GROUP-LIKE ELEMENTS IN QUANTUM GROUPS AND FEIGIN'S CONJECTURE

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Abstract

Let A be an arbitrary symmetrizable Cartan matrix of rank r, and $\mathbf{n} = \mathbf{n}_+$ be the standard maximal nilpotent subalgebra in the Kac-Moody algebra associated with A (thus, $\mathbf{n}$ is generated by $E_1, \dots, E_r$ subject to the Serre relations). Let $\hat{U}_q(\mathbf{n})$ be the completion (with respect to the natural grading) of the quantized enveloping algebra of $\mathbf{n}$. For a sequence $\mathbf{i} = (i_1, \dots, i_m)$ with $1 \le i_k \le r$, let $P_{\mathbf{i}}$ be a skew polynomial algebra generated by $t_1, \dots, t_m$ subject to the relations $t_l t_k = q^{C_{i_k, i_l}} t_k t_l$ $(1 \le k < l \le m)$ where $C = (C_{ij}) = (d_i a_{ij})$ is the symmetric matrix corresponding to A. We construct a group-like element $\mathbf{e}_{\mathbf{i}} \in P_{\mathbf{i}} \bigotimes \hat{U}_q(\mathbf{n})$. This element gives rise to the evaluation homomorphism $\psi_{\mathbf{i}} : \mathbf{C}_q[N] \to P_{\mathbf{i}}$ given by $\psi_{\mathbf{i}}(x) = x(\mathbf{e}_{\mathbf{i}})$, where $\mathbf{C}_q[N] = U_q(\mathbf{n})^0$ is the restricted dual of $U_q(\mathbf{n})$. Under a well-known isomorphism of algebras $\mathbf{C}_q[N]$ and $U_q(\mathbf{n})$, the map $\psi_{\mathbf{i}}$ identifies with Feigin's homomorphism $\Phi(\mathbf{i}) : U_q(\mathbf{n}) \to P_{\mathbf{i}}$. We prove that the image of $\psi_{\mathbf{i}}$ generates the skew-field of fractions $\mathcal{F}(P_{\mathbf{i}})$ if and only if $\mathbf{i}$ is a reduced expression of some element w in the Weyl group W; furthermore, in the latter case, Ker $\psi_{\mathbf{i}}$ depends only on w (so we denote $I_w :=$ Ker $\psi_{\mathbf{i}}$). This result generalizes the results in [5], [6] to the case of Kac-Moody algebras. We also construct an element $\mathcal{R}_w \in \big(\mathbf{C}_q[N]/I_w\big) \bigotimes \hat{U}_q(\mathbf{n})$ which specializes to $\mathbf{e}_{\mathbf{i}}$ under the embedding $\mathbf{C}_q[N]/I_w \hookrightarrow P_{\mathbf{i}}$. The elements $\mathcal{R}_w$ are closely related to the quazi-R-matrix studied by G. Lusztig in [8]. If $\mathbf{i}, \mathbf{i}'$ are reduced expressions of the same element $w \in W$, we have a natural isomorphism $\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} : \mathcal{F}(P_{\mathbf{i}}) \to \mathcal{F}(P_{\mathbf{i}'})$ such that $(\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} \otimes \mathrm{id})(\mathbf{e}_{\mathbf{i}}) = \mathbf{e}_{\mathbf{i}'}$. This leads to identities between quantum exponentials. The maps $\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'}$ are q-deformations of Lusztig's transition maps [8]. The existence of the maps $\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'}$ leads to a surprising combinatorial corollary about skew-symmetric matrices associated with reduced expressions (cf. [12]).

0. Introduction and main results

It is well-known that a quantum group is not a group. One of the goals of this chapter is to introduce group-like elements for quantum deformations of certain nilpotent algebraic groups. In this section, we sketch our main results; more details will be given in subsequent sections.

Consider a maximal unipotent subgroup N in a complex simple algebraic group G. The group-like elements will be obtained as quantum deformation of certain morphisms $\pi_{\mathbf{i}} : \mathbf{C}^m \to N$ defined as follows.

Let $E_1, E_2, \ldots, E_r$ be standard generators of $\mathbf{n}$, the Lie algebra of N. For any sequence $\mathbf{i} = (i_1, \ldots, i_m)$ of indices (possibly with repetitions), one defines a map $\mathbf{C}^m \to N$ by the formula

$$\pi_{\mathbf{i}}(t_1, \ldots, t_m) = \exp(t_1 E_{i_1})\exp(t_2 E_{i_2}) \cdots \exp(t_m E_{i_m}) \tag{0.1}$$

where $\exp : \mathbf{n} \to N$ is the exponential map. Note that $\pi_{\mathbf{i}}$ is a regular (algebraic) map.

It is well-known that $\pi_{\mathbf{i}}$ is a birational isomorphism $\mathbf{C}^m \cong N$ if $\mathbf{i} = (i_1, \ldots, i_m)$ is a reduced expression of w_0, the longest element in the Weyl group W of G. Furthermore, if $\mathbf{i}$ is a reduced expression of $w \in W$, then the closure in N of the image of $\pi_{\mathbf{i}}$ depends only on w.

To construct a q-deformation of $\pi_{\mathbf{i}}$ we interpret the evaluation homomorphism $\pi_{\mathbf{i}}^* : \mathbf{C}[N] \to \mathbf{C}[t_1, \ldots, t_m]$ as follows. First, we think of the product in (0.1) as an element

$$\tilde{\pi}_{\mathbf{i}} \in \mathbf{C}[t_1, \ldots, t_m] \bigotimes \hat{U}(\mathbf{n}),$$

where $\hat{U}(\mathbf{n})$ is the completion of the universal enveloping algebra of $\mathbf{n}$ with respect to the natural grading.

Second, $\mathbf{C}[N]$ can be identified with $U(\mathbf{n})^0$, the restricted dual Hopf algebra. This gives rise to a natural pairing $\mathbf{C}[N] \times \hat{U}(\mathbf{n}) \to \mathbf{C}$. Extending scalars from $\mathbf{C}$ to $P = \mathbf{C}[t_1, \ldots, t_m]$ we see that each $f \in \mathbf{C}[N]$ becomes a linear form on $P \bigotimes \hat{U}(\mathbf{n})$ with values in P. Then we have

$$\pi_{\mathbf{i}}^*(f) = f(\tilde{\pi}_{\mathbf{i}}) \ . \tag{0.2}$$

We construct the deformation of $\tilde{\pi}_{\mathbf{i}}$ in a more general situation when $\mathbf{n}$ is the standard maximal nilpotent Lie subalgebra in a Kac-Moody algebra $\mathbf{g}$. Let us briefly introduce necessary definitions and notation.

Let $A = (a_{ij})$ be a symmetrizable Cartan matrix of size $r \times r$. Denote by $C = (C_{ij}) = (d_i a_{ij})$ the corresponding symmetric matrix. Let $\mathcal{U}$ be the associative algebra over $\mathbf{C}(q)$ generated by $E_1, \ldots, E_r$ subject to the *quantum Serre relations* (this is the quantized universal enveloping algebra $U_q(\mathbf{n})$ of the nilpotent part of the Kac-Moody algebra corresponding to A). The algebra $\mathcal{U}$ is graded by $\mathbf{Z}_+^r$ via $\deg(E_i) = \alpha_i$, the standard basis vector in $\mathbf{Z}_+^r$. Denote by $\hat{\mathcal{U}}$ the completion with respect to the grading. Following [8], Chapter 2, we consider $\mathcal{U}$ with the structure of a *braided bialgebra* with the *braided* coproduct $\Delta : \mathcal{U} \to \mathcal{U} \bigotimes \mathcal{U}$. Namely, $\Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i$, and Δ is a homomorphism of $\mathbf{Z}_+^r$-graded algebras, where the algebra structure on the tensor square of $\mathcal{U}$ differs from the standard one by a twist (see Section 2 below for more details). It follows that $\hat{\mathcal{U}}$ is a *complete bialgebra* with the coproduct $\hat{\Delta} : \hat{\mathcal{U}} \to \hat{\mathcal{U}} \hat{\bigotimes} \hat{\mathcal{U}}$. The *quantum group* $\mathcal{A}$ is the restricted dual algebra of $\mathcal{U}$ (if A is of finite type then $\mathcal{A}$ can be identified with the q-deformed ring of polynomial functions $\mathbf{C}_q[N]$). The natural evaluation pairing $\mathcal{A} \times \hat{\mathcal{U}} \to \mathbf{C}(q)$ will be denoted by $(x, E) \mapsto x(E)$.

Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a sequence of integers with $1 \le i_k \le r$. Denote by $P_{\mathbf{i}}$ the $\mathbf{C}(q)$-algebra generated by $t_1, \ldots, t_m$ subject to the relations

$$t_l t_k = q^{C_{i_k, i_l}} t_k t_l, \quad 1 \le k < l \le m \ . \tag{0.3}$$

Let $\hat{\mathcal{U}}_{\mathbf{i}} = P_{\mathbf{i}} \bigotimes \hat{\mathcal{U}}$ be the space of all series of the form

$$\sum_{\gamma \in \mathbf{Z}_+^r} t_\gamma \otimes E_\gamma$$

where $t_\gamma \in P_{\mathbf{i}}$ and $E_\gamma \in \mathcal{U}$ is a homogeneous element of degree γ. There is a standard algebra structure on $\hat{\mathcal{U}}_{\mathbf{i}}$. We identify $P_{\mathbf{i}} \bigotimes 1$ with $P_{\mathbf{i}}$ and $1 \bigotimes \hat{\mathcal{U}}$ with $\hat{\mathcal{U}}$ so that $tE = Et = t \otimes E$ in $\hat{\mathcal{U}}_{\mathbf{i}}$. Note also that $\hat{\mathcal{U}}_{\mathbf{i}}$ is a $P_{\mathbf{i}}$-bimodule in the standard way. The coproduct $\hat{\Delta}$ on $\hat{\mathcal{U}}$ extends naturally to the $P_{\mathbf{i}}$-bilinear map

$$\hat{\Delta}_{\mathbf{i}} : \hat{\mathcal{U}}_{\mathbf{i}} \to \hat{\mathcal{U}}_{\mathbf{i}} \hat{\bigotimes}_{P_{\mathbf{i}}} \hat{\mathcal{U}}_{\mathbf{i}}$$

by the formula: $\hat{\Delta}_{\mathbf{i}}(\sum_\gamma t_\gamma E_\gamma) = \sum_\gamma t_\gamma \Delta(E_\gamma)$. We call $\mathbf{e} \in \hat{\mathcal{U}}_{\mathbf{i}}$ a *group-like element* if $\hat{\Delta}_{\mathbf{i}}(\mathbf{e}) = \mathbf{e} \otimes \mathbf{e}$.

Finally, we define the q-exponential

$$\exp_q(u) = \sum_{n \geq 0} \frac{u^n}{[n]_q!} \tag{0.4}$$

where $[n]_q! = [1]_q [2]_q \cdots [n]_q$, $[l]_q = 1 + q + \cdots q^{(l-1)}$.

Now we can state our first main result.

Theorem 0.1. *For any sequence* $\mathbf{i} = (i_1, \ldots, i_m)$ *and any* $c_1, \ldots, c_m \in \mathbf{C}(q)$, *the product*

$$\exp_{q_{i_1}}(c_1 t_1 E_{i_1}) \exp_{q_{i_2}}(c_2 t_2 E_{i_2}) \cdots \exp_{q_{i_m}}(c_m t_m E_{i_m})$$

is a group-like *element in* $\hat{\mathcal{U}}_{\mathbf{i}}$, *where* $q_i = q^{C_{ii}}$ *for* $i = 1, \ldots, r$.

We prove Theorem 0.1 in Section 1 for more general *braided bialgebras.*

We denote

$$\mathbf{e}_{\mathbf{i}} = \exp_{q_{i_1}}(t_1 E_{i_1}) \exp_{q_{i_2}}(t_2 E_{i_2}) \cdots \exp_{q_{i_m}}(t_m E_{i_m}) \ . \tag{0.5}$$

As in the commutative case, we extend the evaluation pairing $\mathcal{A} \times \hat{\mathcal{U}} \to \mathbf{C}(q)$ to the $P_{\mathbf{i}}$-linear pairing $\mathcal{A} \times \hat{\mathcal{U}}_{\mathbf{i}} \to P_{\mathbf{i}}$. As an analogue of (0.2) we define the map $\psi_{\mathbf{i}} : \mathcal{A} \to P_{\mathbf{i}}$ by

$$\psi_{\mathbf{i}}(x) := x(\mathbf{e}_{\mathbf{i}}) \ . \tag{0.6}$$

Corollary 0.2. *The map* $\psi_{\mathbf{i}}$ *is an algebra homomorphism.*

Proof. The definition of the pairing $(x, E) \to x(E)$ implies that $(xy)(u) = (x \otimes y)(\hat{\Delta}(u))$ for all $x, y \in \mathcal{A}$ and $u \in \hat{\mathcal{U}}_{\mathbf{i}}$, where $(x \otimes y)(u_1 \otimes u_2) := x(u_1) y(u_2)$ for any $u_1, u_2 \in \hat{\mathcal{U}}_{\mathbf{i}}$. Thus, we have

$$\psi_{\mathbf{i}}(xy) = (xy)(\mathbf{e}_{\mathbf{i}}) = (x \otimes y)(\hat{\Delta}_{\mathbf{i}}(\mathbf{e}_{\mathbf{i}})) = (x \otimes y)(\mathbf{e}_{\mathbf{i}} \otimes \mathbf{e}_{\mathbf{i}}) = x(\mathbf{e}_{\mathbf{i}}) y(\mathbf{e}_{\mathbf{i}}) = \psi_{\mathbf{i}}(x) \psi_{\mathbf{i}}(y) \ .$$

Corollary 0.2 is proved. $\triangleleft$

Expanding (0.5), we obtain the following formula for $\psi_{\mathbf{i}}$:

$$\psi_{\mathbf{i}}(x) = \sum_{a_1, a_2 \ldots, a_m \geq 0} x\big(E_{i_1}^{[a_1]} E_{i_2}^{[a_2]} \cdots E_{i_m}^{[a_m]}\big) t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \tag{0.7}$$

where $E_i^{[n]} = \dfrac{1}{[n]_{q_i}!} E_i^n$. Note that the sum in (0.7) is always finite.

Under a well-known isomorphism $\mathcal{A} \cong \mathcal{U}$ the homomorphism $\psi_{\mathbf{i}}$ becomes *Feigin's homomorphism* $\Phi(\mathbf{i}) : \mathcal{U} \to P_{\mathbf{i}}$ ([5] and Section 2 below).

Using the homomorphism $\psi_{\mathbf{i}}$, we can express the group-like element $\mathbf{e}_{\mathbf{i}}$ in terms of the *universal element* $\mathcal{R} \in \mathcal{A} \bigotimes \hat{\mathcal{U}}$ defined as follows. Under the canonical isomorphism between $\mathcal{A} \bigotimes \hat{\mathcal{U}}$ and the space of linear maps $\mathcal{U} \to \hat{\mathcal{U}}$, the element $\mathcal{R}$ corresponds to the inclusion $\mathcal{U} \hookrightarrow \hat{\mathcal{U}}$.

Proposition 0.3. *For any sequence* $\mathbf{i}$ *as above, we have*

$$(\psi_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}) = \mathbf{e}_{\mathbf{i}} \ . \tag{0.8}$$

The element $\mathcal{R}$ *is uniquely determined by the equations (0.8) for all* $\mathbf{i}$.

Proof. Let B be a homogeneous basis in $\mathcal{U}$ (that is, B is compatible with the $\mathbf{Z}_+^r$-grading), and $\{b^*\}$ be the dual basis in $\mathcal{A}$ (so that $b^*(b') = \delta_{b,b'}$). By definition,

$$\mathcal{R} = \sum_{b \in B} b^* \otimes b \ . \tag{0.9}$$

We have

$$(\psi_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}) = \sum_{b \in B} \psi_{\mathbf{i}}(b^*) \otimes b = \sum_{b \in B} b^*(\mathbf{e}_{\mathbf{i}}) \otimes b = \mathbf{e}_{\mathbf{i}}$$

by definition (0.6) of $\psi_{\mathbf{i}}$ and by the formula $\sum_{b \in B} b^*(u) \otimes b = u$ for any $u \in \hat{\mathcal{U}}_{\mathbf{i}}$.

It remains to check the uniqueness. Assume that there is another $\mathcal{R}'$ satisfying (0.8) for all $\mathbf{i}$. The equations (0.8) imply that $(\mathcal{R} - \mathcal{R}') \in \mathrm{Ker}\ \psi_{\mathbf{i}} \bigotimes \hat{\mathcal{U}}$. By (0.7), an element $x \in \mathcal{A}$ is in the kernel of each $\psi_{\mathbf{i}}$ if and only if x vanishes at all monomials in $E_1, \dots, E_r$. Hence, $\mathcal{R}' = \mathcal{R}$, and we are done. $\triangleleft$

The element $\mathcal{R}$ was studied in [8], Chapter 4 in a slightly different setting; it is a version of the universal R-matrix for the "braided" quantum double of $\mathcal{U}$.

Let W be the Weyl group generated by simple reflections $s_1, \dots, s_r : \mathbf{Z}^r \to \mathbf{Z}^r$ defined by $s_i(\alpha_j) = \alpha_i - a_{ij}\alpha_i$ for all i, j. We say that $\mathbf{i} = (i_1, \dots, i_m)$ is a reduced expression of $w \in W$ if $w = s_{i_1} \cdots s_{i_m}$ and this factorization of w is the shortest possible. We denote by $R(w)$ the set of all reduced expressions of w. We also reserve notation w_0 for the longest element in W if W is finite.

For a sequence $\mathbf{i} = (i_1, \dots, i_m)$ let $\mathcal{U}(\mathbf{i})$ be the subspace in $\mathcal{U}$ spanned by all monomials $E_{i_1}^{a_1} E_{i_2}^{a_2} \cdots E_{i_m}^{a_m}$. It is-well known ([7], Section 4.4, or [9]) that if $\mathbf{i} \in R(w)$ then $\mathcal{U}(\mathbf{i})$ depends only on w. So we denote $\mathcal{U}(w) := \mathcal{U}(\mathbf{i})$ for all $\mathbf{i} \in R(w)$. It is also well-known that $\mathcal{U}(w_0) = \mathcal{U}$ if W is finite. Further, if $\mathbf{i}$ is not reduced then there is a subsequence $\mathbf{i}'$ of $\mathbf{i}$ such that $\mathbf{i}'$ is a reduced expression and $\mathcal{U}(\mathbf{i}) = \mathcal{U}(\mathbf{i}')$. For the convenience of the reader we will prove these assertions in Section 2.

Now we can give a complete description of Ker $\psi_{\mathbf{i}}$ using (0.7) and the above discussion.

Lemma 0.4. *The kernel of* $\psi_{\mathbf{i}}$ *is the orthogonal complement of* $\mathcal{U}(\mathbf{i})$*:*

$$\mathrm{Ker}\ \psi_{\mathbf{i}} = \{x \in \mathcal{A} : x(u) = 0 \text{ for all } u \in \mathcal{U}(\mathbf{i})\} \ .$$

Furthermore,

(i) If $\mathbf{i}$ *is not reduced then there is a reduced subsequence* $\mathbf{i}'$ *of* $\mathbf{i}$ *such that* Ker $\psi_{\mathbf{i}} =$ Ker $\psi_{\mathbf{i}'}$.

(ii) For every $w \in W$ *and* $\mathbf{i}, \mathbf{i}' \in R(w)$ *we have* Ker $\psi_{\mathbf{i}} =$ Ker $\psi_{\mathbf{i}'}$.

(iii) If W *is finite then* Ker $\psi_{\mathbf{i}} = \{0\}$ *for any* $\mathbf{i} \in R(w_0)$.

Denote $I_w :=$ Ker $\psi_{\mathbf{i}}$ for $\mathbf{i} \in R(w)$. Our next main result is the following.

Theorem 0.5. *For every* $w \in W$ *and* $\mathbf{i} \in R(w)$*, the image of* $\psi_{\mathbf{i}}$ *generates the (skew) field of fractions of* $P_{\mathbf{i}}$*. Hence,* $\psi_{\mathbf{i}}$ *induces an isomorphism of the fields of fractions*

$$\overline{\psi}_{\mathbf{i}} : \mathcal{F}\big(\mathcal{A}/I_w\big) \widetilde{\to} \mathcal{F}\big(P_{\mathbf{i}}\big) \ . \tag{0.10}$$

In particular, if W *is finite and* $w = w_0$ *then* $\overline{\psi}_{\mathbf{i}}$ *is an isomorphism between* $\mathcal{F}\big(\mathcal{A}\big)$ *and* $\mathcal{F}\big(P_{\mathbf{i}}\big)$.

Here the symbol $\mathcal{F}$ stands for the skew-field of fractions. We review the necessary definitions and results in the appendix below.

The last statement in Theorem 0.5 coincides with Feigin's conjecture ([5], [6]) (stated for $\mathcal{U}$ rather than for $\mathcal{A}$). The conjecture was proved in [5] for the type A_r and $w = w_0$ by a direct computation involving some specific reduced word $\mathbf{i} \in R(w_0)$. The conjecture was further generalized by A. Joseph ([6]) to any $w \in W$ in the assumption that W is finite, and proved by geometric arguments.

We give an algebraic proof of Theorem 0.5 in Section 2 without the assumption that W is finite. The following proposition plays the crucial role in the proof.

Proposition 0.6. *For every reduced expression* $\mathbf{i}$ *of some* $w \in W$*, there is an element* $x = x(\mathbf{i}) \in \mathcal{A}$ *such that* $\psi_{\mathbf{i}}(x) = t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m}$ *with* $a_1 > 0$.

We prove Proposition 0.6 in Section 3.

Corollary 0.7. *The skew-field* $\mathcal{F}\big(\psi_{\mathbf{i}}(\mathcal{A})\big)$ *coincides with* $\mathcal{F}(P_{\mathbf{i}})$ *if and only if* $\mathbf{i}$ *is a reduced expression.*

Proof. We need to prove the "only if" part (the "if" part is the assertion of Theorem 0.5). Assume that $\mathbf{i} = (i_1, \dots, i_m)$ is not reduced. Then, by Lemma 0.4(i) and Theorem 0.5, there is a reduced subsequence $\mathbf{i}' = (i_{k_1}, \dots, i_{k_l})$ of $\mathbf{i}$ such that $\mathrm{Im}\ \psi_{\mathbf{i}'} \cong \mathrm{Im}\ \psi_{\mathbf{i}}$. It follows that $\mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}}) \cong \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}'}) \cong \mathcal{F}(P_{\mathbf{i}'})$. But $\mathcal{F}(P_{\mathbf{i}'}) \ncong \mathcal{F}(P_{\mathbf{i}})$ since these skew-fields have different Gel'fand-Kirillov dimensions (see [12], Proposition 2.18). Corollary 0.7 is proved. $\triangleleft$

Let us illustrate the above results in the case when A is of type A_{n-1}. In this case, one can show that $\mathcal{A}$ is generated by the elements x_{ij} with $1 \le i < j \le n$ subject to the following relations (cf. [2], [4]):

$$x_{ij}x_{kl} = x_{kl}x_{ij},\ x_{il}x_{jk} = x_{jk}x_{il} \quad (1 \le i < j < k < l \le n)\ , \tag{0.11}$$

$$x_{ik} = \frac{qx_{ij}x_{jk} - x_{jk}x_{ij}}{q - q^{-1}}, \quad x_{ij}x_{ik} = qx_{ik}x_{ij}, \quad x_{jk}x_{ik} = q^{-1}x_{ik}x_{jk}\ (1 \le i < j < k \le n)\ .$$

The elements x_{ij} are q-deformations of the matrix entries considered as polynomial functions on the group N of the unipotent upper-triangular matrices. So we arrange the x_{ij} into the matrix $X = I + \sum\limits_{i<j} x_{ij}E_{ij}$, where I is the identity matrix, and the E_{ij} are the matrix units. Let $\psi_{\mathbf{i}}(X)$ be the $n \times n$-matrix (over $P_{\mathbf{i}}$) obtained from X by applying $\psi_{\mathbf{i}}$ to each matrix entry. The following proposition will be proved in Section 2.

Proposition 0.8. *For any sequence* $\mathbf{i} = (i_1, \dots, i_m)$*, the matrix* $\psi_{\mathbf{i}}(X)$ *admits the following matrix factorization*

$$\psi_{\mathbf{i}}(X) = (I + t_1E_{i_1,i_1+1})(I + t_2E_{i_2,i_2+1}) \cdots (I + t_mE_{i_m,i_m+1})\ . \tag{0.12}$$

If $\mathbf{i} \in R(w_0)$ then, applying the inverse isomorphism $(\overline{\psi}_{\mathbf{i}})^{-1} : \mathcal{F}\big(P_{\mathbf{i}}\big) \to \mathcal{F}\big(\mathcal{A}\big)$ to the factorization (0.12), we obtain the factorization of the matrix X over $\mathcal{F}\big(\mathcal{A}\big)$:

$$X = (I + \tilde{t}_1E_{i_1,i_1+1})(I + \tilde{t}_2E_{i_2,i_2+1}) \cdots (I + \tilde{t}_mE_{i_m,i_m+1})\ ,$$

where $\tilde{t}_k = (\overline{\psi}_{\mathbf{i}})^{-1}(t_k)$. This factorization is a q-deformation of the one studied in [1]; it can be shown that such a factorization is unique. The explicit formulas for $\tilde{t}_k$ in terms of the matrix entries x_{ij} will be given in a separate publication. The above factorizations of the matrix X (and, more generally, $\mathcal{R}$ for quantum groups of finite type) were studied in [11].

Let us return to the general situation and discuss some corollaries of Theorem 0.5. For every $w \in W$ and $\mathbf{i}, \mathbf{i}' \in R(w)$ there is an isomorphism of skew fields

$$\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} : \mathcal{F}\big(P_{\mathbf{i}}\big) \xrightarrow{\sim} \mathcal{F}\big(P_{\mathbf{i}'}\big)$$

defined by $\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} = \overline{\psi}_{\mathbf{i}'} \circ (\overline{\psi}_{\mathbf{i}})^{-1}$. We extend it to the isomorphism $\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} \otimes \mathrm{id} : \mathcal{F}\big(P_{\mathbf{i}}\big) \bigotimes \hat{\mathcal{U}} \to \mathcal{F}\big(P_{\mathbf{i}'}\big) \bigotimes \hat{\mathcal{U}}$. In the following proposition, every element $\mathbf{e}_{\mathbf{i}}$ given by (0.5) is regarded as an element of $\mathcal{F}\big(P_{\mathbf{i}}\big) \bigotimes \hat{\mathcal{U}}$.

Proposition 0.9. *For every* $\mathbf{i}, \mathbf{i}' \in R(w)$*, we have* $(\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} \otimes \mathrm{id})(\mathbf{e}_{\mathbf{i}}) = \mathbf{e}_{\mathbf{i}'}$.

Proof. Let $\mathbf{p}_w : \mathcal{A} \to \mathcal{A}/I_w$ be the canonical projection. Denote $\mathcal{R}_w = (\mathbf{p}_w \otimes \mathrm{id})(\mathcal{R})$. Note that, similarly to $\mathcal{R}$, the element $\mathcal{R}_w \in \mathcal{A}/I_w \bigotimes \hat{\mathcal{U}}$ corresponds to the inclusion $\mathcal{U}(w) \hookrightarrow \mathcal{U}$. Proposition 0.3 implies that

$$(\overline{\psi}_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}_w) = \mathbf{e}_{\mathbf{i}} \tag{0.13}$$

for every $\mathbf{i} \in R(w)$. We are done since $\mathbf{R}_{\mathbf{i}}^{\mathbf{i}'} \otimes \mathrm{id} = (\overline{\psi}_{\mathbf{i}'} \otimes \mathrm{id}) \circ (\overline{\psi}_{\mathbf{i}} \otimes \mathrm{id})^{-1}$. $\triangleleft$

Proposition 0.9 implies some identities between quantum exponentials. For two reduced expressions $\mathbf{i} = (i_1, \dots, i_m)$ and $\mathbf{i}' = (i'_1, \dots, i'_m)$ of an element $w \in W$, let $t_1, \dots, t_m$ (resp. $t'_1, \dots, t'_m$) be the standard generators of $P_{\mathbf{i}}$ (resp. $P_{\mathbf{i}'}$).

Corollary 0.10. *The following identity holds in the algebra* $\mathcal{F}(P_{\mathbf{i}})\bigotimes\hat{\mathcal{U}}$*:*

$$\exp_{q_{i_1}}(t_1E_{i_1})\cdots\exp_{q_{i_m}}(t_mE_{i_m})=\exp_{q_{i'_1}}(p_1E_{i'_1})\cdots\exp_{q_{i'_m}}(p_mE_{i'_m})\ , \tag{0.14}$$

where $p_k=\mathbf{R}^{\mathbf{i}}_{\mathbf{i}'}(t'_k)$ *for* $k=1,\dots,m$*. Identity (0.14) remains true under the rescaling* $E_i\mapsto c_iE_i$ *for any* $c_i\in\mathbf{C}(q)$*,* $i=1,\dots,r$*.*

Example 1. Let A be the Cartan matrix of type A_2 . Then the Weyl group W is the symmetric group S_3, and $w_0=s_1s_2s_1=s_2s_1s_2$. Take $\mathbf{i}=(121)$, $\mathbf{i}'=(212)$, and denote by t_1,t_2,t_3 and t'_1,t'_2,t'_3 the generators of $P_{(121)}$ and $P_{(212)}$ respectively. Then $\mathbf{R}^{(121)}_{(212)}(t'_k)=p_k$ for $k=1,2,3$, where

$$p_1=t_2t_3(t_1+t_3)^{-1},\ p_2=t_1+t_3,\ p_3=(t_1+t_3)^{-1}t_1t_2\ .$$

This is a consequence of the matrix equation

$$(I+p_1E_{23})(I+p_2E_{12})(I+p_3E_{23})=(I+t_1E_{12})(I+t_2E_{23})(I+t_3E_{12})\ ,$$

which follows from the factorization (0.12).

The identity (0.14) takes the form

$$\exp_{q^2}(c_1t_1E_1)\exp_{q^2}(c_2t_2E_2)\exp_{q^2}(c_1t_3E_1)=\exp_{q^2}(c_2p_1E_2)\exp_{q^2}(c_1p_2E_1)\exp_{q^2}(c_2p_3E_2) \tag{0.15}$$

for any $c_1,c_2\in\mathbf{C}(q)$. Expanding both sides of (0.15) and comparing the components of degree $2\alpha_1+\alpha_2$, we obtain the quantum Serre relation

$$E_1^2E_2-(q+q^{-1})E_1E_2E_1-E_2E_1^2=0\ .$$

We also note that setting $c_1=1$ and $c_2=0$ in (0.15) yields the familiar rule

$$\exp_{q^2}(t_1E_1)\exp_{q^2}(t_3E_1)=\exp_{q^2}((t_1+t_3)E_1)\ .$$

The identity (0.15) appeared in [11], Section 10.4; it was proved there by a straightforward computation.

We conclude the introduction by a surprising combinatorial consequence of the above results. To a sequence $\mathbf{i}=(i_1,\dots,i_m)$ we associate a skew-symmetric $m\times m$-matrix $S(\mathbf{i})$ by the formula:

$$S(\mathbf{i})=\sum_{1\le k<l\le m}C_{i_k,i_l}(E_{kl}-E_{lk})\ . \tag{0.16}$$

We say that two $m\times m$ matrices S and S' are *equivalent* if there is a matrix $T\in SL_m(\mathbf{Z})$ such that $S'=TST^t$ (where T^t is the transpose of T).

Proposition 0.11. *For every* w *and* $\mathbf{i},\mathbf{i}'\in R(w)$*, the matrices* $S(\mathbf{i})$ *and* $S(\mathbf{i}')$ *are equivalent.*

This follows from the fact that the skew-fields $\mathcal{F}(P_{\mathbf{i}})$ and $\mathcal{F}(P_{\mathbf{i}'})$ are isomorphic, in view of a general result by A. Panov [12] (see also Section 2 below). Proposition 0.11 essentially says that there exists an isomorphism $\mathcal{F}(P_{\mathbf{i}'})\to\mathcal{F}(P_{\mathbf{i}})$ which takes every generator t'_k to a monomial in $t_1,\dots,t_m$. (Note that the isomorphism $\mathbf{R}^{\mathbf{i}}_{\mathbf{i}'}$ considered above, in general does not have this property).

The material is organized as follows. In Section 1 we introduce braided bialgebras and prove some results about them, including the generalization of Theorem 0.1. The quantum group $\mathcal{A}$ associated with a symmetrizable Cartan matrix is studied in Section 2, which contains the proofs of Lemma 0.4, Theorem 0.5 (modulo Proposition 0.6), and Propositions 0.8 and 0.11. Section 3 is devoted to the proof of Proposition 0.6; our proof is based on the properties of extremal vectors in simple $U_q(\mathbf{g})$-modules. In Appendix we review necessary definitions and results about non-commutative fields of fractions.

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1. Results on braided bialgebras

Let $\mathbf{k}$ be a field and $\mathcal{U}$ be a $\mathbf{Z}_+^r$-graded $\mathbf{k}$-algebra: $\mathcal{U} = \bigoplus \mathcal{U}(\gamma)$, the sum over $\gamma \in \mathbf{Z}_+^r$. We assume that $\mathcal{U}(0) = \mathbf{k}$ and every $\mathcal{U}(\gamma)$ is finite-dimensional. Let $\mathbf{q} = (q_{ij})$, $1 \le i, j \le r$ be a $r \times r$-matrix with all $q_{ij} \in \mathbf{k}, q_{ij} \neq 0$. Following G. Lusztig ([8]) we associate with $\mathbf{q}$ an algebra structure on the vector space $\mathcal{U} \bigotimes \mathcal{U}$. For any two homogeneous elements $b \in \mathcal{U}(m_1, \dots, m_r)$ and $c \in \mathcal{U}(n_1, \dots, n_r)$ we set

$$Q(b, c) = Q((m_1, \dots, m_r), (n_1, \dots, n_r)) = \prod_{i,j=1}^{r} q_{ij}^{m_i n_j} \ . \tag{1.1}$$

We define the $\mathbf{q}$-*braided* multiplication in $\mathcal{U} \bigotimes \mathcal{U}$ by

$$(a \otimes b)(c \otimes d) := Q(b, c)(ac \otimes bd) \tag{1.2}$$

for any homogeneous elements b, c of $\mathcal{U}$ and any $a, d \in \mathcal{U}$.

It is easy to see that (1.2) makes $\mathcal{U} \bigotimes \mathcal{U}$ into a $\mathbf{Z}_+^r$-graded associative algebra (with the standard grading $(\mathcal{U} \bigotimes_{\mathbf{k}} \mathcal{U})(\gamma) = \bigoplus_{\gamma'} \ \mathcal{U}(\gamma') \bigotimes \mathcal{U}(\gamma - \gamma')$). This algebra will be denoted by $\mathcal{U} \bigotimes_{\mathbf{q}} \mathcal{U}$ and called the $\mathbf{q}$-*braided* tensor square of $\mathcal{U}$.

We call $\mathcal{U}$ a $\mathbf{q}$-*braided bialgebra* if

(i) there is a homomorphism of $\mathbf{Z}_+^r$-graded algebras $\Delta : \mathcal{U} \to \mathcal{U} \bigotimes_{\mathbf{q}} \mathcal{U}$ satisfying the coassociativity constrain (we call Δ the *coproduct*);

(ii) There is a *counit* homomorphism of algebras $\varepsilon : \mathcal{U} \to \mathcal{U}(0) = \mathbf{k}$ satisfying

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}, \ \varepsilon(1) = 1. \tag{1.3}$$

This definition implies that for every $u \in \mathcal{U}$,

$$\Delta(u) = u \otimes 1 + 1 \otimes u + \sum_n u_n \otimes u'_n \tag{1.4}$$

where all u_n, u'_n are homogeneous elements of nonzero degrees. In particular, $\Delta(u) = u \otimes 1 + 1 \otimes u$ for every $x \in \mathcal{U}(\alpha_1) \bigoplus \cdots \bigoplus \mathcal{U}(\alpha_r)$ where $\alpha_1, \dots, \alpha_r$ is the standard basis in $\mathbf{Z}_+^r$. Another consequence of this definition is that $\varepsilon(x) = 0$ for any $x \in \mathcal{U}(\gamma), \gamma \neq 0$.

Note that the algebra $\mathcal{U}$ from the introduction, associated to a symmetrizable Cartan matrix A, is a $\mathbf{q}$-braided algebra, where $q_{ij} = q^{C_{ij}}$. Another example is the *free* algebra generated by $E_1, \dots, E_r$, where $\mathbf{q}$ is arbitrary.

Let $\hat{\mathcal{U}}$ be the completion of $\mathcal{U}$ with respect to the grading, that is, the space of all formal series $\hat{u} = \sum_{\gamma \in \mathbf{Z}_+^r} u_\gamma$, where $u_\gamma \in \mathcal{U}(\gamma)$. Clearly, $\hat{\mathcal{U}}$ is an algebra. The coproduct in $\mathcal{U}$ extends to $\hat{\Delta} : \hat{\mathcal{U}} \to \hat{\mathcal{U}} \hat{\bigotimes}_{\mathbf{q}} \hat{\mathcal{U}}$ so $\hat{\mathcal{U}}$ becomes a *complete bialgebra*.

Now we fix a positive integer m and consider a sequence $\mathbf{i} = (i_1, i_2, \dots, i_m)$ of integers with $1 \le i_k \le r$. Let $\mathbf{q} = (q_{ij})$ be the matrix used in the definition of $\mathcal{U}$. Consider a $\mathbf{k}$-algebra $P_{\mathbf{i}} = P_{\mathbf{i},\mathbf{q}}$ generated by $t_1, \dots, t_m$ subject to the following relations:

$$t_l t_k = q_{i_k, i_l} t_k t_l \tag{1.5}$$

for all $1 \le k < l \le m$.

Define $\hat{\mathcal{U}}_{\mathbf{i}} = P_{\mathbf{i}} \bigotimes_{\mathbf{k}} \hat{\mathcal{U}}$, the space of formal series of the form $\sum_\gamma t_\gamma \otimes u_\gamma$, where $t_\gamma \in P_{\mathbf{i}}$ and $u_\gamma \in \mathcal{U}(\gamma)$. We consider $\hat{\mathcal{U}}_{\mathbf{i}}$ with the standard algebra structure (so we can write $tu = ut = t \otimes u$).

Consider the completed tensor square $\mathcal{V}_{\mathbf{i}} = \hat{\mathcal{U}}_{\mathbf{i}} \hat{\bigotimes}_{P_{\mathbf{i}}} \hat{\mathcal{U}}_{\mathbf{i}}$ where the left factor is regarded as a right $P_{\mathbf{i}}$-module and the right factor as a left $P_{\mathbf{i}}$-module. Note that $\mathcal{V}_{\mathbf{i}}$ is a $P_{\mathbf{i}}$-bimodule. In $\mathcal{V}_{\mathbf{i}}$, we can write $t(u \otimes v) = (tu) \otimes v = u \otimes (tv) = (u \otimes v)t$ for any $u, v \in \mathcal{U}, t \in P_{\mathbf{i}}$. Under the standard identification $\mathcal{V}_{\mathbf{i}} \cong P_{\mathbf{i}} \bigotimes \hat{\mathcal{U}} \hat{\bigotimes}_{\mathbf{q}} \hat{\mathcal{U}}$ this bimodule $\mathcal{V}_{\mathbf{i}}$ becomes an algebra.

There is a natural morphism of $P_{\mathbf{i}}$-bimodules

$$\hat{\Delta}_{\mathbf{i}} : \hat{\mathcal{U}}_{\mathbf{i}} \to \mathcal{V}_{\mathbf{i}}$$

which is the $P_{\mathbf{i}}$-linear extension of the coproduct $\hat{\Delta}$ on $\hat{\mathcal{U}}$. Clearly, $\hat{\Delta}_{\mathbf{i}}$ is an algebra homomorphism.

Let $\mathbf{E} = (E_1, \dots, E_m)$ be the family of elements $E_k \in \mathcal{U}(\alpha_{i_k})$. We define an element $\mathbf{e}_{\mathbf{i}} = \mathbf{e}_{\mathbf{i},\mathbf{E}} \in \hat{\mathcal{U}}_{\mathbf{i}}$ as follows:

$$\mathbf{e}_{\mathbf{i},\mathbf{E}} = \exp_{q_1}(t_1 E_1) \exp_{q_2}(t_2 E_2) \cdots \exp_{q_m}(t_m E_m) \tag{1.6}$$

where $q_k = q_{i_k, i_k}$ for $k = 1, \dots, m$, and $\exp_{q_k}$ stands for the quantum exponential defined by (0.4).

The following result extends Theorem 0.1 to arbitrary $\mathbf{q}$-braided algebras.

Theorem 1.1. *For any sequence* $\mathbf{i}$ *and any family* $\mathbf{E} = (E_k)$ *as above the element* $\mathbf{e}_{\mathbf{i}} = \mathbf{e}_{\mathbf{i},\mathbf{E}}$ *is a* group-like *element in* $\hat{\mathcal{U}}_{\mathbf{i}}$*, i.e.,* $\hat{\Delta}_{\mathbf{i}}(\mathbf{e}_{\mathbf{i}}) = \mathbf{e}_{\mathbf{i}} \otimes \mathbf{e}_{\mathbf{i}}$.

Proof . We need the following.

Lemma 1.2.
(a) Each factor $\mathbf{e}_k = \exp_{q_k}(t_k E_k)$ *of* $\mathbf{e}_{\mathbf{i}}$ *is a group-like element in* $\hat{\mathcal{U}}_{\mathbf{i}}$.
(b) $(1 \otimes \mathbf{e}_k)(\mathbf{e}_l \otimes 1) = (\mathbf{e}_l \otimes 1)(1 \otimes \mathbf{e}_k)$ *for any* $1 \le k < l \le m$.

Proof. (a) Denote $E = t_k E_k$. Since $\Delta(E_k) = E_k \otimes 1 + 1 \otimes E_k$, for each k we have

$$\hat{\Delta}_{\mathbf{i}}(E) = t_k(E_k \otimes 1 + 1 \otimes E_k) = E \otimes 1 + 1 \otimes E.$$

Denote $x = E \otimes 1, y = 1 \otimes E$. Let us show that that $yx = qxy$ where $q := q_k = q_{i_k, i_k}$. Indeed,

$$yx = (1 \otimes E)(E \otimes 1) = (1 \otimes t_k E_k)(t_k E_k \otimes 1) = t_k^2 (1 \otimes E_k)(E_k \otimes 1)$$

$$= Q(E_k, E_k) t_k^2 (E_k \otimes E_k) = q_{i_k, i_k}(t_k E_k \otimes t_k E_k) = qxy \ .$$

Further, we obtain

$$\hat{\Delta}_{\mathbf{i}}(\exp_q(E)) = \exp_q(\hat{\Delta}_{\mathbf{i}}(E)) = \exp_q(x + y)$$

and

$$\exp_q(E) \otimes \exp_q(E) = (\exp_q(E) \otimes 1)(1 \otimes \exp_q(E))$$
$$= (\exp_q(E \otimes 1))(\exp_q(1 \otimes E)) = \exp_q(x) \exp_q(y) \ .$$

Then the well-known rule for the quantum exponentials.

$$\exp_q(x + y) = \exp_q(x) \exp_q(y)$$

(provided that $yx = qxy$) implies that $\hat{\Delta}_{\mathbf{i}}(\exp_q(E)) = \exp_q(E) \otimes \exp_q(E)$. Part (a) is proved.

(b) Denote $E = t_k E_k$ and $E' = t_l E_l$. By definition of $\mathcal{U} \bigotimes_{\mathbf{q}} \mathcal{U}$,

$$(1 \otimes E)(E' \otimes 1) = t_k t_l (1 \otimes E_k)(E_l \otimes 1) = q_{i_k, i_l} t_k t_l (E_l \otimes E_k) \ .$$

The commutation relations (1.5) imply that

$$(1 \otimes E)(E' \otimes 1) = t_k t_l (1 \otimes E_k)(E_l \otimes 1) = t_l t_k (E_l \otimes E_k) = E' \otimes E = (E' \otimes 1)(1 \otimes E).$$

It follows that $(1\otimes f(E))(g(E')\otimes 1) = f(E')\otimes g(E) = (f(E')\otimes 1)(1\otimes g(E))$ for any polynomials f and g in one variable. Passing to the completion, we see that f and g can also be power series in the above formula. Taking $f(E) := \mathbf{e}_k = \exp_{q_k}(E)$ and $g(E') := \mathbf{e}_l = \exp_{q_l}(E')$ completes the proof of part (b). Lemma 1.2 is proved. $\triangleleft$

We are ready to complete the proof of Theorem 1.1 now. Recall that we use the shorthand $\mathbf{e}_k = \exp_{q_k}(t_k E_k)$ so $\mathbf{e}_\mathbf{i} = \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_m$.

Using Lemma 1.2 and the fact that $(a\otimes 1)(1\otimes b) = a\otimes b$ for any $a, b\in\hat{\mathcal{U}}_\mathbf{i}$, we obtain

$$\hat{\Delta}_\mathbf{i}(\mathbf{e}_\mathbf{i}) = \hat{\Delta}_\mathbf{i}(\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_m) = \hat{\Delta}_\mathbf{i}(\mathbf{e}_1)\hat{\Delta}_\mathbf{i}(\mathbf{e}_2)\cdots\hat{\Delta}_\mathbf{i}(\mathbf{e}_m) = (\mathbf{e}_1\otimes\mathbf{e}_1)(\mathbf{e}_2\otimes\mathbf{e}_2)\cdots(\mathbf{e}_m\otimes\mathbf{e}_m)$$

$$= (\mathbf{e}_1\otimes 1)(1\otimes\mathbf{e}_1)(\mathbf{e}_2\otimes 1)(1\otimes\mathbf{e}_2)\cdots(\mathbf{e}_m\otimes 1)(1\otimes\mathbf{e}_m)\ .$$

Using the commutativity property in Lemma 1.2(b), we obtain

$$\hat{\Delta}_\mathbf{i}(\mathbf{e}_\mathbf{i}) = \big((\mathbf{e}_1\otimes 1)(\mathbf{e}_2\otimes 1)\cdots(\mathbf{e}_m\otimes 1)\big)\big((1\otimes\mathbf{e}_1)(1\otimes\mathbf{e}_2)\cdots(1\otimes\mathbf{e}_m)\big)$$

Finally, using the identities $(u\otimes 1)(v\otimes 1) = uv\otimes 1, (1\otimes u)(1\otimes v) = 1\otimes uv$ for any $u, v\in\hat{\mathcal{U}}$, we obtain $\hat{\Delta}_\mathbf{i}(\mathbf{e}_\mathbf{i}) = (\mathbf{e}_\mathbf{i}\otimes 1)(1\otimes\mathbf{e}_\mathbf{i}) = \mathbf{e}_\mathbf{i}\otimes\mathbf{e}_\mathbf{i}$. Theorem 1.1 is proved. $\triangleleft$

Now we define the *restricted dual* algebra $\mathcal{A} = \mathcal{U}^0$ of $\mathcal{U}$. As a vector space, $\mathcal{A}$ is the set of all $\mathbf{k}$-linear forms $x : \mathcal{U}\to\mathbf{k}$ such that x vanishes on $\mathcal{U}(\gamma)$ for all but finitely many $\gamma\in\mathbf{Z}_+^r$. In other words, $\mathcal{A}\cong\bigoplus_\gamma\mathcal{A}(\gamma)$ where $\mathcal{A}(\gamma) = \mathrm{Hom}_\mathbf{k}(\mathcal{U}(\gamma),\mathbf{k})$.

We define the multiplication $\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ by the formula $(xy)(u) = (x\otimes y)\big(\Delta(u)\big)$ where $(x\otimes y)\big(u_1\otimes u_2\big) = x(u_1)y(u_2)$. Thus, $\mathcal{A}$ becomes a $\mathbf{Z}_+^r$-graded algebra (with the unit $\mathbf{k}\to\mathcal{A}$ dual to the counit $\varepsilon : \mathcal{U}\to\mathbf{k}$).

Denote by $(x, u)\mapsto x(u)$ the natural non-degenerate evaluation pairing $\mathcal{A}\times\mathcal{U}\to\mathbf{k}$. Furthermore, we define the pairing $\mathcal{A}\times\hat{\mathcal{U}}_\mathbf{i}\to P_\mathbf{i}$ by the formula $x(\sum t_\gamma u_\gamma) = \sum x(u_\gamma)t_\gamma$. (The sum is finite by the definition of $\mathcal{A}$.)

For every family $\mathbf{E}$ as above define a map $\psi_\mathbf{i} = \psi_{\mathbf{i},\mathbf{E}} : \mathcal{A}\to P_\mathbf{i}$ by the formula $\psi_\mathbf{i}(x) := x(\mathbf{e}_\mathbf{i})$. Expanding $\mathbf{e}_\mathbf{i}$ into a power series we obtain

$$\psi_\mathbf{i}(x) = \sum_{a_1,\ldots,a_m\in\mathbf{Z}_+} x\big(E_1^{[a_1]}E_2^{[a_2]}\cdots E_m^{[a_m]}\big)t_1^{a_1}t_2^{a_2}\cdots t_m^{a_m} \tag{1.7}$$

where $E_k^{[n]} = \dfrac{E_k^n}{[n]_{q_k}!}$. Note that the sum in (1.7) is always finite because x vanishes on all but finitely many monomials $E_1^{a_1}\cdots E_{i_m}^{a_m}$. Define a $\mathbf{Z}_+^r$-grading on $P_\mathbf{i}$ by $\deg(t_k) = \alpha_{i_k}$ and denote by $P_\mathbf{i}(\gamma)$ the graded component of degree γ in $P_\mathbf{i}$.

Corollary 1.3. *For any sequence $\mathbf{i} = (i_1,\ldots,i_m)$ and a family $\mathbf{E}$ of elements $E_k\in\mathcal{U}(\alpha_{i_k})$ $(k = 1,\ldots,m)$, the map $\psi_{\mathbf{i},\mathbf{E}} : \mathcal{A}\to P_\mathbf{i}$ defined by (1.7) is a homomorphism of $\mathbf{Z}_+^r$-graded algebras.*

The proof of Corollary 1.3 repeats that of Corollary 0.2. $\triangleleft$

Remark 1. One can prove (see e.g. [10]) that $\mathcal{A}$ is a $\mathbf{q}^t$-braided bialgebra (where $\mathbf{q}^t$ is the transpose of $\mathbf{q}$). Moreover, starting with an arbitrary $\mathbf{q}^t$-braided algebra $\mathcal{A}$, one recovers $\mathcal{U}$ as the restricted dual of $\mathcal{A}$. So the result of Corollary 1.2 holds for any $\mathbf{q}^t$-braided bialgebra $\mathcal{A}$.

Remark 2. Let $\mathcal{A}_1 = \bigoplus_{i=1}^r\mathcal{A}(\alpha_i)$. Corollary 1.2 implies that any morphism $\mathcal{A}_1\to\bigoplus_{i=1}^r P(\alpha_i)$ of $\mathbf{Z}_+^r$-graded vector spaces extends to an algebra homomorphism. If $\mathcal{A}$ is generated by $\mathcal{A}_1$, then this extension is unique. Thus, in the latter case all the homomorphisms $\mathcal{A}\to P_\mathbf{i}$ of $\mathbf{Z}_+^r$-graded algebras are parametrized by the space $\bigoplus_{i=1}^r\big(\mathcal{U}(\alpha_i)\bigotimes P_\mathbf{i}(\alpha_i)\big)$.

We define the *universal* element $\mathcal{R}\in\mathcal{A}\bigotimes\hat{\mathcal{U}}$ as follows. The tensor product $\mathcal{A}\bigotimes\hat{\mathcal{U}}$ is canonically identified with the space of all linear maps $\mathcal{U}\to\hat{\mathcal{U}}$. Then $\mathcal{R}$ is the element in $\mathcal{A}\bigotimes\hat{\mathcal{U}}$ corresponding to the inclusion $\mathcal{U}\hookrightarrow\hat{\mathcal{U}}$.

Proposition 1.4. *The element $\mathcal{R}$ satisfies $(\psi_{\mathbf{i},\mathbf{E}} \otimes \mathrm{id})(\mathcal{R}) = \mathbf{e}_{\mathbf{i},\mathbf{E}}$ for any $\mathbf{i}$ and $\mathbf{E}$ as above.*

The proof of Proposition 1.4 coincides with that of Proposition 0.3. $\triangleleft$

Clearly, the correspondence

$$c_1 t_1 + \cdots + c_m t_m \mapsto \exp_{q_1}(c_1 t_1 E_1) \cdots \exp_{q_m}(c_m t_m E_m)$$

is a map from the m-dimensional "quantum" affine space $(P_{\mathbf{i}})_1 = \oplus_{l=1}^m \mathbf{k} \cdot t_l$ to the set of group-like elements in $\hat{\mathcal{U}}_{\mathbf{i}}$. This map can be regarded as a deformation of the morphism (0.1).

Now let us turn to the fields of fractions. For an algebra $\mathcal{B}$ without zero divisors, $\mathcal{F}(\mathcal{B})$ is a vector space of right fractions (see Appendix). Its elements can be written as ab^{-1}, where $a, b \in \mathcal{B}$ and $b \neq 0$. As shown in the Appendix, for any sequence $\mathbf{i}$ and any subalgebra $\mathcal{B} \subset P_{\mathbf{i}}$, the space $\mathcal{F}(\mathcal{B})$ is a skew-field. Note that $\psi_{\mathbf{i}}$ induces an embedding of skew fields $\overline{\psi}_{\mathbf{i}} : \mathcal{F}(\mathcal{A}/\mathrm{Ker}\ \psi_{\mathbf{i}}) \hookrightarrow \mathcal{F}(P_{\mathbf{i}})$.

Now consider two elements $\mathbf{e}_{\mathbf{i},\mathbf{E}}$ and $\mathbf{e}_{\mathbf{i}',\mathbf{E}'}$ corresponding via (1.6) to two sequences of indices $\mathbf{i} = (i_1, \ldots, i_m)$ and $\mathbf{i}' = (i'_1, \ldots, i'_n)$ and two families of elements $\mathbf{E} = (E_1, \ldots, E_m)$ and $\mathbf{E}' = (E'_1, \ldots, E'_n)$. Let $t_1, \ldots, t_m$ (resp. $t'_1, \ldots, t'_n$) be the standard generators of $P_{\mathbf{i}}$ (resp. $P_{\mathbf{i}'}$).

Proposition 1.5. *Assume that* $\mathrm{Ker}\ \psi_{\mathbf{i},\mathbf{E}} = \mathrm{Ker}\ \psi_{\mathbf{i}',\mathbf{E}'}$, *and* $\overline{\psi}_{\mathbf{i}',\mathbf{E}'}$ *is an isomorphism of skew-fields* $\mathcal{F}(\mathcal{A}/\mathrm{Ker}\ \psi_{\mathbf{i}',\mathbf{E}'})$ *and* $\mathcal{F}(P_{\mathbf{i}'})$. *Then the map* $\mathbf{R} := \overline{\psi}_{\mathbf{i},\mathbf{E}} \circ (\overline{\psi}_{\mathbf{i}',\mathbf{E}'})^{-1}$ *is an embedding* $\mathcal{F}(P_{\mathbf{i}'}) \hookrightarrow \mathcal{F}(P_{\mathbf{i}})$, *and the following identity holds in* $\mathcal{F}(P_{\mathbf{i}}) \bigotimes \hat{\mathcal{U}}$:

$$\exp_{q_1}(t_1 E_1) \cdots \exp_{q_m}(t_m E_m) = \exp_{q'_1}(p_1 E'_1) \cdots \exp_{q'_n}(p_n E'_n)\ , \tag{1.8}$$

where $q_k = q_{i_k, i_k}$ *for* $k = 1, \ldots, m$, *and* $q'_l = q_{i'_l, i'_l}$, $p_l = \mathbf{R}(t'_l)$ *for* $l = 1, \ldots, n$.

Proof. We omit subscripts $\mathbf{E}$ and $\mathbf{E}'$ in the formulas below. Let $\mathbf{p}_{\mathbf{i}} : \mathcal{A} \to \mathcal{A}/\mathrm{Ker}\ \psi_{\mathbf{i}}$ be the canonical projection. Denote $\mathcal{R}_{\mathbf{i}} = (\mathbf{p}_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R})$. Then Proposition 1.4 implies that

$$(\overline{\psi}_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}_{\mathbf{i}}) = \mathbf{e}_{\mathbf{i}} \tag{1.9}$$

for every $\mathbf{i} \in R(w)$. We are done since $\mathbf{R} \otimes \mathrm{id} = (\overline{\psi}_{\mathbf{i}} \otimes \mathrm{id}) \circ (\overline{\psi}_{\mathbf{i}'} \otimes \mathrm{id})^{-1}$. Proposition 1.5 is proved. $\triangleleft$

2. Feigin's conjecture and other results for quantum groups

Throughout this section we will work over the field $\mathbf{k} = k(q)$ where k is a field of characteristic 0 (say, $k = \mathbf{C}$ as in the introduction), and q is a variable (or a purely transcendental element over k). Let $A = (a_{ij})$ be a symmetrizable Cartan matrix of size $r \times r$, and $C = (C_{ij})$ be the corresponding symmetric matrix with integer entries. In this section we consider a matrix $\mathbf{q}$ of the form $\mathbf{q} = (q_{ij}) = (q^{C_{ij}})$. We denote $q_i := q_{ii} = q^{C_{ii}}$ for all i.

Similarly to [8], Chapter 1, we define the *quantized enveloping algebra* $\mathcal{U}$ and the *quantum group* $\mathcal{A}$ associated with A as follows. First, let $\overline{\mathcal{U}}$ be the free algebra over $k(q)$ generated by $E_1, \ldots, E_r$. We make $\overline{\mathcal{U}}$ into a $\mathbf{q}$-braided bialgebra (see Section 1). Second, the restricted dual algebra $\overline{\mathcal{A}}$ of $\overline{\mathcal{U}}$ is defined as in Section 1. Define a homomorphism $f : \overline{\mathcal{U}} \to \overline{\mathcal{A}}$ by $f(E_i) = x_i$ where x_i is the only element in $\overline{\mathcal{A}}(\alpha_i)$ such that $x_i(E_i) = 1$. Finally, define $\mathcal{U} := \overline{\mathcal{U}}/\mathrm{Ker}\ f$ and $\mathcal{A} := \mathrm{Im}\ f$, and keep the above notation for the generators. In particular, $\mathcal{U} \cong \mathcal{A}$ via $E_i \mapsto x_i$. It is well-known that the right kernel of the evaluation pairing $\mathcal{A} \otimes \overline{\mathcal{U}} \to k(q)$ coincides with $\mathrm{Ker}\ f$. Hence the induced pairing

$$\mathcal{A} \otimes \mathcal{U} \to k(q) \tag{2.1}$$

is non-degenerate, so we identify $\mathcal{A}$ with the restricted dual algebra to the $\mathbf{q}$-braided bialgebra $\mathcal{U}$ (and denote the evaluation pairing (2.1) by $(x, E) \mapsto x(E)$). Note that the generators $E_1, \ldots, E_r$ of $\mathcal{U}$ (as well as the generators $x_1, \ldots, x_r$ of $\mathcal{A}$) are subject to the quantum Serre relations ([8], Section 1.4.3, or Section 3 below).

The algebra $\mathcal{U}$ is $\mathbf{Z}_+^r$-graded via $\deg E_i = \alpha_i$. The pairing $\mathcal{A} \times \mathcal{U} \to k(q)$ extends to the $P_{\mathbf{i}}$-linear pairing $\mathcal{A} \times \hat{\mathcal{U}}_{\mathbf{i}} \to k(q)$ (we denote it by $(x, u) \mapsto x(u)$), where $\hat{\mathcal{U}}_{\mathbf{i}} := P_{\mathbf{i}} \bigotimes \hat{\mathcal{U}}$ and $P_{\mathbf{i}}$ is a $k(q)$-algebra generated by $t_1, \ldots, t_m$ subject to the relations $t_l t_k = q^{C_{i_k, i_l}} t_k t_l$ for $1 \leq k < l \leq m$.

For the convenience of the reader, we summarize the results from Section 1 for the quantum groups $\mathcal{A}$ and $\mathcal{U}$ in the following theorem.

Theorem 2.1. *Let* $\mathbf{i} = (i_1, \dots, i_m)$ *be any sequence. Then*

(a) *the element*

$$\mathbf{e}_{\mathbf{i}} = \exp_{q_{i_1}}(t_1 E_{i_1}) \cdots \exp_{q_{i_m}}(t_m E_{i_m})$$

is a group-like element in $\hat{\mathcal{U}}_{\mathbf{i}}$*;*

(b) *the element* $\mathbf{e}_{\mathbf{i}}$ *gives rise to an algebra homomorphism* $\psi_{\mathbf{i}} : \mathcal{A} \to P_{\mathbf{i}}$ *defined by* $\psi_{\mathbf{i}}(x) := x(\mathbf{e}_{\mathbf{i}})$*;*

(c) *there is a unique element* $\mathcal{R} \in \mathcal{A} \bigotimes \hat{\mathcal{U}}$ *satisfying* $(\psi_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}) = \mathbf{e}_{\mathbf{i}}$ *for all* $\mathbf{i}$*;*

(d) *the homomorphism* $\psi_{\mathbf{i}}$ *satisfies*

$$\psi_{\mathbf{i}}(x) = \sum_{a_1, \dots, a_m \geq 0} x(E_{i_1}^{[a_1]} E_{i_2}^{[a_2]} \cdots E_{i_m}^{[a_m]}) t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \ , \tag{2.2}$$

where $E_i^{[n]} = \dfrac{1}{[n]_{q_i}!} E_i^n$*, and* $[n]_q!$ *is defined in (0.4). In particular, for* $i = 1, \dots, r$ *we have* $\psi_{\mathbf{i}}(x_i) = \sum_{k : i_k = i} t_k$*, and this determines* $\psi_{\mathbf{i}}$ *uniquely;*

(e) Ker $\psi_{\mathbf{i}} = \{x \in \mathcal{A} : x(E_{i_1}^{a_1} E_{i_2}^{a_2} \cdots E_{i_m}^{a_m}) = 0 \text{ for all } a_1, \dots, a_m \in \mathbf{Z}_+\}$.

Remark. After the identification $\mathcal{A} \cong \mathcal{U}$ as above, $\psi_{\mathbf{i}}$ coincides with Feigin's homomorphism $\Phi(\mathbf{i}) : \mathcal{U} \to P_{\mathbf{i}}$. B. Feigin introduced this homomorphism in his talk at RIMS in 1992 (see e.g. [5] and [6]).

Let W be the Weyl group associated with the Cartan matrix A. By definition, W is generated by simple reflections $s_1, \dots, s_r : \mathbf{Z}^r \to \mathbf{Z}^r$ where $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. We call a sequence $\mathbf{i} = (i_1, \dots, i_m)$ of indices a *reduced expression* of $w \in W$ if $w = s_{i_1} s_{i_2} \cdots s_{i_m}$, and the above expression of w is the shortest (we call $\mathbf{i}$ simply a reduced expression if w is not specified). We set $l(w) := m$ and call $l(w)$ the *length* of w. Denote by $R(w)$ the set of all reduced expressions of w. It is well-known that W is a Coxeter group, so the defining relations between $s_1, \dots, s_r$ are of the form $(s_i s_j)^l = 1$ where $l \in \{2, 3, 4, 6\}$. (More precisely, $l = a_{ij}a_{ji} + 2$ if $a_{ij}a_{ji} < 3$ and $l = 6$ if $a_{ij}a_{ji} = 3$.) It follows that every two reduced expressions of an element $w \in W$ are connected by a chain of moves

$$(\mathbf{i}_1, (i, j, i, \dots), \mathbf{i}_2) \mapsto (\mathbf{i}_1, (j, i, j \dots), \mathbf{i}_2)$$

where each fragment in parentheses has the length l. If the Weyl group W is finite then there is a unique element of the maximal length in W which we denote by w_0.

Let us study the kernel of $\psi_{\mathbf{i}}$. According to Theorem 2.1(e), Ker $\psi_{\mathbf{i}}$ is the orthogonal complement of the subspace $\mathcal{U}(\mathbf{i}) \subset \mathcal{U}$ spanned by all monomials $E_{i_1}^{a_1} E_{i_2}^{a_2} \cdots E_{i_m}^{a_m}$.

Lemma 2.2.

(a) *For every sequence* $\mathbf{i}$ *there is a reduced expression* $\mathbf{i}'$ *such that* $\mathcal{U}(\mathbf{i}) = \mathcal{U}(\mathbf{i}')$*. Moreover,* $\mathbf{i}'$ *can always be chosen as a subsequence of* $\mathbf{i}$*.*

(b) *For any* $w \in W$ *and* $\mathbf{i}, \mathbf{i}' \in R(w)$*, we have* $\mathcal{U}(\mathbf{i}) = \mathcal{U}(\mathbf{i}')$*.*

(c) $\mathcal{U}(\mathbf{i})$ *contains the subalgebra in* $\mathcal{U}$ *generated by all* E_i *such that* $l(ws_i) = l(w) - 1$*. Therefore,* $\mathcal{U}(\mathbf{i}) = \mathcal{U}$ *for every* $\mathbf{i} \in R(w_0)$ *when* W *is finite.*

Proof. The collection of the subspaces $\{\mathcal{U}(\mathbf{i})\}$ is a multiplicative semigroup with respect to the product of vector subspaces in $\mathcal{U}$. By definition,

$$\mathcal{U}(i_1, i_2, \dots, i_m) = \mathcal{U}(i_1) \cdot \mathcal{U}(i_2) \cdots \mathcal{U}(i_m)$$

where $\mathcal{U}(i)$ is a subalgebra in $\mathcal{U}$ generated by E_i, $i = 1, \dots, r$.

We have $\mathcal{U}(i)\mathcal{U}(i) = \mathcal{U}(i)$, and for every pair (i, j) with $a_{ij}a_{ji} < 4$ the following relation holds:

$$\mathcal{U}(i) \cdot \mathcal{U}(j) \cdot \mathcal{U}(i) \cdots = \mathcal{U}(j) \cdot \mathcal{U}(i) \cdot \mathcal{U}(j) \cdots \tag{2.3}$$

where each product contains l factors. The identity (2.3) can be proved by the standard arguments for the algebras $\mathcal{U}$ whose Cartan matrices are of types $A_1 \times A_1, A_2, B_2$ or G_2.

It follows that every $\mathcal{U}(\mathbf{i})$ equals to $\mathcal{U}(\mathbf{i}')$ for some reduced subsequence $\mathbf{i}'$ of $\mathbf{i}$, which proves (a). Part (b) also follows because the braid relations (2.3) can be used to move from any reduced expression of $w \in W$ to any other one.

(c) Let $J = J_w$ be the the set of all i satisfying $l(ws_i) = l(w) - 1$. For each $i \in J$, there exists $\mathbf{i} \in R(w)$ such that $\mathbf{i}$ ends with i. Using (b) we see that $\mathcal{U}(\mathbf{i})E_i \subset \mathcal{U}(\mathbf{i})$ for any $\mathbf{i} \in R(w)$, $i \in J$. This completes the proof of Lemma 2.2. $\triangleleft$

We define $I_w := \mathrm{Ker}\ \psi_{\mathbf{i}}$ for any $\mathbf{i} \in R(w)$. Since $\mathcal{A}/I_w$ is isomorphic to $\psi_{\mathbf{i}}(\mathcal{A})$, it follows that $\mathcal{F}(\mathcal{A}/I_w)$ is a skew field (see Appendix).

Theorem 2.3. *For every $w \in W$ and $\mathbf{i} \in R(w)$ the map $\psi_{\mathbf{i}}$ induces an isomorphism of skew fields*

$$\overline{\psi}_{\mathbf{i}} : \mathcal{F}(\mathcal{A}/I_w) \cong \mathcal{F}(P_{\mathbf{i}}) \ . \tag{2.4}$$

Taking $w = w_0$ we obtain the following

Corollary 2.4. *(Feigin's conjecture). For any $\mathbf{i} \in R(w_0)$, the homomorphism $\psi_{\mathbf{i}} : \mathcal{A} \to P_{\mathbf{i}}$ is an embedding, and it induces an isomorphism of skew-fields*

$$\overline{\psi}_{\mathbf{i}} : \mathcal{F}(\mathcal{A}) \cong \mathcal{F}(P_{\mathbf{i}}) \ .$$

Proof of Theorem 2.3. It is enough to prove that for any $\mathbf{i} \in R(w)$ the image of $\psi_{\mathbf{i}}$ generates $\mathcal{F}(P_{\mathbf{i}})$, that is, $t_1, \dots, t_m$ belong to $\mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}})$. We will deduce this statement from Proposition 0.6. Then we neeed the following.

Proposition 2.5. *For each element $x \in \mathcal{A}$ satisfying*

$$\psi_{\mathbf{i}}(x) = t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \tag{2.5}$$

with $a_1 > 0$, there is an element $y \in \mathcal{A}$ such that $\psi_{\mathbf{i}}(y) = ct_1^{a_1-1} t_2^{a_2} \cdots t_m^{a_m}$ where $c \in k(q), c \neq 0$.

Proof of Proposition 2.5. For $i = 1, \dots, r$, let $E_i^* : \mathcal{A} \to \mathcal{A}$ be the adjoint operator of the left multiplication operator $E \mapsto E_i E$ in $\mathcal{U}$. Thus, the element $E_i^*(x)$ is determined by the equations $(E_i^*(x))(E) = x(E_i E)$ for every $E \in \mathcal{U}$.

We will show that y can be chosen as $y = E_i^*(x)$.

Indeed, (2.5) means that the right hand side of the expansion (2.2) for $\psi_{\mathbf{i}}(x)$ reduces to one summand or, equivalently,

$$x(E_{i_1}^{[b_1]} \cdots E_{i_m}^{[b_m]}) = 0 \tag{2.6}$$

unless $(b_1, \dots, b_m) = (a_1, \dots, a_m)$.

By (2.2), we have

$$\psi_{\mathbf{i}}(y) = \psi_{\mathbf{i}}(E_{i_1}^*(x)) = \sum_{b_1, \dots, b_m \in \mathbf{Z}_+} (E_{i_1}^*(x)) \big(E_{i_1}^{[b_1]} E_{i_2}^{[b_2]} \cdots E_{i_m}^{[b_m]}\big) t_1^{b_1} t_2^{b_2} \cdots t_m^{b_m}$$

$$= \sum_{b_1, \dots, b_m \in \mathbf{Z}_+} x\big(E_{i_1} E_{i_1}^{[b_1]} E_{i_2}^{[b_2]} \cdots E_{i_m}^{[b_m]}\big) t_1^{b_1} t_2^{b_2} \cdots t_m^{b_m} \ .$$

In view of (2.6),

$$\psi_{\mathbf{i}}(y) = x(E_{i_1} E_{i_1}^{[a_1-1]} E_{i_2}^{[a_2]} \cdots E_{i_m}^{[a_m]}) t_1^{a_1-1} t_2^{a_2} \cdots t_m^{a_m} = ct_1^{a_1-1} t_2^{a_2} \cdots t_m^{a_m}$$

with $c \neq 0$ as desired. $\triangleleft$

Taking x and y as in Proposition 2.5, we see that

$$t_1 = c\psi_{\mathbf{i}}(x)(\psi_{\mathbf{i}}(y))^{-1} \in \ \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}}) \ . \tag{2.7}$$

To complete the proof of Theorem 2.3, we proceed by induction on m. If $m = 1$ then $t_1 \in \mathrm{Im}\ \psi_{\mathbf{i}} = P_{\mathbf{i}}$. So let $m \geq 2$, denote $\mathbf{i}' = (i_2, \ldots, i_m)$ and assume that Theorem 2.3 holds for $\mathbf{i}'$, that is,

$$t_2, t_3, \ldots, t_m \in \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}'})\ . \tag{2.8}$$

Note that $P_{\mathbf{i}'}$ is naturally embedded into $P_{\mathbf{i}}$ as a subalgebra generated by $t_2, \ldots, t_m$.

In view of the formula for $\psi_{\mathbf{i}}(x_i)$ in Theorem 2.1(d),

$$\psi_{\mathbf{i}'}(x_i) = \psi_{\mathbf{i}}(x_i)\ (i \neq i_1),\ \psi_{\mathbf{i}'}(x_{i_1}) = \psi_{\mathbf{i}}(x_{i_1}) - t_1\ . \tag{2.9}$$

Using (2.7). we see that

$$\psi_{\mathbf{i}'}(x_i) \in \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}})\ , (i = 1, \ldots, r)$$

hence

$$\mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}'}) \subset \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}})\ .$$

Combining this with the inductive assumption (2.8), we conclude that $t_2, \ldots, t_m \in \mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}})$. Since t_1 also belongs to $\mathcal{F}(\mathrm{Im}\ \psi_{\mathbf{i}})$, Theorem 2.3 is proved. $\triangleleft$

Proof of Proposition 0.8. let $\mathcal{B}$ be the algebra of the upper triangular $n \times n$-matrices over $\mathbf{C}(q)$ (with the unity I, the identity matrix). Let $\rho : \mathcal{U} \to \mathcal{B}$ be a representation of $\mathcal{U}$ given by $\rho(E_i) = E_{i,i+1}$, where E_{ij} is the matrix unit. The representation ρ extends naturally to $\mathrm{id} \otimes \rho : \mathcal{A} \bigotimes \mathcal{U} \to \mathcal{A} \bigotimes \mathcal{B}$. We identify the latter algebra with $\mathcal{B}(\mathcal{A})$, the algebra of upper triangular matrices over $\mathcal{A}$.

Lemma 2.6. *We have $X = (\mathrm{id} \otimes \rho)(\mathcal{R})$, where $\mathcal{R}$ is the universal element in $\mathcal{A} \bigotimes \hat{\mathcal{U}}$.*

Proof. Note that $\mathcal{B}$ is a $\mathbf{Z}_+^{n-1}$-graded algebra via $\deg(E_{ii}) = 0$, $\deg(E_{ij}) = \alpha_{ij} = \alpha_i + \cdots \alpha_{j-1}$ $(1 \leq i < j \leq n)$, and ρ preserves the $\mathbf{Z}_+^{n-1}$-grading $(n = r + 1)$. Therefore, the formula (0.9) for $\mathcal{R}$ implies

$$(\mathrm{id} \otimes \rho)(\mathcal{R}) = I + \sum_{i<j} \sum_{b \in B_{ij}} b^* \rho(b)\ .$$

where B_{ij} is a basis in $\mathcal{U}(\alpha_{ij})$ and $\{b^*\}$ is the dual basis in $\mathcal{A}(\alpha_{ij})$. We choose B_{ij} to consist of the products (in any order) of the generators $E_i, E_{i+1}, \ldots, E_{j-1}$. It is easy to see that $\rho(b) = E_{ij}$ for the element $b = b_{ij} = E_i E_{i+1} \cdots E_{j-1}$ in B_{ij}, and $\rho(b) = 0$ if $b \in B_{ij}, b \neq b_{ij}$. Denote $x_{ij} = (b_{ij})^*$. To identify these x_{ij} with those in Section 0 we have to verify the relations (0.11). As an algebra, $\mathcal{A}$ is generated by the $x_i := x_{i,i+1}$ $(i = 1, \ldots, r)$ subject to the quantum Serre relations. The relations (0.11) can be verified similarly to those between the t_{ij} in [2], Section 3 (they also follow from the relations in [4]). Thus, $(\mathrm{id} \otimes \rho)(\mathcal{R}) = I + \sum\limits_{i<j} x_{ij} E_{ij} = X$. Lemma 2.6 is proved. $\triangleleft$

To complete the proof of Proposition 0.8, note that, for every $\mathbf{i}$ we have $\psi_{\mathbf{i}}(X) = (\psi_{\mathbf{i}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \rho)(\mathcal{R}) = (\mathrm{id} \otimes \rho) \circ (\psi_{\mathbf{i}} \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \rho)(\mathbf{e}_{\mathbf{i}})$, by (0.8). The formula (0.12) follows since $(\mathrm{id} \otimes \rho)\big(\exp_{q_{i_k}}(t_k E_{i_k})\big) = I + t_k E_{i_k, i_k+1}$ for all k. Proposition 0.8 is proved. $\triangleleft$

We have the following obvious corollary of Theorem 2.3.

Corollary 2.7. *For any $w \in W$ and $\mathbf{i}, \mathbf{i}' \in R(w)$ there is an isomorphism of skew-fields*

$$R_{\mathbf{i}}^{\mathbf{i}'} : \mathcal{F}(P_{\mathbf{i}}) \xrightarrow{\sim} \mathcal{F}(P_{\mathbf{i}'}) \tag{2.10}$$

defined by $R_{\mathbf{i}}^{\mathbf{i}'} := \overline{\psi}_{\mathbf{i}'} \circ (\overline{\psi}_{\mathbf{i}})^{-1}$.

The "transition maps" $R_{\mathbf{i}}^{\mathbf{i}'}$ lead to identities between quantum exponentials given by Corollary 0.10. To compute each p_k in (0.14), it is enough (in principle) to do this for the following pairs:

$$\mathbf{i} = (i, j, i \ldots),\ \mathbf{i}' = (j, i, j, \ldots)\ ,$$

of the length l each, where l is the order of $s_i s_j$ in W. Recall that $l = 2, 3, 4$, or 6 for any Weyl group. In the following proposition, we compute $\mathbf{R}_{\mathbf{i}'}^{\mathbf{i}}$ for these $\mathbf{i}, \mathbf{i}'$ with $l = 2, 3$, or 4 (when $l = 6$ the explicit expressions for p_k are more complicated, so we do not present them here).

Proposition 2.8. *Let $t_1,\dots,t_l$ (resp. $t'_1,\dots,t'_l$) be standard generators of $P_{(iji\dots)}$ (resp. $P_{(jij\dots)}$). We denote $p_k=\mathbf{R}^{(iji\cdots)}_{(jij\cdots)}(t'_k)$ $(k=1,\dots,l)$.*
(a) If $l=2$ then $(p_1,p_2)=(t_2,t_1)$.
(b) If $l=3$ then $(p_1,p_2,p_3)=\big(t_2t_3(t_1+t_3)^{-1},\ t_1+t_3,\ (t_1+t_3)^{-1}t_1t_2\big)$.
(c) If $l=4$ and $a_{ij}=-2, a_{ji}=-1$ then p_1,p_2,p_3,p_4 are determined by the following equations:

$$p_2p_3=t_1t_2+t_1t_4+t_3t_4,\quad p_2p_3p_4=t_1t_2t_3\ ,$$

$$p_2^2p_3=t_1^2t_2+(t_1+t_3)^2t_4,\quad p_1p_2^2p_3=t_2t_3^2t_4\ .$$

Proof. (a) By definition, $\psi_{(ij)}:(x_i,x_j)\mapsto(t_1,t_2)$ and $\psi_{(ji)}:(x_i,x_j)\mapsto(t'_2,t'_1)$. Thus, $p_1=t_2$, and $p_2=t_1$ as claimed.

Part (b) is proved in Section 0 (see Example 1).

(c) Let $\mathcal{U}_{ij}$ be the subalgebra of $\mathcal{U}$ generated by E_i and E_j. Define $\mathcal{B}_{ij}$ as the quotient algebra of $\mathcal{U}_{ij}$ modulo the relations $E_i^3=E_j^2=0$ (we keep the same notation for generators). It is easy to see ([8], or Section 3 below) that the following are all the defining relations in $\mathcal{B}_{ij}$:

$$E_i^3=E_j^2=E_jE_iE_j=0,\ E_i^2E_jE_i=E_iE_jE_i^2\ .$$

Using these relations, it is easy to prove that the homogeneous components $\mathcal{B}_{ij}(\alpha_i+\alpha_j)$, $\mathcal{B}_{ij}(2\alpha_i+\alpha_j)$, and $\mathcal{B}_{ij}(2\alpha_i+2\alpha_j)$ of $\mathcal{B}_{ij}$ have the following bases: $\{E_iE_j,\ E_jE_i\}$ for $\mathcal{B}_{ij}(\alpha_i+\alpha_j)$, $\{E_i^2E_j,\ E_iE_jE_i,\ E_jE_i^2\}$ for $\mathcal{B}_{ij}(2\alpha_i+\alpha_j)$, and $\{E_jE_i^2E_j\}$ for $\mathcal{B}_{ij}(2\alpha_i+2\alpha_j)$.

Applying the projection $\rho:\mathcal{U}_{ij}\to\mathcal{B}_{ij}$ to both sides of (0.14), we obtain the following equation in $\mathcal{F}(P_{ijij})\bigotimes\mathcal{B}_{ij}$:

$$\begin{aligned}&(1+p_1E_j)(1+p_2E_i+\frac{p_2^2E_i^2}{1+q^2})(1+p_3E_j)(1+p_4E_i+\frac{p_4^2E_i^2}{1+q^2})\\ &=(1+t_1E_i+\frac{t_1^2E_i^2}{1+q^2})(1+t_2E_j)(1+t_3E_i+\frac{t_3^2E_i^2}{1+q^2})(1+t_4E_j)\ .\end{aligned}\tag{2.11}$$

The desired expressions for p_2p_3, $p_2p_3p_4$, $p_2^2p_3$, and $p_1p_2^2p_3$ can be obtained from (2.11) by comparing the coefficients of E_iE_j, $E_iE_jE_i$, $E_i^2E_j$, and $E_jE_i^2E_j$ respectively on both sides of (2.11). Proposition 2.9 is proved. $\triangleleft$

Remark. Taking in the identities of Proposition 2.8 the homogeneous components of degrees $\alpha_i+(1-a_{ij})\alpha_j$ and $\alpha_j+(1-a_{ji})\alpha_i$ yields quantum Serre relations between E_i and E_j.

We conclude this section by a proof of Proposition 0.11. For a skew-symmetric $m\times m$-matrix $S=(S_{kl})$ with integer entries let P_S be a $k(q)$-algebra generated by $t_1,\dots,t_m$ subject to the relations

$$t_lt_k=q^{S_{kl}}t_kt_l\ .\tag{2.12}$$

Note that $P_{\mathbf{i}}=P_{S(\mathbf{i})}$ where the matrix $S(\mathbf{i})$ is defined in (0.16). Recall that two $m\times m$-matrices S and S' are called equivalent if there is a matrix $T=T_{kl}\in SL_m(\mathbf{Z})$ such that $S'=TST^t$. It is easy to see that $\mathcal{F}(P_S)\cong\mathcal{F}(P_{S'})$ if S and S' are equivalent: one can choose such an isomorphism $\mathcal{F}(P_{S'})\widetilde{\to}\mathcal{F}(P_S)$ by sending each generator t'_k of $P_{S'}$ to the monomial $t_1^{T_{k,1}}t_2^{T_{k,2}}\cdots t_m^{T_{k,m}}$ in the generators of P_S. The converse statement was proved by A. Panov.

Proposition 2.9 ([11], Theorem 2.19). *Let S,S' be skew-symmetric $m\times m$ matrices with the integer entries. Then $\mathcal{F}(P_S)\cong\mathcal{F}(P_{S'})$ if and only if S and S' are equivalent.*

Thus, Proposition 2.9 means that the existence of *any* isomorphism $\mathbf{R}:\mathcal{F}(P_{S'})\to\mathcal{F}(P_S)$ implies that of a *monomial* isomorphism $\mathbf{M}:\mathcal{F}(P_{S'})\to\mathcal{F}(P_S)$, that is, $\mathbf{M}$ takes each generator t'_k of $P_{S'}$ to a monomial in generators $t_1,\dots,t_m$ of P_S. Taking $S=S(\mathbf{i}),S'=S(\mathbf{i}')$, and $\mathbf{R}=\mathbf{R}^{\mathbf{i}}_{\mathbf{i}'}:\mathcal{F}(P_{\mathbf{i}'})\to\mathcal{F}(P_{\mathbf{i}})$ with $\mathbf{i},\mathbf{i}'\in R(w)$ for some $w\in W$, we obtain, in particular, the statement of Proposition 0.11. Note that $\mathrm{R}^{\mathbf{i}}_{\mathbf{i}'}$ is not monomial in general.

One can prove that there exists a *local* monomial isomorphism $\mathbf{M} = \mathbf{M}_{\mathbf{i},\mathbf{i}'} : \mathcal{F}(P_{\mathbf{i}'}) \to \mathcal{F}(P_{\mathbf{i}})$. Namely, for $\mathbf{i}, \mathbf{i}'$ of the form

$$\mathbf{i} = (i_1, \dots, i_{a-l}; i, j, i, \dots; i_{a+1}, \dots, i_m)$$

$$\mathbf{i}' = (i_1, \dots, i_{a-l}; j, i, j, \dots; i_{a+1}, \dots, i_m) \ ,$$

$\mathbf{M}(t'_k) = t_k$ if $k \le a-l$ or $k > a$, and each $\mathbf{M}(t'_k)$ for $k = a-l+1, \dots, a$ depends only on $t_{a-l+1}, \dots, t_a$. We will present such $\mathbf{M}$ elswhere.

By Proposition 2.9, the equivalence class of $S(\mathbf{i})$ for $\mathbf{i} \in R(w)$ depends only on w. If we choose some representative $\mathbf{S}(w)$ of this class then, by Theorem 2.3, there is an isomorphism

$$\mathcal{F}\big(\mathcal{A}/I_w\big) \widetilde{\to} \mathcal{F}(P_{\mathbf{S}(w)}) \ . \tag{2.16}$$

The well-known normal form for skew-symmetric matrices shows that $\mathbf{S}(w) = (\mathbf{S}_{kl})$ can be chosen uniquely subject to the following requirements:
(i) $\mathbf{S}_{kl} = 0$ unless $k + l \ne m + 1$;
(ii) there is a sequence $c_1, c_2, \dots$ of nonnegative integers such that $\mathbf{S}_{k,m+1-k} = c_1 c_2 \cdots c_k$ for $1 \le k \le \frac{m}{2}$.

It would be interesting to compute the invariants $c_1, \dots, c_m$ in terms of w, and to find a direct way to describe the isomorphism (2.16).

3. Extremal vectors in $\mathcal{A}$ and proof of Proposition 0.6

We retain terminology and notation of Section 2. Recall that $\alpha_1, \dots, \alpha_r$ is the standard basis in $\mathbf{Z}^r_+$. We define a bilinear form in $\mathbf{Z}^r$ by the formula $(\alpha_i, \alpha_j) = C_{ij}$ for all i, j.

Let us fix $\lambda = (l_1, \dots, l_r) \in \mathbf{Z}^r_+$. For $i = 1, \dots, r$ define linear operators $F_i = F_{i,\lambda} : \mathcal{A} \to \mathcal{A}$ by the formula:

$$F_i \cdot x = \frac{v_i^{l_i} q^{-(\gamma,\alpha_i)} x x_i - v_i^{-l_i} x_i x}{v_i - v_i^{-1}} \tag{3.1}$$

for $x \in \mathcal{A}(\gamma)$ (where $v_i = q^{\frac{C_{ii}}{2}}$).

We identify λ with a linear form on the *coroot* lattice $\mathbf{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbf{Z}\alpha_r^\vee$ defined by $\lambda(\alpha_i^\vee) := l_i$ (recall that $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$). For each reduced $\mathbf{i} = (i_1, \dots, i_m)$ we define a sequence of integers $a_1, \dots, a_m$ by the formula

$$a_1 = \lambda(\alpha_{i_1}^\vee), \quad a_2 = \lambda(s_{i_1}(\alpha_{i_2}^\vee)), \dots, a_m = \lambda\big(s_{i_1} s_{i_2} \cdots s_{i_{m-1}}(\alpha_{i_m}^\vee)\big) \ . \tag{3.2}$$

It is well-known that $a_k \in \mathbf{Z}_+$ for all k.

Define the element $v(\mathbf{i}) = v(\mathbf{i})^\lambda \in \mathcal{A}$ by:

$$v(\mathbf{i}) = F_{i_m}^{a_m} F_{i_{m-1}}^{a_{m-1}} \cdots F_{i_1}^{a_1} \cdot 1 \ . \tag{3.3}$$

The following result refines Proposition 0.6.

Theorem 3.1. *In the above notation, we have* $\psi_{\mathbf{i}}(v(\mathbf{i})) = c t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m}$ *where* $c \in k(q), c \ne 0$.

Proposition 0.6 follows by taking any λ with $\lambda(\alpha_{i_1}^\vee) = l_{i_1} > 0$ (and $x := c^{-1} v(\mathbf{i})$).

Proof of Theorem 3.1. Let us reformulate our statement in terms of modules over the quantized enveloping algebra $\mathbf{U} = U_q(\mathbf{g})$. The $k(q)$-algebra $\mathbf{U}$ is generated by $F_1, \dots, F_r$, $E_1, \dots, E_r$ and the invertible pairwise commuting elements $K_1, \dots, K_r$ subject to the following relations (see [8]):

$$K_i E_j K_i^{-1} = q^{C_{ij}} E_j, \ K_i F_j K_i^{-1} = q^{-C_{ij}} E_j K_i, \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K - K^{-1}}{v_i - v_i^{-1}} \ ; \tag{3.4}$$

$$\sum_{p+p'=1-a_{ij}} (-1)^p v_i^{pp'} E_i^{[p]} E_j E_i^{[p']} = 0, \quad \sum_{p+p'=1-a_{ij}} (-1)^p v_i^{pp'} F_i^{[p]} F_j F_i^{[p']} = 0 \ . \tag{3.5}$$

The relations (3.5) are *quantum Serre relations*; they hold for all $i \neq j$ where $E_i^{[n]}$ means the same as in (2.2), and $v_i = q^{\frac{C_{ii}}{2}}$.

The between $E_1, \dots, E_r$ (the same relations between $F_1, \dots, F_r$):

We identify the subalgebra generated by $E_1, \dots, E_r$ with $\mathcal{U}$. Note that $\mathbf{U}$ is a $\mathbf{Z}^r$-graded algebra via $\deg(K_i) = \deg(K_i^{-1}) = 0$, $\deg(E_i) = \alpha_i$, $\deg(F_i) = -\alpha_i$ for $i = 1, \dots, r$. Note also, that each triple (E_i, F_i, K_i) generates the subalgebra in $\mathbf{U}$ isomorphic to $U_{v_i}(sl_2)$.

We will denote by the same symbol E_i the operator $E_i : \mathcal{A} \to \mathcal{A}$ adjoint of the operator of the right multiplication $E \to EE_i$ in $\mathcal{U}$; for every $x \in \mathcal{A}$ the element $E_i \cdot x \in \mathcal{A}$ is defined by the equations $(E_i \cdot x)(E) = x(EE_i)$ for all $E \in \mathcal{U}$. We also define the operator $K_i : \mathcal{A} \to \mathcal{A}$ (depending on λ) by

$$K_i \cdot x = K_{i,\lambda} \cdot x := q^{\lambda(\alpha_i) - (\gamma, \alpha_i)} x \tag{3.6}$$

for all $x \in \mathcal{A}(\gamma)$, $i = 1, \dots, r$.

The following result is well-known, (for type A_r it can be found e.g, in [3]).

Proposition 3.2. *For every λ as above, the operators F_i defined in (3.1) together with the K_i and E_i, give rise to an action $\mathbf{U} \times \mathcal{A} \to \mathcal{A}$.*

Denote by V_λ the cyclic $\mathbf{U}$-submodule in $\mathcal{A}$ generated by the unit $1 \in \mathcal{A}$. It is well-known (cf. [3], [8]) that V_λ is an integrable simple $\mathbf{U}$-module. Note also that the vector $v = 1 \in V_\lambda$ is a *highest weight vector of weight* λ since $E_i \cdot v = 0$ and $K_i(v) = q^{\lambda(\alpha_i)} v$ for all i.

It is also known (see [8]), that the element $v(\mathbf{i})$ given by (3.3) depends only on w. Such elements are called *extremal vectors* in V_λ. Denote $\mathbf{i}_k := (i_1, \dots, i_k)$ for $k = 1, \dots, m$. It is also well-known that for all k we have

$$F_{i_k} \cdot v(\mathbf{i}_k) = 0, \ E_{i_k}^a \cdot v(\mathbf{i}_k) = c_{k;a} F_{i_k}^{a_k - a} v(\mathbf{i}_{k-1}), \ E_{i_k} \cdot v(\mathbf{i}_{k-1}) = 0 \tag{3.7}$$

for some $c_{k;a} \in k(q) \setminus \{0\}$, and $a = 0, 1, \dots, a_k$ (with the agreement $v(\mathbf{i}_0) := v$).

In view of (2.2), Theorem 3.1 is equivalent to the following.

Proposition 3.3. *There is a unique sequence $b = (b_1, \dots, b_m)$ such that*

$$E_{i_1}^{b_1} \cdots E_{i_m}^{b_m} \cdot v(\mathbf{i}) = cv \tag{3.8}$$

for some nonzero scalar $c \in k(q)$, namely, $(b_1, \dots, b_m) = (a_1, \dots, a_m)$, where $a_1, \dots, a_m$ are given by (3.2).

Proof of Proposition 3.3. We proceed by induction on m.

Assume that our statement is true for every reduced expression of length $< m$, in particular, for $\mathbf{i}_{m-1}$.

Step 1. Let us prove that the equality (3.8) implies that $b_k = a_k$ for all k with $i_k \neq i_m$.

We will use the following identity in $\mathbf{U}$, which is a straightforward consequence of the relations (3.4):

$$E_{i_1}^{b_1} E_{i_2}^{b_2} \cdots E_{i_m}^{b_m} F_{i_m}^{a_m} = \sum_{b'} F_{i_m}^{|b'| - |b| + a_m} E_{i_1}^{b'_1} E_{i_2}^{b'_2} \cdots E_{i_m}^{b'_m} p_{b'} \tag{3.9}$$

where the sum is over all $b' = (b'_1, \dots, b'_m) \in \mathbf{Z}_+^m$ such that $b'_k = b_k$ if $i_k \neq i_m$, and $b'_k \leq b_k$ if $i_k = i_m$; each $p_{b'}$ is a Laurent polynomial of K_{i_m}, and $|b| = b_1 + \cdots b_m$.

Using (3.9) and the fact that $v(\mathbf{i}) = F_{i_m}^{a_m} \cdot v(\mathbf{i}_{m-1})$, we rewrite (3.8) as follows:

$$cv = E_{i_1}^{b_1} \cdots E_{i_m}^{b_m} F_{i_m}^{a_m} \cdot v(\mathbf{i}_{m-1}) = \sum F_{i_m}^{|b'| - |b| + a_m} E_{i_1}^{b'_1} \cdots E_{i_m}^{b'_m} p_{b'} \cdot v(\mathbf{i}_{m-1}) \tag{3.10}$$

where the sum is over all $(b'_1, \dots, b'_m)$ such that $b'_k = b_k$ whenever $i_k \neq i_m$.

It follows that, for some b', we have

$$F_i^{|b'| - |b| + a_m} E_{i_1}^{b'_1} \cdots E_{i_{m-1}}^{b'_m} p_{b'} \cdot v(\mathbf{i}_{m-1}) = c'v$$

with $c' \in k(q), c' \neq 0$. By (3.7), we have $|b'| - |b| + a_m = 0$ and $b'_m = 0$; also,

$$E_{i_1}^{b'_1} \cdots E_{i_{m-1}}^{b'_{m-1}} \cdot v(\mathbf{i}_{m-1}) = c''v$$

with $c'' \neq 0$. Remembering the inductive assumption, we see that $b'_k = a_k$ for all $k \leq m-1$. Thus, $b_k = b'_k = a_k$ for all $k \leq m-1$ such that $i_k \neq i_m$. This completes **Step 1**.

Step 2. Let us prove that $b_m = a_m$. If $i_k \neq i_m$ for $k = 1, \dots, m-1$ then the equality $b_m = a_m$ follows by comparing degrees. So we can assume that $i_k = i_m$ for some $k < m$. Let $k < m$ be the maximal index such that $i_k = i_m$. Clearly, $k \leq m-2$ since $\mathbf{i}$ is reduced. By **Step 1**, we have $b_{k+1} = a_{k+1}$, $b_{k+2} = a_{k+2}, \dots, b_{m-1} = a_{m-1}$. Combining this observation with (3.7), we can rewrite the left hand side of (3.10) as follows.

$$cv = dE_{i_1}^{b_1} \cdots E_{i_{m-1}}^{b_{m-1}} F_{i_m}^{a_m - b_m} \cdot v(\mathbf{i}_{m-1}) = dE_{i_1}^{b_1} \cdots E_{i_k}^{b_k} E_{i_{k+1}}^{a_{k+1}} \cdots E_{i_{m-1}}^{a_{m-1}} F_{i_m}^{a_m - b_m} \cdot v(\mathbf{i}_{m-1})$$

for some $d \in k(q), d \neq 0$.

Then, by the commutativity property $E_{i_l} F_{i_m} = F_{i_m} E_{i_l}$ for $k < l < m$, the previous expression is equal to

$$cv = dE_{i_1}^{b_1} \cdots E_{i_k}^{b_k} F_{i_m}^{a_m - b_m} E_{i_{k+1}}^{a_{k+1}} \cdots E_{i_{m-1}}^{a_{m-1}} \cdot v(\mathbf{i}_{m-1}) = d' E_{i_1}^{b_1} \cdots E_{i_k}^{b_k} F_{i_m}^{a_m - b_m} \cdot v(\mathbf{i}_k) \tag{3.11}$$

(we again used the property (3.7) of the extremal vectors). Since $i_k = i_m$, it follows that $F_{i_m} \cdot v(\mathbf{i}_k) = 0$. Hence, the right hand side of (3.11) is zero unless $a_m - b_m = 0$. This completes **Step 2**.

Step 3. Now we are able to complete the proof. Since $b_m = a_m$, (3.7) and (3.11) imply that

$$c_m E_{i_1}^{b_1} \cdots E_{i_{m-1}}^{b_{m-1}} \cdot v(\mathbf{i}_{m-1}) = cv$$

with some nonzero constants c, c_m. We conclude that $(b_1, \dots, b_{m-1}) = (a_1, \dots, a_{m-1})$ by the inductive assumption. Combining this with **Step 2**, we see that $(b_1, \dots, b_m) = (a_1, \dots, a_m)$.

Proposition 3.3 and Theorem 3.1 are proved. $\triangleleft$

Appendix. Skew-fields of fractions and skew polynomials

Let A be an associative ring with unit without zero-divisors. As in [7], A.2, we say that A satisfies the *right Ore* condition if $aA \cap bA \neq \{0\}$ for any non-zero $a, b \in A$. The set of right fractions $\mathcal{F}(A)$ is defined as the set of all pairs (a, b) with $a, b \in A, b \neq 0$ modulo the following equivalence relation: $(a, b) \sim (c, d)$ if there are $f, g \in A \setminus \{0\}$ such that $af = cg$ and $bf = dg$. The equivalence class of (a, b) in $\mathcal{F}(A)$ is denoted by ab^{-1}. The ring A is naturally embedded into $\mathcal{F}(A)$ via $a \mapsto (a, 1)$. It is well known that if A satisfies the right Ore condition then the addition and multiplication in A extend to $\mathcal{F}(A)$ so that $\mathcal{F}(A)$ becomes a skew-field.

Now we suppose that A is an algebra over a field $\mathbf{k}$ with an increasing filtration $\big(\mathbf{k} = A_0 \subset A_1 \subset \cdots\big)$, where each A_k is a finite dimensional $\mathbf{k}$-vector space, $A_k A_l \subset A_{k+l}$, and $A = \cup A_k$. We say that A has *polynomial growth* if for all $n \geq 0$ we have $\dim A_n \leq p(n)$, where $p(x)$ is a polynomial. For the convenience of the reader, we will present a proof of the following well known lemma (see, e.g., [5]).

Lemma A1. *Any algebra of polynomial growth without zero-divisors satisfies the right Ore condition.*

Proof. Assume, on the contrary, that $aA \cap bA = \{0\}$ for some non-zero $a, b \in A$. Denote $I_n = I \cap A_n$ for any subspace $I \subset A$. Choose some k such that $a, b \in A_k$. Then $(aA)_{n+k} \supset aA_n$ and $(bA)_{n+k} \supset bA_n$, which implies

$$\dim (aA)_{n+k} \geq \dim A_n, \ \dim (bA)_{n+k} \geq \dim A_n \ .$$

On the other hand, since $aA \cap bA = \{0\}$, it follows that

$$\dim A_{n+k} \geq \dim (aA)_{n+k} + \dim (bA)_{n+k} \geq 2 \dim A_n$$

for all n. Iterating this inequality, we see that $\dim A_{mk} \geq 2^m$ for $m \geq 0$. This contradicts the condition that A has polynomial growth. Lemma A1 is proved. $\triangleleft$

Lemma A1 implies that any subalgebra of an algebra A of polynomial growth without zero-divisors also satisfies the right Ore condition.

In particular, consider the $\mathbf{k}$-algebra P of skew polynomials generated by $t_1, \ldots, t_m$ subject to the relations $t_l t_k = q_{kl} t_l t_k$ for $1 \leq k < l \leq m$, where the q_{kl} are some non-zero elements of $\mathbf{k}$. It is easy to see that P has no zero-divisors and has polynomial growth with respect to the filtration ($\mathbf{k} = P_0 \subset P_1 \subset \cdots$), where P_n is the linear span of all monomials in $t_1, \ldots, t_m$ of degree $\leq n$. We see that every subalgebra $\mathcal{B}$ of P satisfies the right Ore condition. Therefore, $\mathcal{F}(\mathcal{B})$ is a skew subfield of $\mathcal{F}(P)$.

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