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Transitive and Gallai colorings of the complete graph ☆

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ABSTRACT

A Gallai coloring of the complete graph is an edge-coloring with no rainbow triangle. This concept first appeared in the study of incomparability graphs and anti-Ramsey theory. A directed analogue, called transitive coloring, was introduced by Berenstein, Greenstein and Li in a rather general setting. It is studied here for the acyclic tournament. The interplay of the two notions yields new enumerative results and algebraic perspectives.

We first count Gallai and transitive colorings of the complete graph which use the maximal number of colors. The quasisymmetric generating functions of these colorings, equipped with a natural descent set, are shown to be Schur-positive for any number of colors. Explicit Schur expansions are described when the number of colors is maximal. It follows that descent sets of maximal Gallai and transitive colorings are equidistributed with descent sets of perfect matchings and pattern-avoiding indecomposable permutations, respectively.

Corresponding commutative algebras are also studied. Their dimensions are shown to be equal to the number of Gallai colorings of the complete graph and the number of transitive

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colorings of the acyclic tournament, respectively. Relations to Orlik-Terao algebras are established.

1. Introduction

A *Gallai coloring* of the complete graph K_n on n vertices is an edge-coloring which has no *rainbow triangle*, namely a triangle with edges of (three) different colors. This concept was applied in a seminal paper of Gallai [10] to characterize incomparability graphs, and was named after Gallai by Gyárfás and Simonyi [16]. Gallai colorings have been studied extensively since then; see, e.g., the survey paper [9] and references therein. Asymptotic results about the number of Gallai colorings of complete graphs were obtained recently [2–4]. In particular, it was proved that, for any fixed $k \geq 2$ and sufficiently large n, almost all Gallai colorings of K_n using at most k colors actually use only two colors. Some results regarding precise counting were obtained by Gouge et al. [14].

A directed analogue of a Gallai coloring, called *transitive coloring*, was introduced by Berenstein, Greenstein and Li [5] in a general setting. It is studied here for the acyclic tournament, a natural directed version of the complete graph. The interplay of the two notions yields new perspectives and results regarding both of them. In addition, unlike previous enumerative works dealing with Gallai colorings, we study colorings of a vertex-labeled complete graph rather than their isomorphism classes. As in other classical enumeration problems, such as those of general trees or graphs, the labeled version behaves better and is amenable to q-extensions and additional algebraic structures. Surprising relations to other combinatorial objects, such as perfect matchings, pattern avoiding permutations, hyperplane arrangements and Orlik-Terao algebras emerge. For example, the colorings may be equipped with natural set-valued functions, leading to quasisymmetric generating functions, which are shown to be symmetric and Schur-positive. Equidistribution phenomena with descent sets on distinguished sets of pattern avoiding permutations and perfect matchings follow.

We now define transitive colorings. Let $\vec{K}_n$ be the acyclic tournament with vertex set $[n] := \{1, \ldots, n\}$ and edge set consisting of the ordered pairs

$$E(\vec{K}_n) := \{(i, j) \, : \, i, j \in [n], \; i < j\}.$$

This is an acyclic orientation of the complete graph. Consider the following analogue of Gallai colorings, introduced in [5] (in a general setting).

Definition 1.1. An (edge) m-coloring of $\vec{K}_n$ is a function

$$\varepsilon : E(\vec{K}_n) \longrightarrow \{1, \ldots, m\}.$$

It is *transitive* if

$$\varepsilon(i, k) \in \{\varepsilon(i, j), \varepsilon(j, k)\} \qquad (\forall\, i < j < k).$$

Observe that the set of transitive m-colorings of $\vec{K}_n$, with edge-orientations ignored, is a proper subset of the set of Gallai m-colorings of the complete graph.

Gallai partitions, introduced in [18], are naturally extended to the transitive setting.

Definition 1.2. A *Gallai (transitive) partition* of the complete graph K_n (respectively, the acyclic tournament $\vec{K}_n$) is an equivalence class of Gallai (transitive) colorings, where two colorings are equivalent if one is obtained from the other by renaming the colors. Each color actually used in a coloring defines a (non-empty) *block* of the corresponding partition, consisting of all edges with that color. A Gallai (transitive) partition is *maximal* if its number of blocks is maximal among all Gallai (transitive) partitions of K_n (respectively, $\vec{K}_n$).

Erdős, Simonovits and Sós [8] showed that a maximal Gallai partition of K_n has $n-1$ blocks. It is not difficult to see (Observation 2.15) that this is also the number of blocks in a maximal transitive partition of $\vec{K}_n$. Our first results are the following.

Theorem 1.3. *(Theorem 2.3 below) For every $n > 1$, the number of maximal Gallai partitions of the complete graph K_n is equal to the double factorial $(2n-3)!!$.*

Theorem 1.4. *(Theorem 2.17 below) For every $n > 1$, the number of maximal transitive partitions of the acyclic tournament $\vec{K}_n$ is equal to the Catalan number $C_{n-1} := \frac{1}{n}\binom{2n-2}{n-1}$.*

We prove these results in Section 2; for a q-analogue see Proposition 2.20.

We further associate quasisymmetric generating functions (i.e., refined counts with respect to a certain set-valued function) with Gallai and transitive partitions, and prove that they are symmetric and Schur-positive for any number of blocks.

A symmetric function is called *Schur-positive* if all the coefficients in its expansion in the Schur basis are nonnegative (or polynomials with nonnegative coefficients). Deciding the Schur-positivity of a given symmetric function is equivalent, via the Frobenius characteristic map, to showing that a given class function is actually a character, and is a frequently encountered problem in contemporary algebraic combinatorics; see, e.g., [24, Ch. 3].

Recall the *fundamental quasisymmetric function* indexed by a subset $J \subseteq [n-1]$:

$$\mathcal{F}_{n,J}(\mathbf{x}) := \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in J}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For a set A of combinatorial objects, equipped with a map $\mathrm{Des} : A \to 2^{[n-1]}$, let

$$\mathcal{Q}(A) := \sum_{a \in A} \mathcal{F}_{n,\mathrm{Des}(a)}.$$

The quasisymmetric function $\mathcal{Q}(A)$ was introduced by Gessel in [11]. Gessel was motivated by a well-known conjecture of Stanley [22, III, Ch. 21], which he reformulates as follows: if A is the set of linear extensions of a labeled poset P, then $\mathcal{Q}(A)$ is symmetric if and only if P is isomorphic to the poset determined by a skew semistandard Young tableau. The following problem was posed by Gessel and Reutenauer [12] in the context of permutation sets.

Problem 1.5. For which pairs (A, Des) is $\mathcal{Q}(A)$ symmetric and Schur-positive?

Definition 1.6. The *descent set* of a Gallai (respectively, transitive) partition p of the complete graph K_n (respectively, the acyclic tournament $\vec{K}_n$) on the set of vertices $\{1, \ldots, n\}$ is

$$\mathrm{Des}(p) := \{i \,:\, \text{the edge } (i, i+1) \text{ forms a singleton block in } p\}.$$

Example 1.7. Fig. 1 shows the descent sets of two Gallai partitions of K_4, where the edges in distinct blocks have distinct colors and line-styles. Note that, in the partition on the right, the edge $(1, 3)$ forms a singleton block, but does not yield a descent since its endpoint labels are not consecutive.

A k-partition is a partition into k nonempty blocks. Denote the set of Gallai k-partitions of K_n by $G_{n,k}$ and the set of transitive k-partitions of $\vec{K}_n$ by $T_{n,k}$. We prove the following.

Theorem 1.8. *(Theorem 3.5 below) For every $n > k \ge 1$, the quasisymmetric functions*

$$\mathcal{Q}(G_{n,k}) := \sum_{p \in G_{n,k}} \mathcal{F}_{n,\mathrm{Des}(p)}$$

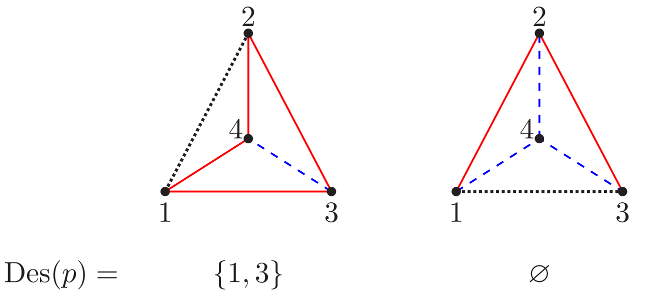

Fig. 1. Descent sets of Gallai partitions.

and

$$\mathcal{Q}(T_{n,k}) := \sum_{p \in T_{n,k}} \mathcal{F}_{n,\mathrm{Des}(p)}$$

are symmetric and Schur-positive.

For maximal (namely, having $k = n - 1$) transitive and Gallai partitions we can describe these functions explicitly. Let ch denote the Frobenius characteristic map, a ring isomorphism from the ring of all class functions on symmetric groups to the ring of symmetric functions; for a definition see Section 3.2.

Theorem 1.9. *(Theorem 3.11 below) For every $n > 1$,*

$$\mathcal{Q}(T_{n,n-1}) = \mathrm{ch}\left(\chi^{(n-1,n-1)} \downarrow^{\mathfrak{S}_{2n-2}}_{\mathfrak{S}_n}\right),$$

where $\chi^{(n-1,n-1)}$ is the irreducible $\mathfrak{S}_{2n-2}$-character indexed by $(n-1, n-1)$.

For the Gallai analogue, with connections to perfect matchings, see Theorem 3.12. It follows that the distribution of sets of singleton blocks of type $\{(i, i+1)\}$ on maximal transitive partitions of the acyclic tournament $\vec{K}_n$ is equal to the distribution of the (standard) descent sets on indecomposable 321-avoiding permutations in the symmetric group $\mathfrak{S}_n$; see Theorem 4.3 below.

Finally, we introduce two families of commutative algebras, and prove that their dimensions are equal to the numbers of transitive and Gallai colorings, respectively. The Hilbert series of these algebras are discussed. In the case of two colors, we construct monomial bases indexed by 2-colorings and show that they are closely related to Orlik-Terao algebras of type A. For the maximal number of colors, the Hilbert series of the Gallai algebra is conjecturally related to the second-order Eulerian numbers.

The rest of the paper is organized as follows. In Section 2 we count maximal Gallai and transitive partitions (of complete graphs and acyclic tournaments, respectively). In Section 3 we equip Gallai and transitive partitions with a natural descent map, determined by singleton blocks. The resulting quasisymmetric functions are shown to be symmetric and Schur-positive, with explicit expressions for maximal partitions. Equidistribution phenomena and bijective proofs are discussed in Section 4. Relevant commutative algebras are introduced and studied in Section 5. Section 6 contains concluding remarks.

2. Enumeration of maximal partitions

2.1. Gallai partitions of the complete graph

Let $K_n = (V, E)$ be the (undirected) *complete graph* on n vertices, with vertex set $V = [n] := \{1, \ldots, n\}$ and edge set $E = \{\{i, j\} : i, j \in V,\ i < j\}$.

Proposition 2.1. *The maximal number of colors in a Gallai coloring of the complete graph K_n is*

$$g(K_n) = n - 1.$$

Proposition 2.1 was proved in [8, Appendix]; see the discussion preceding [14, Theorem JL].
Recall the notion of *Gallai partition* from Definition 1.2.

Corollary 2.2. *A Gallai partition is* maximal *if it has the maximal possible number of blocks, namely $n - 1$.*

The main result of the current subsection is the following.

Theorem 2.3. *The number of maximal Gallai partitions of K_n $(n \geq 2)$ is equal to $(2n-3)!!$.*

Theorem 2.3 will be given two distinct proofs, one using Hamiltonian paths and the other using complete bipartite subgraphs. Both proofs consist of a sequence of lemmas, some of which are of independent interest.

Remark 2.4. Gouge et al. [14] count Gallai colorings of K_n up to renaming the colors as well as the vertices. Gallai partitions, as defined above, correspond to renaming only the colors. Renaming the vertices may result in a different partition.

Definition 2.5. Let ε be a maximal Gallai coloring of K_n. An *ε-rainbow Hamiltonian path* is a (directed) path of length $n-1$, visiting each vertex exactly once, whose edges are assigned $n-1$ different colors by ε.

Lemma 2.6. *Every maximal Gallai coloring of K_n $(n \geq 2)$ has an ε-rainbow Hamiltonian path.*

Proof. Assume that the longest ε-rainbow path $P \subseteq E$ is of length $k-1$, namely visits k vertices. If $k = n$, then P is an ε-rainbow Hamiltonian path, and we are done. Assume that $k < n$.

Using the assumption that the coloring ε is maximal, extend P to an ε-rainbow set $T \subseteq E$ of size $n-1$ by adding edges of the missing colors. The set T contains no cycle, since it is ε-rainbow and ε is Gallai. Having size $n-1$, it is therefore a spanning tree of K_n; in particular, it is connected. Therefore there exists an edge $e \in T \setminus P$ which has precisely one vertex in common with the set of vertices of P. Denote the other vertex of e by v, and the vertices of P, in order, by $v_1, \ldots, v_k$, starting from one of the endpoints of the path P.

The edge e connects v with one of $v_1, \ldots, v_k$ and has a *new* color, namely a color different from those of the edges of P. Let $1 \leq i \leq k$ be the smallest integer such that the edge $\{v, v_i\}$ has a new color. If $i = 1$ then $v, v_1, \ldots, v_k$ is the sequence of vertices of an ε-rainbow path, contradicting the maximality of k. Otherwise $i \geq 2$, and the color of $\{v, v_{i-1}\}$ is not new. Looking at the triangle $\{v, v_{i-1}\}, \{v_{i-1}, v_i\}, \{v, v_i\}$, the color of $\{v, v_i\}$, which is new, is necessarily different from the colors of $\{v, v_{i-1}\}$ and of $\{v_{i-1}, v_i\}$. Since ε is Gallai, the colors of $\{v, v_{i-1}\}$ and of $\{v_{i-1}, v_i\}$ must be equal, and therefore $v_1, \ldots, v_{i-1}, v, v_i, \ldots, v_k$ is the sequence of vertices of an ε-rainbow path, again contradicting the maximality of k. This completes the proof.

Lemma 2.7. *Let ε be a maximal Gallai coloring of K_n, and let $v_1, \ldots, v_n$ be the sequence of vertices of an ε-rainbow Hamiltonian path. Denote $c_i := \varepsilon(\{v_i, v_{i+1}\})$ $(1 \leq i \leq n-1)$. Then:*

(a) For any $1 \leq i < j \leq n$,

$$\varepsilon(\{v_i, v_j\}) \in \{c_i, \ldots, c_{j-1}\}.$$

(b) If $\varepsilon(\{v_1, v_n\}) = c_k$ then, for any $1 \leq i \leq k$ and $k+1 \leq j \leq n$,

$$\varepsilon(\{v_i, v_j\}) = c_k.$$

Proof. (a) Fix $1 \leq i < j \leq n$. If $j - i = 1$ then, by definition, $\varepsilon(\{v_i, v_{i+1}\}) = c_i$. Assume that $j - i \geq 2$. Since ε is Gallai, it assigns the same color to at least two of the edges in the cycle $\{v_i, v_{i+1}\}, \ldots, \{v_{j-1}, v_j\}, \{v_j, v_i\}$. The edges $\{v_i, v_{i+1}\}, \ldots, \{v_{j-1}, v_j\}$ are assigned distinct colors, since they belong to an ε-rainbow path. Therefore $\{v_j, v_i\}$ has the same color as one of the other edges, namely $\varepsilon(\{v_i, v_j\}) \in \{c_i, \ldots, c_{j-1}\}$.

(b) Fix $1 \leq i \leq k$ and $k + 1 \leq j \leq n$, and denote $c := \varepsilon(\{v_i, v_j\})$. Consider the cycle $\{v_1, v_2\}, \ldots, \{v_{i-1}, v_i\}, \{v_i, v_j\}, \{v_j, v_{j+1}\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$. The colors assigned to the edges are, respectively, $c_1, \ldots, c_{i-1}, c, c_j, \ldots, c_{n-1}, c_k$. By (a) above, $c \in \{c_i, \ldots, c_{j-1}\}$. Since ε is Gallai, at least two of the edges in the cycle are assigned the same color. It follows that c_k is equal to one of the other colors, but since $i \leq k \leq j - 1$ the only option is $c_k = c$. Thus $\varepsilon(\{v_i, v_j\}) = c_k$, as claimed.

Definition 2.8. Let $\varepsilon : E \to C$ be a coloring of the edge set E of K_n. A color $c \in C$ is called *singleton* if there is a unique edge with that color.

Lemma 2.9. *Every maximal Gallai coloring of K_n ($n \geq 2$) has a singleton color.*

Proof. We proceed by induction on n. The claim clearly holds for $n = 2$. Let $n > 2$, and assume that the claim holds for K_m for all $2 \leq m < n$. Let ε be a maximal Gallai coloring of K_n, and let $v_1, \ldots, v_n$ be the sequence of vertices of an ε-rainbow Hamiltonian path, which exists according to Lemma 2.6. Denote $c_i := \varepsilon(\{v_i, v_{i+1}\})$ $(1 \leq i \leq n-1)$ and assume, following Lemma 2.7(a), that $v(\{v_1, v_n\}) = c_k$.

By Lemma 2.7(b), $\varepsilon(\{v_i, v_j\}) = c_k$ for any $1 \leq i \leq k$ and $k + 1 \leq j \leq n$. By Lemma 2.7(a), $\varepsilon(\{v_i, v_j\}) \in \{c_1, \ldots, c_{k-1}\}$ for any $1 \leq i < j \leq k$, and $\varepsilon(\{v_i, v_j\}) \in \{c_{k+1}, \ldots, c_{n-1}\}$ for any $k + 1 \leq i < j \leq n$. Since $n > 2$, at least one of k and $n - k$ is larger than 1. If $k \geq 2$ then the restriction of ε to the complete graph K_k on the vertices $v_1, \ldots, v_k$ is a Gallai coloring that uses exactly the colors $c_1, \ldots, c_{k-1}$, and is therefore maximal. Since $k \leq n - 1$, the induction hypothesis implies that this restriction has a singleton color. This color is not used outside K_k, and is therefore a singleton color of ε, as required. Similarly, if $n - k \geq 2$ then the restriction of ε to the complete graph K_{n-k} on the vertices $v_{k+1}, \ldots, v_n$ is maximal Gallai, and has a singleton color which is also singleton for ε itself. This completes the proof.

Lemma 2.7(b) also implies the following well-known result, which will be used in the proof of Lemma 2.14 and in a bijective proof of Theorem 3.15.

Lemma 2.10 (*[14, Corollary 2.5]*). *Any maximal Gallai coloring of K_n ($n \geq 2$) has a unique color c such that the edges colored by c span a complete bipartite graph on n vertices. This is the only color that "touches" every vertex of K_n. On each of the two parts, the induced coloring is also maximal Gallai.*

Surprisingly, the number of ε-rainbow Hamiltonian paths is independent of ε.

Lemma 2.11. *Every maximal Gallai coloring of K_n ($n \geq 2$) has exactly 2^{n-1} (directed) rainbow Hamiltonian paths.*

Proof. We proceed by induction on n. The claim clearly holds for $n = 2$: K_2 has two directed Hamiltonian paths. Let $n > 2$, and assume that the claim holds for $n - 1$. Let ε be a maximal Gallai coloring of K_n, let $v_1, \ldots, v_n$ be the sequence of vertices of an ε-rainbow Hamiltonian path, and denote $c_i := \varepsilon(\{v_i, v_{i+1}\})$ $(1 \leq i \leq n - 1)$. Assume that c_k is a singleton color of ε; its existence is guaranteed by Lemma 2.9.

Let $1 \leq i \leq k - 1$, and consider the triangle $\{v_i, v_k\}, \{v_k, v_{k+1}\}, \{v_i, v_{k+1}\}$. The color $\varepsilon(\{v_k, v_{k+1}\}) = c_k$ is singleton, and is therefore distinct from $\varepsilon(\{v_i, v_k\})$ and $\varepsilon(\{v_i, v_{k+1}\})$. The coloring ε is Gallai, and therefore $\varepsilon(\{v_i, v_k\}) = \varepsilon(\{v_i, v_{k+1}\})$. We can similarly show that $\varepsilon(\{v_k, v_j\}) = \varepsilon(\{v_{k+1}, v_j\})$ for any $k + 2 \leq j \leq n$. If we contract the edge $\{v_k, v_{k+1}\}$ to a single vertex, the coloring ε therefore induces a well-defined coloring ε' of the resulting graph K_{n-1}. This is clearly a maximal Gallai coloring, which does not use the color c_k. By the induction hypothesis, K_{n-1} has 2^{n-2} (directed) ε'-rainbow Hamiltonian paths.

Every ε-rainbow Hamiltonian path in K_n must contain the edge $\{v_k, v_{k+1}\}$, which has a singleton color; it therefore restricts to a ε'-rainbow Hamiltonian path in K_{n-1}. Conversely, every ε'-rainbow Hamiltonian path can be extended to an ε-rainbow Hamiltonian path by blowing the vertex $v_k = v_{k+1}$ to an edge, and this can be done in exactly two ways: the vertex $v_k = v_{k+1}$ cuts the path in K_{n-1} into two sub-paths, each having this vertex as an end-point, and there is a choice which of them to connect (in K_n) to v_k, while connecting the other one to v_{k+1}. Note that at least one of the two sub-paths contains more than one vertex, since $n > 2$. It follows that there are 2^{n-1} ε-rainbow Hamiltonian paths in K_n, as claimed.

We are now nearly at a position to give two proofs of Theorem 2.3. In fact, each proof will require only one additional lemma.

Lemma 2.12. *The number of maximal Gallai partitions of K_n $(n \geq 2)$ for which a given Hamiltonian path is rainbow, is the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.*

Proof. Let $v_1, \ldots, v_n$ be the sequence of vertices of some Hamiltonian path in K_n, and fix a sequence of distinct colors $c_1, \ldots, c_{n-1}$. Maximal Gallai partitions for which this specific path is rainbow correspond bijectively to maximal Gallai colorings ε for which $\varepsilon(\{v_i, v_{i+1}\}) = c_i$ $(1 \leq i \leq n-1)$. Let a_n be the number of such colorings.

Let ε be a maximal Gallai coloring of K_n for which this Hamiltonian path is rainbow, and assume that $\varepsilon(\{v_1, v_n\}) = c_k$, for some $1 \leq k \leq n-1$. By Lemma 2.7(b), $\varepsilon(\{v_i, v_j\}) = c_k$ whenever $1 \leq i \leq k$ and $k+1 \leq j \leq n$. Also, by Lemma 2.7(a), the restriction of ε to the complete graph K_k on the vertices $v_1, \ldots, v_k$ uses only the colors $c_1, \ldots, c_{k-1}$, and is maximal Gallai with the obvious rainbow Hamiltonian path. Similarly for the restriction of ε to the complete graph K_{n-k} on the vertices $v_{k+1}, \ldots, v_n$, using the colors $c_{k+1}, \ldots, c_{n-1}$. It follows that

$$a_n = \sum_{k=1}^{n-1} a_k a_{n-k} \qquad (n \geq 2).$$

This recurrence, together with the initial value $a_1 = 1$, show that $a_n = C_{n-1}$ for all $n \geq 2$, as claimed.

Remark 2.13. A result closely related to Lemma 2.12 is proved in [14, Corollary 2.7].

Proof (*First proof of Theorem 2.3.*)**.** Let p_n be the number of maximal Gallai partitions of K_n $(n \geq 2)$. Consider the pairs (π, P), where π is a maximal Gallai partition of K_n and P is a (directed) π-rainbow Hamiltonian path. By Lemma 2.11, the number of such pairs is $2^{n-1}p_n$. On the other hand, the total number of (directed) Hamiltonian paths in K_n is $n!$. Therefore, by Lemma 2.12, the number of such pairs is $n! \cdot C_{n-1} = \frac{(2n-2)!}{(n-1)!}$. It follows that

$$p_n = \frac{(2n-2)!}{2^{n-1}(n-1)!} = \frac{1 \cdot 2 \cdots (2n-3) \cdot (2n-2)}{2 \cdot 4 \cdots (2n-2)} = (2n-3)!!,$$

as claimed.

Lemma 2.14. *For $n \geq 2$, fix a complete subgraph K_{n-1} of K_n. Then there are exactly $2n-3$ maximal Gallai partitions of K_n which extend any specified maximal Gallai partition of K_{n-1}.*

Proof. The proof is by induction on n. The claim is obvious for $n = 2$. Let $n > 2$, and assume that the claim holds for any $1 \leq k \leq n-1$. Fix a complete subgraph K_{n-1} of K_n and a maximal Gallai partition of K_{n-1}. Fixing colors for each of the $n-2$ blocks of the partition, as well as one additional color to be used outside K_{n-1}, defines a bijection between maximal Gallai partitions and maximal Gallai colorings, of both K_n and K_{n-1}. For convenience we shall use, from now on, the language of colorings.

Let ε be a maximal Gallai coloring of K_{n-1}. By Lemma 2.10, there is a unique color c such that the edges colored by c in ε span a complete bipartite graph on n vertices; call c the *base color* of ε.

Let ε' be a maximal Gallai coloring of K_n which extends ε. It has all the old colors of ε, plus one additional new color. We claim that the base color of ε' is either this new color, or the same as the base color of ε. Indeed, assume that the base color c' of ε' is an old color which is not the base color c of ε. By Lemma 2.10, since $c' \neq c$, there is at least one vertex v of K_{n-1} which the color c' does not touch (in K_{n-1}). Also, since c' is an old color, there is at least one old edge e with this color. The two endpoints of e are on distinct sides of the complete bipartite graph on n vertices colored by c'; therefore one of them is not on the same side as v. The edge connecting this vertex to v is therefore also colored c', contradicting the choice of v. Therefore, indeed, c' is either c, the base color of K_{n-1}, or the new color.

If the base color c' is the new color, then all the old vertices are on one side of the bipartite graph that c' defines, and the new vertex constitutes the other side. It follows that all the new edges are colored c'. This indeed yields a (unique) maximal Gallai partition of K_n extending the old one.

On the other hand, if $c' = c$ then the new vertex of K_n joins one of the two (nonempty) sides of the complete bipartite subgraph of the old K_{n-1}. Assume that the sizes of these sides are k and $n-k-1$ $(1 \le k \le n-2)$. If the new vertex joins the side of size k, then all the edges connecting it to the other side are colored c. The (old) coloring of the complete subgraph K_k is maximal Gallai, by Lemma 2.10, and so is the (new) coloring of K_{k+1} (on this side plus the new vertex). By the induction hypothesis, there are $2k-1$ ways to obtain such a new coloring of K_{k+1}. A similar argument holds if the new vertex belongs to the other side of K_{n-1}, yielding $2(n-k-1)-1$ extensions.

It is easy to see that the above extended colorings yield distinct maximal Gallai partitions. Their number is $1 + (2k-1) + (2(n-k-1)-1) = 2n-3$, as claimed.

Proof (*Second proof of Theorem 2.3*)**.** Since there is a unique (empty) maximal Gallai partition of K_1, the claim follows immediately from Lemma 2.14, by induction on n.

2.2. Transitive partitions of the acyclic tournament

Let $\vec{K}_n = (V, E)$ be the *acyclic tournament* on n vertices. This is a directed graph, with set of vertices $V = \{1, \ldots, n\}$ and set of directed edges $E = \{(i,j) \,:\, 1 \le i < j \le n\}$. It is loopless, and has a unique directed edge between any two distinct vertices, pointing from the vertex with smaller index to the vertex with a larger index.

Observation 2.15. *The maximal number of colors in a transitive coloring of $\vec{K}_n$ is*

$$t(\vec{K}_n) = n - 1.$$

Proof. Since every transitive coloring of $\vec{K}_n$ is a Gallai coloring of the underlying undirected graph K_n, Proposition 2.1 implies that $t(\vec{K}_n) \le g(K_n) = n-1$.

To prove equality note that the function $\varepsilon : E \to [n-1]$ defined by

$$\varepsilon(i,j) := i \qquad (1 \le i < j \le n)$$

is a transitive coloring of $\vec{K}_n$ using exactly $n-1$ distinct colors.

Lemma 2.16. *Maximal transitive colorings of $\vec{K}_n$ correspond bijectively to the maximal Gallai colorings of K_n for which the path $\{1,2\}, \ldots, \{n-1,n\}$ is rainbow.*

Proof. Every transitive coloring of $\vec{K}_n$ yields a Gallai coloring of the underlying undirected graph K_n. Also, in a maximal transitive coloring ε of $\vec{K}_n$, the path $(1,2), \ldots, (n-1,n)$ is rainbow. Indeed, by transitivity (and induction on $j-i$),

$$\varepsilon(i,j) \in \{\varepsilon(i,i+1), \ldots, \varepsilon(j-1,j)\} \subseteq \{\varepsilon(1,2), \ldots, \varepsilon(n-1,n)\} \qquad (\forall\, i<j),$$

so that

$$|\{\varepsilon(i,j) \,:\, 1 \le i < j \le n\}| = |\{\varepsilon(i,i+1) \,:\, 1 \le i \le n-1\}|.$$

Maximality means that the number of colors used is $n-1$, hence $|\{\varepsilon(1,2),\ldots,\varepsilon(n-1,n)\}| = n-1$. Thus the path $(1,2),\ldots,(n-1,n)$ is rainbow.

To complete the proof we need to show that every Gallai coloring $\tilde{\varepsilon}$ of K_n for which the path $\{1,2\},\ldots,\{n-1,n\}$ is rainbow yields a transitive coloring of $\vec{K}_n$. Indeed, by Lemma 2.7(a),

$$\tilde{\varepsilon}(\{i,j\}) \in \{\tilde{\varepsilon}(\{i,i+1\}),\ldots,\tilde{\varepsilon}(\{j-1,j\})\} \qquad (\forall\, i<j).$$

Hence, for every $i<j<k$, $\tilde{\varepsilon}(\{i,j\}) \neq \tilde{\varepsilon}(\{j,k\})$. In order to avoid rainbow triangles we must have

$$\tilde{\varepsilon}(\{i,k\}) \in \{\tilde{\varepsilon}(\{i,j\}),\tilde{\varepsilon}(\{j,k\})\} \qquad (\forall\, i<j<k).$$

Thus $\tilde{\varepsilon}$ yields a transitive coloring of $\vec{K}_n$.

The following theorem is the directed analogue of Lemma 2.12.

Theorem 2.17. *The number of maximal transitive partitions of $\vec{K}_n$ ($n \geq 2$) is equal to the Catalan number $C_{n-1} := \frac{1}{n}\binom{2n-2}{n-1}$.*

Proof. Combine Lemma 2.12 with Lemma 2.16.

Corollary 2.18. *Let $n \geq 2$.*

(a) For every maximal transitive coloring of $\vec{K}_n$, there exists a unique color c, for which the edges colored by c span a complete bipartite graph on n vertices.
(b) There exists $1 \leq k < n$, such that the two sides of this bipartite graph are $\{i : 1 \leq i \leq k\}$ and $\{i : k+1 \leq i \leq n\}$.

Proof. (a) Combine Lemma 2.10 with Lemma 2.16.
(b) Combine the proof of Lemma 2.12 with Lemma 2.16.

We further prove the following refinement.

Definition 2.19. Consider $p \in T_{n,n-1}$, a maximal transitive partition of $\vec{K}_n$.

(a) A directed edge (i,j) is a *minimal edge* in p if every edge (a,b) in the block of (i,j) satisfies $i \leq a$.
(b) Let minimal(p) be the number of minimal edges in p.

Carlitz and Riordan [6] defined a q-Catalan number $C_n(q)$ using the recursion

$$C_{n+1}(q) := \sum_{k=0}^{n} q^{(k+1)(n-k)} C_k(q) C_{n-k}(q) \qquad (n \geq 0)$$

with $C_0(q) := 1$.

Proposition 2.20. *For every $n > 1$,*

$$\sum_{p \in T_{n,n-1}} q^{\mathrm{minimal}(p)} = q^{\binom{n}{2}} C_{n-1}(q^{-1}).$$

Proof. Denote the number of non-minimal edges in a transitive partition p by non-minimal(p). It suffices to prove that for every $n \geq 2$,

$$\sum_{p \in T_{n,n-1}} q^{\mathrm{non-minimal}(p)} = C_{n-1}(q).$$

The proof is by induction on n. For $n = 2$ there are one edge in $\vec{K}_2$ and statement clearly holds.

Assume that the statement is correct for all $k \leq n$. Consider a transitive partition of $\vec{K}_{n+1}$. By Corollary 2.18, there exists a unique $1 \leq t \leq n$, such that the edges in the block containing $(1, n+1)$, which represents a complete bipartite subgraph of $\vec{K}_{n+1}$, are $\{(i,j) \,:\, 1 \le i \le t < j \le n+1\}$. Thus there are $(t-1)(n+1-t)$ non-minimal edges in this block. All other blocks are either in the tournament $\vec{K}_t$ spanned by the first t vertices, or in the tournament $\vec{K}_{n+1-t}$ spanned by the last $n+1-t$ vertices. By the induction hypothesis,

$$\sum_{p \in T_{n+1,n}} q^{\text{non-minimal}(p)} = \sum_{t=1}^{n} q^{(t-1)(n+1-t)} C_{t-1}(q) C_{n-t}(q).$$

Letting $k := n-t$, the RHS is equal to

$$\sum_{k=0}^{n-1} q^{(n-1-k)(k+1)} C_{n-1-k}(q) C_k(q) = C_n(q). \quad \square$$

3. Schur-positivity

Recall Definition 1.6 of a descent set for transitive and Gallai partitions, and the resulting quasisymmetric generating functions $\mathcal{Q}(T_{n,k})$ and $\mathcal{Q}(G_{n,k})$. In this section we prove Theorem 1.8 (Theorem 3.5 below), as well as Theorem 1.9 (Theorem 3.11 below) and its Gallai analogue (Theorem 3.12). Specifically, in Section 3.1 we show that, for any positive integers n and k, the quasisymmetric functions $\mathcal{Q}(T_{n,k})$ and $\mathcal{Q}(G_{n,k})$ are symmetric and Schur-positive; and in Section 3.2 we describe explicitly the symmetric group characters corresponding to $\mathcal{Q}(T_{n,n-1})$ and $\mathcal{Q}(G_{n,n-1})$. A bijection which relates maximal Gallai partitions to perfect matchings will be described in Section 4.1.

3.1. Schur-positivity of Gallai and transitive partitions

In this subsection we prove Theorem 1.8 (Theorem 3.5 below).

Definition 3.1. A subset $J \subseteq [n-1]$ is *sparse* if it does not contain any consecutive pair of elements.

Observation 3.2. *For every Gallai (respectively, transitive) partition p of the complete graph (respectively, acyclic tournament), the set* $\mathrm{Des}(p)$ *is sparse.*

Proof. If $\mathrm{Des}(p)$ is not sparse then there exists an i such that $i, i+1 \in \mathrm{Des}(p)$. By the definition of the descent set, it follows that the edges $(i, i+1)$ and $(i+1, i+2)$ form singleton blocks. Thus $(i, i+1)$, $(i, i+2)$ and $(i, i+2)$ belong to three different blocks, and therefore form a rainbow triangle. This contradicts the assumption of p being a transitive (or Gallai) partition.

Denote $g(n,k) := |G_{n,k}|$ and $t(n,k) := |T_{n,k}|$.

Lemma 3.3. *For every $n > k \ge 1$ and a sparse subset $\varnothing \ne J \subseteq [n-1]$,*

$$|\{p \in G_{n,k} \,:\, J \subseteq \mathrm{Des}(p)\}| = g(n-|J|, k-|J|)$$

and

$$|\{p \in T_{n,k} \,:\, J \subseteq \mathrm{Des}(p)\}| = t(n-|J|, k-|J|).$$

Proof. We prove the lemma for Gallai partitions. The proof for transitive partitions is similar.

Let p be a Gallai partition. For every $i \in \mathrm{Des}(p)$, $(i, i+1)$ is a singleton block. Since p is a Gallai partition, it contains no rainbow triangle. Hence, for every $j \ne i, i+1$, the edges (i,j) and $(i+1,j)$ belong to the same block. It follows that the set of Gallai k-partitions of K_n with a descent at i is in bijection with the set of Gallai $(k-1)$-partitions of $K_n/(i, i+1)$ (edge contraction), which is isomorphic to K_{n-1}. This proves the lemma for $|J| = 1$. Proceed by induction on the size of J.

The following is a weak version of a new criterion of Marmor for Schur-positivity.

Lemma 3.4 (*[19], Theorem 1.8*). *Let A be a set equipped with a descent set map, and assume that for every $a \in A$,* $\mathrm{Des}(a)$ *is sparse. If for every sparse $J \subseteq [n-1]$, the cardinality of the set $\{a \in A : J \subseteq \mathrm{Des}(a)\}$ depends only on the size of J, then $\mathcal{Q}(A)$ is symmetric and Schur-positive.*

Theorem 3.5. *For every $n > k \geq 1$, the quasisymmetric functions*

$$\mathcal{Q}(G_{n,k}) := \sum_{p \in G_{n,k}} \mathcal{F}_{n,\mathrm{Des}(p)}$$

and

$$\mathcal{Q}(T_{n,k}) := \sum_{p \in T_{n,k}} \mathcal{F}_{n,\mathrm{Des}(p)}$$

are symmetric and Schur-positive.

Proof. Combine Lemma 3.4 with Lemma 3.3.

3.2. Schur expansion for maximal transitive partitions

In this subsection we prove Theorem 1.9 (Theorem 3.11 below).

We begin with some necessary background. A *partition* of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_t)$ of positive integers whose sum is n. We denote $\lambda \vdash n$. For a partition $\lambda \vdash n$, let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of shape λ. We use the English convention, according to which row indices increase from top to bottom. See [23, p. 312] for definition and examples.

The *descent set* of a standard Young tableau T of size n

$$\mathrm{Des}(T) := \{i \in [n-1] \,:\, i+1 \text{ appears in } T \text{ in a lower row than } i\}.$$

Let s_λ be the Schur function indexed by the partition λ. The following key result is due to Gessel.

Theorem 3.6. *[23, Theorem 7.19.7] For every integer partition $\lambda \vdash n$,*

$$\mathcal{Q}(\mathrm{SYT}(\lambda)) = s_\lambda.$$

There is a dictionary relating symmetric functions to class functions on the symmetric group. The irreducible characters of $\mathfrak{S}_n$ are indexed by partitions $\lambda \vdash n$ and denoted χ^λ. The *Frobenius characteristic map* ch, from class functions on $\mathfrak{S}_n$ to symmetric functions, is defined by $\mathrm{ch}(\chi^\lambda) := s_\lambda$, and extended by linearity. Theorem 3.6 may then be restated as follows:

$$\mathrm{ch}(\chi^\lambda) = \sum_{T \in \mathrm{SYT}(\lambda)} \mathcal{F}_{n,\mathrm{Des}(T)}.$$

A combinatorial rule for the restriction of irreducible $\mathfrak{S}_n$-characters was given by Young [17, Theorem 9.2]:

Theorem 3.7 (*The Branching Rule*). *For $\lambda \vdash n$*

$$\chi^\lambda \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \sum_{\substack{\mu \vdash n-1 \\ |\lambda/\mu|=1}} \chi^\mu.$$

Viewing tableaux of shape μ as tableaux of shape λ with the entry n "forgotten", the Branching Rule may be restated as

$$\mathrm{ch}(\chi^\lambda \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}) = \sum_{T \in \mathrm{SYT}(\lambda)} \mathcal{F}_{n-1,\mathrm{Des}(T)\cap[n-2]}.$$

Iteration immediately yields the following.

Corollary 3.8. *For every $\lambda \vdash n$ and $m \leq n$,*

$$\mathrm{ch}(\chi^\lambda \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_m}) = \sum_{T \in \mathrm{SYT}(\lambda)} \mathcal{F}_{m,\mathrm{Des}(T)\cap[m-1]}.$$

The following lemma is folklore.

Lemma 3.9. *Let $J \subseteq [2n-3]$ be a subset of size m. Then*

$$|\{T \in \mathrm{SYT}((n-1, n-1)) \,:\, J \subseteq \mathrm{Des}(T)\}| = C_{n-m-1}$$

(the $(n-m-1)$-st Catalan number) if J is sparse, and is zero otherwise.

Proof. The descent set of a standard Young tableau of a two-rowed shape has no consecutive entries. Thus, if J is not sparse, then

$$|\{T \in \mathrm{SYT}((n-1, n-1)) \,:\, J \subseteq \mathrm{Des}(T)\}| = 0.$$

To prove the statement for sparse subsets, recall that a *Dyck path* of length $2n$ is a sequence $(0,0) = t_0, t_1, \ldots, t_{2n} = (2n, 0)$ of points in the plane such that, for each $1 \leq i \leq 2n$, either $t_i = t_{i-1} + (1,1)$ (an up step) or $t_i = t_{i-1} + (1,-1)$ (a down step), and such that no t_i is below the x-axis. Denote by $\mathcal{D}_{2n}$ the set of Dyck paths of length $2n$, and by

$$\mathrm{Peak}(d) := \{1 \leq i \leq 2n-1 \,:\, t_{i-1} \to t_i \text{ is an up step and } t_i \to t_{i+1} \text{ is a down step}\},$$

the set of peaks of $d \in \mathcal{D}_{2n}$.

Recall the bijection from $\mathrm{SYT}(n-1, n-1)$ to $\mathcal{D}_{2n-2}$, determined as follows: the ith step in the Dyck path is an up step if i is in the first row of T, and a down step if i is in the second row. Let $J \subseteq [2n-3]$ be a subset of order m. Assume that J is a subset the peak set of a given Dyck path. Deleting the jth and $(j+1)$-st steps, for every $j \in J$, yields a Dyck path of length $2n-2-2m$. It follows that the set of Dyck paths of length $2n-2$ with peak set containing J is in bijection with Dyck paths of length $2n-2-2m$, whose number is C_{n-m-1}. We conclude that

$$\begin{aligned}|\{T \in \mathrm{SYT}(n-1, n-1) \,:\, J \subseteq \mathrm{Des}(T)\}| &= |\{d \in \mathcal{D}_{2n-2} \,:\, J \subseteq \mathrm{Peak}(d)\}| \\ &= |\mathcal{D}_{2n-2-2m}| = C_{n-m-1}. \quad \square\end{aligned}$$

Lemma 3.10. *Let $J \subseteq [n-1]$ be a subset of size m. Then*

$$|\{p \in T_{n,n-1} \,:\, J \subseteq \mathrm{Des}(p)\}| = C_{n-m-1}$$

if J is sparse, and is zero otherwise.

Proof. Combining Lemma 3.3 with Theorem 2.17 we obtain that if J is sparse then

$$|\{p \in T_{n,n-1} \,:\, J \subseteq \mathrm{Des}(p)\}| = t_{n-m,n-m-1} = C_{n-m-1}.$$

If J is not sparse then, by Observation 3.2, there are no transitive partitions of $\vec{K}_n$ with descent set J, completing the proof.

Theorem 3.11. *For every $n > 1$,*

$$\mathcal{Q}(T_{n,n-1}) = \mathrm{ch}\left(\chi^{(n-1,n-1)} \downarrow^{\mathfrak{S}_{2n-2}}_{\mathfrak{S}_n}\right),$$

where $\chi^{(n-1,n-1)}$ is the irreducible $\mathfrak{S}_{2n-2}$-character indexed by $(n-1, n-1)$.

Proof. Combining Lemma 3.10 with Lemma 3.9 one obtains

$$|\{p \in T_{n,n-1} \,:\, J \subseteq \mathrm{Des}(p)\}| = |\{T \in \mathrm{SYT}(n-1, n-1) \,:\, J \subseteq \mathrm{Des}(T) \cap [n-1]\}|$$
$$(\forall J \subseteq [n-1]),$$

thus also

$$|\{p \in T_{n,n-1} \,:\, J = \mathrm{Des}(p)\}| = |\{T \in \mathrm{SYT}(n-1, n-1) \,:\, J = \mathrm{Des}(T) \cap [n-1]\}| \quad (\forall J \subseteq [n-1]).$$

It follows that

$$\mathcal{Q}(T_{n,n-1}) = \sum_{p \in T_{n,n-1}} \mathcal{F}_{n,\mathrm{Des}(p)} = \sum_{T \in \mathrm{SYT}(n-1,n-1)} \mathcal{F}_{n,\mathrm{Des}(T)\cap[n-1]}.$$

By Corollary 3.8, the RHS is equal to $\mathrm{ch}\left(\chi^{(n-1,n-1)} \downarrow^{\mathfrak{S}_{2n-2}}_{\mathfrak{S}_n}\right)$, completing the proof.

3.3. Schur expansion for maximal Gallai partitions

In this subsection we prove the following result, which is an analogue of Theorem 3.11 for Gallai partitions.

Theorem 3.12. *For every $n > 1$,*

$$\mathcal{Q}(G_{n,n-1}) = \mathrm{ch}\left(\sum_{r=0}^{n-1} a_r \chi^{(n-1+r,n-1-r)} \downarrow^{\mathfrak{S}_{2n-2}}_{\mathfrak{S}_n}\right),$$

where a_r is the number of perfect matchings of $2r$ points on a line with no short chords.

Remark 3.13. For the numbers a_r, see [21, A000806], [19] and references therein.

Lemma 3.14. *Let $J \subseteq [n-1]$ be a subset of size k. Then*

$$|\{p \in G_{n,n-1} \,:\, J \subseteq \mathrm{Des}(p)\}| = (2n-2k-3)!!$$

if J is sparse, and is zero otherwise.

Proof. The proof is the same as the proof of Lemma 3.10, with Theorem 2.17 replaced by Theorem 2.3.

Denote by M_{2n} the set of perfect matchings of $2n$ points on a circle, labeled by $1, \ldots, 2n$. The points labeled by $1, \ldots, n$ are in the upper half-plane. For $m \in M_{2n}$ define the *short match set*

$$\mathrm{Short}(m) := \{1 \le i \le 2n-1 \,:\, (i, i+1) \in m\}.$$

Theorem 3.15. *For every $n > 1$,*

$$\sum_{p \in G_{n,n-1}} \mathcal{F}_{n,\mathrm{Des}(p)} = \sum_{m \in M_{2n-2}} \mathcal{F}_{n,\mathrm{Short}(m)\cap[n-1]}.$$

Proof. Observe that, for every subset $J \subseteq [2n-1]$ of size k,

$$|\{m \in M_{2n} \,:\, J \subseteq \mathrm{Short}(m)\}| = (2n-2k-1)!!$$

if J is sparse, and is zero otherwise. Comparing this observation with Lemma 3.14, one deduces that the descent set distribution on $G_{n,n-1}$ is equal to the distribution of short matches in $[n-1]$ on perfect matching in M_{2n-2}, implying the claim.

For a bijective proof of this result see Section 4.1.

The following result is due to Marmor.

Theorem 3.16 (*[19, Theorem 1.6]*)**.** *The set M_{2n} is symmetric and Schur-positive with respect to* Short*. Furthermore, its Schur expansion is given by the following formula:*

$$\sum_{m\in M_{2n}} \mathcal{F}_{2n,\mathrm{Short}(m)} = \sum_{k=0}^{n} |\{m \in M_{2n-2k} \,:\, \mathrm{Short}(m) = \varnothing\}|\, s_{(2n-k,k)}.$$

Proof of Theorem 3.12. Using the Frobenius characteristic map ch, we can write Theorem 3.16, with n replaced by $n-1$, as

$$\sum_{m\in M_{2n-2}} \mathcal{F}_{2n-2,\mathrm{Short}(m)} = \sum_{k=0}^{n} |\{m \in M_{2n-2-2k} \,:\, \mathrm{Short}(m) = \varnothing\}|\, \mathrm{ch}\left(\chi^{(2n-2-k,k)}\right).$$

By Corollary 3.8, it follows that

$$\sum_{m\in M_{2n-2}} \mathcal{F}_{n,\mathrm{Short}(m)\cap[n-1]} = \sum_{k=0}^{n} |\{m \in M_{2n-2-2k} \,:\, \mathrm{Short}(m) = \varnothing\}|\, \mathrm{ch}\left(\chi^{(2n-2-k,k)} \downarrow^{\mathfrak{S}_{2n-2}}_{\mathfrak{S}_n}\right).$$

Combining this with Theorem 3.15, we obtain

$$\begin{aligned}
\mathcal{Q}(G_{n,n-1}) &= \sum_{p\in G_{n,n-1}} \mathcal{F}_{n,\mathrm{Des}(p)} = \sum_{m\in M_{2n-2}} \mathcal{F}_{n,\mathrm{Short}(m)\cap[n-1]} \\
&= \mathrm{ch}\left(\sum_{k=0}^{n-1} |\{m \in M_{2n-2-2k} \,:\, \mathrm{Short}(m) = \varnothing\}|\, \chi^{(2n-2-k,k)} \downarrow^{\mathfrak{S}_{2n-2}}_{\mathfrak{S}_n}\right) \\
&= \mathrm{ch}\left(\sum_{k=0}^{n-1} a_{n-1-k}\, \chi^{(2n-2-k,k)} \downarrow^{\mathfrak{S}_{2n-2}}_{\mathfrak{S}_n}\right) = \mathrm{ch}\left(\sum_{r=0}^{n-1} a_r\, \chi^{(n-1+r,n-1-r)} \downarrow^{\mathfrak{S}_{2n-2}}_{\mathfrak{S}_n}\right). \qquad \square
\end{aligned}$$

4. Equidistribution phenomena

In this section we compare the distribution of the descent set on Gallai and transitive partitions with its distribution on perfect matchings and permutations.

Theorem 3.15 above is equivalent to the following statement: The distribution of the descent set on maximal Gallai partitions of the complete graph K_n is equal to the distribution of consecutive matches $(i, i+1)$, $1 \le i < n$, in perfect matchings on $2n-2$ points, labeled by $1, \ldots, 2n-2$. A bijective proof of this claim is given in Section 4.1.

In Section 4.2, it is shown that the distribution of the descent set on maximal transitive partitions of the tournament $\vec{K}_n$ is equal to its distribution on indecomposable 321-avoiding permutations in the symmetric group $\mathfrak{S}_n$, see Theorem 4.3 below.

Finally, we point out a bijection from transitive 2-colorings of the tournament $\vec{K}_n$ to permutations in $\mathfrak{S}_n$, which maps the set of edges colored 1 to the inversion set of the corresponding permutation (Observation 4.5 below).

4.1. A bijection from maximal gallai partitions to perfect matchings

The set M_{2n-2} of perfect matchings on $2n-2$ points can be visualized as follows. Draw $2n-2$ points on a circle and label them by $1, \ldots, 2n-2$ clockwise, such that the points labeled 1 and n are the endpoints of the horizontal diameter and the letters $2, \ldots, n-1$ are in the "upper half-plane". Indicate a matched pair by drawing a chord between the corresponding points. Theorem 3.15 then claims that the sets of singletons in maximal Gallai partitions of the complete graph K_n are equidistributed with the sets of short chords contained in the upper half-plane in perfect matchings. The case $n = 3$ is illustrated in Fig. 2. In this subsection, we give a bijective proof of this theorem.

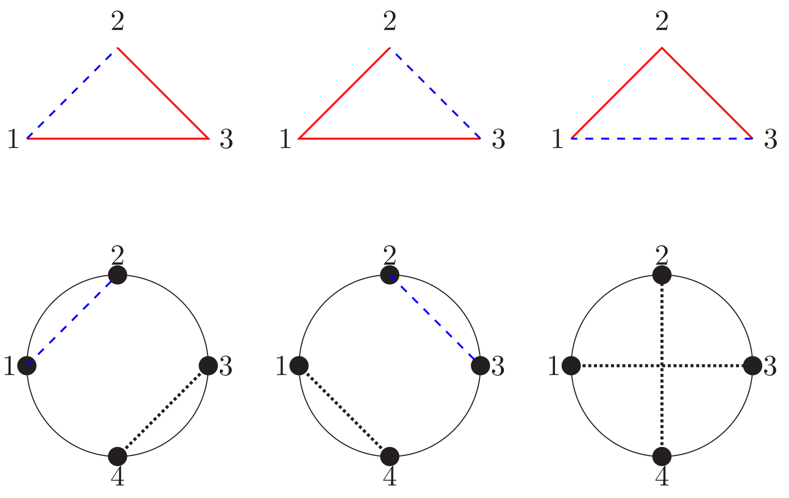

Fig. 2. The case $n = 3$. The top row shows the three maximal Gallai partitions of K_3. Each partition has two blocks, one of which is a singleton. The singleton edge is drawn dashed blue, and the two edges in the other block are drawn solid red. Note that the singleton edge $(1, 3)$ does not have consecutive endpoint labels. The bottom row shows the three perfect matchings in M_4. A chord in a matching is drawn dashed blue if its endpoints have consecutive labels in the set $\{1, 2, 3\}$, namely, it is a short chord in the upper half-plane.

Proof (*A bijective proof of Theorem 3.15*)**.** We describe a bijection

$$\varphi : G_{n,n-1} \to M_{2n-2}$$

from the set $G_{n,n-1}$ of maximal Gallai partitions of K_n to the set M_{2n-2} of perfect matchings of $2n-2$ points labeled by $1, \ldots, 2n-2$, under which

$$\mathrm{Des}(p) = \mathrm{Short}(\varphi(p)) \cap [n-1] \qquad (\forall\, p \in G_{n,n-1}).$$

A *binary total partition tree of* $[n]$ is a rooted complete binary tree with n leaves whose vertices are labeled by subsets of $[n]$, as follows: the leaves are labeled by all distinct singletons, and every internal vertex (father) is labeled by the disjoint union of the sets labeling its two sons. These trees are studied in [23, §5.2]. Denote the set of binary total partition trees of $[n]$ by BTPT_n.

Define a bijection

$$\psi : G_{n,n-1} \to \mathrm{BTPT}_n$$

from maximal Gallai partitions to binary total partition trees of $[n]$, as follows. By Lemma 2.10, translated from the language of Gallai colorings to the language of Gallai partitions, in any maximal Gallai partition of K_n ($n \geq 2$) there is a unique block such that the edges in the block span a complete bipartite graph on n vertices, and the induced partition on the edges in each side of this bipartite graph is also maximal Gallai. Label the root of the tree by the set $[n]$. Label the two sons of the root by the two sides of the bipartition of $[n]$ corresponding to the bipartite graph. Continue labeling the sons of any labeled father, recursively; see Figs. 3 and 4.

The map ψ is a bijection, since the Gallai partition p can be recovered from the labeling of the tree $\psi(p)$, as follows. For every edge $e = (i, j)$ in K_n, there exists a unique pair of brothers (two sons with common father), such that i belong to one of the brothers and j to the other. This pair is called the separating pair of e. Two edges belong to the same block if and only if they have the same separating pair of brothers.

A bijection

$$\phi : \mathrm{BTPT}_n \to M_{2n-2}$$

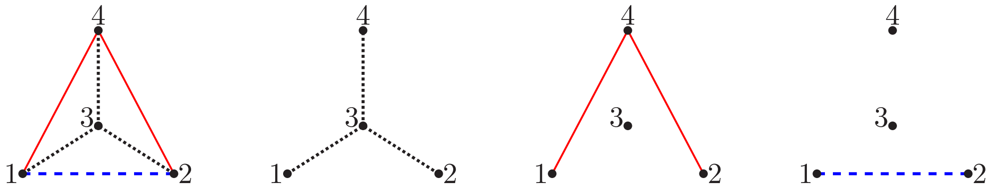

Fig. 3. A maximal Gallai coloring and its bipartitions.

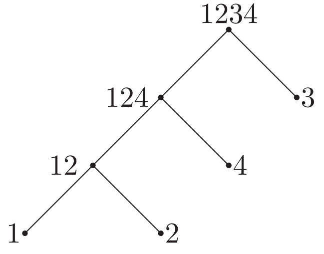

Fig. 4. The corresponding binary partition tree.

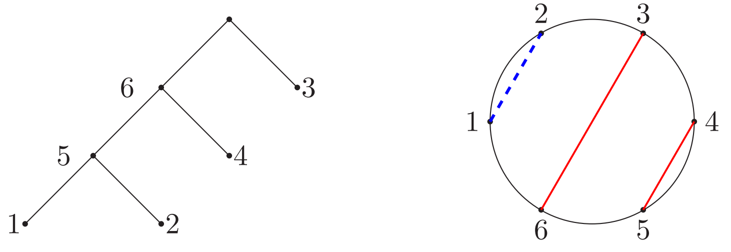

Fig. 5. A relabeled tree and the corresponding perfect matching. The short chord in the upper half-plane is colored dashed blue.

from binary total partition trees of $[n]$ to perfect matchings of $2n-2$ points is described in [23, Example 5.2.6]: Let $T \in \mathrm{BTPT}_n$. Inductively relabel the inner vertices (that is, vertices which are not leaves), excluding the root, by labels $n+1, \ldots, 2n-2$ as follows. If labels $1, \ldots, m$ have already been used, then label by $m+1$ the vertex v satisfying the following condition: among all unlabeled vertices with both sons labeled, the vertex v has a son with minimal labeling. We get a complete binary tree $\hat{T}$ whose vertices (excluding the root) are labeled by $\{1, \ldots, 2n-2\}$. The matched pairs in the perfect matching $\phi(T)$ are the pairs of brothers in $\hat{T}$. For an example see Fig. 5.

Finally, let

$$\varphi := \phi \circ \psi.$$

The map φ is a bijection, since both ψ and ϕ are. Also, the pair $(i, i+1)$ is a (short) match in $\varphi(p)$ if and only if i and $i+1$ are brothers in $\widehat{\phi(p)}$. For $1 \le i < n$ this happens if and only if i and $i+1$ are leaves and brothers in $\phi(p)$. Then the father of the leaves labeled by i and $i+1$ is labeled by $\{i, i+1\}$ in $\phi(p)$. This is equivalent to the edge $(i, i+1)$ being a singleton block in p, namely, $i \in \mathrm{Des}(p)$.

Example 4.1. See Figs. 3, 4 and 5.

4.2. Indecomposable 321-avoiding permutations

A permutation π in the symmetric group $\mathfrak{S}_n$ is *indecomposable* if there is no $1 \le r < n$ for which $\pi(i) < \pi(j)$ for all $i \le r < j$.

Example 4.2. The permutation $\pi = [31254] \in \mathfrak{S}_5$ is decomposable, since for $r = 3$, $\pi(i) < \pi(j)$ for every $i \le 3 < j$. This may be viewed as a non-trivial principal block decomposition of the corresponding permutation matrix. The permutation $\sigma = [43152]$ is indecompsable; indeed, there is no non-trivial principal block decomposition of the corresponding permutation matrix.

$$\pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad \sigma = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Denote by $\mathfrak{S}_n^*(321)$ the set of indecomposable permutations in $\mathfrak{S}_n$ with no decreasing subsequence of length 3. Recall the descent set of a permutation π in the symmetric group $\mathfrak{S}_n$,

$$\mathrm{Des}(\pi) := \{i \,:\, \pi(i) > \pi(i+1)\}.$$

Theorem 4.3. *For every $n > 1$,*

$$\sum_{p \in T_{n,n-1}} \mathbf{x}^{\mathrm{Des}(p)} = \sum_{\pi \in \mathfrak{S}_n^*(321)} \mathbf{x}^{\mathrm{Des}(\pi)},$$

where $\mathbf{x}^J := \prod_{j \in J} x_j$. Equivalently,

$$\mathcal{Q}(T_{n,n-1}) = \mathcal{Q}(\mathfrak{S}_n^*(321)).$$

Proof. By [1, Theorem 1.2],

$$\mathcal{Q}(\mathfrak{S}_n^*(321)) = \mathrm{ch}\left(\chi^{(n-1,n-1)} \downarrow_{\mathfrak{S}_n}^{\mathfrak{S}_{2n-2}}\right).$$

Comparing this result with Theorem 3.11 gives

$$\mathcal{Q}(T_{n,n-1}) = \mathrm{ch}\left(\chi^{(n-1,n-1)} \downarrow_{\mathfrak{S}_n}^{\mathfrak{S}_{2n-2}}\right) = \mathcal{Q}(\mathfrak{S}_n^*(321)). \quad \square$$

Remark 4.4. A bijective proof of Theorem 4.3 will be given elsewhere.

At this point we would like to comment on another bijection to permutations.

A set $J \subseteq A := \{(i,j) \,:\, 1 \le i < j \le n\}$ is *transitive* if it satisfies $(i,j), (j,k) \in J \Rightarrow (i,k) \in J$ for all $i < j < k$. It is *bitransitive* if it is transitive and its complement is also transitive. The *inversion set* of a permutation $\pi \in \mathfrak{S}_n$ is the set $\mathrm{Inv}(\pi) := \{(\pi(i), \pi(j)) \,:\, i < j \text{ and } \pi(i) > \pi(j)\}$. It is well-known that a subset $J \subseteq A$ is an inversion set of a permutation if and only if it is bitransitive; see, e.g., [15].

Observation 4.5. *The map from $\mathfrak{S}_n$ to the set of transitive 2-colorings of $\vec{K}_n$, defined on $\pi \in \mathfrak{S}_n$ by coloring by 1 the edges in $\mathrm{Inv}(\pi)$ and by 2 the edges in its complement, is a bijection.*

Proof. By definition, an edge coloring $\varepsilon \,:\, E(\vec{K}_n) \to [2]$ is transitive if and only if the set $\{(i,j) \,:\, 1 \le i < j \le n \text{ and } \varepsilon(i,j) = 1\}$ is bitransitive.

Remark 4.6. In another paper we introduce transitive colrings of oriented matroids. It is shown that the number of transitive 2-colorings of any vector matroid is equal to the number of chambers in the dual hyperplane arrangement. In particular, the number of transitive 2-colorings of a finite Coxeter root systems is equal to the size of the corresponding Coxeter group, generalizing Observation 4.5.

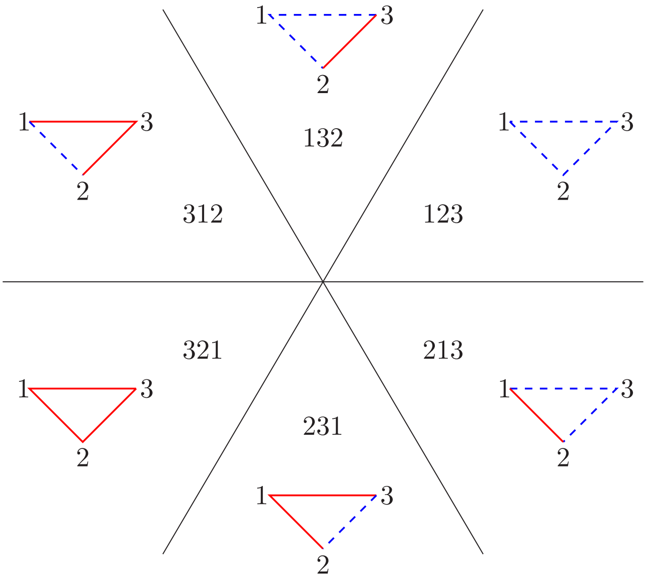

Fig. 6. The bijection from permutations in S_3 to 2-transitive colorings of $\vec{K}_3$. Edges corresponding to inversions are drawn solid red.

Example 4.7. Consider the case $n = 3$. There $3! = 6$ transitive 2-colorings of $\vec{K}_3$. The bijection with the permutations in S_3, described in Observation 4.5, is drawn in Fig. 6.

Each transitive partition with at most 2 blocks consists of two transitive 2-colorings with their colors switched, namely to equivalence classes $\{\pi, \pi w_0\}$ of permutations in S_3, where $w_0(i) := 4-i$ for every i. The reader may notice that these classes correspond, in turn, to opposite chambers of the corresponding hyperplane arrangement.

In addition, for $n = 3$, each maximal transitive partition has $3 - 1 = 2$ blocks. There are $C_2 = 2$ maximal transitive partitions; each of the corresponding equivalence classes contains a unique 321-avoiding indecomposable permutation. These are the permutations 312 and 231, where i is a descent of the permutation if and only if $(i, i+1)$ is a singleton in the corresponding transitive partition.

5. Algebras and Hilbert series

5.1. Definitions and basic properties

In this subsection we introduce two families of algebras, intimately related to transitive and Gallai colorings.

Definition 5.1. Let n and k be positive integers.

(a) The *transitive algebra* $\mathcal{T}_{n,k} := \mathcal{T}_{\vec{K}_n,k}$ is the commutative algebra over $\mathbb{C}$ generated by $\{x_{ij} : 1 \leq i < j \leq n\}$ subject to the relations

$$(x_{im} - x_{ij})(x_{im} - x_{jm}) = 0 \qquad (\forall\, i < j < m),$$
$$x_{ij}^k = 1 \qquad (\forall\, i < j).$$

(b) The *Gallai algebra* $\mathcal{G}_{n,k} := \mathcal{G}_{K_n,k}$ is the commutative algebra over $\mathbb{C}$ generated by $\{x_{ij} : 1 \le i < j \le n\}$ subject to the relations

$$(x_{ij} - x_{im})(x_{ij} - x_{jm})(x_{im} - x_{jm}) = 0 \qquad (\forall\, i < j < m),$$
$$x_{ij}^k = 1 \qquad (\forall\, i < j).$$

Theorem 5.2. *Let k be a positive integer. Then, for any positive integer $n \ge 1$,*

$$\dim \mathcal{T}_{n,k} = \#\{\text{transitive } k\text{-edge-colorings of } \vec{K}_n\}$$

and

$$\dim \mathcal{G}_{n,k} = \#\{\text{Gallai } k\text{-edge-colorings of } K_n\}.$$

Proof. Consider the set V_k of all families $\{x_e\}_{e\in E(\vec{K}_n)}$ of points in $\mathbb{C}^E$ which satisfy the defining relations of $\mathcal{T}_{n,k}$. Clearly, V_k is finite and Zariski closed in $\mathbb{C}^E$. Therefore $\mathcal{T}_{n,k} \cong \mathbb{C}[V_k]$, the algebra of complex-valued functions on V_k. We claim that V_k is in bijection with the set of transitive k-colorings of $\vec{K}_n$. Indeed, let ζ be a fixed primitive complex kth root of unity. For each transitive k-coloring ε of $\vec{K}_n$, define $x_e := \zeta^{\varepsilon(e)}$ $(\forall e \in E(\vec{K}_n))$. It is easy to verify that $\{x_e\}_{e\in E(\vec{K}_n)} \in V_k$, and that the mapping (from transitive k-colorings to elements of V_k) is a bijection. Since $\dim(\mathcal{T}_{n,k}) = \dim \mathbb{C}[V_k] = |V_k|$, we conclude that

$$\dim(\mathcal{T}_{n,k}) = \#\{\text{transitive } k\text{-edge-colorings of } \vec{K}_n\} \qquad (\forall k \ge 1).$$

The proof for the Gallai algebra $\mathcal{G}_{M,k}$ is similar.

Corollary 5.3. *For all $n > 1$,*

$$\dim \mathcal{T}_{n,2} = n! \quad \text{and} \qquad \dim \mathcal{G}_{n,2} = 2^{\binom{n}{2}}.$$

Proof. The claims follow from Theorem 5.2 combined with Observation 4.5 and the fact there is no constraint on Gallai 2-edge colorings.

Recall now the *Hilbert series* of a finitely generated algebra $\mathcal{B}$

$$\mathrm{Hilb}(\mathcal{B}, q) := \sum_{j\ge 0} (\dim(\mathcal{B}_{\le j}) - \dim(\mathcal{B}_{\le j-1}))q^j\ .$$

Here $\mathcal{B}_{\le j}$ is the degree j filtered component of $\mathcal{B}$, where the filtered degree of each generator is 1.

Observation 5.4. *For every $n > 1$,*

$$\mathrm{Hilb}(\mathcal{G}_{n.2}, q) = (1 + q)^{\binom{n}{2}}.$$

Proof. Letting $k = 2$, the relations

$$x_{ij}^2 = 1 \qquad (\forall\, i < j).$$

imply all other relations:

$$(x_{ij} - x_{im})(x_{ij} - x_{jm})(x_{im} - x_{jm}) = 0 \qquad (\forall\, i < j < m).$$

It follows that

$$\mathcal{G}_{n,2} \cong \mathbb{C}[x_{ij} :\ 1 \le i < j \le n]/\langle x_{ij}^2 :\ 1 \le i < j \le n\rangle.$$

One concludes that the set of all multilinear monomials in the x_{ij} form a basis of $\mathcal{G}_{n,2}$, compatible with the natural filtration. Proof is completed.

For the Hilbert series of the transitive algebra $\mathcal{T}_{n,2}$ see Corollary 5.12 below.

Conjecture 5.5. *Let* $[k]_j := \prod_{i=0}^{j-1} \frac{q^{k-i}-1}{q-1}$ $(k, j \geq 1)$*, the q-analogue of the falling factorial.*

(a) For all $n > 1$ *and* $k \geq 1$*,*

$$\mathrm{Hilb}(\mathcal{T}_{n,k}, q) = \sum_{j=1}^{n-1} P_{n,j}(q) \cdot [k]_j,$$

where $P_{n,1}(q), \ldots, P_{n,n-1}(q) \in \mathbb{Z}_{\geq 0}[q]$*. The leading coefficient satisfies* $P_{n,n-1}(q) = q^{\binom{n-1}{2}} C_{n-1}$*, where* C_{n-1} *is the Catalan number.*

(b) For all $n > 1$ *and* $k \geq 1$*,*

$$\mathrm{Hilb}(\mathcal{G}_{n,k}, q) = \sum_{j=1}^{n-1} Q_{n,j}(q) \cdot [k]_j,$$

where $Q_{n,1}(q), \ldots, Q_{n,n-1}(q) \in \mathbb{Z}_{\geq 0}[q]$*.*

Part (a) was checked for $n \leq 8$. Part (b) was checked for $n \leq 5$.

Remark 5.6. For all j, $P_{n,j}(1)$ is equal to the number of *transitive partitions* with j blocks of the edge set of $\vec{K}_n$, and $Q_{n,j}(1)$ is equal to the number of *Gallai partitions* with j blocks of the edge set of K_n.

A *Stirling permutation* of order n is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that, for all m, all entries between two copies of m are larger than m. The *second-order Eulerian number* $E(n, j)$ counts the number of Stirling permutations of order n with j descents, see [13].

Conjecture 5.7. *For any* $n > 1$*,*

$$Q_{n,n-1}(q) = q^{\binom{n}{2}-1} \sum_{j=0}^{n-1} E(n-1, j) q^{-j}.$$

5.2. Transitive 2-colorings and Orlik-Terao algebras

The Orlik-Terao algebra of an hyperplane arrangement was introduced in [20]. For the sake of simplicity, we will only discuss the case of the reflection hyperplane arrangement of type A_{n-1}.

The following definition is for a general simple directed graph.

Definition 5.8. Let G be a simple directed graph with vertex set $V = [n]$ and edge set $E \subseteq \{(i, j) : 1 \leq i < j \leq n\}$. The *Orlik-Terao algebra* of G, denoted $\mathcal{OT}(G)$, is the commutative algebra over $\mathbb{C}$, generated by $\{x_{i,j} = -x_{j,i} : (i, j) \in E\}$ subject to the following relations:

(a) For every directed cycle $(e_1, \ldots, e_t)$ in G,

$$\sum_{j=1}^{t} \prod_{k \neq j} x_{e_k} = 0.$$

(b) For every $e \in E$,

$$x_e^2 = 0.$$

Theorem 5.9 ([20]). *The dimension of the Orlik-Terao algebra of a simple directed graph G is equal to the number of chambers in its dual hyperplane arrangement.*

Let $\mathcal{OT}(A_{n-1})$ be the Orlik-Terao algebra of the hyperplane arrangement of type A_{n-1}, or equivalently, of the acyclic tournament $\vec{K}_n$. Note that $\mathcal{OT}(A_{n-1})$ is a graded algebra (since all its defining relations are homogeneous), while $\mathcal{T}_{n,2}$ is only a filtered algebra. For any algebra A with a generating set S, let $Gr(A)$ be the associated graded of A with respect to the filtration defined by S (where the filtered degree of any element of S is 1).

Theorem 5.10. *For every $n > 1$,*

$$\mathcal{OT}(A_{n-1}) \cong Gr(\mathcal{T}_{n,2})$$

as graded algebras.

For the proof we need the following lemma.

Lemma 5.11. *Let $B_n \subset \mathcal{T}_{n,2}$ be the set of all square-free monomials in x_{ij} not containing products of the form $x_{im}x_{jm}$ for $i < j < m$. Then B_n is a basis of $\mathcal{T}_{n,2}$, compatible with the natural filtration.*

Proof. By Definition 5.1(a), the defining relations of $\mathcal{T}_{n,2}$ are

$$(x_{im} - x_{ij})(x_{im} - x_{jm}) = 0 \qquad (i < j < m) \qquad \text{and} \qquad x_{ij}^2 = 1 \qquad (i < j).$$

Rewrite these relations as

$$x_{im}x_{jm} = x_{ij}x_{jm} - x_{ij}x_{im} + 1 \qquad (i < j < m) \qquad \text{and} \qquad x_{ij}^2 = 1 \qquad (i < j). \tag{5.1}$$

The set of all monomials in the x_{ij} is, obviously, a spanning set for $\mathcal{T}_{n,2}$. Define a weight function on monomials in the x_{ij} by

$$w\left(\prod_{i,j} x_{ij}^{m_{ij}}\right) := \sum_{i,j} m_{ij} \cdot (i + j).$$

Clearly, the weight of (each monomial in) the RHS of each of the relations in (5.1) is strictly smaller than the weight of the LHS. This leads to a (non-deterministic) straightening algorithm (see, e.g. [7, Section 2.2]), as follows: Replace an (arbitrary) occurrence of the LHS in a monomial by the RHS, recursively. Each step of this algorithm leads to a monomial, or a linear combination of three monomials, of strictly smaller weights than the original. Thus the algorithm terminates after a finite number of steps, yielding a linear combination of monomials in B_n. In addition to the weight, the *degree* of each of these monomials is less than or equal to that of the original monomial. This shows that every filtered component of $\mathcal{T}_{n,2}$ is spanned by a subset of B_n, namely, that B_n is compatible with the filtration.

It remains to show that B_n is linearly independent (and, in particular, that its apparently distinct elements are indeed distinct) in $\mathcal{T}_{n,2}$. Since it spans $\mathcal{T}_{n,2}$, it suffices to show that $|B_n| \le \dim \mathcal{T}_{n,2}$. Recall that, by Corollary 5.3, $\dim \mathcal{T}_{n,2} = n!$. We shall prove, by induction on n, that $|B_n| \le n!$. Indeed, $|B_2| = |\{1, x_{12}\}| \le 2$. For $n > 2$, each monomial in B_n is square-free and contains x_{in} for at most one index $1 \le i < n$. Therefore $B_n \subseteq \{1, x_{1n}, \dots, x_{n-1,n}\} \cdot B_{n-1}$, thus $|B_n| \le n \cdot |B_{n-1}|$. This completes the proof.

Proof of Theorem 5.10. By Definition 5.8, the Orlik-Terao algebra $\mathcal{OT}(A_{n-1})$ is generated by $\{x_{ij} = -x_{ji} \,:\, 1 \le i < j \le n\}$, subject to the relations

$$x_{ij}x_{jm} + x_{jm}x_{mi} + x_{mi}x_{ij} = 0 \qquad (i < j < m) \qquad \text{and} \qquad x_{ij}^2 = 0 \qquad (i < j). \tag{5.2}$$

Notice that, due to the relations $x_{ij} = -x_{ji}$, the relations (5.2) are equivalent to

$$(x_{im} - x_{ij})(x_{im} - x_{jm}) = 0 \qquad (i < j < m) \qquad \text{and} \qquad x_{ij}^2 = 0 \qquad (i < j).$$

On the other hand, these relations, with $x_{ij}^2 = 0$ replaced by $x_{ij}^2 = 1$, are defining for $\mathcal{T}_{n,2}$. Therefore, the assignments $x_{ij} \mapsto Gr(x_{ij})$ define a homomorphism of graded algebras

$$\mathcal{OT}(A_{n-1}) \to Gr(\mathcal{T}_{n,2}).$$

To show that this an isomorphism, apply Lemma 5.11. Indeed, since B_n is a basis of $\mathcal{T}_{n,2}$ compatible with the natural filtration, it canonically descends to a basis $Gr(B_n)$ of $Gr(\mathcal{T}_{n,2})$. In particular, each element of $Gr(B_n)$ is a monomial in $Gr(x_{ij})$, hence $Gr(\mathcal{T}_{n,2})$ is generated by $Gr(x_{ij})$ and the above homomorphism is surjective. It is also injective because $\dim \mathcal{OT}(A_{n-1}) = \dim Gr(\mathcal{T}_{n,2}) = n!$.

Corollary 5.12. *For every $n > 1$,*

$$\mathrm{Hilb}(\mathcal{OT}(A_{n-1}), q) = \mathrm{Hilb}(\mathcal{T}_{n,2}, q) = \sum_{k=0}^{n-1} s(n, n-k)\, q^k,$$

where $s(n, k)$ are the unsigned Stirling numbers of the first kind.

Proof. The first equality follows from Theorem 5.10. By Lemma 5.11,

$$\mathrm{Hilb}(\mathcal{T}_{n,2}, q) = \sum_{m \in B_n} q^{\deg(m)} = \prod_{k=0}^{n-1} (1 + kq) = \sum_{k=1}^{n} s(n, n-k)\, q^k,$$

as claimed.

Remark 5.13. Theorem 5.10 may be generalized to any acyclic directed graph whose underlying undirected graph is chordal.

Further connections to the Orlik-Terao algebra will be discussed elsewhere.

6. Concluding remarks

Noting that the edges of the acyclic tournament are in bijection with the positive roots of the reflection group of type A, it is desired to generalize Gallai and transitive colorings, as well as corresponding algebras, to other root systems. A closely related task is to generalize the results in this paper to other families of graphs.

We intend to address these tasks in another paper. We will generalize there the notions of Gallai and transitive colorings to matroids and oriented matroids, respectively, yielding interesting analogues of the main results presented in the current paper. For example, the dimension of the transitive algebra of any (oriented) matroid, with any number of colors, is equal to the corresponding number of Gallai (respectively, transitive) colorings. The number of transitive 2-coloring of any directed graph is equal to the number of chambers in the corresponding graphic hyperplane arrangement. Furthermore, the number of transitive 2-coloring of any finite Coxeter root system is equal to the number of elements in the corresponding Coxeter group. The relations between transitive algebras of directed graphs and Orlik-Terao, Gelfand-Varchenko and other algebras deserve further study.

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