1} T_{n-1}(E_n) = T_{n+1}(E_{n-1})$ so the induction begins. For the inductive step, suppose that $n-i$ is odd, the case of $n-i$ even being similar. Using (3.15), we can write

$$\hat{w}_{\mathbf{i}_{i+1}} s_i \cdots s_{n-1} = s_{n+1} s_n \cdots s_i \hat{w}_{\mathbf{i}_{i+2}} s_{i+1} \cdots s_{n-1}$$

and since both expressions are reduced the induction hypothesis yields

$$\hat{T}_{\mathbf{i}_{i+1}} T_i \cdots T_{n-1}(E_n) = T_{n+1} \cdots T_i T_n T_{n-1} \cdots T_{i+2}(E_{i+1}).$$

The inductive step now follows from the braid relations and Lemma 2.8. Using [2, Lemma 3.5] we obtain from (3.17) that

$$\begin{aligned}\hat{c}_{ii}^{+} x_{ii}^{+} &= (\tilde{E}_{n+1}^{*} \tilde{E}_{n-1}^{*} \cdots \tilde{E}_{i+1}^{*}(E_i))(\tilde{E}_n^{*} \tilde{E}_{n-1}^{*} \cdots \tilde{E}_{i+1}^{*}(E_i)) \\ &= (\tilde{E}_i \cdots \tilde{E}_{n-1}(E_n))(\tilde{E}_i \cdots \tilde{E}_{n-1}(E_{n+1})).\end{aligned}$$

Since by the quantum Serre relations, $\tilde{E}_i^{(2)} \tilde{E}_{i+1} \cdots \tilde{E}_{n-1}(E_r) = 0$, $r \in \{n, n+1\}$, (3.13) follows immediately from (2.21). ■

We can now complete the proof of the Theorem. First, observe that (3.3) implies that $X_{ii} = E_i^{(2)} \cdots E_{n-1}^{(2)}(X_{nn})$, $1 \le i \le n-1$ while

$$X_{ji} = E_j \cdots E_{n-1} E_i \cdots E_{n-2}(X_{n,n-1}), \quad 1 \le i < j \le n.$$

Since $X_{n,n-1} = [2]_q^{-1}(E_{n-1}(w) - q^{-1}z)$, we obtain

$$\tilde{\iota}_{\mathbf{i}}(y_{ij}^{+}) = \frac{E_j \cdots E_{n-1} E_i \cdots E_{n-2}}{(q-q^{-1})^{2n-i-j-1}} \left(\frac{E_{n-1}(w) - q^{-1}z}{q^2 - q^{-2}} \right), \quad 1 \le i < j \le n$$

$$\tilde{\iota}_{\mathbf{i}}(y_{ii}^{+}) = (c_{ii}^{+})^{-1} E_i^{(2)} \cdots E_{n-1}^{(2)}(w), \quad 1 \le i \le n.$$

Note that for all $u \in S_q(V \otimes V)$, $1 \le i \le n-1$ we have $\mu(E_i(u)) = \tilde{E}_i(\mu(u))$ and similarly $\hat{\iota}(E_i(u)) = \tilde{E}_i(u)$. This, together with (3.7), Proposition 3.14 and the multiplicativity of $\tilde{\iota}_{\mathbf{i}}$ and $\iota_{\mathbf{i}}$ immediately implies the assertion for $\mathbf{i} = \mathbf{i}_\circ$.

To complete the proof, it remains to apply Lemma 1.7 and the argument from the proof of Theorem 1.20. ■

This completes the proof of Theorems 1.8 and 1.12.

4 Diagonal Foldings

4.1 Folding $(\mathfrak{sl}_3^{\times n}, \mathfrak{sl}_3)$

Consider the algebra $\mathfrak{s}_n := \mathfrak{sl}_3^{\times n}$ with the diagram automorphism σ which is a cyclic permutation of the components.

Let $\mathcal{A}_{q,3}^{(n)}$ be the associative $\mathbb{C}(q)$-algebra generated by u_1, u_2 and z_k, $1 \le k \le n-1$ subjects to relations given in Theorem 1.21(i).

Theorem 4.1.

(i) The algebra $\mathcal{A}_{q,3}^{(n)}$ is PBW on the totally ordered set

$$\{u_2, u_{21}, u_1\} \cup \{z_k : 1 \le k \le n-1\},$$

where

$$u_{21} = u_1 u_2 - q^{-n} u_2 u_1 - \sum_{k=1}^{n-1} \frac{q - q^{-1}}{q^k - q^{-k}} z_k.$$

(ii) The assignment $u_i \mapsto E_i$, $z_k \mapsto 0$ defines a surjective algebra homomorphism $\mu : \mathcal{A}^{(n)}_{q,3} \to U_q(\mathfrak{s}_n^{\sigma\vee})$. □

Proof. Since $u_{21} = [u_1, u_2]_{q^{-n}} - \sum_{r=1}^{n-1} [r]_q^{-1} z_r$, we have

$$[u_1, u_{21}]_{q^n} = [u_1, [u_1, u_2]_{q^{-n}}]_{q^n} - \sum_{r=1}^{n-1} \frac{[u_1, z_r]_{q^n}}{[r]_q} = \sum_{r=1}^{n-1} \left(q^r (q^{-1} - q) + \frac{q^{2r} - 1}{[r]_q} \right) u_1 z_r = 0.$$

Similarly, we can write

$$u_{21} = -q^{-n}[u_2, u_1]_{q^n} - \sum_{r=1}^{n-1} \tilde{z}_{n-r} = -q^{-n}[u_2, u_1]_{q^n} - \sum_{r=1}^{n-1} [n-r]_q^{-1} z_{n-r}$$

hence

$$\begin{aligned}
[u_2, u_{21}]_{q^{-n}} &= -q^{-n}[u_2, [u_2, u_1]_{q^n}]_{q^{-n}} - \sum_{r=1}^{n-1} [n-r]_q^{-1} [u_2, z_{n-r}]_{q^{-n}} \\
&= -q^{-n}(q^{-1} - q) \sum_{r=1}^{n-1} q^r u_2 z_{n-r} - \sum_{r=1}^{n-1} \frac{1 - q^{2(r-n)}}{[n-r]_q} u_2 z_{n-r} \\
&= (q - q^{-1}) \sum_{r=1}^{n-1} \left(q^{r-n} + \frac{q^{2(r-n)} - 1}{q^{n-r} - q^{r-n}} \right) u_2 z_{n-r} = 0.
\end{aligned}$$

Since clearly u_{21} commutes with the z_r, we obtain the PBW relations from Theorem 1.21(ii) The above computations also show that PBW relations imply Serre-like relations. To prove that $\mathcal{A}^{(n)}_{q,3}$ is PBW, we use Diamond Lemma (Proposition 2.15). It is easy to see that the only situation which needs to be checked is the monomial $u_1 u_{21} u_2$. We have

$$(u_1 u_{21}) u_2 = q^n u_{21} u_1 u_2 = q^n \left(u_2 u_{21} u_1 + u_{21}^2 + \sum_{k=1}^{n-1} [k]_q^{-1} u_{21} z_k \right) = u_1 (u_{21} u_2).$$

The second assertion is obvious. ■

We now proceed to prove that $\mathcal{A}^{(n)}_{q,3}$ is the desired enhanced uberalgebra for this folding.

Given $x_\alpha \in U_q(\mathfrak{sl}_3)$ we denote its copy in the ith component of $U_q(\mathfrak{sl}_3)^{\otimes n}$ by $x_{\alpha,i}$. Let $\mathbf{i}_1 = (121)$ and $\mathbf{i}_2 = (212)$. Define the elements $y_i \in U_q(\mathfrak{s}_n^\sigma)$ and $Y_{i,n} \in U_q(\mathfrak{s}_n)$, $i \in \{1, 2, 12, 21\}$ by

$$X_{\mathbf{i}_1} = \{y_1, y_{21}, y_2\}, \quad \hat{X}_{\mathbf{i}_1} = \{Y_{1,n}, Y_{21,n}, Y_{2,n}\}$$

as ordered sets, in the notation of Section 2.4, and similarly for $\mathbf{i}_2$. It is immediate that $y_1 = E_1$, $y_2 = E_2$ and

$$y_{12} = (q^n - q^{-n})^{-1}(E_1E_2 - q^{-n}E_2E_1), \quad y_{21} = (q^n - q^{-n})^{-1}(E_2E_1 - q^{-n}E_1E_2)$$

while $Y_{1,n} = \prod_{i=1}^n E_{1,i}$, $Y_{2,n} = \prod_{i=1}^n E_{2,i}$ and

$$Y_{21,n} = \prod_{i=1}^n E_{21,i}, \quad Y_{12,n} = \prod_{i=1}^n E_{12,i},$$

where

$$E_{12,i} = \frac{E_{2,i}E_{1,i} - q^{-1}E_{1,i}E_{2,i}}{q - q^{-1}}, \quad E_{21,i} = \frac{E_{1,i}E_{2,i} - q^{-1}E_{2,i}E_{1,i}}{q - q^{-1}}.$$

In particular, $\iota_{\mathbf{i}_1}$ (respectively, $\iota_{\mathbf{i}_2}$) is given by

$$\iota_{\mathbf{i}_1}(y_1^a y_{21}^b y_2^c) = Y_{1,n}^a Y_{21,n}^b Y_{2,n}^c, \quad \iota_{\mathbf{i}_2}(y_2^a y_{12}^b y_1^c) = Y_{2,n}^a Y_{12,n}^b Y_{1,n}^c.$$

We will need some identities for the elements $E_{\alpha,i}$. Clearly

$$qE_{21,i} + E_{12,i} = E_{1,i}E_{2,i}, \quad qE_{12,i} + E_{21,i} = E_{2,i}E_{1,i}. \tag{4.1}$$

It follows from quantum Serre relations that

$$E_{i,r}E_{ij,s} = q^{-\delta_{r,s}}E_{ij,s}E_{i,r}, \quad E_{j,r}E_{ij,s} = q^{\delta_{r,s}}E_{ij,s}E_{j,r}, \quad i \neq j \in \{1, 2\}. \tag{4.2}$$

In particular, this implies that

$$E_{12,i}E_{21,i} = E_{21,i}E_{12,i}.$$

Let $Z_{0,1} = E_{21,1}$, $Z_{1,1} = E_{12,1}$ and define inductively

$$Z_{i,k} = Z_{i,k-1}E_{21,k} + Z_{i-1,k-1}E_{12,k}, \quad 0 \leq i \leq k, \tag{4.3}$$

where we use the convention that $Z_{i,k}=0$ if $i<0$ or $i>k$. In particular, $Z_{0,n}=Y_{21,n}$ and $Z_{n,n}=Y_{12,n}$.

Lemma 4.2. We have for all $0\le k,l\le n$

$$Y_{1,n}Z_{k,n}=q^{n-2k}Z_{k,n}Y_{1,n},\quad Y_{2,n}Z_{k,n}=q^{-n+2k}Z_{k,n}Y_{2,n},\quad Z_{k,n}Z_{l,n}=Z_{l,n}Z_{k,n}, \tag{4.4}$$

$$Y_{1,n}Y_{2,n}=\sum_{k=0}^{n}q^{n-k}Z_{k,n},\quad Y_{2,n}Y_{1,n}=\sum_{k=0}^{n}q^{k}Z_{k,n}. \tag{4.5}$$

□

Proof. The first relation is just (4.2) for $n=1$. Then, using induction on n, we obtain

$$\begin{aligned}Y_{1,n}Z_{k,n}&=Y_{1,n-1}E_{1,n}(Z_{k,n-1}E_{21,n}+Z_{k-1,n-1}E_{12,n})\\&=q^{n-1-2k}Z_{k,n-1}E_{1,n}E_{21,n}Y_{1,n-1}+q^{n+1-2k}Z_{k-1,n-1}E_{1,n}E_{12,n}Y_{1,n-1}\\&=q^{n-2k}Z_{k,n}Y_{k,n}.\end{aligned}$$

The second identity in (4.4) is proved similarly while the last is obvious since $E_{21,r}$ and $E_{12,s}$ commute for all $1\le r,s\le n$. To prove (4.5), we again use the inductive definition of the $Z_{k,n}$. For $n=1$ this relation coincides with (4.1), while

$$\begin{aligned}Y_{1,n}Y_{2,n}=Y_{1,n-1}Y_{2,n-1}E_{1,n}E_{2,n}&=\sum_{k=0}^{n-1}q^{n-k-1}Z_{k,n-1}(E_{12,n}+qE_{21,n})\\&=\sum_{k=0}^{n}q^{n-k-1}Z_{k,n-1}E_{12,n}+\sum_{k=0}^{n}q^{n-k}Z_{k,n-1}E_{21,n}=\sum_{k=0}^{n}q^{n-k}Z_{k,n}.\end{aligned}$$

The remaining identity is proved similarly. ■

As an immediate corollary, we obtain

$$Y_{1,n}Y_{2,n}=q^{-n}Y_{2,n}Y_{1,n}+(q^{n}-q^{-n})Y_{21,n}+\sum_{k=1}^{n-1}(q^{n-k}-q^{k-n})Z_{k,n}. \tag{4.6}$$

Example 4.3. Let $n=3$ and take $\iota=\iota_{\mathbf{i}_1}$. Then in $\langle U_q^+(\mathfrak{sl}_3)\rangle_\iota$ we have

$$Y_{1,3}Y_{2,3}=q^{-3}Y_{2,3}Y_{1,3}+(q^{3}-q^{-3})Y_{21,3}+Y'_{21,3}.$$

Since the terms in $Y'_{21,3}$ quasi-commute with $Y_{1,3}$ with different powers of q and are linearly independent, we obtain an infinite family of generators by taking q-commutators of $Y_{1,3}$ with $Y'_{21,3}$. Thus, this algebra cannot be sub-PBW, since it clearly has polynomial growths. □

Lemma 4.4. The elements $Z_{k,n}$, $1 \le k \le n-1$, are contained in $\langle U_q^+(\mathfrak{sl}_3)\rangle_{\iota_{\mathbf{i}}} \cap \mathcal{F}\mathrm{rac}\, U_q^+(\mathfrak{sl}_3)^{\otimes n}$ for both reduced expressions $\mathbf{i}$ of the longest element in the Weyl group of $\mathfrak{sl}_3$. □

Proof. Using Lemma 4.2, we obtain for all $s > 0$

$$Y_{1,n}^s Y_{2,n} = \left(\sum_{k=0}^{n} q^{(s-1)(n-2k)+n-k} Z_{k,n}\right) Y_{1,n}^{s-1}.$$

Note that one of the $Z_{0,n}$, $Z_{n,n}$ is contained in $\langle U_q^+(\mathfrak{sl}_3)\rangle_{\iota_{\mathbf{i}}}$. Taking $1 \le s \le n$ yields a system of linear equations for the $Z_{k,n}$ in $\langle U_q^+(\mathfrak{sl}_3)\rangle_{\iota_{\mathbf{i}}} \cap \mathcal{F}\mathrm{rac}\, U_q^+(\mathfrak{sl}_3)^{\otimes n}$ with the matrix $(q^{ns+k(1-2s)})$ where $1 \le s \le n+1$, $0 \le k \le n$. This matrix is easily seen to be nondegenerate. ■

Proposition 4.5. The assignment

$$u_i \mapsto Y_{i,n}, \quad i = 1, 2, \quad z_k \mapsto [k]_q (q^{n-k} - q^{k-n}) Z_{k,n}$$

defines an algebra homomorphism $\hat{\iota} : \mathcal{A}_{q,3}^{(n)} \to U_q(\mathfrak{sl}_n)^{\otimes n}$. In particular, the subalgebra of $U_q^+(\mathfrak{sl}_3)^{\otimes n}$ generated by $\iota_{\mathbf{i}}(U_q^+((\mathfrak{sl}_3^{\times n})^{\sigma\vee})$ and $Z_0 = \{Z_{k,n} : 1 \le k \le n-1\}$ is independent of $\mathbf{i}$ and is sub-PBW. □

Proof. It is sufficient to prove that $Y_{1,n}$, $Y_{2,n}$, and $Z_{k,n}$, $1 \le k \le n-1$ satisfy the relations given in Theorem 1.21(i). The first two relations are already obtained in Lemma 4.2. Furthermore, (4.4) implies

$$[Y_{1,n}, [Y_{1,n}, Y_{2,n}]_{q^{-n}}]_{q^n} = \sum_{k=0}^{n-1} h_{n-k} [Y_{1,n}, Z_{k,n}]_{q^n} = -\sum_{k=1}^{n-1} q^k h_{n-k} h_k Y_{1,n} Z_{k,n},$$

where $h_k = q^k - q^{-k}$. Similarly,

$$[Y_{2,n}, [Y_{2,n}, Y_{1,n}]_{q^{-n}}]_{q^n} = \sum_{k=0}^{n-1} h_k [Y_{2,n}, Z_{k,n}]_{q^n} = -\sum_{k=0}^{n-1} q^{n-k} h_{n-k} h_k Y_{2,n} Z_{k,n}.$$

The remaining assertions are trivial. ■

Define $\tilde{\iota}_{\mathbf{i}_1} : U_q(\mathfrak{s}_n^\sigma) \to \mathcal{A}_{q,3}^{(n)}$ by extending multiplicatively the assignments

$$y_1 \mapsto u_1, \quad y_2 \mapsto u_2, \quad y_{21} \mapsto (q^n - q^{-n})^{-1}[u_1, u_2]_{q^{-n}} - \sum_{k=1}^{n-1} [k]_q^{-1} z_k.$$

The map $\tilde{\iota}_{\mathbf{i}_2}$ is defined similarly.

Proposition 4.6. The maps $\tilde{\iota}_{\mathbf{i}_r}$, $r = 1, 2$ split μ and for each of them the diagram (1.3) commutes with $\iota = \iota_{\mathbf{i}}$. In particular, $\mathcal{A}_{q,3}^{(n)}$ is the unique uberalgebra for this quantum folding. □

Proof. We only show this for $\mathbf{i} = \mathbf{i}_1$, the argument for $\mathbf{i}_2$ being similar. Since $\iota_{\mathbf{i}}$ is multiplicative on the modified PBW basis, it is enough to check that the diagram (1.3) commutes on y_1, y_2 and y_{21}, which is straightforward from Lemma 4.2 and from the definition of μ, $\hat{\iota}$, and $\tilde{\iota}_{\mathbf{i}}$. ■

4.2 Folding $(\mathfrak{sl}_4 \times \mathfrak{sl}_4, \mathfrak{sl}_4)$

Now we turn our attention to the folding $(\mathfrak{sl}_4^{\times 2}, \mathfrak{sl}_4)$ with σ being the permutation of the components. Let $\mathcal{A}_{q,4} = \mathcal{A}_{q,4}^{(2)}$ be the algebra with generators u_i, $1 \le i \le 3$, $z_{12} = z_{21}$, $z_{13} = z_{31}$ and $z_{23} = z_{32}$ and relations given in Theorem 1.17(ii).

Theorem 4.7. The algebra $\mathcal{A}_{q,4}$ is PBW on the totally ordered set

$$\{Y_2, Y_{21}, Y_{23}, Z_{21}, Z_{23}, Y_{13}, Z_{123}, Z_{321}, Z_{1232}, Z_{13}, Y_3, Y_1\},$$

where $Y_i = u_i$, $1 \le i \le 3$, $Z_{ij} = (1 - q^2)^{-|i-j|} z_{ij}$, $1 \le i \ne j \le 3$, and

$$Y_{2i} = \frac{u_i u_2 - q^{-2} u_2 u_i + q^{-1} z_{i2}}{q^2 - q^{-2}}, \qquad Z_{i2j} = \frac{[u_j, z_{2i}]_{q^{-2}} + q^{-1} z_{13}}{(1 - q^4)(1 - q^{-2})},$$

$$Y_{213} = \frac{[u_j, Y_{2i}]_{q^{-2}}}{q^2 - q^{-2}} - \frac{Z_{j2i}}{[2]_q}, \qquad Z_{2132} = \frac{q Z_{2j} Y_{2i} - q^{-1} Y_{2i} Z_{2j}}{q - q^{-1}},$$

where $\{i, j\} = \{1, 3\}$. The PBW-type relations are given by the following formulae (where $i \in \{1, 3\}$ and $\{i, j\} = \{1, 3\}$)

$$Y_{2i} Y_2 = q^2 Y_2 Y_{2i}, \quad Z_{2i} Y_2 = Y_2 Z_{2i}, \quad Z_{1232} Y_2 = q^2 Y_2 Z_{1232},$$

$$Y_{13}Y_2 = Y_2Y_{13} + h_2Y_{21}Y_{23} + h_1Z_{1232},$$

$$Y_2Z_{13} = q^2Z_{13}Y_2 + h_1(Y_2Z_{123} + q^2Y_2Z_{321} - q^{-2}Y_{23}Z_{21} - qZ_{21}Z_{23} - Y_{21}Z_{23}) - h_1h_2Z_{1232},$$

$$Y_iY_2 = q^{-2}Y_2Y_i + h_2Y_{2i} + h_1Z_{2i}, \quad Y_{2i}Z_{13} = Z_{13}Y_{2i} + h_1(Y_{2i}Z_{i2j} - Z_{2i}Y_{13}),$$

$$Y_{23}Y_{21} = Y_{21}Y_{23}, \quad Z_{2i}Y_{2i} = Y_{2i}Z_{2i}, \quad Y_{13}Y_{2i} = q^2Y_{2i}Y_{13},$$

$$Z_{j2i}Y_{2i} = Y_{2i}Z_{j2i}, \quad Z_{i2j}Y_{2i} = q^2Y_{2i}Z_{i2j}, \quad Z_{1232}Y_{i2} = Y_{i2}Z_{1232}, \quad Y_iY_{2i} = q^2Y_{2i}Y_i,$$

$$Y_iY_{2j} = q^{-2}Y_{2i}Y_j + h_2Y_{13} + h_1Z_{j2i}, \quad Z_{2j}Y_{2i} = q^{-2}Y_{2i}Z_{2j} + q^{-1}h_1Z_{1232},$$

$$Z_{23}Z_{21} = Z_{21}Z_{23} + h_1(Y_2Z_{123} - Y_2Z_{321} + q^{-2}Y_{21}Z_{23} - q^{-2}Y_{23}Z_{21}),$$

$$Y_{13}Z_{2i} = Z_{2i}Y_{13} + qh_1Y_{2i}Z_{i2j},$$

$$Z_{2i}Z_{j2i} = q^2Z_{j2i}Z_{2i} + h_1(Z_{2i}Y_{13} - Y_{2i}Z_{i2j} - qY_{2i}Z_{13}),$$

$$Z_{i2j}Z_{2i} = q^2Z_{2i}Z_{i2j} + h_1(Z_{2i}Y_{13} - Y_{2i}Z_{i2j} - Z_{1232}Y_i),$$

$$Z_{1232}Z_{2i} = Z_{2i}Z_{1232} + h_1(q^2Y_2Y_{2i}Z_{i2j} + q^{-2}Y_{2i}Z_{1232} - q^{-2}Y_{21}Y_{23}Z_{2i}),$$

$$\begin{aligned} Z_{13}Z_{2i} = {} & Z_{2i}Z_{13} + h_1q^{-2}(qY_2Z_{j2i}Y_i + q^{-1}Y_{2i}Z_{2j}Y_i - Y_{2i}Z_{13} + Z_{2i}Z_{j2i}) \\ & + h_1^2q^{-2}(Z_{1232}Y_i + Y_{2i}Z_{i2j} - Z_{2i}Y_{13}), \end{aligned}$$

$$Y_iZ_{2j} = q^{-2}Z_{j2}Y_i + h_1Z_{13} + h_2Z_{j2i}, \quad Z_{13}Y_{13} = q^2Y_{13}Z_{13} - qh_1Z_{123}Z_{321},$$

$$Y_iZ_{2i} = Z_{2i}Y_i, \quad Z_{i2j}Y_{13} = Y_{13}Z_{i2j}, \quad Z_{1232}Y_{13} = q^{-2}Y_{13}Z_{1232},$$

$$Y_iY_{13} = q^2Y_{13}Y_i, \quad Z_{321}Z_{123} = Z_{123}Z_{321},$$

$$Z_{1232}Z_{i2j} = Z_{i2j}Z_{1232} + h_1(Y_{21}Y_{23}Z_{i2j} - Y_{2j}Z_{2i}Y_{213} + q^{-2}Y_{213}Z_{1232}),$$

$$Z_{i2j}Z_{13} = Z_{13}Z_{i2j} + h_1(Y_{213}Z_{213} - Z_{123}Z_{321} + q^{-1}Y_{2j}Z_{j2i}Y_i - q^{-1}Z_{2j}Y_{213}Y_i),$$

$$\begin{aligned} Z_{1232}Z_{13} = {} & Z_{13}Z_{1232} + h_1(q^{-1}Y_2Y_{213}Z_{123} + qY_2Y_{213}Z_{321} + Y_2Z_{123}Z_{321} - Y_{21}Y_{23}Z_{123} \\ & - q^{-1}Y_{21}Y_{23}Z_{321} - Z_{21}Z_{23}Y_{213}) + h_1^2(q^{-2}Y_{23}Z_{21}Y_{213} - q^{-4}Y_{213}Z_{2132}), \end{aligned}$$

$$Y_iZ_{1232} = Z_{1232}Y_i - h_1q(Y_{2i}Z_{i2j} - Z_{2i}Y_{13}),$$

$$Y_iZ_{i2j} = Z_{i2j}Y_i, \quad Y_iZ_{j2i} = q^2Z_{j2i}Y_i, \quad Z_{13}Y_i = Y_iZ_{13}, \quad Y_3Y_1 = Y_1Y_3.$$

where we abbreviate $h_k = q^k - q^{-k}$. □

Proof. Since the proof is rather computational, we only provide a sketch. First, we define an algebra $\mathcal{A}'$ with generators Y_α, $\alpha \in \{1, 2, 3, 21, 23, 13\}$ and Z_β, $\beta \in \{21, 23, 13, 123, 321, 1232\}$ and relations as above. Using the Diamond Lemma (see Proposition 2.15) we show that $\mathcal{A}'$ is PBW on these generators with the total order as defined in the theorem. Next, we introduce generators $u_i = Y_i$, $1 \le i \le 3$, $z_{ij} = (1-q^2)^{-|i-j|} Z_{ij}$ $1 \le i, j \le 3$ and show that they satisfy the relations in Theorem 1.17(ii). In particular, this yields a surjective homomorphism of algebras $\mathcal{A}_{q,4} \to \mathcal{A}'$. To prove that it is an isomorphism, we use Lemma 2.21. We define a grading on $\mathcal{A}_{q,4}$ by $\deg u_i = 1$, $\deg z_{12} = \deg z_{23} = 2$, and $\deg z_{13} = 3$. It is easy to see that specializations of defining relations of $\mathcal{A}_{q,4}$ are defining relations in the universal enveloping algebra of a nilpotent Lie algebra $\mathfrak{n}$ of dimension 12 generated by u_i, $1 \le i \le 3$ and $z_{ij} = z_{ji}$, $1 \le i < j \le 3$ subject to the relations

$$[u_i, [u_i, u_j]] = 0, \quad |i-j| = 1, \quad [z_\alpha, z_\beta] = 0, \alpha, \beta \in \{12, 13, 23\},$$

$$[u_i, z_{i2}] = [u_2, z_{i2}] = [u_i, z_{13}] = 0, \quad i \neq 2,$$

$$[u_2, [u_2, z_{13}]] = [z_{13}, [z_{13}, u_2] = 0, \quad [u_1, u_3] = 0,$$

$$[z_{i2}, [z_{i2}, u_j]] = [u_j, [u_j, z_{i2}]] = 0, \quad [z_{i2}, [u_j, u_2]] = [u_2, z_{13}], \quad \{i, j\} = \{1, 3\}.$$

In particular, $\dim U(\mathfrak{n})_k = \dim \mathcal{A}'_k$ for all k, where we endow $\mathcal{A}'_k$ with the induced grading. It remains to apply Lemma 2.21. ■

Using the above and Proposition 2.18, we immediately obtain the following theorem.

Theorem 4.8. The algebra $\mathcal{A}_{q,4}$ is optimal specializable. In particular, the following formulae define a Poisson structure on its specialization (only nonzero brackets are shown), where $i \in \{1, 3\}$ and $\{i, j\} = \{1, 3\}$.

$$\{Y_2, Y_i\} = 2Y_2Y_i - 4Y_{2i} - 2Z_{i2},$$

$$\{Y_2, Y_{2i}\} = -2Y_{i2}Y_2,$$

$$\{Y_2, Y_{13}\} = -4Y_{12}Y_{32} - 2Z_{2132},$$

$$\{Y_2, Z_{i2j}\} = -2Z_{2i}Y_{2j} + 2Z_{2132},$$

$$\{Y_2, Z_{13}\} = 2Y_2Z_{123} + 2Y_2Z_{13} + 2Y_2Z_{321} - 2Y_{21}Z_{23} - 2Y_{23}Z_{21} - 2Z_{21}Z_{23},$$

$$\{Y_2, Z_{2132}\} = -2Y_2 Z_{2132},$$

$$\{Y_i, Y_{2i}\} = 2Y_i Y_{2i},$$

$$\{Y_j, Y_{2i}\} = -2Y_{2i} Y_j + 4Y_{13} + 2Z_{j2i},$$

$$\{Y_{2i}, Y_{13}\} = -2Y_{2i} Y_{13},$$

$$\{Y_{2i}, Z_{2j}\} = 2Y_{i2} Z_{j2} - 2Z_{2132},$$

$$\{Y_{2i}, Z_{i2j}\} = -2Y_{2i} Z_{i2j},$$

$$\{Y_{2i}, Z_{13}\} = 2Y_{2i} Z_{13} - 2Z_{2i} Y_{13},$$

$$\{Y_i, Y_{13}\} = 2Y_{13} Y_i,$$

$$\{Y_{13}, Z_{13}\} = -2Y_{13} Z_{13} + 2Z_{123} Z_{321},$$

$$\{Y_{13}, Z_{1232}\} = 2Y_{13} Z_{2132},$$

$$\{Y_i, Z_{2j}\} = -2Z_{2j} Y_i + 4Z_{j2i} + 2Z_{13},$$

$$\{Y_{13}, Z_{i2}\} = 2Y_{i2} Z_{i2j},$$

$$\{Z_{21}, Z_{23}\} = -2Y_2 Z_{123} + 2Y_2 Z_{321} - 2Y_{21} Z_{23} + 2Y_{23} Z_{21},$$

$$\{Z_{i2}, Z_{i2j}\} = 2Z_{1232} Y_i + 2Y_{2i} Z_{i2j} - 2Z_{2i} Z_{i2j} - 2Z_{2i} Y_{13},$$

$$\{Z_{2i}, Z_{13}\} = 2(Y_2 Z_{j2i} Y_i - Y_{2i} Z_{2j} Y_i + Y_{2i} Z_{13} - Z_{2i} Z_{321}),$$

$$\{Z_{2i}, Z_{j2i}\} = -2Y_{2i} Z_{i2j} - 2Y_{2i} Z_{13} + 2Z_{2i} Y_{13} + 2Z_{2i} Z_{j2i},$$

$$\{Z_{2i}, Z_{1232}\} = -2Y_2 Y_{2i} Z_{i2j} - 2Y_{2i} Z_{1232} + 2Y_{21} Y_{23} Z_{2i},$$

$$\{Z_{13}, Z_{i2j}\} = 2Z_{23} Y_{13} Y_1 - 2Y_{23} Z_{321} Y_1 + 2Z_{123} Z_{321} - 2Y_{13} Z_{13},$$

$$\{Z_{13}, Z_{2132}\} = 2(Y_{21} Y_{23} Z_{123} - Y_2 Y_{13} Z_{123} - Y_2 Z_{123} Z_{321} - Y_2 Y_{13} Z_{321} + Z_{21} Z_{23} Y_{13} + Y_{21} Y_{23} Z_{321}),$$

$$\{Y_i, Z_{j2i}\} = 2Y_i Z_{j2i},$$

$$\{Z_{1232}, Y_i\} = 2Y_{2i} Z_{i2j} - 2Y_{13} Z_{2i},$$

$$\{Z_{1232}, Z_{i2j}\} = 2Y_{21} Y_{23} Z_{i2j} - 2Y_{2i} Z_{2j} Y_{13} + 2Y_{13} Z_{1232}.$$

□

It remains to prove that $\mathcal{A}_{q,4}$ is the uberalgebra for our quantum folding. For, let $\mathbf{i} = \{2, 1, 3, 2, 1, 3\} \in R(w_o)$ and define the elements $x_\alpha \in U_q^+(\mathfrak{sl}_4)$ by

$$X_{\mathbf{i}} = \{x_2, x_{21}, x_{23}, x_{13}, x_1, x_3\}$$

as ordered sets, in the notation of Section 2.4. We identify the x_α with the elements of the first copy of $U_q^+(\mathfrak{sl}_4)$ in $U_q^+(\mathfrak{sl}_4)^{\otimes 2}$ and denote by x'_α the corresponding elements of the second copy. Then the quantum folding $\iota_{\mathbf{i}}$ is given by extending multiplicatively

$$\iota_{\mathbf{i}}(x_\alpha) = x_\alpha x'_\alpha.$$

The following Lemma is checked by direct computations.

Lemma 4.9. Let $\tilde{y}_\alpha = x_\alpha x'_\alpha$ and let

$$\begin{aligned}
\tilde{z}_{2i} &= q^{-1}(x_{2i}x'_2x'_i + x_2x_ix'_{2i} - 2\tilde{y}_{2i}),\\
\tilde{z}_{13} &= q^{-2}(4\tilde{y}_{13} - 2x_{21}x_3x'_{13} - 2x_{23}x_1x'_{13} - 2x_{13}x'_{21}x'_3 - 2x_{13}x'_{23}x'_1\\
&\qquad + x_{21}x_3x'_{23}x'_1 + x_{23}x_1x'_{21}x'_3 + x_2x_1x_3x'_{13} + x_{13}x'_2x'_1x'_3),\\
\tilde{z}_{i2j} &= q^{-1}(x_{13}x'_{2j}x'_i + x_{2j}x_ix'_{13} - 2\tilde{y}_{13}),\\
\tilde{z}_{1232} &= x_2x_{13}x'_{21}x'_{23} + x_{21}x_{23}x'_2x'_{13} - 2q^{-1}\tilde{y}_{21}\tilde{y}_{23}.
\end{aligned}$$

Then the assignment $Y_\alpha \mapsto \tilde{y}_\alpha$ and $Z_\alpha \mapsto \tilde{z}_\alpha$ define a surjective algebra homomorphism $\mathcal{A}_{q,4} \to \langle U_q^+(\mathfrak{sl}_4)\rangle_{\iota_{\mathbf{i}}}$. Moreover, the folding $\iota_{\mathbf{i}}$ is tame liftable. □

In this case as well it can be shown that the diagram (1.3) commutes for a suitable choice of $\tilde{\iota}_{\mathbf{i}}$. We conclude this section with the following problem.

Problem 4.10. Construct the uberalgebra for the quantum folding $(\mathfrak{sl}_n^{\times k}, \mathfrak{sl}_n)$ for all n and $k \geq 2$. □

5 Folding $(\mathfrak{so}_8, G_2)$

In this section we let $\mathfrak{g} = \mathfrak{so}_8$ with $I = \{0, 1, 2, 3\}$ so that σ is a cyclic permutation of $\{1, 2, 3\}$ and $I/\sigma = \{0, 1\}$. In this numbering we have $(\alpha_0, \alpha_0) = 2$ and $(\alpha_1, \alpha_1) = 6$ in

$\mathfrak{g}^{\sigma\vee}$ which we abbreviate as $\mathfrak{g}^{\sigma}$ since its Langlands dual is obtained by renumbering the simple roots. Let $\mathcal{U}_{q,G_2}$ be the associative $\mathbb{C}(q)$-algebra generated by $U_q(\mathfrak{sl}_2)$ with Chevalley generators E_0, F_0, and $K_0^{\pm 1}$ and w, z_1, and z_2 satisfying the relations given in Theorem 1.23(iii) (with u replaced by E_0) as well as

$$[F_0, w] = 0 = [F_0, z_j], \quad K_0 w K_0^{-1} = q^{-3} w, \quad K_0 z_j K_0^{-1} = q^{-1} z_j, \quad j = 1, 2.$$

Theorem 5.1.

(i) The algebra $\mathcal{U}_{q,G_2}$ is isomorphic to the cross product $A_q \rtimes U_q(\mathfrak{sl}_2)$, where A_q is a flat deformation of the symmetric algebra of the nilpotent Lie algebra $\mathfrak{n}_{G_2}$ defined in Theorem 1.23(ii).

(ii) The assignment $w \mapsto E_1$, $z_j \mapsto 0$, $j = 1, 2$ defines a homomorphism $\hat{\iota} : \mathcal{U}_{q,n} \to U_q(\mathfrak{g}^{\sigma})$. Its image is the (parabolic) subalgebra of $U_q(\mathfrak{g}^{\sigma})$ generated by $U_q^+(\mathfrak{g}^{\sigma})$ and $K_0^{\pm 1}$, F_0.

(ii) The assignments $w \mapsto E_1 E_2 E_3$ and

$$z_1 \mapsto [E_1 E_2 E_3, E_0]_{q^{-3}} - \frac{q^2 + 1 + q^{-2}}{(q - q^{-1})^2} [E_1, [E_2, [E_3, E_0]_{q^{-1}}]_{q^{-1}}]_{q^{-1}},$$

$$z_2 \mapsto [E_0, E_1 E_2 E_3]_{q^{-3}} - \frac{q^2 + 1 + q^{-2}}{(q - q^{-1})^2} [[[E_0, E_1]_{q^{-1}}, E_2]_{q^{-1}}, E_3]_{q^{-1}}$$

define an algebra homomorphism $\mu : \mathcal{U}_{q,G_2} \to U_q(\mathfrak{g})$. Its image is contained in the (parabolic) subalgebra of $U_q(\mathfrak{g})$ generated by $U_q^+(\mathfrak{g})$ and $K_0^{\pm 1}$, F_0.

(iv) The assignments

$$T_0(w) = ((q^2 + 1 + q^{-2})(q + q^{-1}))^{-1} [[w, E_0]_{q^{-3}}]_{q^{-1}}]_q, \quad T_0(z_j) = [z_j, E_0]_{q^{-1}}$$

extend Lusztig's action (2.9) of the braid group $Br_{\mathfrak{sl}_2}$ on $U_q(\mathfrak{sl}_2)$ to an action on $\mathcal{U}_{q,G_2}$ by algebra automorphisms. Moreover, μ and $\hat{\iota}$ are $Br_{\mathfrak{sl}_2}$-equivariant. □

Most of the computations necessary to prove this theorem were performed on a computer and were involving rather heavy computations (for example, it took about 22 h for the UCR cluster to check that the Diamond Lemma holds). Otherwise, the structure of the proof is rather similar to the ones discussed above.

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A Naive Quantum Folding does not Exist

In this appendix, we show that the classical additive folding does not admit a quantum deformation even in the simplest possible case of $\mathfrak{sl}_4$. We use the standard numbering of the nodes of its Dynkin diagram. Let $u_1 = E_1 + E_3$, $u_2 = E_2$. We obviously have

$$u_2^2 u_1 - (q + q^{-1}) u_2 u_1 u_2 + u_1 u_2^2 = 0.$$

On the other hand, suppose that we have a relation

$$\sum_{j=0}^{3} c_j u_1^j u_2 u_1^{3-j} = 0. \tag{A.1}$$

Retain the notation of Section 2.7. Applying r_2 and $r_2 r_1$ to the left-hand side of (A.1) and considering the coefficients of linearly independent monomials we obtain a system of linear equations for the c_i, $0 \le i \le 3$ with the matrix

$$\begin{pmatrix} 1 & q^{-1} & q^{-2} & q^{-3} \\ 3q^2 & q(q^2+2) & 2q^2+1 & 3q \\ q^2+1+q^{-2} & q+2q^{-1} & 2+q^{-2} & q+q^{-1}+q^{-3} \\ 3(q^2+1) & 5q+q^{-1} & q^2+5 & 3(q+q^{-1}) \end{pmatrix}$$

of determinant $-(q-q^{-1})^6$. Thus, there is no relation of the form (A.1). Clearly, replacing u_1 and u_2 by $E_1 + E_3 + (q-1)a$ and $E_2 + (q-1)b$, where a and b are σ-invariant elements of degree greater than 1 will not affect the above calculation. Thus, there is no embedding of $U_q^+(\mathfrak{so}_5)$ into $U_q^+(\mathfrak{sl}_4)$ which deforms the embedding of $\mathfrak{so}_5$ into $\mathfrak{sl}_4$.

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