The Quarterly Journal of Mathematics
Quart. J. Math. 00 (2022), 1–18; doi:https://doi.org/10.1093/qmath/haac033

TWISTS OF RATIONAL CHEREDNIK ALGEBRAS

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[Received 10 October 2022]

Abstract

We show that braided Cherednik algebras introduced by Bazlov and Berenstein are cocycle twists of rational Cherednik algebras of the imprimitive complex reflection groups $G(m,p,n)$, when m is even. This gives a new construction of mystic reflection groups which have Artin–Schelter regular rings of quantum polynomial invariants. As an application of this result, we show that a braided Cherednik algebra has a finite-dimensional representation if and only if its rational counterpart has one.

1. Introduction

Cocycle twists of associative (and Lie) algebras have their origin in physics literature. Situations when a parameterized family of isomorphic groups of symmetry have a non-isomorphic group as a limit were formalized as 'contractions' in Inonu and Wigner [13]: for example, the Galilei group of classical mechanics is a limit of relativistic Lorentz groups. Moody and Patera [18] show that for graded Lie algebras, contractions are determined by 2-cocycles on the grading group. See also Vafa and Witten [20] where twists by cocycles of a finite abelian group are held as examples of mirror symmetry. There are various applications of cocycle twists within mathematics, for example in non-commutative geometry (see Davies [8]) and colour Lie algebras (see Chen, Silvestrov and Oystaeyen [6]). Cocycle twists also find a generalization in the language of Hopf algebras, leading to a twist originally due to Drinfeld [9]. The Drinfeld twist has been well-studied, see Majid [17], and has also found applications in representation theory, see Giaquinto and Zhang [12] and Jordan [14]. To ascribe physical meaning to twists of an algebra, representations of the algebra also need to be twisted; but this proves to be more difficult, and there is no general approach to this so far.

We see the main result of this work that rational and braided Cherednik algebras are related via a twist, as a stepping stone towards a better understanding of the representation theory of braided Cherednik algebras [2], for which very little is currently known. In this work we present one result in this direction, showing that finite-dimensional representations of one algebra exist if and only if they do for the twisted partner. We also give an example using one-dimensional representations of the

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Cherednik algebra, to show that twisting can non-trivially permute the characters of the underlying reflection group.

The contents of this paper are laid out as follows. Sections 2, 3 and 4 introduce the main definitions and objects of the paper, with the only new result being Theorem 2.3, where we give a presentation of the mystic reflection groups $\mu(G(m,p,n))$. These groups arose independently in the work of Bazlov and Berenstein [**2**], as the groups over which the braided Cherednik algebras are defined, and of Kirkman, Kuzmanovich and Zhang [**16**] as a class of groups with Artin–Schelter regular rings of quantum polynomial invariants. Mystic reflection groups were comprehensively studied in page [4]. The main result of the paper is found in Section 5, where the braided Cherednik algebra over $\mu(G(m,p,n))$ is shown to be the twist of the rational Cherednik algebra over the imprimitive complex reflection group $G(m,p,n)$ by a cocycle (in fact, a quasitriangular (QT-)structure) on a finite abelian group. A key step in the proof is to verify that the twist preserves the braid relations between the mystic reflection generators of $\mu(G(m,p,n))$—this fact turns out to be related to the Clifford Braiding Theorem of Kauffman and Lomonaco [**15**]. Finally in Section 6 we use this twisting construction to obtain examples of non-trivial and finite-dimensional representations of a braided Cherednik algebra out of representations of a rational Cherednik algebra.

Note added in proof. The subject of this paper is being further developed in the upcoming paper [**5**] by Y. Bazlov and E. Jones-Healey. In [**5**] we establish, as a corollary of the main result of this paper, a non-trivial isomorphism between the rational Cherednik algebra of $G(m,p,n)$ and the negative braided Cherednik algebra of $\mu(G(m,p,n))$ in the case $\frac{m}{p}$ even. Currently we do not believe such an isomorphism exists in the case $\frac{m}{p}$ odd. In [**5**] we also explore the implications of twisting on the representations of these algebras.

2. Reflection groups

2.1. *Complex reflection groups*

Let V be an n-dimensional $\mathbb{C}$-vector space, with dual space V^*. If $y \in V^*$, $x \in V$, we denote the evaluation of y on x by $\langle y,x\rangle$. If G is a finite subgroup of $\mathrm{GL}(V)$, then for $g \in G$, $x \in V$ we denote the action of g on x by $g(x)$. Via the contragradient representation we have an action of G on V^*: if $g \in G, v \in V^*$, then $\langle g(y),x\rangle = \langle y, g^{-1}(x)\rangle\ \forall x \in V$.

A complex reflection on V is an element $s \in \mathrm{GL}(V)$ that has finite order and satisfies $\mathrm{rank}(s-\mathrm{id}) = 1$. Equivalently, the characteristic polynomial for s is $(t-1)^{n-1}(t-\lambda)$ for some root of unity $\lambda \neq 1$. Note that in this case s acts on V^* also as a complex reflection with characteristic polynomial $(t-1)^{n-1}(t-\lambda^{-1})$. A complex reflection group on V is a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections on V.

We fix a basis $\{x_1,\dots,x_n\}$ for V and dual basis $\{y_1,\dots,y_n\}$ of V^*. This allows us to identify $\mathrm{GL}(V)$ with the group $\mathrm{GL}_n(\mathbb{C})$ of $n\times n$-invertible matrices. The groups of permutation matrices and diagonal matrices are given respectively as

$$\mathbb{S}_n = \{w \in \mathrm{GL}(V) \mid \forall i\ w(x_i) \in \{x_1,\dots,x_n\}\}, \qquad \mathbb{T}_n = \{t \in \mathrm{GL}(V) \mid \forall i\ t(x_i) \in \mathbb{C}x_i\}.$$

We see that

$$\mathbb{S}_n \cap \mathbb{T}_n = \{\mathrm{id}\}, \qquad \mathbb{S}_n \text{ normalizes } \mathbb{T}_n \text{ within } \mathrm{GL}_n(\mathbb{C}), \qquad \text{so that } \mathbb{S}_n \ltimes \mathbb{T}_n \subseteq \mathrm{GL}_n(\mathbb{C}).$$

In particular, $\mathbb{S}_n$ acts on $\mathbb{T}_n$ by conjugation inside $\mathrm{GL}_n(\mathbb{C})$; we will write this action as $w(t)$ for $w \in \mathbb{S}_n$, $t \in \mathbb{T}_n$.
For parameters $m,p \in \mathbb{N}$ with $p \mid m$, let $C_{\frac{m}{p}} \subseteq C_m \subset \mathbb{C}^\times$ be the finite multiplicative subgroups of $\mathbb{C}^\times$ of $\frac{m}{p}$-th, respectively m-th, roots of unity. Besides the 34 exceptional cases, every irreducible complex reflection group belongs to the following family of imprimitive subgroups of $\mathbb{S}_n \ltimes \mathbb{T}_n$,

$$G(m,p,n) = \mathbb{S}_n \ltimes T(m,p,n),$$

where

$$T(m,p,n) = \{t \in \mathbb{T}_n \mid t^m = \mathrm{id}, \det(t) \in C_{\frac{m}{p}}\}$$

is the group of diagonal matrices with diagonal entries in C_m whose product is in $C_{\frac{m}{p}}$. The complex reflections of $G(m,p,n)$ are given by

$$S = \{s_{ij}^{(\epsilon)} \mid i,j \in [n], i \neq j,\ \epsilon \in C_m\} \cup \{t_i^{(\zeta)} \mid i \in [n], \zeta \in C_{\frac{m}{p}} \backslash \{1\}\}$$

where $[n] := \{1,2,\dots,n\}$, and for general $\epsilon \in \mathbb{C}^\times$,

$$s_{ij}^{(\epsilon)}(x_k) := \begin{cases} x_k, & k \neq i,j, \\ \epsilon^{-1} x_j, & k = i, \\ \epsilon x_i, & k = j. \end{cases} \qquad t_i^{(\epsilon)}(x_k) = \epsilon^{\delta_{ik}} x_k.$$

Note that the groups $G(1,1,n)$, $G(2,1,n)$, $G(2,2,n)$ correspond to the Coxeter groups of type A_{n-1}, B_n, D_n respectively, whilst $G(p,p,2)$ corresponds to the dihedral group $I_2(p)$.
It will be relevant to the construction of the rational Cherednik algebras in which conjugacy class each complex reflection lies in. The reflections $s_{ij}^{(\epsilon)}$ with $\epsilon \in C_m$ are involutions and form a single conjugacy class in $G(m,p,n)$, unless $n = 2$ and p is even. Additionally for each $\zeta \in C_{\frac{m}{p}} \backslash \{1\}$, the $\{t_i^{(\zeta)} \mid i \in [n]\}$ forms a separate conjugacy class.
Recall also that complex reflection groups are characterized in terms of their polynomial invariants. If G is a finite subgroup of $\mathrm{GL}(V)$, the action of G on V extends naturally to algebra automorphisms of the symmetric algebra $S(V)$. The invariant set $S(V)^G$ forms a subalgebra of $S(V)$.

THEOREM 2.1 (Chevalley–Shephard–Todd). *For V and G as above, the invariant ring $S(V)^G$ is a polynomial algebra if and only if G is a complex reflection group.*

2.2. *Mystic reflection groups*

Each of the complex reflection groups $G(m,p,n)$, for m even, has a so-called 'mystic partner', which is another subgroup of $\mathbb{S}_n \ltimes \mathbb{T}_n$, defined as follows:

$$\mu(G(m,p,n)) := \{wt \in \mathbb{S}_n \ltimes T(m,1,n) \mid \det(wt) \in C_{\frac{m}{p}}\}.$$

These are examples of mystic reflection groups, which are defined by generalizing the Chevalley–Shephard–Todd characterization of complex reflection groups (Theorem 2.1), see [**16**]. The same class of groups was obtained independently in [**2**], see also [4].

DEFINITION 2.2 For a matrix $q \in \mathrm{Mat}_n(\mathbb{C})$ with $q_{ij}q_{ji} = 1$, $q_{ii} = 1$, let $S_q(V)$ be the algebra generated by V subject to relations $x_ix_j = q_{ij}x_jx_i$ for $i,j \in [n]$. A finite group G is a *mystic reflection group* if it has a faithful action by degree-preserving automorphisms on $S_q(V)$ such that the invariant subalgebra $S_q(V)^G$ is isomorphic to $S_{q'}(V)$ for some q'.

The groups $G(m,p,n)$ are related to their mystic partners in the following ways:

$$\mu(G(m,p,n)) \begin{cases} = G(m,p,n) & \text{if } \frac{m}{p} \text{ is even,} \\ \cong G(m,p,n) & \text{if } \frac{m}{p} \text{ and } n \text{ are odd,} \\ \ncong G(m,p,n) & \text{if } \frac{m}{p} \text{ is odd and } n \text{ is even.} \end{cases} \tag{1}$$

Even though the groups $\mu(G(m,p,n))$ and $G(m,p,n)$ in some cases coincide as a subgroup of $\mathbb{S}_n \ltimes \mathbb{T}_n$, the generating set for $\mu(G(m,p,n))$ relevant for what follows is the set

$$\underline{S} := \{\sigma_{ij}^{(\epsilon)} \mid i,j \in [n], i \neq j,\ \epsilon \in C_m\} \cup \{t_i^{(\zeta)} \mid i \in [n], \zeta \in C_{\frac{m}{p}} \setminus \{1\}\}, \tag{2}$$

where

$$\sigma_{ij}^{(\epsilon)}(x_k) = \begin{cases} x_k & k \neq i,j, \\ \epsilon^{-1}x_j & k = i, \\ -\epsilon x_i & k = j. \end{cases} \tag{3}$$

We call the elements of $\underline{S}$ *mystic reflections*. That $\underline{S}$ indeed generates $\mu(G(m,p,n))$, for all even m, all divisors p of m and all $n \geq 1$, follows for example from Theorem 2.3. Notice $\sigma_{ij}^{(\epsilon)} \in \mathbb{S}_n \ltimes \mathbb{T}_n$ are of order 4, with characteristic polynomial $(x-1)^{n-2}(x^2+1)$. Similarly to above, when $n \geq 3$, the $\sigma_{ij}^{(\epsilon)}$ form a single conjugacy class, whilst for each $\zeta \in C_{\frac{m}{p}} \setminus \{1\}$ the $t_i^{(\zeta)}$ again form separate conjugacy classes.

2.3. *A presentation of mystic reflection groups*

It turns out that for fixed n, each mystic reflection group $\mu(G(m,p,n))$ contains the Tits group of type A_{n-1}, introduced in [**19**, Section 4.6]. The Tits group is realized as $\mu(G(2,2,n))$, the mystic partner of the Coxeter group of type D_n; it is the group of even elements in the Coxeter group $G(2,1,n)$ of type B_n.

THEOREM 2.3 (A presentation of $\mu(G(m,p,n))$). *For all even m and all divisors p of m, the abstract group generated by symbols $\sigma_1, \ldots, \sigma_{n-1}$ and the abelian group $T(m,p,n)$, subject to the relations*

(i) $\sigma_i^2 = t_i^{(-1)}t_{i+1}^{(-1)}$,
(ii) *the braid relations* $\sigma_i\sigma_j = \sigma_j\sigma_i$, $i - j > 1$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $1 \leq i \leq n-2$ *and*
(iii) $\sigma_i t \sigma_i^{-1} = s_{i,i+1}(t)$ *for all* $t \in T(m,p,n)$

is isomorphic to $\mu(G(m,p,n))$ *via the map* $\sigma_i \mapsto \sigma_{i,i+1}^{(1)}$ *and the identity map on* $T(m,p,n)$.

Proof. Let W be the quotient of the free product of a free group on $\{\sigma_i \mid i \in [n-1]\}$ with $T(m,p,n)$ by relations (i)–(iii). If $m = p = 2$, then by [**1**, Lemma 2.1], W is the Tits group $\mathcal{T} \subset \mathrm{SL}_n(\mathbb{C})$ of type A_{n-1}, which surjects onto $\mathbb{S}_n$ with kernel $T(2,2,n)$. Therefore, for m even, the subgroup of W generated by $\sigma_1,\dots,\sigma_{n-1}$ and $T(2,2,n)$ is some quotient $\overline{\mathcal{T}}$ of $\mathcal{T}$. Moreover, by rearranging generators using relation (iii) we can write W as $\overline{\mathcal{T}} \cdot \overline{Q}$ where Q is a transversal of $T(2,2,n)$ in $T(m,p,n)$. Hence

$$|W| \leq |\mathcal{T}| \cdot |Q| = |\mathbb{S}_n||T(2,2,n)| \cdot |T(m,p,n)|/|T(2,2,n)| = |\mathbb{S}_n||T(m,p,n)|.$$

Now observe that relations (i)–(iii) hold in $\mu(G(m,p,n))$: one checks them using the factorization $\sigma^{(1)}_{i,i+1} = s_{i,i+1}t^{(-1)}_{i+1}$ in $\mathbb{S}_n \ltimes \mathbb{T}_n$, the Coxeter relations for the $s_{i,i+1}$ and the semidirect product relations. Hence the map $W \to \mu(G(m,p,n))$, given in the Theorem, is well-defined. We show that this map is surjective, that is, $\sigma^{(1)}_{i,i+1}$, $i \in [n-1]$ and elements of $T(m,p,n)$ generate the group $\mu(G(m,p,n))$. The composite homomorphism $\mu(G(m,p,n)) \hookrightarrow \mathbb{S}_n \ltimes \mathbb{T}_n \xrightarrow{\mathrm{proj}_1} \mathbb{S}_n$ carries the $\sigma^{(1)}_{i,i+1}$ to the generators $s_{i,i+1}$ of $\mathbb{S}_n$ so is surjective with kernel $T(m,p,n)$. Hence the $\sigma^{(1)}_{i,i+1}$ generate $\mu(G(m,p,n))$ modulo $T(m,p,n)$, as required. We thus have

$$|W| \leq |\mathbb{S}_n||T(m,p,n)| = |\mu(G(m,p,n))|,$$

so the surjective homomorphism $W \to \mu(G(m,p,n))$ is a bijection. □

3. Rational and braided Cherednik algebras

3.1. *Rational Cherednik algebras*

For complex reflection group $G \subseteq \mathrm{GL}(V)$, let S denote the complex reflections in G. Let $\mathfrak{t} \in \mathbb{C}$ and $c : S \to \mathbb{C}, s \mapsto c_s$ be such that $c_{gsg^{-1}} = c_s$ $\forall g \in G, s \in S$. Using $\mathfrak{t}$ and c, we define a bilinear map

$$\kappa_{\mathfrak{t},c} : V^* \times V \to \mathbb{C}G,\ (y,x) \mapsto \mathfrak{t}\langle y,x\rangle 1 + \sum_{s\in S} c_s \langle y, (\mathrm{id}-s)x\rangle s.$$

The following algebras were introduced by Etingof and Ginzburg in [**11**]:

Definition 3.1 The *rational Cherednik algebra* $H_c(G)$ is generated by $V, \mathbb{C}G, V^*$, subject to the relations: $\forall x,x' \in V,\ y,y' \in V^*,\ g \in G$,

- $xx' - x'x = yy' - y'y = 0$
- $gx = g(x)g,\ yg = g \cdot g^{-1}(y)$
- $yx - xy = \kappa_{\mathfrak{t},c}(y,x)$

For brevity, we do not use $\mathfrak{t}$ as an extra subscript in $H_c(G)$. There are two essentially different cases, $\mathfrak{t} \neq 0$ and $\mathfrak{t} = 0$. The rational Cherednik algebra at $\mathfrak{t} = 0$ is finite over its centre, whereas at $\mathfrak{t} \neq 0$, $H_0(G) = \mathcal{A}(V)\#\mathbb{C}G$, the smash product of the Weyl algebra with the group algebra $\mathbb{C}G$. Whilst this algebra has no finite-dimensional modules, $H_c(G)$ can have finite-dimensional modules for special values of c.

Theorem 3.2 (The Poincaré-Birkhoff-Witt (PBW) theorem for rational Cherednik algebras, [11, Theorem 1.3]). *Let $x_1, \dots, x_n$ be the basis of V, and $y_1, \dots, y_n$ a dual basis for V^*. As a $\mathbb{C}$-vector space, $H_c(G)$ has basis*

$$\{x_1^{k_1} \dots x_n^{k_n} g y_1^{l_1} \dots y_n^{l_n} \mid g \in G,\ k_1, \dots, k_n, l_1, \dots, l_n \in \mathbb{Z}_{\geq 0}\}.$$

In other words, as vector spaces, we have $H_c(G) \cong \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}G \otimes \mathbb{C}[y_1, \dots, y_n]$.

In the following we restrict to the case $G = G(m,p,n)$ in which m is even, and either $n \geq 3$, or p is odd and $n = 2$. This means the $s_{ij}^{(\epsilon)}$ form a single conjugacy class, and we denote the value of $c : S \to \mathbb{C}$ on this class as c_1. The groups with odd m are excluded because they have no mystic partner. If $n = 2$ and p is even, the algebra defined below is not the most general case of the rational Cherednik algebra because it only has a single parameter for the set of complex reflections of the form $s_{ij}^{(\epsilon)}$, although this set is split into two conjugacy classes in $G(m,p,2)$.

Definition 3.3 The *rational Cherednik algebra* $H_c(G(m,p,n))$ is the algebra generated by $x_1, \dots, x_n \in V$, $g \in G(m,p,n)$, $y_1, \dots, y_n \in V^*$, subject to relations:

$$x_i x_j - x_j x_i = 0 \qquad y_i y_j - y_j y_i = 0 \qquad g x_i = g(x_i) g \qquad g y_i = g(y_i) g$$

$$y_i x_j - x_j y_i = c_1 \sum_{\epsilon \in C_m} \epsilon s_{ij}^{(\epsilon)} \qquad y_i x_i - x_i y_i = \mathfrak{t} - c_1 \sum_{j \neq i} \sum_{\epsilon \in C_m} s_{ij}^{(\epsilon)} - \sum_{\zeta \in C_{\frac{m}{p}} \setminus \{1\}} c_\zeta t_i^{(\zeta)}$$

where c_ζ is the value of c on conjugacy class of $t_i^{(\zeta)}$, for each $\zeta \in C_{\frac{m}{p}}$.

3.2. *Braided Cherednik algebras*

Consider the mystic reflection group $\mu(G(m,p,n))$, with mystic reflections $\underline{S}$ as in (2). We require m to be even in order for the mystic reflection group $\mu(G(m,p,n))$ to be defined; we also assume that $n \geq 3$ or p is odd and $n = 2$ as above, so that the mystic reflections $\sigma_{ij}^{(\epsilon)}$ form a single conjugacy class. Similarly to above, we consider a function $c' : \underline{S} \to \mathbb{C}$ that is invariant under conjugation in $\mu(G(m,p,n))$, and let $\underline{x}_1, \dots, \underline{x}_n$ be a basis of V, with $\underline{y}_1, \dots, \underline{y}_n$ a dual basis for V^*. The following algebras were introduced in [2] in the special case $\mathfrak{t} = 1$:

Definition 3.4 The *negative braided Cherednik algebra* $\underline{H}_{c'}(\mu(G(m,p,n)))$ is the algebra generated by $\underline{x}_1, \dots, \underline{x}_n \in V$, $g \in \mu(G(m,p,n))$, $\underline{y}_1, \dots, \underline{y}_n \in V^*$, subject to relations:

$$\underline{x}_i \underline{x}_j + \underline{x}_j \underline{x}_i = 0 \qquad \underline{y}_i \underline{y}_j + \underline{y}_i \underline{y}_j = 0 \qquad g \underline{x}_i = g(\underline{x}_i) g \qquad g \underline{y}_i = g(\underline{y}_i) g$$

$$\underline{y}_i \underline{x}_j + \underline{x}_j \underline{y}_i = c'_1 \sum_{\epsilon \in C_m} \epsilon \sigma_{ij}^{(\epsilon)} \qquad \underline{y}_i \underline{x}_i - \underline{x}_i \underline{y}_i = \mathfrak{t} + c'_1 \sum_{j \neq i} \sum_{\epsilon \in C_m} \sigma_{ij}^{(\epsilon)} + \sum_{\zeta \in C_{\frac{m}{p}} \setminus \{1\}} c'_\zeta t_i^{(\zeta)}$$

The key property of $\underline{H}_{c'}(\mu(G(m,p,n))$ is the PBW-type theorem, proved in [2, Theorem 0.2] for $\mathfrak{t} = 1$. Below we obtain a new proof of this result for all $\mathfrak{t}$, see Remark 5.10.

4. The cocycle twist

Although we will only use cocycles on a finite abelian group, we will work in a more general Hopf algebra setting as it provides the useful language of duality. Our notation generally follows [17]. If H is a Hopf algebra over $\mathbb{C}$, $\triangle\colon H\to H\otimes H$ will denote the coproduct of H and $\epsilon\colon H\to\mathbb{C}$, the counit. An example is the group algebra $H=\mathbb{C}T$ of a group T, with $\triangle(t)=t\otimes t$ and $\epsilon(t)=1$, extended from T to $\mathbb{C}T$ by linearity. The action of $h\in H$ on $a\in A$ where A is an H-module will be written as $h\triangleright a$. Recall that in an H-module algebra A, the product map $m\colon A\otimes A\to A$ is a morphism in the category H-Mod of H-modules, and $h\triangleright 1_A=\epsilon(h)1_A$ for all $h\in H$.

4.1. *Quasitriangular structures and 2-cocycles*

We begin with two well-known definitions, see [17, Definition 2.1.1 and Example 2.3.1].

Definition 4.1 A *2-cocycle* on a Hopf algebra H is an invertible $\chi\in H\otimes H$ such that $(\chi\otimes 1)\cdot(\triangle\otimes\mathrm{id})(\chi)=(1\otimes\chi)\cdot(\mathrm{id}\otimes\triangle)(\chi)$ and $(\epsilon\otimes\mathrm{id})(\chi)=1=(\mathrm{id}\otimes\epsilon)(\chi)$.

Definition 4.2 A *QT-structure* on a Hopf algebra H is an invertible element $\mathcal{R}\in H\otimes H$ satisfying:

(QT1) $\mathcal{R}\cdot(\triangle x)=(\triangle^{\mathrm{op}}x)\cdot\mathcal{R}\ \forall x\in H$,
(QT2) $(\triangle\otimes\mathrm{id})\mathcal{R}=\mathcal{R}_{13}\cdot\mathcal{R}_{23}$, $(\mathrm{id}\otimes\triangle)\mathcal{R}=\mathcal{R}_{13}\cdot\mathcal{R}_{12}$,

where $\mathcal{R}_{12}:=\mathcal{R}\otimes 1$, $\mathcal{R}_{23}:=1\otimes\mathcal{R}$ and $\mathcal{R}_{13}$ similarly has 1 inserted in the middle leg.

QT-structures satisfy the quantum Yang–Baxter equation [17, Lemma 2.1.4]

$$\mathcal{R}_{12}\cdot\mathcal{R}_{13}\cdot\mathcal{R}_{23}=\mathcal{R}_{23}\cdot\mathcal{R}_{13}\cdot\mathcal{R}_{12}. \tag{4}$$

(QT2) and (4) imply that a QT-structure is a Hopf algebra 2-cocycle [17, Example 2.3.1].

4.2. *Twists*

It is natural to complement Majid's description of a twisted H-module algebra [17, Section 2.3] by the observation that twisting by χ is functorial:

Proposition 4.3 *A 2-cocycle $\chi\in H\otimes H$ for a Hopf algebra H gives rise to the functor*

$$(\ )_\chi:\mathrm{Alg}(H\text{-Mod})\to\mathrm{Alg}(H_\chi\text{-Mod})$$

which takes an object (A,m) to $(A,m_\chi=m(\chi^{-1}\triangleright -))$, and an arrow $(A,m)\xrightarrow{\phi}(B,m')$ to $(A,m_\chi)\xrightarrow{\phi_\chi}(B,m'_\chi)$, where $\phi_\chi=\phi$ as H-module morphisms. The twisted Hopf algebra H_χ is defined as having the same algebra structure, and counit, as H, but with coproduct $\triangle_\chi(h)=\chi\cdot\triangle(h)\cdot\chi^{-1}$ where $\triangle$ is the coproduct on H.

Proof. That (A,m_χ) is in $\mathrm{Alg}(H_\chi\text{-Mod})$ is [17, Proposition 2.3.8], so we only need to check functoriality. Since $A\otimes A\xrightarrow{\phi\otimes\phi}B\otimes B$ is an $H\otimes H$-module morphism, it commutes with the action of

$\chi^{-1} \in H \otimes H$, so $\phi \circ m_\chi = m \circ (\phi \otimes \phi)(\chi^{-1} \triangleright) = m(\chi^{-1} \triangleright)(\phi \otimes \phi)$ shows that ϕ_χ is an algebra morphism if ϕ is. Also, ϕ_χ is an H_χ-module morphism because the actions of H and of H_χ are the same. Therefore, ϕ_χ is indeed an arrow in $\mathrm{Alg}(H_\chi\text{-Mod})$. □

REMARK 4.4 The functor given in Proposition 4.3 is essentially the restriction of the monoidal equivalence of categories H-Mod $\to H_\chi$-Mod, see [**10**, Remark 5.14.3], to algebras.

4.3. *The cocycle $\mathcal{F}$*

We will now define the cocycle $\mathcal{F}$ which will be used for twisting in the rest of the paper. Let T be the abelian group

$$T = \langle \gamma_1, \dots, \gamma_n \mid \gamma_i^2 = 1, \gamma_i\gamma_j = \gamma_j\gamma_i, i,j \in [n] \rangle,$$

isomorphic to $T(2,1,n) \cong (C_2)^n$. Define

$$f_{ij} = \frac{1}{2}\left(1 \otimes 1 + \gamma_i \otimes 1 + 1 \otimes \gamma_j - \gamma_i \otimes \gamma_j\right), \qquad \mathcal{F} = \prod_{1 \le j < i \le n} f_{ij}\,.$$

Let a, b be elements of some associative algebra. It is easy to check that

$$\text{if } a, b \text{ are commuting involutions, then } \frac{1}{2}(1 + a + b - ab) \text{ is an involution,} \tag{5}$$

which implies that the f_{ij} and $\mathcal{F}$ are pairwise commuting involutions in $\mathbb{C}T \otimes \mathbb{C}T$. That they are cocycles follows from

LEMMA 4.5 *$\mathcal{F}$ and f_{ij} for all i,j are QT-structures on $\mathbb{C}T$.*

Proof. (QT1) is vacuous as $\mathbb{C}T$ is a commutative and cocommutative Hopf algebra. Rewriting f_{ij} in the form $2^{-1}\sum_{a,b=0}^{1}(-1)^{ab}\gamma_i^a \otimes \gamma_j^b$, one checks (QT2) for f_{ij} in the same way as in the case $n=2$ of [**17**, Example 2.1.6]. Since (QT2) is multiplicative in $\mathcal{R}$, (QT2) also holds for $\mathcal{F}$. □

4.4. *$\mathcal{F}$-twisted product of T-eigenvectors*

Since T is a finite abelian group, a $\mathbb{C}T$-module is the same as a comodule for the dual Hopf algebra $\mathbb{C}\hat{T}$, where the dual group $\hat{T}$ of T is

$$\hat{T} = \{\alpha : T \to \mathbb{C}^\times \mid \alpha \text{ is a group homomorphism}\} = \langle \alpha_1, \dots, \alpha_n \rangle, \qquad \alpha_i(\gamma_j) = (-1)^{\delta_{ij}}.$$

A coaction by the group algebra of $\hat{T}$ manifests itself as a $\hat{T}$-grading, so for a $\mathbb{C}T$-module algebra A we have $A = \bigoplus_{\alpha \in \hat{T}} A_\alpha$ where $A_\alpha := \{a \in A \mid \forall \gamma \in T,\ \gamma \triangleright a = \alpha(\gamma)a\}$ is the α-eigenspace of T. Denote by $\bigstar$ the twisted product on A induced by the cocycle $\mathcal{F} \in \mathbb{C}T \otimes \mathbb{C}T$ as in Proposition 4.3. Then

$$a \bigstar b = (\alpha \otimes \beta)(\mathcal{F}^{-1}) a \cdot b \qquad \text{for } a \in A_\alpha, b \in A_\beta. \tag{6}$$

Here $\alpha, \beta \in \hat{T}$ induce an algebra homomorphism $\alpha \otimes \beta \colon \mathbb{C}T^{\otimes 2} \to \mathbb{C}$. Observe that

$$(\alpha_i \otimes \alpha_j)(f_{ab}) = 1 \text{ if } (i,j) \neq (a,b), \qquad (\alpha_i \otimes \alpha_j)(f_{ij}) = -1. \tag{7}$$

Every one of the 2^n elements of $\hat{T}$ is of the form

$$\alpha_I := \prod_{i \in I} \alpha_i \qquad \text{where } I \subseteq [n],$$

so (6) and (7) imply the following

Lemma 4.6 *Let I, J be subsets of $[n]$. If $a \in A_{\alpha_I}$ and $b \in A_{\alpha_J}$, then $a \bigstar b = (-1)^{d(I,J)} ab$ where $d(I,J) = |\{(i,j) : i \in I, j \in J, i > j\}|$.*

Remark 4.7 In fact, Lemma 4.6 is a particular case of the construction in [**3**, Lemma 3.6] where a twist $A_{\mathcal{F}}$ of a G-graded algebra A by a 2-cocycle $\mathcal{F}$ on the group G is realized as the image of the coaction $A \to A \otimes \mathbb{C}G$ viewed as a subalgebra of $A \otimes (\mathbb{C}G)_{\mathcal{F}}$. In the case $G = \hat{T}$ and the cocycle $\mathcal{F}$ given above, the twisted group algebra $(\mathbb{C}\hat{T})_{\mathcal{F}}$ is isomorphic to the complex Clifford algebra Cl_n of a space with an orthonormal basis $\alpha_1, \ldots, \alpha_n$, [**3**, Example 1.7]. The calculations done in the next section can be interpreted as embedding the negative braided Cherednik algebra $\underline{H}_{\underline{c}}(\mu(G(m,p,n)))$ in $H_c(G(m,p,n)) \otimes Cl_n$, although we do not explicitly write the Clifford algebra generators.

5. The Main Result

5.1. *Statement of the main theorem*

To state Main Theorem 5.2, we need to define the action of the abelian group T on the rational Cherednik algebra $H_c(G(m,p,n))$. This will allow us to twist the associative product in $H_c(G(m,p,n))$.

Proposition 5.1 *Let m be even and $n \geq 2$. The rational Cherednik algebra $H_c(G(m,p,n))$ is a $\mathbb{C}T$-module algebra with respect to the action $\triangleright$ given by:*

$$\gamma_i \triangleright g = t_i^{(-1)} g t_i^{(-1)} \qquad \gamma_i \triangleright x_j = t_i^{(-1)}(x_j), \qquad \gamma_i \triangleright y_j = t_i^{(-1)}(y_j)$$

for all $i,j \in [n]$, $g \in G(m,p,n)$.

Proof. The PBW theorem 3.2 and the defining relations in Definition 3.3 imply that $H_c(G(m,p,n))$ embeds as a subalgebra in $\widetilde{H} = H_{\tilde{c}}(G(m,1,n))$, where $\tilde{c}$ is defined by $\tilde{c}_1 = c_1$, $\tilde{c}_\zeta = c_\zeta$ for $\zeta \in C_{\frac{m}{p}} \setminus \{1\}$ and $\tilde{c}_\zeta = 0$ whenever $\zeta \in C_m \setminus C_{\frac{m}{p}}$. This subalgebra is generated by x_i, y_i for $i \in [n]$ and by the subgroup $G(m,p,n)$ of $\widetilde{G} = G(m,1,n)$. Observe that $\widetilde{H}$ is a $\mathbb{C}\widetilde{G}$-module algebra where $a \in \widetilde{G}$ acts by conjugation, carrying $u \in \widetilde{H}$ to $aua^{-1} \in \widetilde{H}$. Since $G(m,p,n)$ is a normal subgroup of $\widetilde{G}$, this action of

$\widetilde{G}$ preserves the subalgebra $H_c(G(m,p,n))$ of $\widetilde{H}$. Now the embedding of T in $\widetilde{G}$ via $\gamma_i \mapsto t_i^{(-1)}$ defines the T-action on $H_c(G(m,p,n))$ given in the Proposition. □

We now twist the rational Cherednik algebra by the cocycle $\mathcal{F} \in \mathbb{C}T \otimes \mathbb{C}T$ from Section 4 and denote the result $(H_c(G(m,p,n)), \bigstar)$.

Theorem 5.2 *There exists an isomorphism*

$$\phi\colon \underline{H}_{\underline{c}}(\mu(G(m,p,n))) \to (H_c(G(m,p,n)), \bigstar)$$

of associative algebras, where $\underline{c} : \underline{S} \to \mathbb{C}$ *is defined by* $\underline{c}_1 := -c_1$, $\underline{c}_\zeta := -c_\zeta$ $\forall \zeta \in C_{\frac{m}{p}} \setminus \{1\}$, *and*

$$\phi(\sigma_i) = \bar{s}_i, \qquad \phi(t) = t, \qquad \phi(\underline{x}_j) = x_j, \qquad \phi(\underline{y}_j) = y_j,$$

for all $i \in [n-1]$, $j \in [n]$, $\sigma_i := \sigma^{(1)}_{i,i+1}$, $\bar{s}_i := s^{(-1)}_{i,i+1}$, $t \in T(m,p,n)$.

5.2. *Outline of the proof of the theorem*

We fix the triple (m,p,n) and denote $G = G(m,p,n)$, $H_c = H_c(G)$, $\underline{H}_{\underline{c}} = \underline{H}_{\underline{c}}(\mu(G))$. The theorem defines the algebra homomorphism ϕ on generators of $\underline{H}_{\underline{c}}$, and so ϕ, if exists, is unique. We first extend ϕ from the generators σ_i and t of $\mu(G)$ to a homomorphism from $\mu(G)$ to a subgroup of the twisted group algebra $(\mathbb{C}G, \bigstar)$. Then, by checking that the defining relations of $\underline{H}_{\underline{c}}$ from Definition 3.4 are satisfied by the elements $\phi(\underline{x}_i)$, $\phi(\underline{y}_i)$ and $\phi(g)$, $g \in \mu(G)$ of the algebra $(H_c, \bigstar)$, we show that ϕ extends from the generators to the whole algebra $\underline{H}_{\underline{c}}$. We then use the PBW theorem for H_c to argue that ϕ is bijective.

5.3. $\bigstar$*-multiplication by simple generators*

We need several lemmas where we express the new, $\mathcal{F}$-twisted associative product $\bigstar$ of certain elements of H_c in terms of the usual product (written as $\cdot$ or omitted).

Lemma 5.3 *For all* $t \in T(m,p,n)$, $u \in H_c$, $t \bigstar u = tu$ *and* $u \bigstar t = ut$.

Proof. As t is invariant under the action of T, that is, is a T-eigenvector with eigencharacter $1 = \alpha_\varnothing$, by Lemma 4.6, $t \bigstar u = tu$ and $u \bigstar t = ut$ for all T-eigenvectors u, and so by linearity for all u in H_c. □

We denote

$$s_i := s_{i,i+1} \in \mathbb{S}_n, \quad r_{ij} := t_i^{(-1)} t_j^{(-1)} \in T(m,p,n), \quad p_i = \frac{1}{2}(1+\gamma_i), \quad q_i = \frac{1}{2}(1-\gamma_i) \in \mathbb{C}T$$

and let

$$\bar{s}_i := s^{(-1)}_{i,i+1}, \quad s_i^+ := p_i \triangleright s_i = \frac{1}{2}(s_i + \bar{s}_i), \quad s_i^- := q_i \triangleright s_i = \frac{1}{2}(s_i - \bar{s}_i) \in \mathbb{C}G.$$

Observe that $\gamma_a \rhd s_i = s_i$ if $a \notin \{i, i+1\}$, so s_i^+ is T-invariant, and $s_i^- \in (H_c)_{\alpha_i\alpha_{i+1}}$.

LEMMA 5.4 *For all $i \in [n-1]$, $u \in H_c$,*

$$s_i \bigstar u = s_i \cdot (p_i \rhd u) + \bar{s}_i \cdot (q_i \rhd u) \qquad \textit{and} \qquad u \bigstar s_i = (p_{i+1} \rhd u)s_i + (q_{i+1} \rhd u)\bar{s}_i.$$

Proof. Since the expressions are linear in u, we may assume that u is a T-eigenvector with eigencharacter α_J, $J \subseteq [n]$. Apply Lemma 4.6 to $s_i^- \in (H_c)_{\alpha_i\alpha_{i+1}}$ and u. In the string $i, i+1, J$, the pair $i, i+1$ forms zero or two inversions with every element of J except possibly i and forms exactly one inversion with i if $i \in J$, so $s_i^- \bigstar u = s_i^- u$ if $i \notin J$, and $s_i^- \bigstar u = -s_i^- u$ if $i \in J$. That is, $s_i^- \bigstar u = s_i^- \cdot (\gamma_i \rhd u)$.

Also by Lemma 4.6, $s_i^+ \bigstar u = s_i^+ u$. The formula for $s_i \bigstar u = (s_i^+ + s_i^-) \bigstar u$ follows. The proof for $u \bigstar s_i$ is similar. □

LEMMA 5.5 *The simple transpositions s_i, $i \in [n-1]$, obey the relations $s_i \bigstar s_i = r_{i,i+1}$ and the braid relations with respect to the $\bigstar$-product on the group algebra of $G(m,p,n)$.*

Proof. By Lemma 5.4, $s_i \bigstar s_i = s_i s_i^+ + \bar{s}_i s_i^- = \frac{1}{2}(1 + r_{i,i+1}) + \frac{1}{2} r_{i,i+1}(1 - r_{i,i+1}) = r_{i,i+1}$.

If $|i-j| > 1$, then s_j is γ_i-invariant, hence $q_i \rhd s_j = 0$. Then by Lemma 5.4, $s_i \bigstar s_j = s_i s_j$. This is symmetric in i,j, so s_i and s_j $\bigstar$-commute.

To check the braid relation for s_i and s_{i+1}, we can assume $i = 1$. We calculate the product

$$s_2 \bigstar s_{1,3} \bigstar s_1 = (s_2 \bigstar s_2 s_1 s_2) \bigstar s_1 = s_2 \bigstar (s_1 s_2 s_1 \bigstar s_1) \tag{8}$$

in two ways. First, since the transposition $s_{1,3} = s_2 s_1 s_2$ is γ_2-invariant, by Lemma 5.4 $s_2 \bigstar s_2 s_1 s_2 = s_2 s_2 s_1 s_2 = s_1 s_2$. This is the same as $s_1 \bigstar s_2$ because s_1 is γ_3-invariant, so (8) equals $s_1 \bigstar s_2 \bigstar s_1$.

On the other hand, by γ_2-invariance of $s_{1,3} = s_1 s_2 s_1$ and the second part of Lemma 5.4, $s_1 s_2 s_1 \bigstar s_1 = s_1 s_2 s_1 s_1 = s_1 s_2 = s_1 \bigstar s_2$, so (8) equals $s_2 \bigstar s_1 \bigstar s_2$. The $\bigstar$-braid relations are proved. □

REMARK 5.6 If $(H_c, \bigstar)$ is embedded in $H_c \otimes Cl_n$ as in Remark 4.7, the simple transposition s_i becomes $s_i^+ \otimes 1 + s_i^- \otimes \alpha_i \alpha_{i+1}$. The calculation to prove Lemma 5.5 is then equivalent to verifying part of the Clifford Braiding Theorem of Kauffman and Lomonaco [**15**] which states that the $e_i := \frac{1+\alpha_i\alpha_{i+1}}{\sqrt{2}}$, $i \in [n-1]$, obey the braid relations in Cl_n. The Clifford Braiding Theorem goes further to assert the circular braid relations involving $e_n := 1 + \alpha_n \alpha_1$, but these do not arise from Lemma 5.5.

5.4. *The extension of ϕ from the generators to the group algebra $\mathbb{C}\mu(G)$*

To prove that the assignment $\phi(\sigma_i) = \bar{s}_i$, $\phi(t) = t$ extends to a homomorphism $\phi\colon \mathbb{C}\mu(G) \to (\mathbb{C}G, \bigstar)$ of algebras, we check relations (i)–(iii) from the presentation of $\mu(G)$ given in Theorem 2.3.

(i) $\sigma_i \sigma_i = r_{i,i+1}$. We check that the relation $\phi(\sigma_i) \bigstar \phi(\sigma_i) = \phi(r_{i,i+1})$ holds in $(H_c, \bigstar)$. The left-hand side is $\bar{s}_i \bigstar \bar{s}_i = (s_i r_{i,i+1}) \bigstar (s_i r_{i,i+1})$, which by Lemma 5.3 is $s_i \bigstar s_i$. By Lemma 5.5, this is $r_{i,i+1}$, the same as $\phi(r_{i,i+1})$.

(ii) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i - j > 1$. We need to check that $\phi(\sigma_i) = \bar{s}_i$ and $\phi(\sigma_j) = \bar{s}_j$ $\bigstar$-commute. This follows from Lemma 5.3 and Lemma 5.5.

$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. By Lemma 5.5, $s_i \bigstar s_{i+1} \bigstar s_i = s_{i+1} \bigstar s_i \bigstar s_{i+1}$, and by Proposition 4.3, $(H_c, \bigstar)$ is a T-module algebra. Acting by γ_{i+1} on both sides gives the required relation $\bar{s}_i \bigstar \bar{s}_{i+1} \bigstar \bar{s}_i = \bar{s}_{i+1} \bigstar \bar{s}_i \bigstar \bar{s}_{i+1}$.

(iii) $\sigma_i t \sigma_i^{-1} = s_i(t)$. We need to check that $\bar{s}_i \bigstar t \bigstar s_i = s_i(t)$. The left-hand side rewrites by Lemma 5.3 as $(\bar{s}_i \bigstar s_i)s_i(t) = r_{i,i+1}(\bar{s}_i \bigstar \bar{s}_i)s_i(t)$, which simplifies to $s_i(t)$ by (i).

We can now describe the map ϕ on the whole of $\mathbb{C}\mu(G)$ using a special basis of $\mathbb{C}\mu(G)$:

Proposition 5.7 *There exist involutions $\theta_w \in \mathbb{C}T(2,1,n)$, indexed by $w \in \mathbb{S}_n$, such that $\{w\theta_w t : w \in \mathbb{S}_n, t \in T(m,p,n)\}$ is a basis of $\mathbb{C}\mu(G)$. In this basis,*

$$\phi(w\theta_w t) = wt. \tag{9}$$

Proof. First of all, we observe that, for each $i \in [n-1]$ and $w \in \mathbb{S}_n$,

$$\bar{s}_i \bigstar w = s_i w \cdot \eta_i(w)$$

for some involution $\eta_i(w) \in \mathbb{C}T(2,2,n)$. Indeed, denote $t_i^{(-1)} \cdot w^{-1}(t_i^{(-1)})$ by r so that $\gamma_i \triangleright w = wr$. By Lemma 5.3 and Lemma 5.4,

$$\bar{s}_i \bigstar w = s_i r_{i,i+1} \cdot \frac{1}{2}(w + wr) + s_i \cdot \frac{1}{2}(w - wr) = s_i w \cdot \frac{1}{2}(1 - r + r' + rr')$$

with $r' = w^{-1}(r_{i,i+1})$. Thus, $\eta_i(w) = \frac{1}{2}(1 + r + r' - rr')$, which is an involution by (5).

Factorize w into simple transpositions as $w = s_{i_k} \dots s_{i_2} s_{i_1}$, and let $\sigma_w = \sigma_{i_k} \dots \sigma_{i_2}\sigma_{i_1}$. Since $\sigma_i = s_i t_{i+1}$ and $\det \sigma_i = 1$, in the group $\mathbb{S}_n \ltimes T(2,1,n)$ one has $\sigma_w = wt_w$ with $t_w \in T(2,1,n)$ such that $\det(t_w) = \det(w)$. Therefore,

$$\phi(wt_w) = \bar{s}_{i_k} \bigstar \dots \bigstar \bar{s}_{i_2} \bigstar \bar{s}_{i_1} = s_{i_k} \dots s_{i_2} s_{i_1} \cdot \eta_{i_2}(s_{i_1})\eta_{i_3}(s_{i_2}s_{i_1}) \dots \eta_{i_k}(s_{i_{k-1}} \dots s_{i_1}),$$

so (9) holds with $\theta_w = \eta_{i_2}(s_{i_1}) \dots \eta_{i_k}(s_{i_{k-1}} \dots s_{i_1})t_w$. Note that wt_w and $\eta_{i_j}(s)$, hence $w\theta_w$, lie in $\mathbb{C}G(2,2,n)$ and so $\{w\theta_w t : w \in \mathbb{S}_n, t \in T(m,p,n)\}$ is a subset of $\mathbb{C}\mu(G)$. It is a basis of the space $\mathbb{C}\mu(G)$, because this set is carried by the linear map ϕ to the basis $\{wt\}$ of the space $\mathbb{C}G$ of the same dimension. $\square$

5.5. *Commutation relations between the $\underline{x}_i$ and between the $\underline{y}_j$*

We need to show that $\phi(\underline{x}_i) \bigstar \phi(\underline{x}_j) = -\phi(\underline{x}_j) \bigstar \phi(\underline{x}_i), \phi(\underline{y}_i) \bigstar \phi(\underline{y}_j) = -\phi(\underline{y}_j) \bigstar \phi(\underline{y}_i)$ whenever $1 \le i < j \le n$. This is immediate by the following

Corollary 5.8 *For all $i,j \in [n]$, $i < j$,*

$$x_i \bigstar x_i = x_i^2, \quad x_i \bigstar x_j = -x_j \bigstar x_i = x_i x_j, \quad x_i \bigstar y_i = x_i y_i, \quad x_i \bigstar y_j = x_i y_j, \quad y_j \bigstar x_i = -y_j x_i.$$

The same holds where the letters x and y are swapped.

Proof. Since the x_i and y_i are T-eigenvectors with eigencharacter α_i this follows by Lemma 4.6. $\square$

5.6. *The main commutator relation between $\underline{x}_i$ and $\underline{y}_j$*

We will now check the relation obtained by applying ϕ to both sides of the main commutator relation in $\underline{H}_{\underline{c}}$ for $i \neq j$:

$$\phi(\underline{y}_i) \star \phi(\underline{x}_j) + \phi(\underline{x}_j) \star \phi(\underline{y}_i) = \underline{c}_1 \sum_{\epsilon \in C_m} \epsilon \phi(\sigma_{ij}^{(\epsilon)}). \tag{10}$$

To calculate the right-hand side, we need

Lemma 5.9 *$\phi(\sigma_{ij}^{(\epsilon)}) = s_{ij}^{(-\epsilon)}$ if $i < j$, and $\phi(\sigma_{ij}^{(\epsilon)}) = s_{ij}^{(\epsilon)}$ if $i > j$.*

Proof. Since $\sigma_{ij}^{(\epsilon)} = \sigma_{ij}^{(1)} t_i^{(\epsilon^{-1})} t_j^{(\epsilon)}$, $s_{ij}^{(\pm\epsilon)} = s_{ij}^{(\pm 1)} \star t_i^{(\epsilon^{-1})} t_j^{(\epsilon)}$ and $\phi(t_i^{(\epsilon^{-1})} t_j^{(\epsilon)}) = t_i^{(\epsilon^{-1})} t_j^{(\epsilon)}$, it is enough to prove the Lemma for $\epsilon = 1$.

The case $i<j$: if $j = i+1$, the statement becomes $\phi(\sigma_i) = \bar{s}_i$ which is true by definition of ϕ. To proceed by induction in j, we consider the identity $s_j s_{i,j+1}^{(-1)} = s_{ij}^{(-1)} s_j$. Since $s_{i,j+1}^{(-1)}$ is γ_j-invariant, and $s_{ij}^{(-1)}$ is γ_{j+1}-invariant, this rewrites by Lemma 5.4 as $s_j \star s_{i,j+1}^{(-1)} = s_{ij}^{(-1)} \star s_j$. Using $\bar{s}_j \star s_j = 1$,

$$s_{i,j+1}^{(-1)} = \bar{s}_j \star s_{ij}^{(-1)} \star s_j = \phi(\sigma_j \sigma_{ij}^{(1)} \sigma_j^{-1}) = \phi(\sigma_{i,j+1}^{(1)}),$$

so the inductive step is done, and the case $i<j$ follows.

The case $i>j$: $\phi(\sigma_{ij}^{(\epsilon)}) = \phi(\sigma_{ji}^{(-\epsilon^{-1})})$, which by the first part of the Lemma is $s_{ji}^{(\epsilon^{-1})} = s_{ij}^{(\epsilon)}$. □

Now, by Corollary 5.8 and Lemma 5.9 relation (10) for $i<j$ is rewritten as

$$y_i x_j - x_j y_i = \underline{c}_1 \sum_{\epsilon \in C_m} \epsilon s_{ij}^{(-\epsilon)}.$$

Substituting $\underline{c}_1 = -c_1$ and recalling that $-1 \in C_m$, and so $-\epsilon \in C_m$ iff $\epsilon \in C_m$, transforms this equation into the relation between y_i and x_j in Definition 3.3. If $i>j$, (10) becomes $-(y_i x_j - x_j y_i) = -c_1 \sum_{\epsilon \in C_m} \epsilon s_{ij}^{(\epsilon)}$, which is again true by Definition 3.3. Thus, (10) is proved.

5.7. *The main commutator relation between $\underline{x}_i$ and $\underline{y}_i$*

We need to show that

$$\phi(\underline{y}_i) \star \phi(\underline{x}_i) - \phi(\underline{x}_i) \star \phi(\underline{y}_i) = \mathfrak{t} + \underline{c}_1 \sum_{j \neq i} \sum_{\epsilon \in C_m} \phi(\sigma_{ij}^{(\epsilon)}) + \sum_{\zeta \in C_{\frac{m}{p}} \setminus \{1\}} \underline{c}_\zeta \phi(t_i^{(\zeta)}) \tag{11}$$

where the left-hand side is $y_i x_i - x_i y_i$ by Corollary 5.8. Apply Lemma 5.9, note that $\sum_{\epsilon \in C_m} s_{ij}^{(-\epsilon)}$ is the same as $\sum_{\epsilon \in C_m} s_{ij}^{(\epsilon)}$ because $-1 \in C_m$, and substitute $\phi(t_i^{(\zeta)}) = t_i^{(\zeta)}$, $\underline{c}_\zeta = -c_\zeta$ to rewrite the right-hand side of (11) as $\mathfrak{t} - c_1 \sum_{j \neq i} \sum_{\epsilon \in C_m} s_{ij}^{(\epsilon)} - \sum_{\zeta \in C_{\frac{m}{p}} \setminus \{1\}} c_\zeta t_i^{(\zeta)}$. This shows that (11) is true by the relation for $y_i x_i - x_i y_i$ in Definition 3.3.

5.8. *The semidirect product relations*

We need to prove

$$\phi(g)\star x_i = g(x_i)\star\phi(g), \qquad \phi(g)\star y_i = g(y_i)\star\phi(g) \tag{12}$$

for all $g\in\mu(G)$ and all $i\in[n]$. We will prove this only for x_i, as the proof for y_i will be similar. Moreover, observe that if (12) holds for $g=g_1$ and for $g=g_2$, then it holds for $g=g_1g_2\in\mu(G)$:

$$\phi(g)\star x_i=\phi(g_1)\star\phi(g_2)\star x_i=\phi(g_1)\star g_2(x_i)\star\phi(g_2)=g_1(g_2(x_i))\star\phi(g_1)\star\phi(g_2)=g(x_i)\star\phi(g).$$

By Theorem 2.3, every element of $\mu(G)$ is a product of some generators σ_j, $j\in[n-1]$, and of some $t\in T(m,p,n)$. If $g=t\in T(m,p,n)$, one can omit ϕ and $\star$ in (12) by Lemma 5.3, and then (12) clearly holds by the semidirect product relations in H_c. Hence it is enough to prove (12) when $g=\sigma_j$:

$$\phi(\sigma_j)\star x_i=\sigma_j(x_i)\star\phi(\sigma_j), \quad \text{equivalently } \bar{s}_j\star x_i=\sigma_j(x_i)\star\bar{s}_j. \tag{13}$$

If $i\notin\{j,j+1\}$, then $x_i=\sigma_j(x_i)$ is both γ_j and γ_{j+1}-invariant, so by Lemma 5.4, (13) is rewritten as $\bar{s}_jx_i=x_i\bar{s}_j$.
If $i=j+1$, x_{j+1} is γ_j-invariant and $\sigma_j(x_{j+1})=-x_j$ is γ_{j+1}-invariant, so by Lemma 5.4, (13) is $\bar{s}_jx_{j+1}=-x_j\bar{s}_j$.
If $i=j$, x_j is a T-eigenvector with eigencharacter α_j and $\sigma_j(x_j)=x_{j+1}$, with α_{j+1}, so by Lemma 5.4 (13) is $s_jx_j=x_{j+1}s_j$. In all three cases, (13) is true by the semidirect product relations in H_c.

5.9. *Bijectivity of* ϕ.

Hence all the relations are satisfied and ϕ is a well-defined algebra homomorphism. We are left to prove that ϕ is bijective. It is enough to construct a spanning set of $\underline{H}_c$ which is carried by ϕ to a basis of H_c.
Let $w\in\mathbb{S}_n$. Consider the coset $wT(m,p,n)$ of $T(m,p,n)$ inside G and let $\langle wT(m,p,n)\rangle$ denote the span of this coset, a subspace of $\mathbb{C}G$. Observe that $\langle wT(m,p,n)\rangle$ is a T-submodule of $\mathbb{C}G$, because, if $i\in[n]$, $t\in T(m,p,n)$, $\gamma_i\triangleright(wt)=wrt$ where $r=t_i^{(-1)}\cdot w^{-1}(t_i^{(-1)})\in T(2,2,n)\subset T(m,p,n)$. Therefore, $\langle wT(m,p,n)\rangle$ has T-eigenbasis $wb_1(w),\dots,wb_N(w)$ where $N=|T(m,p,n)|$ and $b_1(w),\dots,b_N(w)\in\mathbb{C}T(m,p,n)$. It follows that $\{wb_m(w): w\in\mathbb{S}_n, m\in[N]\}$ is a basis of the group algebra $\mathbb{C}G=\bigoplus_{w\in\mathbb{S}_n}\langle wT(m,p,n)\rangle$. The PBW-type tensor product factorization of H_c, see Theorem 3.2, implies that

$$\mathcal{B}=\{x_1^{k_1}\dots x_n^{k_n}wb_m(w)y_1^{l_1}\dots y_n^{l_n}: w\in\mathbb{S}_n, m\in[N], k_i,l_i\geq 0\} \tag{14}$$

is a basis of H_c.
We replace the basis $\{w\theta_wt\}$ of $\mathbb{C}\mu(G)$, given by Proposition 5.7, by the following alternative basis: $\{w\theta_wb_m(w): w\in\mathbb{S}_n, m\in[N]\}$. It is a basis of $\mathbb{C}\mu(G)$ because by Proposition 5.7, it is carried by ϕ to the basis $\{wb_m(w)\}$ of $\mathbb{C}G$, and $\dim\mathbb{C}\mu(G)=\dim\mathbb{C}G$. It then follows from the defining relations in $\underline{H}_c$ that the set

$$\underline{\mathcal{B}}=\{\underline{x}_1^{k_1}\dots\underline{x}_n^{k_n}w\theta_wb_m(w)\underline{y}_1^{l_1}\dots\underline{y}_n^{l_n}: w\in\mathbb{S}_n, m\in[N], k_i,l_i\geq 0\} \tag{15}$$

spans $\underline{H}_c$.

It is immediate from Corollary 5.8 that $\phi(\underline{x}_1^{k_1} \dots \underline{x}_n^{k_n}) = x_1^{k_1} \dots x_n^{k_n}$ and $\phi(\underline{y}_1^{l_1} \dots \underline{y}_n^{l_n}) = y_1^{k_1} \dots y_n^{l_n}$. We can now view how a general basis element of (15) is mapped under ϕ,

$$\begin{aligned}\phi(\underline{x}_1^{k_1} \dots \underline{x}_n^{k_n} w\theta_w b_m(w)\underline{y}_1^{l_1} \dots \underline{y}_n^{l_n}) &= x_1^{k_1} \dots x_n^{k_n} \bigstar w b_m(w) \bigstar y_1^{l_1} \dots y_n^{l_n} \\ &= \pm x_1^{k_1} \dots x_n^{k_n} w b_m(w) y_1^{l_1} \dots y_n^{l_n},\end{aligned}$$

where the second equality follows by Lemma 4.6, since $x_1^{k_1} \dots x_n^{k_n}$, $wb_m(w)$ and $y_1^{l_1} \dots y_n^{l_n}$ are T-eigenvectors. On noting that a spanning set of $\underline{H}_{\underline{c}}$ has been mapped (up to a scalar multiple of ± 1) to the basis of H_c given in (14), we conclude that $\underline{\mathcal{B}}$ is a basis of $\underline{H}_{\underline{c}}$ and that ϕ is bijective, as required.

Remark 5.10 The above argument showing that $\underline{\mathcal{B}}$ is a basis of $\underline{H}_{\underline{c}}$ is a new proof of the PBW-type theorem for negative braided Cherednik algebras, extending the result obtained earlier in [2] to arbitrary $\mathsf{t} \in \mathbb{C}$.

6. Twists of representations

From Theorem 5.2 we know rational Cherednik algebras can be twisted, and the result is isomorphic to a negative braided Cherednik algebra. Next we show that representations of rational Cherednik algebras can also be twisted, generating a representation of the corresponding negative braided Cherednik algebra. A systematic approach to twists of representations, going beyond the examples considered in this section, will be explored in the upcoming paper [5].

6.1. *Twisting and finite-dimensional representations*

For the purposes of this section, we assume $\frac{m}{p}$ to be even. Recall that by Proposition 5.1, $H_c(G(m,p,n))$ is a $\mathbb{C}T$-module algebra under the conjugation action of $T = (C_2)^n$. When $\frac{m}{p}$ is even, T is embedded as the subgroup $T(2,1,n)$ in $G(m,p,n)$, so a representation

$$\rho : H_c(G(m,p,n)) \to \mathrm{End}(V)$$

of $H_c(G(m,p,n))$ induces a T-action on $\mathrm{End}(V)$ via $t \blacktriangleright f = \rho(t) f \rho(t)^{-1}$ for all $t \in T, f \in \mathrm{End}(V)$. With this, ρ becomes a $\mathbb{C}T$-module algebra homomorphism. Denote by $\mathrm{End}(V)_{\mathcal{F}}$ the twist of the $\mathbb{C}T$-module algebra $\mathrm{End}(V)$ by the cocycle $\mathcal{F}$ defined in Section 4.3.
Recall that the underlying vector space is unchanged by twisting, therefore ρ can be viewed as a linear map $\rho : H_c(G(m,p,n))_{\mathcal{F}} \to \mathrm{End}(V)_{\mathcal{F}}$.

Proposition 6.1 *The linear map $\rho : H_c(G(m,p,n))_{\mathcal{F}} \to End(V)_{\mathcal{F}}$ is an algebra homomorphism.*

Proof. Let m denote the product on $H_c(G(m,p,n))$ and m' be the product on $\mathrm{End}(V)$, so that $\rho \circ m = m' \circ (\rho \otimes \rho)$ because ρ is a homomorphism between the untwisted algebras. The twisted product maps are $m_{\mathcal{F}} = m \circ (\mathcal{F}^{-1} \rhd)$ and $m'_{\mathcal{F}} = m' \circ (\mathcal{F}^{-1} \blacktriangleright)$, and since ρ is a $\mathbb{C}T$-module algebra morphism, so that $(\rho \otimes \rho)(\mathcal{F}^{-1} \rhd) = (\mathcal{F}^{-1} \blacktriangleright)(\rho \otimes \rho)$, we conclude that $\rho \circ m_{\mathcal{F}} = m'_{\mathcal{F}} \circ (\rho \otimes \rho)$, as required. □

We use this to deduce the following:

THEOREM 6.2 *For $\frac{m}{p}$ even, if $H_c(G(m,p,n))$ has finite-dimensional representations, then so does the negative braided Cherednik algebra $\underline{H}_{\underline{c}}(\mu(G(m,p,n)))$.*

Proof. If $\rho : H_c(G(m,p,n)) \to \mathrm{End}(V)$ is a finite-dimensional representation, the algebra $\mathrm{End}(V)_{\mathcal{F}}$ is finite-dimensional, so it has finite-dimensional modules on which $\underline{H}_{\underline{c}}(\mu(G(m,p,n))) \cong H_c(G(m,p,n))_{\mathcal{F}}$ acts via the algebra homomorphism $\rho : H_c(G(m,p,n))_{\mathcal{F}} \to \mathrm{End}(V)_{\mathcal{F}}$. □

6.2. *Finite-dimensional representations at* $\mathfrak{t} = 1$*: a general construction*

Let W be an irreducible complex reflection group with reflection representation V, and $H_c(W)$ be a rational Cherednik algebra over W. In the rest of the paper, we assume that $\mathfrak{t} = 1$.
We recall a general approach which produces finite-dimensional representations of $H_c(W)$. Start with a simple $\mathbb{C}W$-module τ and extend τ to an $S(V)\#\mathbb{C}W$-module where V acts by zero. The standard $H_c(W)$-module $M_c(\tau)$ is defined as

$$M_c(\tau) = H_c(W) \otimes_{S(V)\#\mathbb{C}W} \tau.$$

The underlying vector space of $M_c(\tau)$ is $S(V^*) \otimes \tau$, hence these are infinite-dimensional representations of $H_c(W)$. The standard module $M_c(\mathbb{C})$ given by $\tau = \mathbb{C}$, the trivial $\mathbb{C}W$-module, is the famous Dunkl (or polynomial) representation of $H_c(W)$. Every $M_c(\tau)$ has a unique simple quotient, denoted $L_c(\tau)$. For some τ and some values of c, $L_c(\tau)$ are finite-dimensional.
If x_i is a basis of V and y_i the dual basis of V^*, one has the following important element of $H_c(W)$:

$$h = \sum_i x_i y_i + \frac{n}{2} - \sum_{s\in S} \frac{2c_s}{1-\lambda_s} s \tag{16}$$

where $n = \dim(V)$, S is the set of complex reflections in W and λ_s is the non-trivial eigenvalue of s in the dual reflection representation. By [7, Section 2.1], h satisfies the commutator relations

$$[h,x] = x \ \forall x \in V, \qquad [h,y] = -y \ \forall y \in V^*. \tag{17}$$

6.3. *Twisting one-dimensional representations of* $H_c(G(2,1,n))$

We restrict the discussion above to the group $G = G(2,1,n) \cong \mathbb{S}_n \ltimes (C_2)^n$, the Coxeter group of type B_n, and consider the modules $L_c(\tau)$ over $H_c = H_c(G)$ which are one-dimensional. Such modules correspond to the four linear characters triv, $\varkappa$, det and $\varkappa$det of G; each character is determined by its values on $s = s_i$ and $t = t_i^{(-1)}$ as follows:

$$\mathrm{triv}\colon (s,t) \mapsto (1,1), \quad \varkappa\colon (s,t) \mapsto (1,-1), \quad \det\colon (s,t) \mapsto (-1,-1), \quad \varkappa\det\colon (s,t) \mapsto (-1,1). \tag{18}$$

To find the parameters c where $\dim L_c(\tau) = 1$, we note that commutators must act on a 1-dimensional module by zero, so (17) implies that the generators x_i and y_i, $i \in [n]$, act by 0. Most relations in

Definition 3.3 are satisfied by $x_i = y_i = 0$ automatically: the relation $y_i x_j - x_j y_i = c_1 \sum_{\epsilon\in\{\pm 1\}} \epsilon s_{ij}^{(\epsilon)}$ holds because s_{ij} and $s_{ij}^{(-1)}$ act on $L_c(\tau)$ by the same scalar. The only constraint on the parameter $c = (c_1, c_{-1})$ arises from the last relation which reads

$$0 = 1 - c_1 \cdot (n-1) \cdot 2\tau(s_i) - c_{-1}\tau(t_i^{(-1)}). \tag{19}$$

Let τ be one of the four characters of G given in (18), and assume that c satisfies (19) so that $L_c(\tau)$ is 1-dimensional. We apply the twisting procedure from Section 6.1 to the H_c-module $L_c(\tau)$. The action of the group T on $\mathrm{End}(L_c(\tau))$ is via conjugation; however, $\mathrm{End}(L_c(\tau)) \cong \mathbb{C}$ is commutative. Hence T acts trivially, and $\mathrm{End}(L_c(\tau))_{\mathcal{F}} = \mathrm{End}(L_c(\tau))$.
We obtain the following 1-dimensional representation of the negative braided Cherednik algebra $\underline{H}_{\underline{c}} = \underline{H}_{\underline{c}}(\mu(G))$,

$$\underline{H}_{\underline{c}} \xrightarrow{\phi} (H_c)_{\mathcal{F}} \xrightarrow{\rho_\tau} \mathrm{End}(L_c(\tau))_{\mathcal{F}} \cong \mathbb{C}. \tag{20}$$

Here ρ_τ is the algebra homomorphism we arrive at from Proposition 6.1. Denote the 1-dimensional $\underline{H}_{\underline{c}}$-module given by (20) by $L_c(\tau)_{\mathcal{F}}$.
Recall from (1) that the group $\mu(G)$ is the same as the group G. To characters τ of $\mu(G) = G$ there correspond 1-dimensional representations $\underline{L}_{\underline{c}}(\tau)$ of the negative braided Cherednik algebra $\underline{H}_{\underline{c}}(G)$ where the $\underline{x}_i$ and $\underline{y}_i$ act by 0 and elements of G act via τ. We can now identify the twists $L_c(\tau)_{\mathcal{F}}$ as certain representations $\underline{L}_{\underline{c}}(\tau')$ of $\underline{H}_{\underline{c}}(G)$, as follows:

$$\tau'(\sigma_i) = \rho_\tau(\phi(\sigma_i)) = \rho_\tau(\bar{s}_i) = \rho_\tau(\sigma_i t_i) = \tau(\sigma_i)\tau(t_i), \qquad \tau'(t_i) = \rho_\tau(\phi(t_i)) = \rho_\tau(t_i) = \tau(t_i).$$

This means that twisting induces a non-trivial permutation of linear characters of the group $G(2,1,n)$, resulting in the following theorem which concludes the paper:

THEOREM 6.3 (Twists of 1-dimensional representations of $H_c(G(2,1,n))$). *The twisting procedure outlined above maps* 1*-dimensional representations of* $H_c(G(2,1,n))$ *to* 1*-dimensional representations of* $\underline{H}_{\underline{c}}(G(2,1,n))$ *as follows,*

$$L_c(\mathrm{triv})_{\mathcal{F}} = \underline{L}_{\underline{c}}(\mathrm{triv}), \quad L_c(\kappa)_{\mathcal{F}} = \underline{L}_{\underline{c}}(\det), \quad L_c(\det)_{\mathcal{F}} = \underline{L}_{\underline{c}}(\kappa), \quad L_c(\kappa\det)_{\mathcal{F}} = \underline{L}_{\underline{c}}(\kappa\det).$$

Acknowledgement

We thank Kieran Calvert for useful discussions and the referee for helpful comments and suggestions.

Funding

E.J.H. acknowledges support by the Engineering and Physical Sciences Research Council, UK.

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