VALUATIONS, BIJECTIONS, AND BASES

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ABSTRACT. The aim of this paper is to build a theory of commutative and noncommutative *injective* valuations of various algebras. The targets of our valuations are (well-)ordered commutative and noncommutative (partial or entire) semigroups including any sub-semigroups of the free monoid F_n on n generators and various quotients. In the case when the (partial) valuation semigroup is finitely generated, we construct a generalization of the standard monomial bases for the so-valued algebra, which seems to be new in noncommutative case. Quite remarkably, for any pair of well-ordered valuations one has canonical bijections between the valuation semigroups, which serve as analogs of the celebrated Jordan-Hölder correspondences and these bijections are "almost" homomorphisms of the involved (partial and entire) semigroups.

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1. INTRODUCTION

The aim of this paper is twofold:

• To initiate and systematically study *injective* valuations of algebras into (partial) semigroups.

• For pairs of such injective valuations of a given algebra establish and study a bijection between their images, which we refer to as *Jordan-Hölder* (JH) bijection.

Recall (cf. [35] Ch. 18) that a valuation ν on a k-vector space V is a map $V \setminus \{0\} \to P$ where P is a totally ordered set with an order \preceq such that $\nu(av) = \nu(v)$ for all $v \in V$, $a \in \mathbb{k}^{\times}$ and

(1.1)
$$\nu(u+v) \preceq \max(\nu(u), \nu(v))$$

whenever $u + v \neq 0$ (in some papers max is replaced by min and \leq by \geq). It is immediate that (1.1) becomes an equality if $\nu(u) \neq \nu(v)$.

In addition, if V is a k-algebra, we require P to be a (partial) semigroup (see e.g. [1]) with the operation \circ (see Section 2 for details) and

• If $\nu(u) \circ \nu(v)$ is defined in P for some $u, v \in V \setminus \{0\}$ then $uv \neq 0$ and

$$\nu(uv) = \nu(u) \circ \nu(v)$$

In particular, if (P, \circ) is an entire (rather than partial) semigroup then the algebra V is necessarily a domain, and conversely if V is not a domain, then (P, \circ) is necessarily a partial semigroup.

Also we impose the following condition on the order in P: if $c, c', d, d' \in P$ satisfy inequalities $c \leq d, c' \leq d'$ then

$$(1.2) c \circ c' \preceq d \circ d'$$

provided that $c \circ c', d \circ d' \in P$.

If P was an entire semigroup then the axiom (1.2) would follow from weaker ones: $a \leq b$ implies $c \circ a \leq c \circ b$ and $a \circ b \leq b \circ c$ (for ordered entire semigroups one can look in [7]). However, in partial semigroups, (1.2) is not always derived (see, e.g., Example 2.47).

Note that this axiomatic is rather strong: if P is an (entire) ordered monoid then for any non-unital invertible element c, the unit is strictly between c and c^{-1} .

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Additionally, we say that the order \prec on a partial semigroup P satisfies the *strict* property if $c \prec d$ or $c' \prec d'$ implies that $c \circ c' \prec d \circ d'$ in (1.2).

One can show (see e.g., Remark 2.25) that for any valuation ν of V the image $P_{\nu} := \nu(V \setminus \{0\})$ is always a (partial) subsemigroup of P.

Following [19, 22] we say that a valuation ν on (a vector space or an algebra) V is *injective* if there exists a basis **B** of V such that $\nu|_{\mathbf{B}}$ is an injective map $\mathbf{B} \hookrightarrow P$ (we refer to such a basis as *adapted* to ν).

Note also that for some finitely generated commutative domains A there is no injective valuation to an entire semigroup (see Lemma 3.32, Theorem 3.33).

For contrast, we claim that injective valuations to reasonable partial semigroups always exist (in Section 2 we construct a class of *coideal* partial semigroups, see Definition 2.1, which will provide such reasonable valuations).

Theorem 1.1 (Theorem 2.38). Any finitely generated commutative algebra \mathcal{A} admits an injective valuation onto a coideal partial subsemigroup of $\mathbb{Z}_{\geq 0}^m$. Thereby, the order in the coideal partial subsemigroup satisfies the strict property.

In view of above, this means that even finite-dimensional algebras are "domains." Our next result extends Theorem 1.1 to the noncommutative case.

Theorem 1.2 (Theorem 2.38). Any finitely generated algebra \mathcal{A} admits an injective valuation onto a coideal partial subsemigroup of F_m (the free monoid on m generators) with respect to (strict) deglex order (Lemma 2.13 and Remark 2.14).

We can refine Theorem 1.1 by employing Gröbner basis-like approach (cf. [20]). Nameley, consider a commutative algebra \mathcal{A} generated by a finite set X and a finite set **S** of monomials in X. We say that a monomial b in X is *standard* if b does not contain elements of **S** as submonomials. If the set **B** of all standard monomials is a basis of \mathcal{A} , it is referred as a *standard monomial* one.

The following is a restatement of a well-known result (asserting that any finitely generated commutative algebra admits a standard monomial basis) in terms of injective valuation into partial semigroups.

Theorem 1.3. Let \mathcal{A} be a finitely generated commutative algebra. Then

(a) (Proposition 2.45) Any standard monomial basis **B** defines a structure of an ordered partial semigroup on $\mathbb{Z}_{\geq 0}^X$ and an injective valuation $\nu : \mathcal{A} \setminus \{0\} \to \mathbb{Z}_{\geq 0}^X$ such that **B** is adapted to ν .

(b) (Theorem 2.36). Conversely, let $\nu : \mathcal{A} \setminus \{0\} \to \mathbb{Z}_{\geq 0}^m$ be an injective valuation such that $\nu(\mathcal{A} \setminus \{0\}) = \mathbb{Z}_{\geq 0}^m \setminus [\mathbf{S}]$, where $[\mathbf{S}] := \bigcup_{s \in \mathbf{S}} (s + \mathbb{Z}_{\geq 0}^m)$ for some finite subset \mathbf{S} of $\mathbb{Z}_{\geq 0}^m$. Then there exists a standard monomial basis \mathbf{B} of \mathcal{A} adapted to ν .

A notable example is the (generalized) Stanley-Reisner algebra $\mathcal{A}_{\mathbf{S}}$ defined for any finite subset $\mathbf{S} \subset \mathbb{Z}_{\geq 0}^m$ as follows. $\mathcal{A}_{\mathbf{S}}$ is generated by $X = \{x_1, \ldots, x_m\}$ and has a presentation $x^s = 0$ for all $s \in \mathbf{S}$. Clearly, the set $x^c, c \in \mathbb{Z}_{\geq 0}^n \setminus [\mathbf{S}]$ form a standard monomial basis **B**.

The complement $\mathbb{Z}_{\geq 0}^n \setminus [\mathbf{S}]$ has a structure of a partial semigroup via: for $c, c' \in \mathbb{Z}_{\geq 0}^n \setminus [\mathbf{S}]$ their sum c + c' is defined unless it belongs to $[\mathbf{S}]$, which is an ideal of $\mathbb{Z}_{\geq 0}^n$. Then the lexicographic (or deglex) order on $\mathbb{Z}_{>0}^n$ gives rise to a (unique) injective valuation $\nu_{\mathbf{S}}$ on $\mathcal{A}_{\mathbf{S}}$ via

$$\nu_{\mathbf{S}}(x^a) = a$$

for all $a \in \mathbb{Z}_{\geq 0}^n \setminus [\mathbf{S}]$. Clearly, **B** is adapted to $\nu_{\mathbf{S}}$.

A theory generalizing Buchberger's algorithm producing Gröbner bases of ideals was developed for certain classes of noncommutative algebras (see e.g. [24, 27]). In our version we construct bases for injectively valued (noncommutative) algebras as follows (some difference in approach is that we focus mainly on bases of quotient algebras rather than of ideals).

First, we will replace $\mathbb{Z}_{\geq 0}^n \setminus [\mathbf{S}]$ by a (not necessarily commutative) partial semigroup (P, \circ) , second, fix an injective valuation $\nu : \mathcal{A} \setminus \{0\} \to P$ (and denote by P_{ν} the image of ν , which is automatically a partial sub-semigroup of P), and, third, construct a linearly independent set $\mathbf{B} := \{b_c, c \in P\}$ of certain monomials in \mathcal{A} which is *adapted* to ν , i.e., $\nu(\mathbf{B}) = P_{\nu}$. It is critical for the construction that \mathbf{B} is a basis of \mathcal{A} which is guaranteed for any adapted set whenever P is well-ordered, see Corollary 4.15 (the assumption of well-orderness seems to be indispensible, see Remark 4.16).

More precisely, let P be a partial semigroup and P_0 be a generating set of P. A factorization of $c \in P$ is any sequence $\vec{c} = (c_1, \ldots, c_\ell) \in P_0^\ell$ such that

$$c = c_1 \circ \cdots \circ c_\ell$$

in P; denote by F(c) the set of all factorizations of c of the shortest length (to be denoted $\ell(c)$). Now suppose that P is well-ordered. We refer to $\vec{c} \in R(c)$ as standard if it is the smallest in the lexicographic order on R(c) and denote it by \vec{c}^{st} .

Then fix an injective valuation $\nu : \mathcal{A} \setminus \{0\} \to P$, a generating set of $(P_{\nu})_0$, and choose $x_{c_0} \in \mathcal{A} \setminus \{0\}$ for all $c_0 \in (P_{\nu})_0$ such that $\nu(x_{c_0}) = c_0$ for all $c \in (P_{\nu})_0$. For any $c \in P_{\nu}$ and any $\vec{c} = (c_1, \ldots, c_\ell) \in D(c)$ denote $x_{\vec{c}} := x_{c_1} \cdots x_{c_\ell}$.

Theorem 1.4 (Theorem 2.36). In assumptions as above, the set of all $x_c := x_{\vec{c}^{st}}$, $c \in P$ is a basis of \mathcal{A} adapted to ν .

This is a generalization of Theorem 1.3(b). Note that unlike in the commutative case, P can be any small category, in particular, with countably many arrows. In that case, its generating set P_0 is a quiver (possibly with relations) so that P is the set of all paths in P_0 (cf. Examples 2.16, 2.17).

A large class of injective valuations on \mathcal{A} can be constructed by restriction of a given injective valuation $\hat{\nu}$ on a larger algebra \mathcal{B} for various injective homomorphisms $\mathbf{j} : \mathcal{A} \hookrightarrow \mathcal{B}$ by $\nu := \hat{\nu} \circ \mathbf{j}$ with some care. We say that a valuation $\nu : V \setminus \{0\} \to P$ is well-ordered if its range P_{ν} is a well-ordered set.

Lemma 1.5. (cf. Lemma 3.13.) Suppose that $\hat{\nu}$ is an injective well-ordered valuation of an algebra \mathcal{B} . Then the restriction $\hat{\nu}$ to any subalgebra \mathcal{A} is also a well-ordered injective valuation.

In fact, for any ordered partial semigroup P we can construct a "default" invective valuation onto P as follows.

First, recall that for any partial semigroup P and any field \Bbbk we equip the linear span $\Bbbk P = \bigoplus_{c \in P} \Bbbk \cdot [c]$ with an associative algebra structure via

$$[c][c'] = \begin{cases} [c \circ c'] & \text{if } c \circ c' \text{ is defined in } P \\ 0 & \text{otherwise} \end{cases}$$

for all $c, c' \in P$. In particular, if P is a group then $\Bbbk P$ is the group algebra of P. In the case when $P = \mathbb{Z}_{>0} \setminus [\mathbf{S}]$, $\Bbbk P$ is the Stanley-Reisner algebra $\mathcal{A}_{\mathbf{S}}$.

Second, suppose that P is ordered and \Bbbk is of characteristic 0. Clearly, the assignments $[c] \mapsto c, c \in P$ define an injective valuation $\nu_P : \Bbbk P \setminus \{0\} \to P$ (Definition 2.28). We refer to ν_P as the *tautological valuation* of $\Bbbk P$. For instance, if $P = \mathbb{Z}_{\geq 0}^n$, i.e., $\Bbbk P = \Bbbk[x_1, \ldots, x_n]$ and the order is lexicographic, then ν_P is just the usual leading degree valuation of $\Bbbk[x_1, \ldots, x_n]$.

Using this, we extend Theorems 1.1, 1.2 as follows.

Theorem 1.6 (Theorem 2.38). Let P be a well-ordered partial semigroup and I be a two-sided ideal of &P.

i) Then $J := \nu_P(I \setminus \{0\})$ is an ideal in P (i.e., it is invariant under left and right compositions), thus $P \setminus J$ is a well-ordered (coideal) partial semigroup. Moreover, the assignments $\nu(a + I) := \min_{j \in I} \nu_P(a + j)$ define an injective well-ordered valuation $\nu : A/I \setminus \{0\} \twoheadrightarrow P \setminus J$.

ii) Then $\mathbf{B} := \{ [c] + I : c \in P \setminus J \}$ is a standard monomial basis of $\mathbb{k}P/I$ with respect to ν .

Thus, combining constructions of Theorem 1.6 with Lemmas 1.5 and Definition 2.28, we obtain a large class of injective valuations of commutative and noncommutative algebras into various entire and partial semigroups.

The following are examples of partial semigroups involved in some valuations.

Example 1.7. For any Coxeter group $W = \langle s_i, i \in I \rangle$ its Nil-Coxeter monoid W_0 is a partial monoid generated by $s_i, i \in I$ and has a presentation $s_i \circ s_i$ is undefined and $\underbrace{s_i \circ s_j \circ s_i...}_{m_{ij}} = \underbrace{s_j \circ s_i \circ s_j...}_{m_{ij}}$ It is well-known that $W_0 = W$ as a set and its

multiplication table is given by $w \circ w' = ww'$ iff $\ell(ww') = \ell(w) + \ell(w')$, where ℓ denotes the length of words in generators, otherwise, $w \circ w'$ is not defined. For example, if $W = S_3$, then $W_0 = \{1, s_1, s_2, s_1 \circ s_2, s_2 \circ s_1, s_1 \circ s_2 \circ s_1 = s_2 \circ s_1 \circ s_2\}$.

More generally, let C be a monoid, (D, \bullet) is an ordered monoid, and $\ell : C \to D$ be a map such that $\ell(cc') \leq \ell(c) \bullet \ell(c')$. This defines a partial monoid structure on C via $c \circ c' = cc'$ iff $\ell(cc') = \ell(c) \bullet \ell(c')$.

In Example 2.18 we construct an order on W_0 when |I| = 2 (and expect that such an order does not exist if |I| > 2).

Example 1.8. It turns out that one can construct (injective) valuations from any finite-dimensional algebra \mathcal{A} to the groupoid (e.g., a partial semigroup) M_n , which is the set of all pairs $(i, j), i, j \in \{1, \ldots, n\}$ with the composition $(ij) \circ (jk) = (ik)$ (see Example 2.47 for details). Fix a total ordering on M_n (compatible with the composition):

$$(1,n) < \ldots < (1,1) < (2,n) < \ldots < (2,1) < \cdots < (n,n) < \ldots < (n,1)$$

(i.e., the lexicographic ordering on pairs (i, n-j)) and define a valuation $\nu_0 : \mathbb{k}M_n = Mat_n(\mathbb{k}) \setminus \{0\} \to M_n$, that is, $\nu_0(e_{ij}) = (ij)$ where e_{ij} is the (ij)-th matrix unit.

Given a finite-dimensional algebra k-algebra \mathcal{A} , any faithful *n*-dimensional representation ρ of \mathcal{A} defines an injective valuation $\nu_0 \circ \rho : \mathcal{A} \setminus \{0\} \to M_n$ (This, in particular, applies to $\Bbbk G$ for any finite group G, even though G has no compatible total ordering, see Section 2 for details).

By varying total orderings on M_n compatible with the operation, will give a large class of new injective valuations on A. In Section 2 we construct some such orderings (the symmetric group S_n permutes them) and pose a problem of their classification.

One can show that any finitely generated partial semigroup P can be covered by a coideal of an entire semigroup \widehat{P} (Proposition 2.11). For instance, we can take \widehat{M}_n to be generated by all $\widehat{(ij)}$, $i, j = 1, \ldots, n$ subject to $\widehat{(ij)} \circ \widehat{(jk)} = \widehat{(ik)}$. However, in a contrast with free coideal semigroup in Proposition 2.11, we do not know whether (an appropriate coideal of) \widehat{M}_n is ordered in a compatible way. It would be interesting to classify all ordered partial semigroups P which admit such a lifting to (coideals of) ordered entire semigroups \widehat{P} .

On the other hand, one can apply Theorem 1.6 to the free semigroup \widehat{P} freely generated by (ij), $1 \leq i, j \leq n$, tautological valuation ν_0 on $\mathbb{k}\widehat{P}$, and the ideal

$$I := \langle \{e_{ij}e_{pq} : p \neq q\} \cup \{e_{ij}e_{jl} - e_{il}\} \rangle \subset \mathbb{k}\widehat{P}.$$

Then one obtains an injective valuation $\nu : (\Bbbk \widehat{P}/I) \setminus \{0\} = A \setminus \{0\} \twoheadrightarrow \widehat{P} \setminus J$, where $J = \nu_0(I \setminus \{0\})$ is the corresponding ideal of \widehat{P} . By definition, $\widehat{P} \setminus J$ is a partial semigroup consisting of n^2 elements $\{(ij) : 1 \leq i, j \leq n\}$ such that no composition of them is defined. Thus, $\widehat{P} \setminus J$ differs from M_n .

Returning to general partial semigroups, we can construct new valuations from P-filtered algebras and vice versa. We say that \mathcal{A} is filtered by an ordered (partial) semigroup P if $\mathcal{A} = \sum_{c \in P} \mathcal{A}_{\leq c}, \ \mathcal{A}_{\leq c'} \subset \mathcal{A}_{\leq c}$ whenever $c' \leq c$, and $\mathcal{A}_{\leq c} \mathcal{A}_{\leq c'} \subset \mathcal{A}_{\leq coc'}$ whenever $c \circ c'$ is defined in P.

Proposition 1.9 (Proposition 2.46). (a) For any valuation $\nu : \mathcal{A} \setminus \{0\} \to P$ the subsets $\mathcal{A}_{\prec c} := \{x \in \mathcal{A} \setminus \{0\} : \nu(x) \preceq c\}$ define a P_{ν} -filtration of \mathcal{A} .

(b) Conversely, for a well-ordered P, given a P-filtration $\mathcal{A}_{\preceq \bullet}$ of \mathcal{A} , setting $\nu(x) := \min\{c : x \in \mathcal{A}_{\prec c}\}$ for any nonzero $x \in \mathcal{A}$, defines a P-valuation of \mathcal{A} .

In particular, if P is well-ordered, by the standard procedure gr this defines a P-graded algebra (recall that \mathcal{A} is graded by a (partial) semigroup P if $\mathcal{A} = \bigoplus_{c \in P} \mathcal{A}_c$ so that $\mathcal{A}_c \mathcal{A}_{c'} \subset \mathcal{A}_{coc'}$ whenever $c \circ c'$ is defined in P). Thus, Proposition 1.9 applied to $\mathcal{A} := \Bbbk P$ recovers the tautological valuation $\nu_P : \Bbbk P \setminus \{0\} \to P$.

It turns out that having a pair of injective valuations ν, ν' of an algebra (or even a vector space) \mathcal{A} to partial semigroups P and P' gives an interesting information about both the algebra and the pair $P_{\nu} = \nu(\mathcal{A} \setminus \{0\}), P'_{\nu'} = \nu'(\mathcal{A} \setminus \{0\}).$

Theorem 1.10. [Theorem 4.24] Suppose that $\nu : \mathcal{A} \setminus \{0\} \to P$ and $\nu' : \mathcal{A} \setminus \{0\} \to P'$ are injective valuations and P and P' are well-ordered. Then the assignments $a \mapsto \min\{\nu'(\nu^{-1}(a))\}$ define a bijection $\mathbf{K}_{\nu',\nu} : P_{\nu} \xrightarrow{\sim} P'_{\nu'}$. Moreover, $\mathbf{K}_{\nu',\nu}^{-1} = \mathbf{K}_{\nu,\nu'}$. We call $\mathbf{K}_{\nu',\nu}$ a Jordan-Hölder bijection (JH bijection). It can be reformulated in terms of the generalized Jordan-Hölder correspondence on matroids developed by Abels in 1991 [8], cf. also Remark 4.25.

In addition, under the same assumptions there is a common adapted basis for both valuations.

Theorem 1.11. [Theorem 4.24] Under assumptions of Theorem 1.10, there exists a basis $\mathbf{B}_{\nu,\nu'}$ of \mathcal{A} adapted to both ν and ν' and such that $\mathbf{K}_{\nu',\nu}(\nu(b)) = \nu'(b)$ for all $b \in \mathbf{B}_{\nu,\nu'}$.

We sometimes refer to such a basis as an *JH*-basis of \mathcal{A} .

The following result asserts that any JH-bijection is almost a homomorphism of partial semigroups.

Theorem 1.12. [Proposition 3.83] Under assumptions of Theorem 1.10 the JHbijection $\mathbf{K} := \mathbf{K}_{\nu',\nu}$ is sub-multiplicative in the following sense:

$$\mathbf{K}(c \circ c') \preceq \mathbf{K}(c) \circ \mathbf{K}(c')$$

whenever $c \circ c'$ and $\mathbf{K}(c) \circ \mathbf{K}(c')$ are defined in P and P', respectively.

This implies that $\mathbf{K}_{\nu,\nu'} = \mathbf{K}_{\nu',\nu}^{-1}$ is also sub-multiplicative, which will, in particular, allow to stratify both P_{ν} and $P'_{\nu'}$ into "multiplicativity domains" (see Examples 3.88, 3.89, 3.90 and 3.91).

In fact, JH bijection as well as any sub-multiplicative maps $P \to Q$ can be viewed as "homomorphisms" of partial semigroups in the following sense.

We say that a map $f: P \to Q$ is a *partial* homomorphism of partial semigroups if the operation \circ_f determined by the requirement: $c \circ_f c' = c \circ c'$ whenever the latter one is defined and $f(c \circ c') = f(c) \circ f(c')$, gives a structure of partial semigroup on P (In particular, f becomes a homomorphism of partial semigroups $(P, \circ_f) \to Q$).

Partial homomorphisms are abound in "nature", for instance, Proposition 2.35 asserts that for any valuation $\nu : \mathbb{k}P \setminus \{0\} \to Q$, the assignments $c \mapsto \nu([c])$ define a partial homomorphism $P \to Q$.

Proposition 1.13 (Proposition 2.10). Let P, Q be ordered partial semigroups and $f: P \to Q$ be sub-multiplicative. Suppose that the order on Q has a strict property (see Definition 2.9). Then f is a partial homomorphism.

Thus, the JH bijection $\mathbf{K} : P_{\nu} \rightarrow P'_{\nu'}$ is a partial isomorphism (whenever the orders on P_{ν} and $P'_{\nu'}$ have strict property) and is an "honest," rather than partial, isomorphism $(P_{\nu}, \circ_{\mathbf{K}}) \rightarrow (P'_{\nu'}, \circ_{\mathbf{K}^{-1}})$.

In fact, if $\nu' = f \circ \nu$ for some isomorphism $f : P \xrightarrow{\sim} P'$ of ordered partial semigroups, then $\mathbf{K} = f|_{P_{\nu}}$ is also an isomorphism of ordered partial semigroups $P_{\nu} \xrightarrow{\sim} P'_{\nu'}$.

2. Injective valuations on algebras with zero divisors

In this section we extend the concept of valuations to algebras with (possibly) zero divisors.

2.1. Partial semigroups.

Definition 2.1. We say that (P, \circ) is a (not necessary commutative) partial semigroup if for some elements $c, d \in P$ their composition $c \circ d \in P$ is defined (in this case we say that c, d are composable) satisfying the following property (of associativity). If two elements $c \circ c', (c \circ c') \circ c'' \in P$ are defined then $c' \circ c'', c \circ (c' \circ c'') \in P$ are also defined and $(c \circ c') \circ c'' = c \circ (c' \circ c'')$. Vice versa also holds (this is equivalent to that $\hat{P} = P \sqcup \{\mathbf{0}\}$ is an entire semigroup with the requirement that $\mathbf{0} \cdot P = P \cdot \mathbf{0} = \{\mathbf{0}\}$).

We say that a subset $J \subset P$ is an ideal in P if for any elements $c \in P, d \in J$ it holds $c \circ d \in J$, provided that $c \circ d \in P$, and similarly, $d \circ c \in J$, provided that $d \circ c \in P$. Then $P \setminus J$ is a partial semigroup. If M is a semigroup and $J \subset M$ is an ideal we call $M \setminus J$ a coideal partial semigroup.

We assume that P is endowed with a linear order \prec satisfying the following property. For $c, c', d, d' \in P$ the inequalities $c \preceq d, c' \preceq d$ imply that $c \circ c' \preceq d \circ d'$ (sometimes we consider partial semigroups without apriori linear order which we introduce afterwards).

A mapping $f : P \to Q$ of partial semigroups P, Q preserving the orders is called a homomorphism if for any $c, d \in P$ it holds that $c \circ d$ is defined iff $f(c) \circ f(d)$ is defined as well, and in this case the equality $f(c \circ d) = f(c) \circ f(d)$ is true.

Remark 2.2. i) For a partial semigroup P we call $P_0 \subset P$ a subsemigroup of P if for any $c, d \in P_0$ it holds $c \circ d \in P_0$ whenever $c \circ d \in P$. Any subset $R \subset P$ generates the uniquely defined minimal subsemigroup $\overline{R} \subset P$ such that $R \subset \overline{R}$.

ii) If $f: P \to Q$ is a homomorphism of partial semigroups then the image f(P) is a subsemigroup of Q.

For any partial semigroup P we say that a subset $S \subset P \times P$ is *admissible* if it defines a partial semigroup on P. The following is immediate.

Lemma 2.3. The intersection of any family of admissible subsets of P is admissible

This defines an admissible closure of any $X \subset P \times P$ to be the intersection of all admissible subsets containing X. This means that any $X \subset P \times P$ defines a canonical partial semigroup on P so that pairs $(a, b) \in X$ are composable which we denote by P_X .

Definition 2.4. We say that a mapping $f : P \to Q$ of partial semigroups P, Q is a *partial* homomorphism if the set S_f of all pairs $(a, b) \in P \times P$ such that a, b are composable in P and f(a), f(b) are composable in Q is admissible, in addition we require that $f(a \circ b) = f(a) \circ f(b)$ for $(a, b) \in S_f$.

Remark 2.5. If $f : P \to Q$ is a partial homomorphism of partial semigroups then $f : P_{S_f} \to Q$ is a homomorphism (see Definition 2.1).

Remark 2.6. If \prec is a linear order on a semigroup M then it induces a linear order on a coideal partial semigroup $P \subset M$.

The following is immediate.

Lemma 2.7. Let $f : P \rightarrow Q$ be an (ordered) epimorphism of (ordered) partial semigroups. Suppose that the fibers of f are well-ordered. Then the assignments

 $x \mapsto \min\{f^{-1}(x)\}$ define a section $f^* : Q \hookrightarrow P$ of f. In turn, this defines a vector space decomposition $\mathbb{k}P = \mathbb{k}f^*(Q) \oplus I$ where I is the kernel of the canonical homomorphism $\mathbb{k}P \to \mathbb{k}Q$. Also, all elements $y - f^*(x)$, $y \in f^{-1}(x)$, $y \neq f^*(x)$ form a basis \mathbf{B}_I of I.

One can easily verify the following proposition.

Proposition 2.8. Let P be an (ordered) partial semigroup and Q be a partial semigroup (without apriori an order). Suppose that there is an order on Q viewed as a set and let f be a surjective order-preserving map $P \rightarrow Q$. Suppose that f is also a homomorphism of partial semigroups. Then Q is an ordered partial semigroup.

Definition 2.9. We say that the order \prec in a partial semigroup Q fulfills a *strict* property if for any elements $a, b, c, d \in Q$ such that $a \prec b, c \preceq d$ it holds $a \circ c \prec b \circ d$ (respectively, $c \circ a \prec d \circ b$), provided that $a \circ c, b \circ d \in Q$ (respectively, provided that $c \circ a, d \circ b \in Q$), cf. Definition 2.1.

Given ordered partial semigroups P and Q, we say that a map $f : P \to Q$ is sub-multiplicative if $f(c \circ c') \preceq f(c) \circ f(c')$ whenever c, c' are composable in P and f(c), f(c') are composable in Q.

Proposition 2.10. Let P, Q be ordered partial semigroups and $f : P \to Q$ be submultiplicative. Suppose that the order on Q has a strict property. Then f is a partial homomorphism.

Proof. It suffices to show that that S_f is admissible. Indeed, suppose that $(c, c') \in S_f$ and $(c \circ c', c'') \in S_f$. Then

$$f(c \circ c') = f(c) \circ f(c'), \ f((c \circ c') \circ c'') = f(c \circ c') \circ f(c'') = f(c) \circ f(c') \circ f(c'').$$

Thus, c', c'' and $c, c' \circ c''$ are composable, as well as f(c'), f(c'') and $f(c), f(c') \circ f(c'')$ are composable. Therefore

$$f((c \circ c') \circ c'') = f(c \circ (c' \circ c'')) \preceq f(c) \circ f(c' \circ c'') \preceq f(c) \circ f(c') \circ f(c'').$$

Since both the latter inequalities are actually, equalities, we get that $(c, c' \circ c'') \in S_f$, finally the strict property of Q implies that $f(c' \circ c'') = f(c') \circ f(c'')$, thus $(c', c'') \in S_f$. The admissibility of S_f is established.

Denote by F_k the free semigroup generated freely by k elements.

The following result provides a converse statement to Proposition 2.8 under the strict property of the order.

Theorem 2.11. Let Q be a partial semigroup generated by k elements u_1, \ldots, u_k and let \prec be an order on Q satisfying the strict property. Then there exists a coideal partial semigroup $F \subset F_k$ and an epimorphism $f : F \twoheadrightarrow Q$ of (ordered) partial semigroups.

Proof. By definition, $\mathbb{k}F_k = \mathbb{k} < u_1, \ldots, u_k >$ and one has a epimorphism of algebras $\overline{f} : \mathbb{k}F_k \to \mathbb{k}Q$ which sends any monomial to element of Q or 0. Denote by F the set of all elements of F_k whose image is not 0. Clearly, F is a coideal of F_k .

This induces a natural epimorphism of partial semigroup $f: F \rightarrow Q$.

Introduce an order \triangleleft on F as follows. We say that $F \ni v := u_{i_1} \circ \cdots \circ u_{i_m} \triangleleft w := u_{j_1} \circ \cdots \circ u_{j_p} \in F$ iff either

• $f(v) \prec f(w)$, either

• f(v) = f(w) and m < p, or

• f(v) = f(w), m = p and the word v is less than w in the lexicographical order (defined on u_1, \ldots, u_k in an arbitrary way). Denote the length l(v) := m.

We claim that the epimorphism f fulfills Definition 2.1. Indeed, let $v, w, v_1, w_1 \in F$; $v \leq w, v_1 \leq w_1; v \circ v_1, w \circ w_1 \in F$. If either $f(v) \prec f(w)$ or $f(v_1) \prec f(w_1)$ then $f(v \circ v_1) = f(v) \circ f(v_1) \prec f(w) \circ f(w_1) = f(w \circ w_1)$ due to the assumption in the theorem. If $f(v) = f(w), f(v_1) = f(w_1)$ and either l(v) < l(w) or $l(v_1) < l(w_1)$ then $l(v \circ v_1) < l(w \circ w_1)$. Finally, if $l(v) = l(w), l(v_1) = l(w_1)$ then the word $v \circ v_1$ is less than $w \circ w_1$ in the lexicographical order, unless $v = w, v_1 = w_1$.

Using an argument similar to that of the proof of Theorem 2.11, we establish the following.

Lemma 2.12. Let P_1 and P_2 be any partial semigroups. Then

(a) their direct product $P_1 \times P_2$ is also a (partial) semigroup. Moreover, if P_1 and P_2 are ordered so that the ordering on P_1 fulfills the strict property then $P_1 \times P_2$ is ordered as well via $(p_1, p_2) \preceq (p'_1, p'_2)$ iff either $p_1 \prec p'_1$ or $p_1 = p'_1$ and $p_2 \preceq p'_2$.

(b) Moreover, if P_2 also fulfills the strict property then $P_1 \times P_2$ fulfills the strict property as well.

We say that a function $\ell : P \to \mathbb{Z}_{>0}$ is *length* if $\ell(c \circ c') = \ell(c) + \ell(c')$ for all composable $c, c' \in P$. We say that (P, ℓ) is a graded partial semigroup if ℓ is a length on P (sometimes we omit ℓ).

We say that an order \prec on a graded partial semigroup is *length compatible* if $\ell(c) < \ell(c')$ implies that $c \prec c'$.

The following is immediate

Lemma 2.13. (Generalized deglex) Let P be a free semigroup freely generated by a set X. Then

(a) Any function $f : X \to \mathbb{Z}_{>0}$ defines (unique) length function on P and vice versa.

(b) For any length function $\ell : P \to \mathbb{Z}_{>0}$ any total order \prec of X such that $\ell(x) < \ell(y)$ implies $x \prec y$, determines a unique length compatible order (fulfilling the strict property) on P such that $xa \prec yb$ whenever $x, y \in X$, $\ell(xa) = \ell(yb)$ and $x \prec y$ or x = y and $a \prec b$.

(c) If for any $m \in \mathbb{Z}_{>0}$ the preimage $f^{-1}(m) \subset X$ is finite then \prec is a well ordering.

Remark 2.14. If $\ell(x) = 1$ for any generator of P (e.g., when $P = F_+^n$ or $P = \mathbb{Z}_{\geq 0}^n$) this becomes an ordinary deglex on P.

Denote by $P_1 * P_2 = P_2 * P_1$ the free product of (partial) semigroups P_1 and P_2 .

Suppose that P_1 and P_2 are ordered. We say that that an order on the partial semigroup $P_1 * P_2$ is *compatible* if $p \prec p'$ implies $p * c \prec p' * c$ and $c * p \prec c * p'$ for any $c \in P_1 * P_2$ and any $p, p' \in P_i$, i = 1, 2.

If P_1 and P_2 are entire then there are several constructions of the order on $P_1 * P_2$ (see e.g. [6]). In particular, for an ideal J_1 in P_1 and an ideal J_2 of P_2 , any such order restricts to an order on the free product $(P_1 \setminus J_1) * (P_2 \setminus J_2)$ of coideal partial semigroups.

The following immediate fact gives another construction of an order on free products of graded partial semigroups (making use of Lemma 2.13).

Lemma 2.15. Let P_1 and P_2 be any graded partial semigroups. Then their free product $P_1 * P_2 = P_2 * P_1$ is also a graded (partial) semigroup.

Example 2.16. i) For a monoid $M := \mathbb{Z}_{\geq 0}^n = \{x_1^{i_1} \cdots x_n^{i_n} : i_1, \ldots, i_n \geq 0\}$ and a family of monomials u_1, \ldots, u_s in the variables x_1, \ldots, x_n , the set of monomials not dividing any of u_1, \ldots, u_s , forms a coideal partial monoid $P(u_1, \ldots, u_s)$. Then $P(u_1, \ldots, u_s)$ coincides with the complement to the monomial ideal $J(u_1, \ldots, u_s) := \bigcup_{1 \leq j \leq s} (u_j + \mathbb{Z}_{\geq 0}^n).$

ii) For a free monoid $M_n := \langle x_1, \ldots, x_n \rangle$ consider an ideal $J := \langle x_i x_j x_k : 1 \leq i, j, k \leq n \rangle$. We define a well-ordering \prec on M as follows. For a pair of words $u, v \in M$ we say that $u \prec v$ if either u is shorter than v or they have the same length and u is lower than v with respect to the lexicographical order in which $x_n \prec \cdots \prec x_1$ (see Lemma 2.13). Then $P := M \setminus J$ is a finite coideal partial monoid.

iii) For a free monoid $M_n := \langle x_1, \ldots, x_n \rangle$ consider an ideal $J_n := \langle x_j x_i : j \ge i \rangle$. Then $P_n := M_n \setminus J_n$ is a finite coideal partial monoid consisting of 2^n elements of the form $u := x_{i_1} \cdots x_{i_k}, 1 \le i_1 < \cdots < i_k \le n$. For an element $v := x_{j_1} \cdots x_{j_l}, 1 \le j_1 < \cdots < j_l \le n$ the composition $u \circ v \in P_n$ iff $i_k < j_1$.

Example 2.17. i) Now we modify the construction of Example 2.16 iii) and produce a partial monoid Q_n coinciding as a set with P_n and equipped with the same ordering \prec . The composition law in Q_n differs from the one in P_n : let $u := x_{i_1} \cdots x_{i_k}, 1 \le i_1 < \cdots < i_k \le n, v := x_{j_1} \cdots x_{j_l}, 1 \le j_1 < \cdots < j_l \le n$ be two elements of Q_n , then

$$u \circ v = x_{i_1} \cdots x_{i_k} \circ x_{j_2} \cdots x_{j_k}$$

iff $i_k = j_1$; otherwise the composition is not defined.

The partial monoid Q_n is isomorphic to the following partial monoid R_n . The generators of R_n are $\{y_{i,j} : 1 \le i < j \le n\}$. The composition $y_{i,j} \circ y_{k,l}$ is defined iff j = k. The isomorphism of R_n and Q_n is established by mapping of $y_{i_1,i_2} \circ y_{i_2,i_3} \circ \cdots \circ y_{i_{k-1},i_k}$ to $x_{i_1} \circ \cdots \circ x_{i_k}$.

ii) One can yield a family of partial submonoids of R_n as follows. Consider a directed acyclic graph G with n vertices numbered by $\{1, \ldots, n\}$ in such a way that for any arrow (i, j) of G it holds i < j. Then one can consider a partial submonoid R_G of R_n generated by the elements $\{y_{k,l}\}$ for which there is a path from a vertex k to a vertex l in G.

iii) Alternatively, one can consider a partial submonoid T_G of R_G of paths in G. A partial monoid T_G is generated by the elements $\{z_{k,l}\}$ where (k, l) is an arrow in G (cf. [30]).

More generally, one can consider a partial monoid T_G of paths in an arbitrary directed graph G (when G contains cycles, T_G is infinite). One can treat T_G as a coideal partial submonoid of the free monoid M_G generated by $\{z_{k,l}\}$ where (k, l) is an arrow in G. Then $T_G = M_G \setminus J_G$ where the ideal J_G is generated by all compositions of the form $z_{k,l} \circ z_{i,j}$ where $l \neq i$. **Example 2.18.** Denote by $W_0(m), m \ge 1$ the nil-Coxeter semigroup generated by s_1, s_2 satisfying the following relations:

$$s_1 \circ s_1, s_2 \circ s_2 \notin W_0(m), \underbrace{s_1 \circ s_2 \circ s_1 \circ s_2 \dots}_m = \underbrace{s_2 \circ s_1 \circ s_2 \circ s_1 \dots}_m$$

Then $W_0(m)$ consists of 2m-1 elements: for each $1 \leq k < m$ it contains two elements

$$c_k := \underbrace{s_1 \circ s_2 \circ s_1 \circ s_2 \dots}_k, \ d_k := \underbrace{s_2 \circ s_1 \circ s_2 \circ s_1 \dots}_k$$

of length k, and in addition the element $c_m = d_m$.

The following compositions are defined in $W_0(m)$:

 $c_{2k} \circ c_l = c_{2k+l}, c_{2k+1} \circ d_l = d_{2k+l}, d_{2k} \circ d_l = d_{2k+l}, d_{2k+1} \circ c_l = d_{2k+l+1},$

provided that $2k + l \leq m$ or respectively, $2k + l + 1 \leq m$. All other compositions are not defined. One can verify that $W_0(m)$ is a partial semigroup with an order defined by $d_k \prec c_k \prec d_{k+1}, 1 \leq k < m$.

Observe that in case of $W_0(m)$ one cannot replace the axiom from Definition 2.1 by weaker axioms that $c \leq d$ implies that $b \circ c \leq b \circ d, c \circ b \leq d \circ b$, provided that $b \circ c, b \circ d, c \circ b, d \circ b$ are defined.

If one applies Theorem 2.11 to the partial semigroup $W_0(m)$, then one obtains the partial semigroup $\overline{W_0(m)}$ generated by two elements $\overline{s_1}, \overline{s_2}$ such that $\overline{s_1} \circ \overline{s_1}, \overline{s_2} \circ \overline{s_2}$ and $\underbrace{\overline{s_1} \circ \overline{s_2} \circ \cdots}_{m+1}$, $\underbrace{\overline{s_2} \circ \overline{s_1} \circ \cdots}_{k}$ are not defined; thus consisting of 2m elements of the form either $\overline{c_k} = \underbrace{s_1 \circ s_2 \circ \cdots}_{k}$ or $\overline{d_k} = \underbrace{s_2 \circ s_1 \circ \cdots}_{k}$, $1 \le k \le m$. The epimorphism

 $f: \overline{W_0(m)} \twoheadrightarrow W_0(m)$ sends $f(\overline{c_k}) = c_k, f(\overline{d_k}) = d_k, 1 \leq k \leq m$. Thus, f is not injective just on two elements: $f(\overline{c_m}) = f(\overline{d_m}) = c_m = d_m$. The order on $\overline{W_0(m)}$ is defined by $\overline{c_k} \triangleleft \overline{d_k} \triangleleft \overline{c_{k+1}}, 1 \leq k < m$ and in addition, $\overline{c_m} \triangleleft \overline{d_m}$.

The following two propositions provide constructions of extending partial semigroups.

Proposition 2.19. Let P and Q be partial semigroups. Then $P' = P \sqcup Q$ is a partial semigroup with the composition inherited from P and Q and pq = qp = q for all $p \in P$, $q \in Q$. Suppose that P and Q are ordered such that

i) $q \leq qq'$ and $q \leq q'q$ for all $q, q' \in Q$ (this property is called positive ordering, see, e.g. [32], [30]). Then the assignments $p \prec q$ for $p \in P$ and $q \in Q$ turn P' into an ordered partial semigroup;

ii) $q \succeq qq'$ and $q \succeq q'q$ for all $q, q' \in Q$. Then the assignments $p \succ q$ for $p \in P$ and $q \in Q$ turn P' into an ordered partial semigroup.

Note that when P, Q are commutative partial semigroups, the resulting P' is commutative as well. The next proposition allows one to construct non-commutative partial semigroups from arbitrary (in particular, commutative) ones.

Proposition 2.20. Let P, Q be partial semigroups. Consider a partial semigroup $R := Q \sqcup \{x\} \sqcup \{y\} \sqcup P$ defined as follows:

$$xz = x, yz = y, z \in \{P, Q, x, y\}; Pz_1 = y, z_1 \in \{x, y, Q\}; Qz_2 = x, z_2 \in \{x, y, P\}.$$

Then R is a partial semigroup with an ordering $Q \prec x \prec y \prec P$.

When in Propositions 2.19, 2.20 P, Q are entire semigroups, the results of constructions are entire semigroups as well. In contrast, there are no entire finite semigroups satisfying the strong property.

Proposition 2.21. There are no entire finite semigroups (with more than one element) satisfying the strong property.

Proof. Suppose the contrary. If for some element a of the semigroup it holds $a \prec a^2$ then $a^i \prec a^{i+1}, i \ge 1$, which leads to a contradiction. By the same token an assumption $a \succ a^2$ leads to a contradiction as well. Thus, $a = a^2$ for any element a.

For any pair of elements $a \prec b$ it holds $a = a^2 \prec ab$, hence $ab \prec ab^2 = ab$. The obtained contradiction completes the proof.

Now we concoct a construction for extending partial monoids satisfying the strict property.

Proposition 2.22. Let P be a partial monoid with an order \prec^0 satisfying the strict property. For $x \notin P$ construct a partial monoid

$$Q := P \sqcup P \circ x \sqcup \cdots \sqcup P \circ x^{\circ k}$$

such that $x \circ P, x^{\circ(k+1)}$ are not defined. We set the order \prec on Q as follows:

 $P \circ x^{\circ i} \prec P \circ x^{\circ (i+1)}, 0 \le i < k \text{ and}$

 $c \circ x^{\circ j} \prec d \circ x^{\circ j} iff c \prec^0 d, c, d \in P, 0 \le j \le k.$

Then Q is a partial monoid satisfying the strict property. Alternatively, one could set the order as $P \circ x^{\circ i} \succ P \circ x^{\circ (i+1)}$.

Proof. To verify the strict property consider elements $u, v, w, t \in Q$ such that $u \leq v, w \leq t$ and at least one of two latter inequalities is strict. We assume that $u \circ w, v \circ t \in Q$. When $u, v, w, t \in P$, the strict property follows from the strict property for P. Otherwise, $v \in P$, $t = t_0 \circ x^{\circ i}$ for some $1 \leq i \leq k$. Therefore, $u \in P, w = w_0 \circ x^{\circ j}$ for suitable $0 \leq j \leq i$. If j < i then $u \circ w \prec v \circ t$. Otherwise, if j = i then it holds $w_0 \preceq^0 t_0$. Since one of two inequalities $u \preceq^0 v, w_0 \preceq^0 t_0$ is strict, we deduce from the strict property for P that $u \circ w_0 \prec^0 v \circ t_0$, which implies the strict property for Q:

$$u \circ w = u \circ w_0 \circ x^{\circ i} \prec v \circ t_0 \circ x^{\circ i} = v \circ t.$$

By the same token one considers an alternative order $P \circ x^{\circ i} \succ P \circ x^{\circ (i+1)}$. \Box

Remark 2.23. One can generalize Proposition 2.22 to partial semigroups (rather than monoids). For a partial semigroup P satisfying the strict property consider a partial semigroup $Q := \bigsqcup_{0 \le i \le k} (i, P)$ (where (i, P) is a copy of P), in which the product is defined as $(0, p_0) \circ (i, p) := (i, p_0 \circ p), p_0, p \in P, 0 \le i \le k$, and $(j, P) \circ (i, P)$ is not defined when j > 0. The order in Q is lexicographical with respect to i and to the order in P. As in the proof of Proposition 2.22 one can verify that Q fulfills the strict property.

2.2. Valuations of algebras in partial semigroups.

Definition 2.24. For a k-algebra \mathcal{A} a mapping $\nu : A \setminus \{0\} \twoheadrightarrow P$ onto a partial semigroup P is a *valuation* if for any $a, b \in A \setminus \{0\}$ it holds the following:

i) $\nu(\mathbb{k}^*a) = \nu(a);$ ii) $\nu(a+b) \preceq \max\{\nu(a), \nu(b)\},$ provided that $a+b \neq 0;$

iii) $\nu(ab) = \nu(a) \circ \nu(b)$, provided that $\nu(a) \circ \nu(b) \in P$ (in particular, in this case it holds $ab \neq 0$).

Remark 2.25. Alternatively, one could consider a mapping $\nu : A \setminus \{0\} \to P_0$ satisfying the properties similar to i), ii), iii) where P_0 is a partial semigroup. Then $P_{\nu} := \nu(A \setminus \{0\}) \subset P_0$ is also a partial semigroup.

Remark 2.26. In this section we do not suppose that A is unital or P contains 1.

We say that a valuation $\nu : A \setminus \{0\} \to P$ onto a partial semigroup P is *injective* if there exists a k-basis $\mathbf{B} \subset A$ of A such that $\nu : \mathbf{B} \to P$ is a bijection. A basis fulfilling the latter property is called *adapted* with respect to ν .

The proof of the following proposition is straightforward.

Proposition 2.27. Let $\nu : A \setminus \{0\} \to P$ be an injective valuation and let $f : P \to Q$ be an ordered homomorphism of partial semigroups. Then $f \circ \nu : A \setminus \{0\} \to Q$ is also an injective valuation.

Definition 2.28. For a partial semigroup P define a semigroup algebra & P as having a basis $\{[u] : u \in P\}$. We define $[u][v] = [u \circ v]$, provided that $u \circ v \in P$, otherwise [u][v] = 0.

Then a tautological valuation $\nu := \nu_P : \mathbb{k}P \setminus \{0\} \twoheadrightarrow P$ is defined by

$$\nu(\sum_{u\in P}\alpha_u[u]):=\max\{u\},\,\alpha_u\in \Bbbk^*.$$

Observe that the order on P is not necessary to be a well order: still, we get an injective valuation with an adapted basis $\{[u] : u \in P\}$.

It is immediate that a homomorphism $f: P \to Q$ of partial semigroups induces a natural homomorphism of algebras $\Bbbk P \to \Bbbk Q$. In addition, it induces a (not necessary injective) valuation $f \circ \nu : \Bbbk P \setminus \{0\} \to Q$.

Note however that if P is a partial semigroup and Q is a coideal in P, there is a homomorphism of algebras $\Bbbk P \to \Bbbk Q$ given by $[c] \to \begin{cases} [c] & \text{if } c \in Q \\ 0 & \text{otherwise} \end{cases}$ but, in general, there is no corresponding partial homomorphism from P to Q.

Remark 2.29. In the conditions of Lemma 2.7 $\mathbf{B} = f^*(Q) \sqcup \mathbf{B}_I$ is a basis of $\mathbb{k}P$ adapted to the tautological valuation of $\nu_P : \mathbb{k}P \setminus \{0\} \to P$ (produced in Definition 2.28).

Example 2.30. Following Example 2.16 i) one can consider the monoidal algebra $\mathbb{k}P(u_1 \dots, u_s)$. It is called a Stanley-Reisner algebra in case when all u_1, \dots, u_s are square-free.

The proof of the following proposition is similar to the proof of Theorem 3.1 ii), iv).

Proposition 2.31. Let $\nu : A \setminus \{0\} \twoheadrightarrow P$ be a valuation onto a partial semigroup P. When P is well-ordered and dim $(\mathcal{A}_u) = 1$ for any $u \in P$, the valuation ν is injective. Every set $\mathbf{B} \subset A$ such that the mapping $\nu : \mathbf{B} \to P$ is a bijection, is an adapted basis of A (with respect to ν).

Vice versa, if ν is injective then dim $(\mathcal{A}_u) = 1$ for any $u \in P$.

Proposition 2.32. Let $\nu : A \setminus \{0\} \to P$ be an injective valuation of an algebra A into a well-ordered partial semigroup P, and B be a subalgebra of A. Then the restriction of ν on $B \setminus \{0\}$ is also an injective valuation.

Remark 2.33. In view of Remark 4.18, it is interesting whether an analog of Proposition 2.32 holds without assumption of well-orderness of P.

The following proposition for vector spaces is established in Proposition 4.4 (b). The extension to algebras is straight-forward.

Proposition 2.34. Let \mathcal{A}_i , i = 1, 2 be algebras and $\nu_i : \mathcal{A}_i \setminus \{0\} \to P_i$ be their valuations to respective partial semigroups. Then the assignments $a_1 \otimes a_2 \mapsto (\nu_1(a_1), \nu_2(a_2))$ extend to a valuation $\nu : \mathcal{A}_1 \otimes \mathcal{A}_2 \setminus \{0\} \to P_1 \times P_2$ (an order in $P_1 \times P_2$ is defined in Lemma 2.12). If both valuations ν_1, ν_2 are injective then ν is injective as well.

Proposition 2.35. Consider a partial semigroup P and an ordered partial semigroup Q. Let $\nu : \mathbb{k}P \setminus \{0\} \to Q$ be a valuation. Then the mapping $c \mapsto \nu([c])$ is a partial homomorphism $P \to Q$. In particular, P acquires a new structure of a partial semigroup P_{S_f} in notation of Remark 2.5.

Proof. Take $c, c', c'' \in P$ such that $\nu([c]), \nu([c'])$ are composable and that $\nu([c]) \circ \nu([c']), \nu([c''])$ are also composable. Let S_{ν} be the set of all $(c, c') \in P \times P$ such that $\nu([c]), \nu([c'])$ are composable. Then due to Definition 2.1 it holds that $\nu([c']), \nu([c''])$ are composable and that $\nu([c]), \nu([c']) \circ \nu([c''])$ are composable as well. Thus, S_{ν} is admissible due to Definition 2.24.

If $(c, c') \in S_{\nu}$ then c, c' are composable in $P_{S_{\nu}}$ and $\nu([c \circ c']) = \nu([c]) \circ \nu([c'])$ again due to Definition 2.24.

In the following theorem we consider different words in generators of a partial semigroup representing the same element of the partial semigroup, among them we choose the minimal with respect to deglex (also for non-commutative partial semigroups), cf. Lemma 2.13, and call this word canonical. The following theorem can be easily deduced from Corollary 4.15.

Theorem 2.36. Let $\nu : A \setminus \{0\} \to P_{\nu} \subset P$ be an injective well-ordered valuation into a partial semigroup P, generated by P_0 , and let X_0 be a generating set of Asuch that $\nu|_{X_0}$ is a bijection $X_0 \to P_0$. Then the set of all monomials $x_u := \prod_{x \in X_0} x$

corresponding to the canonical factorization of $u \in P_{\nu}$ is an adapted to ν basis in A, and $\nu(x_u) = u$ (we will refer to the elements x_u as standard monomials).

Remark 2.37. X_0 is not always a minimal generating set for \mathcal{A} . The same applies to P_0 . In principle we can require that P_0 is minimal by inclusion. In some cases, including submonoids of $\mathbb{Z}_{\geq 0}^m$, P^{ind} of indecomposable elements of P generate P, in which case we can choose $P_0 := P^{ind}$.

In the following theorem we show that given an injective valuation on an algebra, how one can define it on its quotient algebra.

Theorem 2.38. Let A be a k-algebra and $\nu_0 : A \setminus \{0\} \rightarrow P$ be an injective valuation onto a well-ordered partial semigroup P. Let $I \subset A$ be an ideal.

i) Then $\nu_0(I \setminus \{0\})$ is an ideal in P. For $a \in (A/I) \setminus \{0\}$ the formula from Proposition 4.21, i.e.,

(2.1)
$$\nu(a) := \min \nu_0(a+I)$$

defines an injective valuation $\nu : (A/I) \setminus \{0\} \twoheadrightarrow (P \setminus \nu_0(I \setminus \{0\}))$. If $\nu_0(a) \in P \setminus \nu_0(I \setminus \{0\})$ then $\nu(a) = \nu_0(a)$.

ii) If $u \in P \setminus \nu_0(I \setminus \{0\})$ is indecomposable then u is also indecomposable in P.

iii) Let $\{x_u : u \in P\}$ be a standard monomial basis of A with respect to ν_0 (cf. Theorem 2.36). Then $\mathbf{B} := \{q(x_u) : u \in P \setminus \nu_0(I \setminus \{0\})\}$ is a standard monomial basis of A/I with respect to ν where $q : A \to A/I$ is the natural projection.

Proof. i) First, we note that if $\nu_0(a) \in P \setminus \nu_0(I \setminus \{0\})$ then $\nu(a) = \nu_0(a)$. Indeed, suppose that on the contrary it holds $\nu_0(a+f) \prec \nu_0(a)$ (cf. (2.1)). Then $\nu_0(f) = \nu_0(a)$ which contradicts the supposition.

Observe that for any $a \in (A/I) \setminus \{0\}$ it holds $\nu(a) \notin \nu_0(I \setminus \{0\})$. Indeed, otherwise $\nu(a) = \nu_0(a+f) \in \nu_0(I \setminus \{0\})$ for suitable $f \in I \setminus \{0\}$. Then there exists $g \in I \setminus \{0\}$ such that $\nu_0(g) = \nu_0(a+f)$. Due to the injectivity of ν_0 there exists $\alpha \in k^*$ for which holds $\nu_0(a+f+\alpha g) \prec \nu_0(a+f)$, this contradicts to the equality $\nu(a) = \nu_0(a+f)$ and to (2.1).

Now let $a, b \in (A/I) \setminus \{0\}$ and $f, g \in I$ be such that $\nu(a) = \nu_0(a+f), \nu(b) = \nu_0(b+g)$ according to (2.1). Then

 $\nu(a+b) \leq \nu_0(a+f+b+g) \leq \max\{\nu_0(a+f), \nu_0(b+g)\} = \max\{\nu(a+f), \nu(b+g)\}$ which justifies Definition 2.24 ii) for ν .

To verify Definition 2.24 iii) for ν assume that $\nu(a) \circ \nu(b) \in P \setminus \nu_0(I \setminus \{0\})$. Since

$$\nu(ab) \preceq \nu_0(ab + ag + fb + fg) = \nu_0(a + f) \circ \nu_0(b + g)$$

due to (2.1) and to Definition 2.24 iii) for ν_0 , we get $\nu(ab) \leq \nu(a) \circ \nu(b)$. Suppose that $\nu(ab) \prec \nu(a) \circ \nu(b)$. Let $\nu(ab) = \nu_0((a+f)(b+g) + f_0)$ for appropriate $f_0 \in I \setminus \{0\}$ (see (2.1)). Hence

$$\nu_0((a+f)(b+g)+f_0) \prec \nu(a) \circ \nu(b) = \nu_0((a+f)(b+g))$$

and thereby, $\nu_0((a+f)(b+g)) = \nu_0(f_0) \in \nu_0(I^*)$. The obtained contradiction shows that $\nu(ab) = \nu(a) \circ \nu(b)$.

Finally, we prove that ν is injective. Let $a, b \in (A/I) \setminus \{0\}$ and $f, g \in I$ be such that $\nu_0(a+f) = \nu(a) = \nu(b) = \nu_0(b+g)$ (see (2.1)). Since ν_0 is injective there exists $\alpha \in \mathbb{k}^*$ such that either $\nu_0((a+f) + \alpha(b+g)) \prec \nu_0(a+f)$ or $a+f+\alpha(b+g) = 0$. In the former case $\nu(a+\alpha b) \preceq \nu_0((a+f) + \alpha(b+g)) \prec \nu(a)$, while in the latter case $(A/I) \ni a + \alpha b = 0$.

ii) Suppose the contrary, then $u = u_1 u_2$ for suitable $u_1, u_2 \in P$. It holds $u_1, u_2 \notin \nu_0(I \setminus \{0\})$, this contradicts to that $u \in P \setminus \nu_0(I \setminus \{0\})$ is indecomposable.

iii) Due to i) it holds $\nu(q(x_u)) = \nu_0(x_u) = u$ for $x_u \in \mathbf{B}$ (cf. Theorem 2.36) and $\nu((A/I) \setminus \{0\}) = P \setminus \nu_0(I \setminus \{0\}) = \nu(\mathbf{B})$. Therefore, Proposition 2.31 implies that \mathbf{B}

is an adapted basis of A/I with respect to ν . Finally, ii) entails that **B** is a standard monomial basis. \Box

Example 2.39. Let an algebra $A := \mathbb{k}[x, y]/(x^2 - y^3)$. Following Theorem 3.21 one produces an injective valuation $\nu : A \setminus \{0\} \twoheadrightarrow C$ onto a semigroup $C := \{(i, j) : 0 \le i < \infty, j = 0, 1\}$ where $(0, 1) \circ (0, 1) = (3, 0)$, and $\nu(y^i) = (i, 0), \nu(xy^i) = (i, 1)$ (cf. Example 3.38).

On the other hand, applying Theorem 2.38 one obtains an injective valuation $\underline{\nu} : A \setminus \{0\} \twoheadrightarrow P \subset \mathbb{Z}_{\geq 0}^2$ onto a partial semigroup P which coincides with C as a set, while $(0,1) \circ (0,1)$ is not defined in P. The values of $\underline{\nu}$ coincide with the corresponding values of ν , i.e. $\underline{\nu}(y^i) = (i,0), \underline{\nu}(xy^i) = (i,1)$.

Remark 2.40. One can study the following inverse issue to Theorem 2.38. Let $J \subset A$ be an ideal in a commutative algebra A, and let $\nu : J \setminus \{0\} \to P$ be a valuation (not necessary injective) in a partial semigroup P whose ordering \prec fulfills the strict property. Assume in addition that for any element $a \in A$ it holds $aJ \neq \{0\}$ and that for any $c \in P$ there exists $d \in P$ such that c, d are composable.

When one can extend the valuation $\overline{\nu}: A \setminus \{0\} \to Q$ for a suitable partial semigroup $Q \supset P$ such that $\overline{\nu}|_{J \setminus \{0\}} = \nu$? To define Q consider a set $P \times P$ with the componentwise composition $(c_1, c_2) \circ_Q (d_1, d_2) := (c_1 \circ d_1, c_2 \circ d_2)$ (provided that both c_1, d_1 and c_2, d_2 are composable) and impose the following relations (the idea is to treat Q as a set of "fractions" with numerators and denominators from P). Firstly, we identify pairs $(c \circ d_1, d_1), (c \circ d_2, d_2) \in P \times P$ (provided that both c, d_1 and c, d_2 are composable). Secondly, we identify in Q pairs $(c_1, c_2), (d_1, d_2)$ if $c_1 \circ d_2 = c_2 \circ d_1$ (provided that both c_1, d_2 and c_2, d_1 are composable in P). Thirdly, for any $a \in A$ if $ab_1 = b_2, ab_3 = b_4$ for non-zero $b_1, b_2, b_3, b_4 \in J$, we identify in Q the pairs $(\nu(b_2), \nu(b_1))$ and $(\nu(b_4), \nu(b_3))$. We define an order on Q as follows: $(c_1, c_2) \prec_Q (d_1, d_2)$ if $c_1 \circ d_2 \prec c_2 \circ d_1$ (provided that both $c_1, d_2 \circ d_1$ (provided that both $c_1, d_2 \circ d_2 \circ d_1$).

If the resulting semigroup Q is ordered and contains P embedded for $c \in P$ by $(c \circ d, d) \in Q$ such that c, d are composable, then one can define an extension $\overline{\nu}(a) := (\nu(b_2), \nu(b_1))$. Moreover, in this case the order \prec_Q fulfills the strict property.

Denote by $A_1 * A_2 = A_2 * A_1$ the free product of algebras A_1 and A_2 . The following is immediate.

Lemma 2.41. Let A_i , i = 1, 2 be algebras and let $\nu_i : A_i \setminus \{0\} \to P_i$, i = 1, 2 be a valuation of A_i to a (partial) semigroup P_i . Suppose that $P_1 * P_2$ has a compatible order (see the definition after Remark 2.14). Then the free product $A_1 * A_2$ has a natural valuation $\nu_1 * \nu_2 : A_1 * A_2 \setminus \{0\} \to P_1 * P_2$.

Example 2.42. Let W be a finite reflection group on the space V, recall that its coinvariant algebra $A_W = S(V)/\langle S(V)^W_+ \rangle$ has dimension |W|. Also if W is a Weyl group of a complex semisimple group G, then $A_W \cong H^*(G/B)$, where B is the Borel subgroup of G. In this case, A_W has a canonical Schubert basis $X_w, w \in W$.

If $W = \langle s_1, s_2 | s_1^2 = s_2^2 = 1$, $(s_1 s_2)^n = 1 \rangle$ is dihedral of order 2n, then $A_W = \mathbb{C}[z, \overline{z}] / \langle z\overline{z}, z^n + \overline{z}^n \rangle$ because W acts on $V = \mathbb{C} \cdot z \oplus \mathbb{C} \cdot \overline{z}$ by $s_1(z) = \overline{z}$, $s_1 s_2(z) = \zeta z$, $s_1 s_2(\overline{z}) = \zeta^{-1}\overline{z}$, where $\zeta = e^{\frac{2\pi i}{n}}$, therefore $z\overline{z}$ and $z^n + \overline{z}^n$ are basic W-invariants. Writing z = x + iy we expect that the Schubert basis is $\{Re \ z^k = \frac{z^k + \overline{z}^k}{2}, Im \ z^k = z^k = z^k + \overline{z}^k\}$ $\frac{z^{k}-\overline{z^{k}}}{2i}, k = 0, \ldots, n\} \setminus \{0\}.$ Note that in this case $A_{W} \cong \mathbb{C}[z, \overline{z}] / \langle z\overline{z}, z^{n} - \overline{z}^{n} \rangle = \mathbb{C}P$, where P is a partial additive monoid on $M_{n} \sqcup_{0,n} M_{n}$ where M_{n} is the partial monoid on [0, n] with $a \circ b$ defined iff $a + b \leq n$, in which case the composition is a + b and $\sqcup_{0,n}$ stands for disjoint union with identified unit 0 and identified n. Namely, the first copy of M_{n} consists of $z^{k}, 0 \leq k \leq n$, while the second copy consists of $\overline{z}^{k}, 0 \leq k \leq n$, note that $z^{n+1} = \overline{z}^{n+1} = 0$.

Note also that $A_{S_3} \cong \mathbb{C}[x_1, x_2, x_3] / \langle e_1, e_2, e_3 \rangle = \mathbb{C}[x_1, x_2] / \langle x_1^2 + x_1 x_2 + x_2^2, x_1 x_2(x_1 + x_2) \rangle$ where $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3$, $e_3 = x_1 x_2 x_3$. An S_3 -equivariant isomorphism is given by $z = x_1 - \zeta x_2$, $\overline{z} = x_2 - \zeta x_1$. The latter algebra has a Schubert basis $\{1, x_1, x_1 + x_2, x_1^2, x_1 x_2, x_1^2 x_2\}$.

Note also that $A_{I_2(4)} \cong \mathbb{C}[x_1, x_2] / \langle x_1^2 + x_2^2, x_1^2 x_2^2 \rangle$ with the action given by $s_1(x_1) = x_2, s_2(x_2) = -x_2, s_2(x_1) = x_1$. An $I_2(4)$ -equivariant isomorphism is given by $z = x_1 - ix_2, \overline{z} = x_1 + ix_2$. The latter algebra has a Schubert basis $\{1, x_1, x_1 + x_2, x_1^2, x_1 x_2, x_1^2 x_2 + x_1 x_2^2, x_1^3, x_1^3 x_2\}$.

When n is odd P admits the following ordering fulfilling the strict property (see Definition 2.9):

$$1 \prec z \prec \cdots \prec z^{(n-1)/2} \prec \overline{z} \prec \cdots \prec \overline{z}^{n-1} \prec z^{(n+1)/2} \prec \cdots \prec z^{n-1} \prec z^n (=\overline{z}^n).$$

In contrast, when *n* is even there is no ordering of *P* fulfilling the strict property since if $z^{n/2} \prec \overline{z}^{n/2}$ (or, respectively $z^{n/2} \succ \overline{z}^{n/2}$) then $z^n \prec \overline{z}^n$ (respectively, $z^n \succ \overline{z}^n$). On the other hand, any ordering on *P* merging the orderings $1 \prec z \prec \cdots \prec z^n$ and $1 \prec \overline{z} \prec \cdots \prec \overline{z}^n$ satisfies Definition 2.1.

One can consider another representation $A_W = \mathbb{C}Q$ where Q is a partial monoid

$$Q := \{ c_k := z^k + \overline{z}^k : 0 \le k \le n \} \sqcup \{ d_k := z^k - \overline{z}^k : 0 < k < n \}$$

with the following composition rules:

• $c_k \circ c_l = d_k \circ d_l = c_{k+l}$ iff $k+l \le n$;

•
$$c_k \circ d_l = d_{k+l}$$
 iff $k+l < n$.

Then Q satisfies Definition 2.1 with an ordering

$$1 \prec c_1 \prec d_1 \prec \cdots \prec c_{n-1} \prec d_{n-1} \prec c_n,$$

while Q does not admit an ordering fulfilling the strict property.

Now we construct a common adapted basis of A_W for a pair of injective valuations $\nu_P : A_W \setminus \{0\} (= \mathbb{C}P \setminus \{0\}) \twoheadrightarrow P$ and $\nu_Q : A_W \setminus \{0\} (= \mathbb{C}Q \setminus \{0\}) \twoheadrightarrow Q$ (see Theorem 1.11). When n is odd, the common basis consists of

$$\{1, z^n\} \sqcup \{z^k, z^k + \overline{z}^k \ : \ 1 \le k < n/2\} \sqcup \{\overline{z}^k, z^k + \overline{z}^k \ : \ n/2 < k < n\}.$$

When n is even, fix the following ordering in P:

$$1 \prec z \prec \overline{z} \prec \cdots \prec z^k \prec \overline{z}^k \prec \cdots \prec z^{n-1} \prec \overline{z}^{n-1} \prec z^n (= \overline{z}^n).$$

Then the common basis consists of

$$\{1, z^n\} \sqcup \{z^k, z^k + \overline{z}^k : 1 \le k < n\}.$$

Consider the tautological injective valuation $\nu_0 : \mathbb{C}[z,\overline{z}] \setminus \{0\} \twoheadrightarrow \mathbb{Z}^2_{\geq 0}$ where $\mathbb{Z}^2_{\geq 0}$ is endowed with lex ordering in which $\overline{z} \prec z$. If we apply Theorem 2.38 to an ideal

 $I := \langle z\overline{z}, z^n - \overline{z}^n \rangle \subset \mathbb{C}[z, \overline{z}]$ then we obtain an injective valuation $\nu : A_W \setminus \{0\} \twoheadrightarrow P'$, where $P' = \{1, z, \dots, z^{n-1}, \overline{z}, \dots, \overline{z}^n\}$. Thus, P' contains also 2n elements as P, but differs from P as a partial monoid since z^n is not defined in P' unlike P. Nevertheless, ν coincides with ν_P element-wise.

Example 2.43. Recall that the nil Hecke algebra \mathcal{H}_{S_2} of S_2 is generated by α , x subject to

$$x^2 = 0, x\alpha + \alpha x = -2$$

In particular, $s = \alpha x + 1 = -x\alpha - 1$ is an involution.

More generally, let $W = \langle s_i, i \in I | s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$ be a Coxeter group and $V = \bigoplus \Bbbk \alpha_i$ be its reflection representation with a basis α_i so that the action is given by

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$$

where A is the corresponding Cartan matrix.

Then \mathcal{H}_W is generated by x_i, α_i subject to

$$x_i^2 = 0, x_i \alpha_j = s_i(\alpha_j) x_i - a_{ij}$$

and the braid relation

$$\underbrace{x_i x_j x_i \cdots}_{m_{ij}} = \underbrace{x_j x_i x_j \cdots}_{m_{ij}}$$

It is easy to see that $\Bbbk W$ embeds into \mathcal{H}_W via $s_i \mapsto \alpha_i x_i + 1$.

This embedding extends to an embeddings $S(V) \rtimes \Bbbk W \hookrightarrow \mathcal{H}_W$ and $\mathcal{H}_W \hookrightarrow Frac(S(V)) \rtimes \Bbbk W$ and

$$\mathcal{H}_W \cong (\Bbbk W)_0 \otimes S(V)$$

as a vector space where $(\Bbbk W)_0 = \langle x_i \rangle$ is the nil-Coxeter algebra.

 \mathcal{H}_W admits a quotient $\underline{\mathcal{H}}_W$ by the ideal in S(V) generated by W-invariants so that

$$\underline{\mathcal{H}}_W \cong (\Bbbk W)_0 \otimes A_W$$

It is proved in [14], [9] that if |W| = N, then the algebra \mathcal{H}_W is isomorphic to $Mat_N(S(V)^W)$, hence $\underline{\mathcal{H}}_W \cong Mat_N(\Bbbk)$. Therefore, one can apply to the algebra \mathcal{H}_W Proposition 2.34 and Example 2.47, and thereby produce an injective valuation on \mathcal{H}_W .

Example 2.44. [Galois extensions] Let \mathbb{K} be a finite Galois extension of \mathbb{k} and let $G = Gal(\mathbb{K}/\mathbb{k})$. Then the assignments $g \otimes a \mapsto g \circ L_a$ for all $g \in G$, $a \in \mathbb{K}$ define an isomorphism of algebras $\mathbb{K} \rtimes \mathbb{k} G \xrightarrow{\sim} End_{\mathbb{k}}(\mathbb{K})$ (where $L_a : \mathbb{K} \to \mathbb{K}$ is the multiplication by $a \in \mathbb{K}$), this again follows from [14], [9]. In particular, any choice of basis \mathbb{k} -basis $\{b_1, \ldots, b_n\}$ of \mathbb{K} canonically identifies the algebra $\mathbb{K} \rtimes \mathbb{k} G$ with $Mat_n(\mathbb{k})$. Similar to Example 2.43 one can produce an injective valuation on $\mathbb{K} \rtimes \mathbb{k} G$.

One can verify the following proposition.

Proposition 2.45. For a commutative k-algebra A let a set P of monomials in elements $a_1, \ldots, a_n \in A$ form a k-basis in A, and P be a partial semigroup (in particular, P is endowed with a linear order). For an element

$$a = \sum_{u \in P} \alpha_u u \in A \setminus \{0\}, \, \alpha_u \in \mathbb{K}^{2}$$

define $\nu(a) := \max\{u\}$ where max ranges over u from the latter sum. Then $\nu : A \setminus \{0\} \twoheadrightarrow P$ is an injective valuation onto P. Moreover, P is adapted with respect to ν .

2.3. Injective valuations onto coideal partial semigroups via tropical geometry and adapted bases.

Proposition 2.46. i) Let $\nu : A \setminus \{0\} \to P$ be a valuation in a partial semigroup P. For $u \in P$ denote a k-linear space $A_{\preceq u} := \{a \in A \setminus \{0\} : \nu(a) \preceq u\} \cup \{0\}$ (see Definition 2.24 i), ii)). Then $A_{\preceq u}A_{\preceq v} \subset A_{\preceq uov}$, provided that $u \circ v \in P$. Thus the family $\{A_{\preceq u} : u \in P\}$ forms a filtration of A.

ii) Let P be a well-ordered partial semigroup and $\{A_u : u \in P\}$ be a filtration of an algebra A. For $a \in A \setminus \{0\}$ setting $\nu(a)$ to be the minimal $u \in P$ such that $a \in A_u$, defines a valuation $\nu : A \setminus \{0\} \to P$.

Example 2.47. Consider a partial semigroup $P_k := \{(i, j) : 1 \leq i, j \leq k\}$ where $(i, j) \circ (j, l) := (i, l)$ and $(i, j) \circ (m, l)$ is not defined when $m \neq j$. We define a linear order \prec on $(i, j) \in P_k$ being lexicographical with respect to a vector

i) (j - i, -i) or

ii) (-i, j).

In both cases P_k endowed with \prec satisfies Definition 2.1. Moreover, \prec satisfies the strict property (see Definition 2.9).

Note that the axiom of the order from Definition 2.1 for P_k cannot be deduced from weaker axioms: $c \leq d$ implies that $a \circ c \leq a \circ d$ and $c \circ a \leq d \circ a$, provided that $a \circ c, a \circ d, c \circ a, d \circ a \in P_k$.

It would be interesting to clarify whether P_k can be represented as a coideal semigroups (with a compatible ordering).

Clearly, $Mat_k(\mathbb{k}) = \mathbb{k}P_k$ and the tautological valuation $\nu : Mat_k(\mathbb{k}) \setminus \{0\} \to P_k$ is injective and given by $\nu(e_{ij}) = (i, j)$ from Definition 2.28.

Observe that the valuation of the unit of the algebra $Mat_k(\mathbb{k})$ equals $\nu(e_{1,1} + \cdots + e_{k,k}) = (1,1)$ in both cases i), ii).

Note that one cannot take a vector (i, j) in place of vectors from either i) or ii) since the induced ordering does not satisfy Definition 2.1.

Consider a partial semigroup P' with an ordering \triangleleft . One can construct (see Lemma 2.12) a partial semigroup $P_k \times P'$ in which the ordering is given by the lexicographical pair (\prec , \triangleleft), where \prec is one of the described above orderings on P_k . If \triangleleft fulfills the strict property then the resulting ordering fulfills the strict property as well (cf. Lemma 2.12). Thus, if an algebra A admits an injective valuation onto P' then the matrix algebra $Mat_k(\Bbbk) \otimes A$ admits an injective valuation onto $P_k \times P'$, see Proposition 2.34.

Denote by T_k the partial monoid of paths in the complete directed graph having k vertices and loops (cf. Example 2.17 iii)). So, T_k is generated by the set $\{z_{i,j} : 1 \leq i, j \leq k\}$. Following the construction in the proof of Theorem 2.11 we produce an epimorphism $f: T_k \twoheadrightarrow P_k$ such that $f(z_{i,j}) = (i, j)$, thus $f(z_{i_1,i_2} \circ z_{i_2,i_3} \circ \cdots \circ z_{i_{s-1},i_s}) = (i_1, i_s)$. Denote by \prec one of the introduced above orders on P_k (say, i) or ii)). Now define an order \triangleleft on T_k as follows. We say that $z_{i_1,i_2} \circ \cdots \circ z_{i_{s-1},i_s} \triangleleft z_{j_1,j_2} \circ \cdots \circ z_{j_{l-1},j_l}$ if either

- $f(z_{i_1,i_2} \circ \cdots \circ z_{i_{s-1},i_s}) \prec f(z_{j_1,j_2} \circ \cdots \circ z_{j_{l-1},j_l})$, either
- $f(z_{i_1,i_2} \circ \cdots \circ z_{i_{s-1},i_s}) = f(z_{j_1,j_2} \circ \cdots \circ z_{j_{l-1},j_l})$ and s < l, or

• $f(z_{i_1,i_2} \circ \cdots \circ z_{i_{s-1},i_s}) = f(z_{j_1,j_2} \circ \cdots \circ z_{j_{l-1},j_l}), s = l$ and the vector (i_1, \ldots, i_s) is less than (j_1, \ldots, j_l) in the lexicographical order (in which, say, $1 < \cdots < n$). One can verify that f satisfies Proposition 2.8.

Note that f induces a natural epimorphism of semigroup algebras $\mathbb{k}T_k \twoheadrightarrow \mathbb{k}P_k = Mat_k(\mathbb{k})$.

Consider $P_{\infty} := \{(i, j) : 1 \leq i, j < \infty\}$, this is naturally an ordered partial semigroup of infinite matrices, and inclusions $P_k \subset P_{\infty}$ are ordered, moreover P_{∞} is their injective limit. Despite \prec is not a well ordering, the semigroup algebra $\mathbb{k}P_{\infty}$ is the algebra $Mat_{\infty}(\mathbb{k})$ of infinite matrices with finite support, and the injective valuation onto P_{∞} provides the tautological valuation $Mat_{\infty}(\mathbb{k}) \setminus \{0\} \to P_{\infty}$ with an adapted basis $\{e_{i,j} : 1 \leq i, j < \infty\}$.

Denote by $F := \mathbb{k} \langle \{e_{i,j} : 1 \leq i, j \leq k\} \rangle$ the free algebra with the natural injective valuation ν_0 onto the free semigroup $P := \langle (i, j), 1 \leq i, j \leq k \rangle$. We assume that P is equipped with the well ordering produced in Lemma 2.13. Denote by

$$I := \langle \{e_{i,j}e_{p,q} : j \neq p, \ 1 \le i, j, p, q \le k\} \cup \{e_{i,l} - e_{i,j}e_{j,l} : 1 \le i, j, l \le k\} \rangle$$

an ideal in F. When we apply Theorem 2.38 we obtain an injective valuation ν : $(F/I)\setminus\{0\} = Mat_k(\Bbbk)\setminus\{0\} \twoheadrightarrow P\setminus\nu_0(I\setminus\{0\})$. Observe that $P\setminus\nu_0(I\setminus\{0\})$ is a partial semigroup consisting of k^2 elements $\{(i, j) : 1 \leq i, j \leq k\}$ such that no composition of them is defined since $\nu_0(e_{i,j}e_{p,q}) = (i, j) \circ (p, q)$ and $\nu_0(e_{i,l} - e_{i,j}e_{j,l}) = (i, j) \circ (j, l)$. Thus, $P \setminus \nu_0(I \setminus \{0\})$ differs from P_k .

Problem 2.48. Describe all possible orderings on P_k .

Example 2.49. Let us apply the construction from Proposition 2.35 to the symmetric group $P := S_k$ and $Q := P_k$ taking as ν the valuation from Example 2.47 ii). We consider the standard representation of S_k in GL_k . This provides a partial homomorphism $f : S_k \to P_k$. Then following Remark 2.5 one obtains a partial semigroup $R_k := (S_k)_{S_f}$ and a homomorphism from R_k to P_k . One can explicitly describe R_k as follows. Two permutations $p, q \in P_k$ are composable in R_k iff p(1) = 1 taking into account that $\nu(p) = (1, p^{-1}(1))$.

In case of a coideal partial semigroup P the following construction allows one to obtain a stronger property of filtrations.

Definition 2.50. For an algebra A we say that $\nu : A \setminus \{0\} \rightarrow P$ is a valuation onto a coideal partial semigroup $P \subset M$ if in addition to Definition 2.24 for any elements $a, b \in A \setminus \{0\}$ an inequality $\nu(ab) \preceq \nu(a) \circ \nu(b) \in M$ holds, provided that $ab \neq 0$.

Recall (cf. the definition prior to Remark 3.51) that an order \prec on a semigroup M is archimedian if for any $u \in M$ the set all elements of M less than u is finite.

Proposition 2.51. Let $\nu : A \setminus \{0\} \rightarrow P$ be a valuation on a k-algebra onto a coideal partial semigroup $P \subset M$. We assume that M is endowed with an archimedian order \prec . In this case for the filtration: $A_{\leq u}$ it holds

$$A_{\preceq u}A_{\preceq v} \subset A_{\max\{P \ni w \preceq u \circ v\}}$$

owing to Definition 2.50. Observe that the latter maximum exists since \prec is archimedian.

One can also denote $\mathcal{A}_u := A_{\leq u}/A_{\prec u}$ and the natural projection $p_u : A_{\leq u} \twoheadrightarrow \mathcal{A}_u$. The following remark extends Remark 3.52 to coideal partial semigroups.

Remark 2.52. Let A be a (not necessary commutative) k-algebra and $\nu : A \rightarrow P \subset M$ be an injective valuation onto a coideal partial semigroup P. We assume that M is endowed with a linear order \prec and a function $f : M \rightarrow \mathbb{Z}_{\geq 0}$ such that $c_1 \prec c_2$ implies that $f(c_1) \leq f(c_2)$, and $f(c_1 + c_2) \leq f(c_1) + f(c_2)$ for $c_1, c_2 \in M$, moreover the set $C_n := \{c \in M : f(c) \leq n\}$ is finite for any $n \in \mathbb{Z}_{\geq 0}$. Note that the latter implies that the order \prec is archimedian. Then the k-subspaces $A_n := \{a \in A \setminus \{0\} : f(\nu(a)) \leq n\} \cup \{0\}, n \in \mathbb{Z}_{\geq 0}$ provide a filtration of A such that $\dim(A_n) = |C_n|$.

Proposition 2.53. When $\nu : A \setminus \{0\} \twoheadrightarrow P$ is a valuation onto a coideal partial semigroup $P \subset M$, one can define a graded associated algebra $\mathcal{A} := \bigoplus_{u \in P} \mathcal{A}_u$ as follows. Let $u, v \in P, c \in \mathcal{A}_u, d \in \mathcal{A}_v$. If $u + v \in P, a \in \mathcal{A}_{\preceq u}, b \in \mathcal{A}_{\preceq v}$ such that $p_u(a) = c, p_v(b) = d$ then we define the product $cd := p_{u+v}(ab) \in \mathcal{A}_{u+v}$. It holds $cd \neq 0$. Otherwise, if $u + v \notin P$ then we define cd := 0.

Proof. The correctness of the definition of the product cd and that $cd \neq 0$ in case when $u + v \in P$ follows from Definitions 2.24 iii), 2.50. The associativity of \mathcal{A} can be verified taking into account Definition 2.1. \Box

The following proposition generalizes Theorem 3.21 to partial semigroups. We utilize the notations from Theorem 3.21.

Proposition 2.54. Let $A = \Bbbk[x_1, \ldots, x_n]/I$ be an algebra and $I_{trop} \subset I$ be a (n-d)dimensional subideal satisfying the following properties. Assume that there exists a common (n-d)-dimensional rational plane $H \subset \mathbb{R}^n$ of the tropical variety $Trop(I_{trop})$ such that H is prop and I_{trop} is saturated with respect to H. Then there exists a coideal partial monoid $P \subset \mathbb{Z}_{\geq 0}^n/H_{\mathbb{Z}}$ and an injective valuation $\nu : A \setminus \{0\} \rightarrow P$. A linear order on P is induced by a linear order on $\mathbb{Z}_{\geq 0}^n/H_{\mathbb{Z}}$ which in its turn, is determined by a hyperplane from $ETrop(I_{trop})$.

Proof. Apply Theorem 3.21 to the ideal I_{trop} and obtain an injective valuation

$$\nu_0: (\Bbbk[x_1,\ldots,x_n]/I_{trop}) \twoheadrightarrow \mathbb{Z}_{\geq 0}^n/H_{\mathbb{Z}}.$$

Then apply Theorem 2.38 to ν_0 which results in ν . Observe that $P = (\mathbb{Z}^n_{\geq 0}/H_{\mathbb{Z}}) \setminus \nu_0((I/I_{trop}) \setminus \{0\})$. \Box

The following proposition is inverse to Proposition 2.45 and extends Theorem 3.39 to valuations onto coideal partial monoids.

Proposition 2.55. Let A be a commutative k-algebra and $\nu : A \setminus \{0\} \rightarrow P \subset M$ be an injective valuation onto a finitely-generated coideal partial commutative monoid P, where the monoid M is endowed with a linear well-ordering \prec . Assume that $c_1, \ldots, c_s \in P$ is a family of generators of P. Take $a_1, \ldots, a_s \in A$ such that $\nu(a_i) = c_i, 1 \leq i \leq s$. Then $A = k[a_1, \ldots, a_s]/I$ for a suitable ideal $I \subset k[a_1, \ldots, a_s]$. Consider a linear ordering \triangleleft on monomials in a_1, \ldots, a_s such that $a_1^{i_1} \cdots a_s^{i_s} \triangleleft a_1^{j_1} \cdots a_s^{j_s}$ if either $M \ni i_1\nu(a_1) \circ \cdots \circ i_s\nu(a_s) \prec j_1\nu(a_1) \circ \cdots \circ j_s\nu(a_s) \in M$

or $i_1\nu(a_1) \circ \cdots \circ i_s\nu(a_s) = j_1\nu(a_1) \circ \cdots \circ j_s\nu(a_s)$ and the monomial $a_1^{i_1} \cdots a_s^{i_s}$ is less than $a_1^{j_1} \cdots a_s^{j_s}$ in deglex. Then the monomials in a_1, \ldots, a_s belonging to the complement of the monomials ideal J of leading monomials of the Gröbner basis of I (relatively to \triangleleft), constitute an adapted basis of A with respect to ν .

Proof. For a monomial $a := a_1^{i_1} \cdots a_s^{i_s}$ such that $i_1\nu(a_1) \circ \cdots \circ i_s\nu(a_s) \in M \setminus P$ it holds $\nu(a) \prec i_1\nu(a_1) \circ \cdots \circ i_s\nu(a_s)$ due to Definitions 2.24 iii), 2.50. Therefore Jcontains all monomials $a_1^{i_1} \cdots a_s^{i_s}$ for which $i_1\nu(a_1) \circ \cdots \circ i_s\nu(a_s) \in M \setminus P$.

On the other hand, among all monomials $a_1^{j_1} \cdots a_s^{j_s}$ with a fixed valuation $v := j_1\nu(a_1) \circ \cdots \circ j_s\nu(a_s) \in P$ all these monomials belong to J except of a single one being minimal in deglex since ν is injective (cf. the proof of Theorem 3.39 ii) and remark 3.46). Denote the latter monomial by a_v . Then the set $\{a_v : v \in P\}$ constitutes an adapted basis of A with respect to ν . \Box

For a commutative partial monoid P we define its rank rk(P) to be the maximal number of elements $c_1, \ldots, c_r \in P$ such that all the elements $i_1c_1 \circ \cdots \circ i_rc_r \in P, i_1, \ldots, i_r \geq 0$ are pairwise distinct. In this case we call elements c_1, \ldots, c_r independent. The following corollary extends Corollary 3.47 to coideal partial monoids.

Corollary 2.56. Let A be a commutative \Bbbk -algebra and $\nu : A \setminus \{0\} \rightarrow P \subset M$ be an injective valuation onto a finitely-generated coideal partial commutative monoid, where monoid M is endowed with a linear well-ordering. Then dim(A) = rk(P).

Proof. Denote r := rk(P) and let $c_1, \ldots, c_r \in P$ be independent. Take $a_1, \ldots, a_r \in A$ for which $\nu(a_i) = c_i, 1 \leq i \leq r$. Then monomials $a_1^{i_1} \cdots a_r^{i_r}, i_1, \ldots, i_r \geq 0$ are k-linearly independent, hence $d := \dim(A) \geq r$.

Conversely, among monomials belonging to the complement of J (see Proposition 2.55) there are monomials b_1, \ldots, b_d such that all monomials in b_1, \ldots, b_d belong to the complement of J taking into account the property of the Gröbner basis (cf. the proof of Corollary 3.47). Then $\nu(b_1), \ldots, \nu(b_d) \in P$ are independent due to Proposition 2.55, hence $r \geq d$. \Box

3. Injective valuations on domains

In this section we consider (more familiar) valuations of algebras in semigroups (rather than in partial semigroups as in section 2).

3.1. Valuations of domains into semigroups. Let C be a semigroup endowed with a linear ordering < compatible with the semigroup operation + (not necessary commutative). For a k-algebra A its valuation we define as a mapping $\nu : A \setminus \{0\} \to C$ such that

$$\nu(\alpha a) = \nu(a), \nu(a + a_0) \le \max\{\nu(a), \nu(a_0)\}, \nu(aa_0) = \nu(a) + \nu(a_0),$$

$$a, a_0, a + a_0 \in A \setminus \{0\}, \alpha \in \mathbb{k}^*.$$

Denote $C_{\nu} := \nu(A \setminus \{0\})$. An example is provided by a semigroup algebra &C with a valuation (see Proposition 2.45)

 $\nu(\alpha_1c_1+\cdots+\alpha_kc_k):=\max\{c_1,\ldots,c_k\},c_1,\ldots,c_k\in C,\alpha_1,\ldots,\alpha_k\in \Bbbk^*.$

Let $C := \langle c_1, \ldots, c_n \rangle$ be a free semigroup generated by c_1, \ldots, c_n . One can define a linear ordering on C as follows (see Lemma 2.13): $c_{i_1} \cdots c_{i_m} < c_{j_1} \cdots c_{j_s}, 1 \leq$ $i_1, \ldots, i_m, j_1, \ldots, j_s \leq n$ iff either m < s or m = s and the vector (i_1, \ldots, i_m) is less than the vector (j_1, \ldots, j_m) with respect to lex. Note that this provides a well-ordering on C.

We say that a valuation ν is *injective* if there exists a k-basis $\{a_c : c \in C_{\nu}\}$ of A, where $\nu(a_c) = c, c \in C_{\nu}$ (such a basis we call *adapted* with respect to ν). Then ν has one-dimensional leaves ([18]). Observe that

$$a_{c_1}a_{c_2} = \alpha(c_1, c_2)a_{c_1+c_2} + \sum_{c < c_1+c_2} \alpha_c c$$

for suitable $\alpha(c_1, c_2) \in \mathbb{k}^*, \alpha_c \in \mathbb{k}$. Observe that due to the associativity in A the following relations are fulfilled:

$$\alpha(c_1, c_2)\alpha(c_1 + c_2, c_3) = \alpha(c_2, c_3)\alpha(c_1, c_2 + c_3).$$

For example, $\{c \in C\}$ is an adapted basis of &C with respect to the tautological valuation (see Definition 2.28).

More generally, for an arbitrary valuation ν on A we say that $\{a_i \in A\}_i$ is an adapted basis [20] (with respect to ν) if for any $a = \sum_j \alpha_j a_j \in A \setminus \{0\}, \alpha_j \in \mathbb{k}^*$ it holds $\nu(a) = \max_j \{\nu(a_j)\}$. In particular, if A has a Khovanskii basis [20] then one can produce relying on it an adapted basis.

Theorem 3.1. Let A be a k-algebra, $\nu : A \setminus \{0\} \twoheadrightarrow C_{\nu}$ be a mapping onto a linearly ordered semigroup C_{ν} such that $\nu(\alpha a) = \nu(a), \nu(a+a_0) \leq \max\{\nu(a), \nu(a_0)\}, a, a_0, a+a_0 \in A \setminus \{0\}, \alpha \in \mathbb{k}^*$. Denote

$$A_c := \{ a \in A \setminus \{ 0 \} : \nu(a) \le c \} \cup \{ 0 \}$$

and $G_c := A_c/A_{< c}$. Consider an associated graded algebra $G := \bigoplus_{c \in C_u} G_c$.

i) ν is a valuation iff G is a domain. In this case $\nu_0(g) = c$ for $g \in G_c^*$ defines a valuation on G^* .

ii) Let C_{ν} be well-ordered and ν be a valuation. Then ν is an injective valuation iff $\dim_{\mathbb{K}}(G_c) = 1$ for any $c \in C_{\nu}$.

iii) Let ν be an injective valuation and $\mathcal{C} = \{c_i\} \subset C_{\nu}$ be a set of generators of a well-ordered C_{ν} . Then A has an adapted basis of the form $\{a_{i_1} \cdots a_{i_k} : (i_1, \ldots, i_k) \in S\}$ for an appropriate set S, where $\nu(a_i) = c_i \in \mathcal{C}$ and $C_{\nu} = \{c_{i_1} \cdots c_{i_k} : (i_1, \ldots, i_k) \in S\}$. Note that \mathcal{C} can be infinite.

iv) For a valuation ν on A and a well-ordered C there is an adapted basis of A.

Proof. i) Let ν be a valuation. Denote by $p_c : A_c \twoheadrightarrow G_c$ the projection. For any $g \in G_c \setminus \{0\}, g_0 \in G_{c_0} \setminus \{0\}$ take $a \in p_c^{-1}(g), a_0 \in p_{c_0}^{-1}(g_0)$. Then $\nu(a) = c, \nu(a_0) = c_0$. Since $\nu(aa_0) = \nu(a) + \nu(a_0)$, it holds $aa_0 \notin A_{<(c+c_0)}$. Therefore $gg_0 = p_{c+c_0}(aa_0) \neq 0$, i.e. G is a domain.

In a similar manner one can verify the inverse statement.

ii) Let $\dim(G_c) = 1$ for any $c \in C_{\nu}$. For each $c \in C_{\nu}$ pick $a_c \in A_c$ such that $\nu(a_c) = c$. We claim that the elements $\{a_c : c \in C_{\nu}\}$ constitute an adapted basis of A with respect to ν (this implies the injectivity of ν).

Clearly, the elements $\{a_c : c \in C_\nu\}$ are linearly independent. For any element $a \in A \setminus \{0\}$ with $\nu(a) = c_0$ there exists (and unique) $\alpha \in \mathbb{k}^*$ such that $\nu(a - \alpha a_{c_0}) < c_0$ since dim $(G_{c_0}) = 1$. Applying a similar argument to $a - \alpha a_{c_0}$ (in place of a), unless

 $a - \alpha a_{c_0} = 0$, and continuing in this way, we arrive eventually at a decomposition of a in a linear combination of the elements from $\{a_c : c \in C_{\nu}\}$, taking into account that C_{ν} is well-ordered. The claim is proved.

In a similar manner one can verify the inverse statement.

iii) follows from ii).

iv) Choose a basis $\{b_{c,i} : i \in I_c\}$ of G_c and elements $a_{c,i} \in A_c$ such that $p_c(a_{c,i}) = b_{c,i}$. We claim that $\mathcal{A} := \{a_{c,i} : c \in C, i \in I_c\}$ constitute an adapted basis of \mathcal{A} .

Indeed, consider $a = \sum_{c,j} \alpha_{c,j} a_{c,j} \in A \setminus \{0\}, \alpha_{c,j} \in \mathbb{k}^*$. Denote a subsum $e := \sum_l \alpha_{c_0,l} a_{c_0,l}$ which ranges over all l such that $c_0 := \nu(a_{c_0,l}) = \max_{c,j} \{\nu(a_{c,j})\}$. Then $\nu(e) = c_0$ owing to the choice of $a_{c,i}$. Hence $\nu(a) = c_0$. In particular, the elements of \mathcal{A} are independent.

Similar to the proof above of ii) one can express any element of A as a linear combination of elements from \mathcal{A} which proves the claim. \Box

Remark 3.2. If ν is an injective valuation on an algebra A over an radically closed field onto a well-ordered finitely-generated monoid C then one can treat A as a deformation of & C (see Proposition 3.53 below).

Now we describe a construction which starting with a valuation on an algebra, produces a valuation on its quotient (cf. Theorem 2.38). Let $A = \bigoplus_{c \in C} A_c$ be a domain over a field k graded by an ordered monoid C. For $a \in A \setminus \{0\}$ denote by $lt(a) \in A_{c_0}$ the leading term of a for suitable $c_0 \in C$, i.e. $a - lt(a) \in \bigoplus_{c < c_0} A_c$. Note that $\nu_0(a) := c_0$ defines a valuation on $A \setminus \{0\}$ (not necessary injective). For an ideal $J \subset A$ denote by $lt(J) \subset A$ the homogeneous ideal generated by lt(f) for $f \in J$.

Theorem 3.3. Let $A = \bigoplus_{c \in C} A_c$ be a domain over a field k graded by an ordered monoid C. For an ideal $J \subset A$ one can define a mapping ν on the algebra $(A/J) \setminus \{0\}$ filtered by C as follows. For $g \in (A/J) \setminus \{0\}$ denote

$$\nu(g) := \min\{\nu_0(g+J)\} \in C.$$

i) $\nu(\alpha g_1) = \nu(g_1), \nu(g_1 + g_2) \le \max\{\nu(g_1), \nu(g_2)\};$

ii) $\nu(g_1g_2) = \nu(g_1) + \nu(g_2)$ for any $g_1, g_2 \in (A/J) \setminus \{0\}$ iff the ideal $lt(J) \subset A$ is prime;

iii) ν is injective iff C is well-ordered and $\dim_{\Bbbk}(A_c/(lt(J) \cap A_c)) = 1$ for each $c \in C$.

Thus, when the conditions in ii), iii) are satisfied, ν is an injective well-ordered valuation of $(A/J) \setminus \{0\}$.

Proof. i) is straight-forward.

One can verify the following lemma.

Lemma 3.4. Assume that for $g \in (A/J) \setminus \{0\}$ it holds $\nu(g) = \nu_0(g+f_0) = c_0, f_0 \in J$. Then $lt(g + f_0) \notin lt(J)$. In addition, for any $c > c_0$ and $lt(g + f) \in A_c, f \in J$ it holds $lt(g + f) \in lt(J)$.

ii) Let lt(J) be prime, and $\nu(g_1) = \nu_0(g_1 + f_1), \nu(g_2) = \nu_0(g_2 + f_2)$ for appropriate $f_1, f_2 \in J$. It holds $lt(g_1 + f_1)lt(g_2 + f_2) \notin lt(J)$ since lt(J) is prime and employing Lemma 3.4. Therefore $\nu(g_1g_2) = \nu(g_1) + \nu(g_2)$ again due to Lemma 3.4.

One can prove ii) in the opposite direction in a similar way.

iii) Let $\dim(A_c/(lt(J)\cap A_c)) = 1$ for any $c \in C$. Then for every $g_1, g_2 \in (A/J) \setminus \{0\}$ such that $\nu(g_1) = \nu(g_2) = c, lt(g_1 + f_1), lt(g_2 + f_2) \in A_c$, we have $lt(g_1 + f_1) - \alpha \cdot lt(g_2 + f_2) \in lt(J)$ for a suitable $\alpha \in \mathbb{k}$. Hence there exists $f \in J, lt(f) = lt(g_1 + f_1) - \alpha \cdot lt(g_2 + f_2)$ for which $\nu_0((g_1 + f_1) - \alpha \cdot (g_2 + f_2) - f) < c$ that establishes the injectivity of ν , taking into account that C is well-ordered (cf. the proof of Theorem 3.1 ii)).

One can prove iii) in the opposite direction in a similar way. \Box

Remark 3.5. Assume that $A = \Bbbk \langle x_1, \ldots, x_n \rangle / J$ for a prime ideal $J \subset \Bbbk \langle x_1, \ldots, x_n \rangle$ such that $x_i \notin J, 1 \leq i \leq n$, and ν is a valuation (not necessary injective) on $A \setminus \{0\}$. Then one can define $\nu_0(x_i) := \nu(x_i), 1 \leq i \leq n$ which provides a grading on $\Bbbk \langle x_1, \ldots, x_n \rangle$. Now if we apply the construction from Theorem 3.3 to the latter graded algebra $\Bbbk \langle x_1, \ldots, x_n \rangle$ and to the ideal J, we arrive at the initial valuation ν on $A \setminus \{0\}$.

Let $\nu : A \setminus \{0\} \to C$ be a well-ordered injective valuation of an algebra A.

Given an ideal J in A, we say that a generating set B of J is a ν -Gröbner basis of J if $(AbA, b \in B)$ is a ν -ensemble, that is,

$$\nu(J \setminus \{0\}) = \bigcup_{b \in B} C_{\nu} \cdot \nu(b) \cdot C_{\nu}$$

3.2. Examples of injective valuations on algebras of dimension 2.

Example 3.6. Consider the following injective valuation on the ring $A := \Bbbk[x, y] \setminus \{0\}$. One can uniquely represent an arbitrary polynomial $f \in A$ as $f = g(y, y^3 - x^2) + xh(y, y^3 - x^2)$ for some polynomials g, h. Define ν and its adapted basis as follows:

$$\nu(y^k(y^3 - x^2)^l) := (2k, l), \ \nu(xy^k(y^3 - x^2)^l) := (2k + 3, l), \ k, l \ge 0.$$

Therefore, $\nu(f) = \max\{\nu(g(y, y^3 - x^2)), \nu(xh(y, y^3 - x^2))\}$. The valuation monoid is $\{(u, v) \in \mathbb{Z}_{\geq 0}^2 : u \neq 1\}$. We consider its linear ordering with respect to deglex, say, with u being higher than v. Thus, ν is not induced by a minimal generating set of $\Bbbk[x, y]$.

One can straightforwardly verify the following proposition.

Proposition 3.7. Let ν be an injective valuation ν on an algebra $A \setminus \{0\}$ with a finitely generated valuation in a well-ordered semigroup C. Consider a partition of C according to [21]. Namely, each element of the partition has a form c + D where $c \in C$ and a semigroup $D \subseteq C$ is isomorphic to $\mathbb{Z}_{\geq 0}^k$ for some k with basis vectors $c_1, \ldots, c_k \in D$. Let $a, a_1, \ldots, a_k \in A$ be such that $\nu(a) = c, \nu(a_1) = c_1, \ldots, \nu(a_k) = c_k$. Then the elements $aa_1^{i_1} \cdots a_k^{i_k}$, $i_1, \ldots, i_k \in \mathbb{Z}_{\geq 0}$ for all the elements of the partition of C form an adapted basis of A with respect to ν .

Let A be a finitely generated k-algebra of dimension d. Let $\nu : A \setminus \{0\} \to C$ be a valuation on $A \setminus \{0\}$ and C be a well-ordered semigroup of a rank r.

For each $c \in C$ pick an arbitrary element $a_c \in A$ such that $\nu(a_c) = c$. Then the elements $\{a_c : c \in C\}$ are k-linearly independent. Therefore, $r \leq d$. Indeed, otherwise take linearly independent $c_0, \ldots, c_d \in C$ (in Grothéndieck group of C), then all the monomials in the elements a_{c_0}, \ldots, a_{c_d} are linearly independent. The obtained contradiction justifies the inequality $r \leq d$. Note that for the latter inequality we did not use the injectivity of ν .

Obviously, one can yield a well-ordered injective valuation on $\mathbb{k}[x_1, \ldots, x_d] \setminus \{0\}$ (in a unique manner) by means of assigning linearly independent vectors $\nu(x_1), \ldots, \nu(x_d) \in \mathbb{Z}_{>0}^d$ and defining a well-ordering on $\mathbb{Z}_{>0}^d$.

Below we produce a different family of well-ordered injective valuations of rank 2 on the polynomial ring $\mathbb{k}[x, y] \setminus \{0\}$ generalizing Example 3.6 in which $\nu(x), \nu(y)$ are linearly dependent.

Proposition 3.8. Let $f = x^n + \sum_{0 \le i < n} f_i x^i$, $f_i \in \mathbb{k}[y]$ be a polynomial such that $m := \deg_y(f_0)$ is relatively prime with n, and $mi + n\deg_y(f_i) \le mn$ for $0 \le i \le n$. Then there is a well-ordered injective valuation $\nu : (\mathbb{k}[x, y] \setminus \{0\}) \to \mathbb{Z}^2_{\ge 0}$ defined as follows on its adapted basis:

(3.1)
$$\nu(x^i y^k f^l) := (mi + nk, l), \ 0 \le i < n, \ 0 \le k, l.$$

Proof. We observe that $\Bbbk[x, y]$ is a finite $\Bbbk[f, y]$ -module with a basis $1, x, \ldots, x^{n-1}$ with an irreducible monic polynomial f(x, y) - f defining x^n . This justifies that in (3.1) we have a basis of $\Bbbk[x, y]$. The right-hand sides of (3.1) are pairwise distinct due to relative primality of m, n.

To verify the multiplicativity of ν note that $mi + ndeg_y(f_i) < mn$ for 0 < i < n, hence

$$\nu(x^{j}) + \nu(x^{n-j}) = (mj + m(n-j), 0) = \nu(y^{m}) = \nu(\sum_{0 \le i < n} f_{i}(y)x^{i} - f) = \nu(x^{n}).$$

One can extend this construction.

Corollary 3.9. Let a ring B be a finite A-module with an integral basis $1, x, \ldots, x^{n-1}$. Let ν be a well-ordered injective valuation on $A \setminus \{0\}$ with a valuation semigroup $C \subseteq \mathbb{Z}_{\geq 0}^d$. Assume that x^n satisfies a polynomial $f = x^n + \sum_{0 \leq i < n} f_i x^i$ where $f_i \in A, 0 \leq i < n$ such that

$$\frac{i\nu(f_0)}{n} \notin G(C), \ 0 < i < n, \ n\nu(f_i) < (n-i)\nu(f_0), \ 0 \le i \le n$$

where G(C) denotes Grothéndieck group of C. Then one can uniquely extend ν to a well-ordered injective valuation ν_1 on $B \setminus \{0\}$ such that $\nu_1(x) = \nu(f_0)/n$. Clearly, the valuation semigroup of ν_1 has the same rank as of ν .

Remark 3.10. Let $\nu : \Bbbk[x, y] \setminus \{0\} \to C$ be a well-ordered injective valuation. When the values $\nu(x), \nu(y)$ are independent, the semigroup C is isomorphic to $\mathbb{Z}^2_{\geq 0}$, while in Proposition 3.8 the semigroup of the produced valuation consists of n copies of (shifted) $\mathbb{Z}^2_{\geq 0}$.

Example 3.11. In Corollary 3.9 we have provided a construction of an extension of a domain with an injective valuation. In the course of this construction the Grothéndieck group of the valuation monoid is also extended. Now we give an example of an extension of a domain with an injective valuation when the Grothéndieck group of monoids does not change.

Let a domain $A_0 := \mathbb{k}[x, y]$ and ν be its valuation onto $\mathbb{Z}_{\geq 0}^2$ such that $\nu(x) = (1, 0), \nu(y) = (0, 1)$ (one can take an arbitrary linear well-ordering on $\mathbb{Z}_{\geq 0}^2$). Consider polynomials $a, b \in A_0$ such that the leading monomial (with respect to ν) of a equals x^k for some $k \geq 1$, while the leading monomial of b equals y^l for some $l \geq 1$. Denote $A := A_0[b/a] \subset \mathbb{k}(x, y)$. Therefore, the extension of ν on $A \setminus \{0\}$ is inherited uniquely from ν . Observe that $\nu(A \setminus \{0\}) \subset \mathbb{Z}^2$ is well-ordered since $l \geq 1$. The following set forms an adapted basis of A:

$$\{x^i y^j : i, j \ge 0\} \bigsqcup \{(b/a)^s x^i y^j : s \ge 1, 0 \le i < k, 0 \le j\}.$$

Indeed, this set spans A. On the other hand, $\nu(x^i y^j) = (i, j)$, $\nu(b/a)^s x^i y^j = (-sk + i, sl + j)$, and these values are pairwise distinct for different i, j, s.

Example 3.12. Consider an injective homomorphism $\mathbb{k}[x, y] \hookrightarrow \mathbb{k}[x - y^{3/2}, y^{1/2}]$ and an injective well-ordered valuation ν_1 on the latter algebra defined by $\nu_1(x - y^{3/2}) :=$ $(-3, 1), \nu_1(y^{1/2}) := (1, 0)$. Then $\nu_1(x - y^{3/2}), \nu_1(y^{1/2})$ are linearly independent (cf. Remark 3.10). One can verify that the restriction of ν_1 to $\mathbb{k}[x, y] \setminus \{0\}$ coincides with ν .

3.3. Valuations on polynomial algebras. The following is a particular case of Corollary 4.17.

Lemma 3.13. Let $\varphi : \mathbb{k}[x_1, \ldots, x_n] \to \mathbb{k}[t_1, \ldots, t_m]$ be an injective homomorphism of algebras. Then the composition $\nu_0 \circ \varphi$ is an injective valuation $\nu_{\varphi} : \mathbb{k}[x_1, \ldots, x_n] \setminus \{0\} \to \mathbb{Z}_{\geq 0}^m$ (here ν_0 denotes the tautological injective valuation $\nu_0 : \mathbb{k}[t_1, \ldots, t_m] \setminus \{0\} \to \mathbb{Z}_{\geq 0}^m$ given by $\nu_0(t_1^{i_1} \cdots t_m^{i_n}) = (i_1, \ldots, i_m) \in \mathbb{Z}_{\geq 0}^m$ with respect to the lexicographical ordering on $\mathbb{Z}_{\geq 0}^n$).

Problem 3.14. Classify all injective valuations $\mathbb{k}[x_1, \ldots, x_n] \setminus \{0\} \to \mathbb{Z}_{\geq 0}^m$.

Problem 3.15. Given $N \geq m$ and a valuation $\nu : \mathbb{k}[x_1, \ldots, x_N] \setminus \{0\} \to \mathbb{Z}_{\geq 0}^m$, describe all subalgebras \mathcal{A} of $\mathbb{k}[x_1, \ldots, x_N]$ such that

- $\mathcal{A} \cong \mathbb{k}[t_1, \ldots, t_m]$
- The restriction of ν to $\mathcal{A} \setminus \{0\}$ is injective.

For instance, if N = 2, m = 1, and ν is given by a locally nilpotent derivation E (see Lemma 4.9), then $\mathcal{A} = \Bbbk[t]$, where E(t) = 1. More generally, if a nilpotent group U acts on a variety X, and ν is a string valuation based on the action of Lie(U) on $\Bbbk[X]$, then we search for subgroups U' of U such that $\mathcal{A} = \Bbbk[X]^{Lie(U')}$.

This problem is related to the following linear algebra problem.

Problem 3.16. Let $F = (V_1 \subset \cdots \subset V_m = \mathbb{C}^N)$ be a partial flag in \mathbb{C}^N . Describe the set $Gr(m, N)_F$ of all $U \in Gr(m, N)$ such that $\dim(V_i \cap U) = i$ for $i = 1, \ldots, m$.

It is obvious that any $A \in Mat_{m \times N}(\mathbb{Z})$ such that $A \cdot \mathbb{Z}_{\geq 0}^N \subset \mathbb{Z}_{\geq 0}^m$ must belong to $Mat_{m \times N}(\mathbb{Z})$

Problem 3.17. Given a finite subset S of $\mathbb{Z}_{\geq 0}^N$, classify all $A \in Mat_{m \times N}(\mathbb{Z}_{\geq 0})$ such that the restriction of the map $x \to Ax$ to the complement $\mathbb{Z}_{\geq 0}^N \setminus \left(\bigcup_{v \in S} (v + \mathbb{Z}_{\geq 0}^N)\right)$ is injective.

Problem 3.18. Classify Zariski closed subsets $X \subset \mathbb{A}^N$ by injective valuations on $\Bbbk[X]$ and vice versa.

3.4. Injective well-ordered valuations on varieties based on tropical geometry. In the sequel we provide a realization of the construction from Theorem 3.3. Let $I \subseteq \mathbb{k}[X_1, \ldots, X_n]$ be a prime ideal where \mathbb{k} is a field of zero characteristic. Our purpose is to construct injective well-ordered valuations on the quotient ring $A \setminus \{0\} = (\mathbb{k}[X_1, \ldots, X_n]/I) \setminus \{0\}$ which are induced from the tautological valuation $\nu_0(X_1^{j_1} \cdots X_n^{j_n}) := (j_1, \ldots, j_n) \in \mathbb{Z}_{\geq 0}^n$ on $\mathbb{k}[X_1, \ldots, X_n] \setminus \{0\}$. Note that to determine ν_0 completely, one has to fix also a linear ordering on $\mathbb{Z}_{\geq 0}^n$.

For the sake of convenience we need to describe linear orders \prec on monomials $X^J = X_1^{j_1} \cdots X_n^{j_n}$ compatible with the product, i.e. $X^J \prec X^K$ implies $X^{J+L} \prec X^{K+L}$, in a different language than in section 4.6. To this end, we introduce an infinitesimal ε , i.e. $0 < \varepsilon < y$ for any $0 < y \in \mathbb{R}$. Then $\mathbb{R}[\varepsilon]$ is an ordered ring. Assign weights $w_i = w(X_i) \in \mathbb{R}_{\geq 0}[\varepsilon], 1 \leq i \leq n$. This induces a linear (non-strict) order on monomials X^J according to the value of $w_1j_1 + \cdots + w_nj_n \in \mathbb{R}_{\geq 0}[\varepsilon]$. This determines a well-ordering, in other words, there does not exist a strictly decreasing sequence of monomials. It is proved in [31] and in Theorem 9 [11] (in a different language) that any linear order on monomials can be obtained in the described manner. For instance, for two variables lex corresponds to the vector of weights $(1, \varepsilon)$, and deglex corresponds to $(1 + \varepsilon, 1)$.

Definition 3.19. Denote $d := \dim A$. Consider the tropical variety $T := Trop(I) \subseteq \mathbb{R}^n$ [25]. One can view each element of T as a hyperplane in \mathbb{R}^n which supports from above Newton polytope $N(f) \subset \mathbb{R}^n$ at least at two points (thus, at least at an edge) for every $f \in I$. In such a case we say that this edge is located on the roof of N(f). Then T is equidimensional of dimension d [25] being a finite union of polyhedra each of dimension d. Every polyhedron corresponds to a union of hyperplanes containing a (unique) common subplane of dimension n - d which is dual to the polyhedron (we call these subplanes **common** for the tropical variety T). Every such common subplane $H \subset \mathbb{R}^n$ is supporting to N(f) for any $f \in I$ and is definable by linear equations with rational coefficients.

We extend Trop(I) considering

$$ETrop(I) := Trop(I) \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon] \subset (\mathbb{R}[\varepsilon])^n$$

where ETrop(I) satisfies the same linear inequalities as Trop(I). Thus, one can view Etrop(I) still as a finite union of polyhedra. Each hyperplane from ETrop(I) contains $H \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$ for some common subplane H of Trop(I) and supports $N(f) \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$ at least at two points.

We call a subplane H **prop** if $H_0 \cap \mathbb{R}^n_{\geq 0} = \{X_{l_1} = \cdots = X_{l_m} = 0\} \cap \mathbb{R}^n_{\geq 0}$ for suitable $1 \leq l_1, \ldots, l_m \leq n$ where H_0 is parallel to H and contains the origin $(0, \ldots, 0)$.

Let us fix a common subplane H for the time being. We say that the ideal I is saturated (with respect to H) if for any pair of integer points $u, v \in \mathbb{Z}_{\geq 0}^n$ such that $v-u \in H$ there exists a polynomial $f \in I$ whose Newton polytope N(f) possesses an edge (u, v) on its roof. Below (see Theorem 3.21) under the condition of saturation we obtain an injective valuation, so this condition is stronger than the property that an initial ideal corresponding to H is prime in $\Bbbk[X_1, \ldots, X_n]$ (cf. Theorem 3.3 ii) and [20]).

Remark 3.20. In fact, one can reduce the condition of saturation to a finite number of conditions. Indeed, consider a semigroup

$$G := \{ (u, v) : u, v \in \mathbb{Z}_{>0}^n, u - v \in H \} \subset \mathbb{Z}_{>0}^{2n}.$$

Due to Gordan's lemma [13] G is finitely generated. Among its generators select all (u, v) such that $u \neq v$. Denote a vector $(w'_1, \ldots, w'_n) =: u - v$ and a vector $w := (w_1, \ldots, w_n) =: (w'_1, \ldots, w'_n)/GCD(w'_1, \ldots, w'_n)$. Introduce points (3.2) $u_0 := (\max\{w_1, 0\}, \ldots, \max\{w_n, 0\}), v_0 := (\max\{-w_1, 0\}, \ldots, \max\{-w_n, 0\}) \in \mathbb{Z}^n_{\geq 0}$.

Then $u_0 - v_0 = w$.

One can verify that it suffices for the saturation to impose for all the constructed pairs of points u_0, v_0 (3.2) the existence of a polynomial $f \in I$ such that N(f) has an edge (u_0, v_0) on its roof.

From now on we assume that the subplane H is prop and I is saturated (with respect to H). Our aim is to produce a valuation $\nu := \nu_H$ on $A \setminus \{0\}$. Denote $H_{\mathbb{Z}} := H \cap \mathbb{Z}^n$. The following construction of a valuation is similar to [20].

Consider an epimorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n/H_{\mathbb{Z}}$. Then the image $C := \varphi(\mathbb{Z}_{\geq 0}^n)$ is a semigroup cone (since H is prop). The valuation ν under production will have Cas its valuation cone. Choose some linear ordering < on C for definiteness by fixing a prop hyperplane determined by a vector $(w_1, \ldots, w_n) \in (\mathbb{R}_{\geq 0}[\varepsilon])^n$ from ETrop(I)which contains $H \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$.

Take $0 \neq a \in A$. Assume that there exists $f \in a + I$ such that its Newton polytope N(f) contains no edge in H. Then there is a unique vertex v of N(f) with the maximal value of the ordering of $\varphi(v) \in C$. Put $\nu(a) := \varphi(v)$.

Let us establish the correctness of this definition. If otherwise, for some $f_1 \in a + I$ its Newton polytope $N(f_1)$ has a unique vertex v_1 with the maximal value of the ordering of $\varphi(v_1)$, then $\varphi(v) = \varphi(v_1)$ taking into account that $f - f_1 \in I$.

Next we show that for any $0 \neq a \in A$ there exists $f \in a + I$ for which N(f) contains no edge in H. Indeed, take $f \in a + I$ such that the vertices v of N(f) with the maximal value of the ordering of $\varphi(v) \in C$ are minimal among all $f \in a + I$. If u is another vertex of N(f) for which $\varphi(v) = \varphi(u)$, i. e. an interval (u, v) lies in H, then due to the saturation condition there exists $g \in I$ whose Newton polytope N(g) contains an edge (u, v) on its roof. Therefore, for a suitable $\alpha \in \mathbb{k}$ the support of the polynomial $f + \alpha g$ does not contain u. Continuing in this way, we arrive eventually to a polynomial $f_1 \in a + I$ such that its Newton polytope $N(f_1)$ contains a single vertex w_0 with the maximal ordering of $\varphi(w_0) \in C$ greater than the orderings of $\varphi(w)$ for all other vertices w of $N(f_1)$. Clearly, $\varphi(w_0) = \varphi(v)$ due to the choice of f satisfying the minimality property.

Observe that we have proved at the same time that one can equivalently define

(3.3)
$$\nu(a) = \min_{f \in a+I} \max_{v \in N(f)} \{\varphi(v)\}$$

where $v \in N(f)$ means that v is a vertex of N(f).

Thus, the valuation ν on $A \setminus \{0\}$ is defined correctly. If $0 \neq a_1, a_2 \in A$ then take polynomials $f_1 \in a_1 + I$, $f_2 \in a_2 + I$ such that Newton polytope $N(f_1)$ (respectively, $N(f_2)$) contains a unique vertex v_1 (respectively, v_2) such that $\varphi(v_1)$ (respectively, $\varphi(v_2)$) has a greater ordering than $\varphi(w)$ for all other vertices w of $N(f_1)$ (respectively, $N(f_2)$). Then $\nu(a_1 + a_2) \leq \max\{\varphi(v_1), \varphi(v_2)\} = \max\{\nu(a_1), \nu(a_2)\}$ because of (3.3). In addition, for a polynomial $f_1 f_2 \in a_1 a_2 + I$ its Newton polytope $N(f_1 f_2)$ contains a unique vertex $v_1 + v_2$ such that $\varphi(v_1 + v_2) \in C$ has a greater ordering than all other vertices of $N(f_1 f_2)$, hence $\nu(a_1 a_2) = \varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$.

Now we verify the injectivity of ν . Let $\nu(a_1) = \nu(a_2)$ for $0 \neq a_1, a_2 \in A$. Take $f_1 \in a_1 + I$, $f_2 \in a_2 + I$ with vertices $v_1 \in N(f_1)$, $v_2 \in N(f_2)$ as above. Thus, $\varphi(v_1) = \varphi(v_2)$. Therefore, there exists $g \in I$ such that its Newton polytope N(g) contains an edge (v_1, v_2) on its roof due to the saturation condition. Hence, for any vertex $w \in N(f_1 + \alpha f_2 + \beta g)$ we have $\varphi(w) < \varphi(v_1) = \nu(a_1)$ for appropriate $\alpha, \beta \in \mathbb{k}$. Thus, $\nu(a_1 + \alpha a_2) < \nu(a_1)$, see (3.3), which justifies the injectivity of ν .

We summarize the proved above in the following theorem.

Theorem 3.21. Let $A = \Bbbk[X_1, \ldots, X_n]/I$ be a domain of dimension d. Let $H \subset \mathbb{R}^n$ be one of a finite number of (rationally definable) common subplanes of dimension n-d dual to a (highest dimensional) polyhedron of dimension d of the tropical variety $Trop(I) \subset \mathbb{R}^n$ (see Definition 3.19). Assume that H is prop and I is saturated with respect to H. Consider a natural epimorphism $\varphi : (\mathbb{R}[\varepsilon])^n \to (\mathbb{R}[\varepsilon])^n/(H \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon])$. Fix a prop hyperplane from ETrop(I) which contains $H \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$, it determines a linear order on $\varphi(\mathbb{Z}^n_{\geq 0})$. Then (3.3) defines a well-ordered injective valuation ν on $A \setminus \{0\}$ having a valuation cone $\varphi(\mathbb{Z}^n_{>0})$.

Remark 3.22. The constructions of injective valuations on $\Bbbk[x, y] \setminus \{0\}$ from Example 3.6 and Proposition 3.8 are particular cases of Theorem 3.21 when one represents $\Bbbk[x, y] \simeq \Bbbk[x_1, \ldots, x_n]/I$ for suitable ideals $I \subset \Bbbk[x_1, \ldots, x_n]$.

Let (M, \cdot) be a (not necessary commutative) monoid. We say that an equivalence relation \sim is admissible if $u \sim v$ implies $wu \sim wv, uw \sim vw$ for any $u, v, w \in M$. Then one can define a quotient monoid M/\sim on equivalence classes. A linear order \prec on equivalence classes $U \prec V$ (or on M/\sim) is defined as $u \prec v$ for any $u \in U, v \in V$, we require that this linear order on M/\sim is correct. The latter linear order is admissible if $U \prec V$ implies $UW \prec VW, WU \prec WV$ (cf. Definition 2.1). Below we consider only admissible equivalence relations and linear orders.

Denote by $M := \langle a_1, \ldots, a_s \rangle$ the free monoid generated by a_1, \ldots, a_s . Let $A := \&\langle a_1, \ldots, a_s \rangle / I$ be a (not necessary commutative) algebra where $I \subset \&\langle a_1, \ldots, a_s \rangle$ is an ideal. We say that an equivalence relation \sim on M and a linear order \prec on M / \sim are **compatible with** I if for any element

(3.4)
$$f = \sum_{u \in supp(f) \subset M} \alpha_u u \in I, \, \alpha_u \in \mathbb{k}^*$$

there are elements $u_1, u_2 \in supp(f)$ such that $u_1 \sim u_2$ and for every $u \in supp(f)$ it holds $u \leq u_1$. One can treat this concept as a generalization of the tropical variety of I to the non-commutative case.

We say that I is saturated with respect to \sim, \prec if for any pair $u_1 \sim u_2, u_1 \neq u_2$ there exists f of the form (3.4) such that $u_1, u_2 \in supp(f)$ and for every $u \in$

 $supp(f), u \neq u_1, u_2$ it holds $u \prec u_1$. Similarly to the proof of Theorem 3.21 one can verify the following proposition.

Proposition 3.23. Let $A := \Bbbk \langle a_1, \ldots, a_s \rangle / I$ be an algebra, \sim be an admissible equivalence relation on the free monoid $M := \langle a_1, \ldots, a_s \rangle$, and \prec be an admissible well order on M / \sim . Assume that \sim, \prec are compatible with I, and I is saturated with respect to \sim, \prec . Then there is an injective valuation $\nu : A \setminus \{0\} \twoheadrightarrow M / \sim$ defined as follows: for $f \in A \setminus \{0\}$ put $\nu(f)$ as the minimal equivalence class U_0 such that

$$f = \alpha_{u_0} u_0 + \sum_{u \in M} \alpha_u u, \, \alpha_{u_0}, \alpha_u \in \mathbb{k}^*,$$

where $u_0 \in U_0$ and $u \prec u_0$ for all u (cf. (3.3)). In addition, one can pick an adapted basis among monomials from M.

3.5. Injective well-ordered valuations on algebraic curves.

Remark 3.24. Assume that the field k is algebraically closed and $A = \mathbb{k}[X_1, \ldots, X_n]/I$ is a domain. Let $\nu : A \setminus \{0\} \to C$ be a valuation and let $\psi : C \to \mathbb{Q}$ be a homomorphism preserving the order. There exist Puiseux series $(x_1, \ldots, x_n) \in \mathbb{k}((\varepsilon^{1/\infty}))^n$ satisfying I of the form $\alpha_0 \varepsilon^{s_0/q} + \alpha_1 \varepsilon^{s_1/q} + \cdots \in \mathbb{k}((\varepsilon^{1/\infty}))^n$ where integers $s_0 > s_1 > \cdots$ decrease, such that $ord(x_i) = \psi \circ \nu(X_i), 1 \leq i \leq n$ [25].

Proposition 3.25. Let a valuation ν on an irreducible curve $A \setminus \{0\} := \mathbb{k}[x, y]/(f) \setminus \{0\}, f \in \mathbb{k}[x, y]$ fulfill Theorem 3.21, i.e. ν is injective and $\nu(a) \geq 0$ for any $a \in A \setminus \{0\}$, where \mathbb{k} is algebraically closed. Then there exists an injective homomorphism $\eta : A \hookrightarrow \mathbb{k}((\varepsilon^{1/\infty}))$ such that for every $a \in A \setminus \{0\}$ it holds $\nu(a) = \operatorname{ord}(\eta(a))$, provided that $\nu(x) = 1$ for normalization.

Proof. According to Theorem 3.21 Newton polygon N_f has an edge with endpoints (p, 0), (0, q) for relatively prime $p, q \ge 1$. Let $p \le q$ for definiteness. Then the equation f(x, y) = 0 has a Puiseux series solution of the form $y(x) = \alpha x^{p/q} + \ldots$ where $\alpha \in \mathbb{k}^*$ and the terms in dots contain powers of x less than p/q.

One can define $\eta(x) := \varepsilon, \eta(y) := y(\varepsilon)$. Then η is injective since f is irreducible. The monomials $\mathbf{B} := \{x^i y^j : 0 \le i < \infty, 0 \le j < q\}$ constitute a basis of A. The orders $ord(\eta(x^i y^j)) = i + jp/q = \nu(x^i y^j)$ are pairwise distinct for the monomials from \mathbf{B} . \Box

One can prove a certain converse statement to Proposition 3.25.

Remark 3.26. Let for a polynomial $f \in \Bbbk[x, y]$ where \Bbbk is algebraically closed, its Newton polygon $N_f \subset \mathbb{R}^2$ is not of the shape from Proposition 3.25, i.e. N_f does not contain an edge with vertices (p, 0), (0, q) with relatively prime p, q. Then the imbedding $\eta : A := \Bbbk[x, y]/(f) \hookrightarrow \Bbbk((x^{1/\infty}))$ into the field of Puiseux series induces a valuation $\nu : A \setminus \{0\} \to \mathbb{Q}$ by a formula $\nu(a) := ord(\eta(a))$, being not an injective well-ordered.

Any automorphism φ of $\Bbbk[x, y]$ produces an injective valuation on the algebra $\Bbbk[x, y]/(f \circ \varphi) \setminus \{0\}.$

Consider an algebra $A := \mathbb{k}[x, y]/(f)$ of a curve where f is irreducible. Let $A \hookrightarrow \mathbb{k}((x^{1/\infty}))$ be an injective homomorphism into the field of Newton-Puiseux series. We investigate when this induces an injective well-ordered valuation $\nu(= ord)$ on $A \setminus \{0\}$. W.l.o.g. one can suppose that ord(x) = 1 and $f = y^d + f_1$ is normalized, i.e. $deg_y(f_1) < d$.

Lemma 3.27. Let $M \subset \mathbb{k}((x^{1/\infty}))$ be a free $\mathbb{k}[x]$ -module of a rank d. Then $M \setminus \{0\}$ admits a $\mathbb{k}[x]$ -basis s_1, \ldots, s_d such that $ord(s_1), \ldots, ord(s_d)$ are non-negative and $ord(s_i) - ord(s_j) \notin \mathbb{Z}$ for each pair $1 \leq i \neq j \leq d$ iff for any $s \in M \setminus \{0\}$ it holds $ord(s) \geq 0$.

Proof. In one direction the lemma is evident, so assume that $ord(s) \ge 0$ for any $s \in M \setminus \{0\}$. Let $p_1, \ldots, p_d \in M$ be a $\Bbbk[x]$ -basis of M. If $ord(p_i) - ord(p_j) \in \mathbb{Z}_{\ge 0}$ for some $1 \le i \ne j \le d$ and $ord(p_i - \alpha x^{ord(p_i)}) < ord(p_i)$, $ord(p_j - \beta x^{ord(p_j)}) < ord(p_j)$ for suitable $\alpha, \beta \in \Bbbk^*$, one can replace p_j by $p'_j := p_j - (\beta/\alpha) x^{ord(p_i) - ord(p_j)} p_i$. Clearly, $ord(p'_j) < ord(p_j)$. Continuing in this way, we arrive to a required basis s_1, \ldots, s_d .

Remark 3.28. Let $f = y^d + f_1 \in \mathbb{Z}[x, y]$ be normalized. Assume that the bit-sizes of the integer coefficients of f do not exceed L. Here we agree that the field $\mathbb{k} = \overline{\mathbb{Q}}$.

For a root $Y \in \mathbb{k}((x^{1/\infty}))$ of f consider a free $\mathbb{k}[x]$ -module $M \subset \mathbb{k}((x^{1/\infty}))$ with a basis $1, Y, \ldots, Y^{d-1}$. Then the algorithm designed in the proof of Lemma 3.27 either yields a basis s_1, \ldots, s_d of M such that $ord(s_i) \geq 0$ and $ord(s_i) - ord(s_j) \notin \mathbb{Z}$ for every pair $1 \leq i \neq j \leq d$ or the algorithm discovers an element $s \in M$ such that ord(s) < 0.

The complexity of the algorithm is polynomial in d, $deg_x(f)$, L. It follows from the polynomial complexity bound for developing Newton-Puiseux series [10].

Now we are able to summarize the obtained above in the following corollary.

Corollary 3.29. Let $A = \Bbbk[x, y]/(f)$ be an algebra of an irreducible curve. Let $Y \in \Bbbk((x^{1/\infty}))$ be a root of f in the field of Newton-Puiseux series. Denote by $M \subset \Bbbk((x^{1/\infty}))$ the $\Bbbk[x]$ -module generated by $1, Y, \ldots, Y^{d-1}$. The valuation ord on $A \setminus \{0\}$ induced by means of an injective homomorphism $A \hookrightarrow \Bbbk((x^{1/\infty}))$ where $y \to Y$, is injective and well-ordered iff for any $s \in M \setminus \{0\}$ it holds $ord(s) \ge 0$ (agreeing ord(x) = 1).

In the case of $f \in \mathbb{Z}[x, y]$ and $\mathbb{k} = \overline{\mathbb{Q}}$ there is an algorithm which either yields an adapted (with respect to ord) $\mathbb{k}[x]$ -basis of A or discovers an element $s \in M \setminus \{0\}$ such that ord(s) < 0.

Remark 3.30. Due to Lemma 3.27 an adapted basis yielded in Corollary 3.29 has a form $\{s_i x^j : 1 \le i \le d, 0 \le j\}$ for appropriate elements $s_i \in M, 1 \le i \le d$.

Example 3.31. Let $f := (y^2 - x)^3 - 8x^2$. The Newton-Puiseux expansion of its root is $Y = x^{1/2} + x^{1/6} + \cdots$. Denote $a := y^2 - x$. Newton polygon N_f has an edge with the endpoints (3,0), (0,6). Therefore, it does not fulfill the conditions of Theorem 3.21. Nevertheless, the algebra $A := \overline{\mathbb{Q}}[x,a] \setminus \{0\}$ admits an injective well-ordered valuation *ord* with an adapted basis of a form

$$x^j, yx^j, ax^j, ayx^j, a^2x^j, a^2yx^j, j \ge 0$$

due to Corollary 3.29. It holds ord(y) = 1/2, ord(a) = 2/3, ord(ay) = 7/6, $ord(a^2) = 4/3$, $ord(a^2y) = 11/6$.

Now we proceed to a proof of a converse statement to Corollary 3.29: if an algebra $A := \mathbb{k}[x, y]/(f) \setminus \{0\}$ of an irreducible curve admits an injective well-ordered valuation ν , then ν is inherited from an injective homomorphism $A \hookrightarrow \mathbb{k}((x^{1/\infty}))$ under which y is mapped to a root of f (and in addition, ν does not depend on a choice of a root). We agree that $\nu(x) = 1$. Denote $f = y^d + f_1, deg_y(f_1) < d$.

We will repeatedly make use of the following easy observation. Let $a = \sum_{0 \le i < d} \alpha_i y^i \in A$ and g(a) = 0 for a suitable polynomial $g \in k[x, z], deg_z(g) \le d$. Then the value $\nu(a)$ is among the slopes of the edges of Newton polygon N_q .

First, we recall some properties of Newton-Puiseux expansions of the roots in $\mathbb{k}((x^{1/\infty}))$ of f (see e.g. [36]). There is a partition of the roots of f into classes of cardinalities d_1, \ldots, d_k where $d_1 + \cdots + d_k = d$. For each class of a cardinality d_i every root from this class has a form

(3.5)
$$Y = \sum_{j \ge 0} \beta_j x^{p_j/d_i} \in \mathbb{k}((x^{1/\infty}))$$

where integers $p_0 > p_1 > \cdots$ decrease. Moreover, all the roots from this class are exhausted by Newton-Puiseux series

(3.6)
$$\sum_{j\geq 0} \beta_j \omega^{p_j} x^{p_j/d_i}$$

where ω ranges over the roots of unity of the degree d_i . In the process of Newton-Puiseux expanding of Y for any intermediate current polynomial $h \in \mathbb{k}[x, y]$ for the slope $p/q \in \mathbb{Q}$ of each edge of Newton polygon N_h it holds $q|d_i$.

Lemma 3.32. If an algebra $A = k[x, y]/(f) \setminus \{0\}$ of a curve admits an injective well-ordered valuation ν then the roots of f in the field of Newton-Puiseux series constitute a single class.

Proof. Denote by d_1, \ldots, d_k the cardinalities of the classes of the roots of f. Consider an element $a = \sum_{0 \le i < d} \alpha_i y^i \in A \setminus \{0\}$. Let g(a) = 0 for an appropriate polynomial $g \in \Bbbk[x, y], \deg_y(h) \le d$. Then $g(\sum_{0 \le i < d} \alpha_i Y^i) = 0$ for any root $Y \in \Bbbk((x^{1/\infty}))$ of f. Therefore, for the slope p/q of every edge of Newton polygon N_g it holds $q|d_l$ for suitable $1 \le l \le k$.

Hence the values of ν on $A \setminus \{0\}$ are contained in a set

$$\mathbb{Z}_{>0}/d_1\cup\cdots\cup\mathbb{Z}_{>0}/d_k.$$

Here we use that ν is well-ordered, so non-negative on $A \setminus \{0\}$. Denote by $L_N \subset A$ for an integer $N \ge 0$ the k-linear space with a basis $y^i x^j : 0 \le i < d, 0 \le j < N$. Then $\dim(L_N) = Nd$. On the other hand, ν attains on L_N the values from a set

$$\{0,\ldots,N+const\} \cup \bigcup_{1 \le l \le k, 1 \le p < d_l} (\{0,\ldots,N+const\} + p/d_l).$$

The cardinality of the latter set does not exceed (N+const)(d-k+1). Thus, if $k \geq 2$ then the valuation ν attains on L_N less than $\dim(L_N)$ values, which contradicts to the injectivity of ν . This completes the proof of the lemma.

For any $a = \sum_{0 \le i < d} \alpha_i y^i \in A \setminus \{0\}$ due to Lemma 3.32 we have

$$\sum_{0 \le i < d} \alpha_i Y^i = \gamma x^{p/q} + \dots \in \mathbb{k}((x^{1/\infty}))$$

where Y is a root of f(3.5), p/q is the leading exponent of Newton-Puiseux expansion, and $\gamma \in \mathbb{k}^*$, $p \in \mathbb{Z}$. Let g(a) = 0 for a polynomial $g \in \mathbb{k}[x, y]$, $deg_y(g) \leq d$. All the roots of g have an expansion of the form $\gamma \omega^p x^{p/d} + \cdots$, where ω ranges over the roots of unity of the degree d. Hence Newton polygon N_g has a unique edge with the slope p/d, thus $\nu(a) = p/d$.

Summarizing, we have established the following theorem.

Theorem 3.33. If an algebra $A = \Bbbk[x, y]/(f) \setminus \{0\}$ of an irreducible curve admits an injective well-ordered valuation ν then ν is inherited from the valuation ord on $\Bbbk((x^{1/\infty}))$ by means of an injective homomorphism $A \hookrightarrow \Bbbk((x^{1/\infty}))$ where y is mapped to a root $Y \in \Bbbk((x^{1/\infty}))$ of f. The value of ν does not depend on a choice of a root.

Remark 3.34. Corollary 3.29 and Theorem 3.33 together describe all the injective well-ordered valuations on a curve, and moreover, provide an algorithm to yield all such valuations.

Example 3.35. We provide a complete description when an algebra A = k[x, y]/(g) where g is a quadratic polynomial, admits an injective well-ordered valuation ν .

First, if g = xy + px + qy + t then either $\nu(x) = 0$ or $\nu(y) = 0$ (cf. Theorem 3.21). In both cases we get a contradiction with the injectivity of ν .

Now we assume that $g = x^2 + exy + by^2 + px + qy + t$ and either $b \neq 0$ or $e \neq 0$. Then $\nu(x) = \nu(y)$ (unless $b = 0, e \neq 0$ when one should consider in addition, another possibility $\nu(x) = 0$, which contradicts to the injectivity, cf. above). Therefore, due to the injectivity, there exists $\alpha \in k$ such that for $u := x + \alpha y$ it holds $\nu(u) < \nu(y)$. Substituting $u - \alpha y$ for x in g, we deduce that $\alpha^2 - \alpha e + b = 0$ (being the coefficient at the highest monomial y^2 in g) and $2\alpha - e = 0$ (being the coefficient at the next highest monomial uy in g). Hence $\alpha = e/2$ and $e^2 - 4b = 0$ (being the discriminant of the highest form of g). Thus, $g = u^2 + pu + (q - ep/2)y + t$.

If $q - ep/2 \neq 0$, we fall in the conditions of Theorem 3.21, therefore A admits an injective well-ordered valuation ν , and the monomials in u constitute an adapted basis of A with respect to ν . By the same token this arguments covers also the case b = e = 0.

Else if q - ep/2 = 0, we have $g = u^2 + pu + t$, hence $\nu(u) = 0$ which contradicts to the injectivity of ν (cf. above).

Thus, A admits an injective well-ordered valuation iff (the discriminant of the highest form of g) $e^2 - 4b = 0$, while $q - ep/2 \neq 0$.

Consider a domain $A = \Bbbk[x, y]/(g)$ where $g \in \Bbbk[x, y]$. We study necessary conditions when $A \setminus \{0\}$ admits an injective well-ordered valuation ν (cf. the sufficient conditions from Theorem 3.21). There exists an edge e of the roof of Newton polygon $\mathcal{N}(g)$ such that for any points (i, j), (k, l) from the edge e it holds $\nu(x^i y^j) = \nu(x^k y^l)$. In this case we say that ν goes along the edge e. **Proposition 3.36.** Let ν be an injective well-ordered valuation on $\mathbb{k}[x, y]/(g) \setminus \{0\}$ which goes along an edge of $\mathcal{N}(g)$ being parallel to the line $\{x = -y\}$, and $\deg(g) > 1$. Then the discriminant of the leading homogeneous form of g vanishes.

Proof. We have $\nu(x) = \nu(y)$ because ν goes along the edge parallel to the line $\{x = -y\}$. The injectivity implies the existence of $0 \neq \alpha \in \mathbb{k}$ such that for $z := x - \alpha y \in A$ it holds $\nu(z) < \nu(x)$. Since $d := \deg(g) > 1$ the element $z \notin \mathbb{k}$, hence $\nu(z) > 0$.

Denote by $h(x, y) := b_0 x^d + b_1 x^{d-1} y + \dots + b_d y^d$ the leading homogeneous form of g, where $b_0, \dots, b_d \in \mathbb{k}$. Replace x in g by $z + \alpha y$ and the resulting polynomial denote by $\tilde{g} \in \mathbb{k}[z, y]$. In \tilde{g} the monomial y^d has the higher valuation than the other monomials. Therefore, the coefficient in \tilde{g} at this monomial, which equals $h(\alpha, 1)$, vanishes. The monomial zy^{d-1} has the higher valuation than the other monomials in \tilde{g} (except of the monomial y^d). Therefore, the coefficient in \tilde{g} at the monomial zy^{d-1} which equals the derivative $h_x(\alpha, 1)$, vanishes as well. Since h and its derivative have a common root, its discriminant vanishes. \Box

3.6. Adapted bases in domains with injective well-ordered valuations.

Remark 3.37. In case when $A = \mathbb{k}[X_1, \ldots, X_n]/(g)$ is a ring of regular functions on an irreducible hypersurface, we consider an edge of Newton polytope N(g) with the endpoints $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathbb{Z}_{\geq 0}^n$. Denote by H the line passing through u, v. The principal ideal (g) is saturated with respect to H iff $\min\{u_i, v_i\} = 0, 1 \leq i \leq n$ and in addition, $u_1, \ldots, u_n, v_1, \ldots, v_n$ have no nontrivial common divisor, cf. Remark 3.20 and (3.2). Moreover, H is prop iff either $0 \neq u, v$ or one of vectors u, v equals 0 and the other one has a single non-zero coordinate equal 1. When H is prop and I is saturated with respect to H, there exists a well-ordered injective valuation ν on $A \setminus \{0\}$ with a valuation cone $\varphi(\mathbb{Z}_{\geq 0}^n) \subset \mathbb{Z}^n/H_{\mathbb{Z}}$ according to Theorem 3.21.

Observe that in this way one can obtain a well-ordered valuation ν on $\Bbbk[x, y] \setminus \{0\} \simeq (\Bbbk[x, y, z]/(z - y^3 + x^2)) \setminus \{0\}$ produced in Example 3.6 (see also Proposition 3.8). Indeed, Newton polytope of the polynomial $f := z - y^3 + x^2$ is a triangle. As H we take the line passing through the edge (2, 0, 0), (0, 3, 0). The principal ideal (f) is saturated with respect to H (cf. Proposition 3.25 and Example 3.38). Therefore, Theorem 3.21 provides just the valuation ν as in Example 3.6.

Example 3.38. Let $g \in X^3 + Y^2 + \mathcal{L}\{1, Y, X, XY, X^2\}$ where \mathcal{L} denote the linear hull. The domain $A := \mathbb{k}[X, Y]/(g)$ defines a curve. Then the line $H = \{2X + 3Y = 0\}$ and $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ is given by $\varphi(i, j) = 2i + 3j$, the valuation cone $\varphi(\mathbb{Z}_{\geq 0}^2) = \mathbb{Z}_{\geq 0} \setminus \{1\}$. The valuation $\nu(X^{i_0}Y^{j_0} + \mathcal{L}\{X^iY^j : 2i + 3j < 2i_0 + 3j_0\}) = 2i_0 + 3j_0$ on $A \setminus \{0\}$ is well-ordered and injective.

Theorem 3.39. Let A be a \Bbbk -algebra.

i) Then for any finite set of its generators x_1, \ldots, x_m there is a finite set of vectors $S \subset \mathbb{Z}_{\geq 0}^m$ such that all monomials $x^w, w \in \mathbb{Z}_{\geq 0}^m$ for which holds $(w - S) \cap \mathbb{Z}_{\geq 0}^m = \emptyset$ form a basis **B** of A;

ii) let $\nu : A \setminus \{0\} \to C$ be an injective valuation onto a monoid C generated by c_1, \ldots, c_m endowed with a linear well ordering \prec . Let $a_1, \ldots, a_m \in A$ be such that
$\nu(a_i) = c_i, 1 \leq i \leq m$. Similar to i) there exists a finite set S of monomials in a_1, \ldots, a_m such that **B** consisting of monomials off the monomial ideal generated by S, form an adapted basis of A with respect to ν .

Proof. i) Choose a finite presentation $A = \mathbb{k}[x_1, \ldots, x_m]/J$ and fix an injective linear weight function $q : \mathbb{Z}_{\geq 0}^m \to \mathbb{R}$ inducing a well-ordering on $\mathbb{Z}_{\geq 0}^m$ and being compatible with the addition: if $q(v_1) < q(v_2)$ then $q(v_1 + v) < q(v_2 + v)$ for any $v, v_1, v_2 \in \mathbb{Z}_{\geq 0}^m$. In particular, one can take $q(u_1, \ldots, u_m) = \alpha_1 u_1 + \cdots + \alpha_m u_m$ for $0 < \alpha_1, \ldots, \alpha_m \in \mathbb{R}$ being Q-linearly independent.

Take a Gröbner basis of J (with respect to the ordering q). In each element $a = \sum_i \beta_i x^{v_i}, \beta_i \in \mathbb{k}$ of the basis choose v_{i_0} with the biggest value of $q(v_{i_0})$ among $q(v_i)$. We call $v_{i_0} := lev(a)$ the leading exponent vector of a. Put S to consist of the leading exponent vectors of all the elements of the basis.

First, we verify that the elements of $\mathbf{B} := \{x^w : w \notin S + \mathbb{Z}_{\geq 0}^m\}$ are k-linearly independent in A. Indeed, otherwise let

$$\sum_{j} \gamma_j x^{w_j} \in J, \, \gamma_j \in \mathbb{k}, w_j \notin S + \mathbb{Z}_{\geq 0}^m.$$

This contradicts to the property of Gröbner bases that the monomial ideal $S + \mathbb{Z}_{\geq 0}^m$ coincides with the ideal of the leading monomials of all the elements of J.

Now we show that any element of the form x^v , $v \in \mathbb{Z}_{\geq 0}^m$ is a k-linear combination of the elements of **B**. If $x^v \notin \mathbf{B}$ then, again due to the property of Gröbner bases, there exists an element $a_0 \in J$ such that its leading monomial coincides with x^v . Consider a linear combination $x^v + \alpha a_0$ for an appropriate (unique) $\alpha \in \mathbb{k}$ for which $q(lev(x^v + \alpha a_0)) < q(v)$. Then we continue in a similar way, taking the biggest monomial in $x^v + \alpha a_0$ which does not belong to **B**, provided that it does exist. This process terminates due to the well-ordering with respect to q. i) is proved.

ii) Again pick positive reals $\alpha_1, \ldots, \alpha_m$ being \mathbb{Q} -linearly independent. Introduce a well-ordering q on the monomials in a_1, \ldots, a_m as follows. We say that $q(a_1^{j_1} \cdots a_m^{j_m}) < q(a_1^{i_1} \cdots a_m^{i_m})$ iff either $\nu(a_1^{j_1} \cdots a_m^{j_m}) \prec \nu(a_1^{i_1} \cdots a_m^{i_m})$ or $\nu(a_1^{j_1} \cdots a_m^{j_m}) = \nu(a_1^{i_1} \cdots a_m^{i_m})$ and $\alpha_1 j_1 + \cdots + \alpha_m j_m < \alpha_1 i_1 + \cdots + \alpha_m i_m$.

The elements a_1, \ldots, a_m are generators of A since ν is injective and \prec is wellordered. Therefore, $A = \Bbbk[a_1, \ldots, a_m]/J$ for certain ideal J. Consider a Gröbner basis of J with respect to q.

We claim that the basis **B** of A (consisting of some monomials in a_1, \ldots, a_m) produced in i), is adapted with respect to ν . Suppose the contrary. Let $\nu(a_1^{i_1} \cdots a_m^{i_m}) = \nu(a_1^{j_1} \cdots a_m^{j_m})$ for two different monomials from the basis **B**. Let $\alpha_1 i_1 + \cdots + \alpha_m i_m > \alpha_1 j_1 + \cdots + \alpha_m j_m$ for definiteness. There exists (and unique) $\beta \in \mathbb{k}$ for which $\nu(a_1^{i_1} \cdots a_m^{i_m} + \beta a_1^{j_1} \cdots a_m^{j_m}) \prec \nu(a_1^{i_1} \cdots a_m^{i_m})$ holds, because ν is injective. There exists an element $a_1^{l_1} \cdots a_m^{l_m} \in \mathbf{B}$ such that $\nu(a_1^{i_1} \cdots a_m^{i_m} + \beta a_1^{j_1} \cdots a_m^{j_m}) = \nu(a_1^{l_1} \cdots a_m^{l_m})$. We continue the process this way. Due to well-ordering of ν the process terminates, and we arrive at an element of the form

(3.7)
$$a_1^{i_1} \cdots a_m^{i_m} + \beta a_1^{j_1} \cdots a_m^{j_m} + \sum_K \beta_K a^K \in J$$

for appropriate $\beta_K \in \mathbb{k}$, where for all the monomials from the latter sum in (3.7) it holds $\nu(a^K) \prec \nu(a_1^{i_1} \cdots a_m^{i_m})$. Thus, $a_1^{i_1} \cdots a_m^{i_m}$ is the highest (with respect to q) monomial in (3.7), hence $a_1^{i_1} \cdots a_m^{i_m} \notin \mathbf{B}$ due to the construction of **B** in i). The obtained contradiction proves the claim and ii).

Remark 3.40. The elements a_1, \ldots, a_m produced in the proof of Theorem 3.39 ii) constitute a Khovanskii basis of A [20].

Remark 3.41. i) The proof of Theorem 3.39 provides an inverse to the construction from Theorem 3.3. Namely, let ν be an injective well-ordered valuation on $A \setminus \{0\}$ with a valuation semigroup C, and a set of generators a_1, \ldots, a_m of A be produced as in the proof of Theorem 3.39. One can represent the polynomial algebra $\Bbbk[a_1, \ldots, a_m] = \bigoplus_{c \in C} D_c$ as a graded domain where a \Bbbk -basis of D_c consists of all the monomials $p = a_1^{i_1} \cdots a_m^{i_m}$ such that $\nu(p) = c$. Then we fall in the conditions of Theorem 3.3.

ii) Theorem 3.39 implies that one can view A as a deformation of &C.

Definition 3.42. Given a basis **B** of an algebra \mathcal{A} we say that a map $\nu : \mathbf{B} \to \mathbb{Z}^m$ is a **B**-prevaluation of \mathcal{A} if

• $\nu(bb') = \nu(b) + \nu(b')$ for any $b, b' \in \mathbf{B}$ such that $bb' \in \mathbf{B}$.

• $\nu(\mathbf{B})$ is a submonoid in \mathbb{Z}^m .

If ν is injective, then, clearly, the basis **B** is naturally labeled by the monoid $\nu(\mathbf{B})$. It is also clear that if $\nu : \mathcal{A} \setminus \{0\} \to \mathbb{Z}^m$ is a valuation, then $\nu|_{\mathbf{B}}$ is a **B**-prevaluation of \mathcal{A} .

The following are immediate

Lemma 3.43. If ν, ν' are injective **B**-prevaluations, then the assignments $a \rightarrow \nu(\nu^{-1}(b))$ define a bijection $\mathbf{K}_{\nu,\nu'}$: $\nu(\mathbf{B}) \xrightarrow{\sim} \nu'(\mathbf{B})$ (we refer to it as a generalized JB bijection).

Problem 3.44. Suppose that \Bbbk is a ring and A is a finitely generated and finitely presented commutative algebra over \Bbbk . If A is a free \Bbbk -module, does it admit a standard basis **B** (i.e., as in Theorem 3.39 i)?

Problem 3.45. Using an adapted basis **B** of Theorem 3.39 i), we can define a multivariate Hilbert series of A by

$$Hilb(A) = \sum_{b \in \mathbf{B}} b$$

By definition, this is a rational function with denominator being the product of $(1 - x_i)$.

Therefore, we can define a multivariate Hilbert polynomial of A as the "numerator" of Hilb(A). The question is whether this definition gives more information about Gr A and A than the ordinary Hilbert series Hilb(A, t).

Remark 3.46. The adapted basis **B** produced in Theorem 3.39 ii) consists of the following elements: for each $c \in C$ take the monomial M in a_1, \ldots, a_m being minimal (with respect to f) among the monomials for which $\nu(M) = c$ holds.

Another description is that **B** consists of all the monomials being k-linearly independent (in A) from less (with respect to f) monomials. For any monomial $M_0 \in \mathbf{B}$ consider the next (with respect to f) monomial $M_1 \in \mathbf{B}$. Then for any monomial M such that $f(M_0) \leq f(M) < f(M_1)$ it holds $\nu(M) = \nu(M_0)$.

Corollary 3.47. Let A be a commutative \Bbbk -algebra, $\nu : A \setminus \{0\} \rightarrow C$ be an injective valuation onto a finitely-generated monoid C of rank r endowed with a linear well ordering. Then $r = d := \dim(A)$.

Proof. First we show that $r \leq d$. Pick independent elements $c_1, \ldots, c_r \in C$ and $a_1, \ldots, a_r \in A$ such that $\nu(a_i) = c_i, 1 \leq i \leq r$. Then all monomials in a_1, \ldots, a_r have pairwise distinct valuations ν , therefore a_1, \ldots, a_r are algebraically independent, thus $r \leq d$. Now we prove the opposite inequality.

Due to Theorem 3.39 **B** is the complement of a monomial ideal generated by the leading monomials of Gröbner basis of the ideal J in the representation $A = \mathbb{k}[a_1, \ldots, a_m]/J$. Therefore, there exist $1 \leq l_1 < \cdots < l_d \leq m$ such that all the monomials in a_{l_1}, \ldots, a_{l_d} belong to **B**, see Proposition 3 in Chapter 9.1 and Proposition 4 in Chapter 9.3 [12]. Hence the elements $\nu(a_{l_1}), \ldots, \nu(a_{l_d})$ are independent in C, taking into account that the basis **B** is adapted to ν due to Theorem 3.39 ii). Thus, $r \geq d$.

Remark 3.48. Assume that C is a (not necessary commutative) monoid generated by c_1, \ldots, c_r . We call the length |c| of $c \in C$ the minimal length of words in c_1, \ldots, c_r equal c. Let C be endowed with a linear well-ordering \prec compatible with the length, i.e. $|c_0| < |c|, c_0, c \in C$ implies $c_0 \prec c$. For example, the ordering described prior to Theorem 3.1 of the free monoid is compatible with the length.

Consider an algebra A having an injective valuation $\nu : A \to C$, and pick elements $a_1, \ldots, a_r \in A$ such that $\nu(a_i) = c_i, 1 \leq i \leq r$. For each $c \in C$ choose a monomial a_c in a_1, \ldots, a_r for which $\nu(a_c) = c$. Then $\{a_c : c \in C\}$ form an adapted basis of A (and a_1, \ldots, a_r form a Khovanskii basis of A). Then the linear subspaces $A_k := \{a \in A : |\nu(a)| \leq k\}, k \geq 0$ constitute a filtration of A, and dim A_k coincides with the cardinality of the set $C_k := \{c \in C : |c| \leq k\}$, moreover $\nu(A_k) = C_k$. We recall that in the commutative case the latter cardinality grows polynomially in k (being a Hilbert polynomial, see e.g. [21]).

The following remark is inverse to Theorem 3.39 ii) and to Remark 3.46. According to Theorem 3.39 ii) and to Remark 3.46 every injective well-ordered valuation on an algebra can be obtained as described in the remark.

Remark 3.49. Let a_1, \ldots, a_m be generators of a commutative k-algebra A endowed with a linear order f on monomials in a_1, \ldots, a_m (compatible with the product).

Consider the family **B** of all the monomials being k-linearly independent in A from the less ones (with respect to f). Then **B** forms a basis of A. For each $M_1, M_2 \in \mathbf{B}$ denote by $h(M_1, M_2)(=h(M_2, M_1)) \in \mathbf{B}$ the leading monomial in the k-linear expansion of the product M_1M_2 in **B**. Assume that for any pair of monomials $M_0, M_1 \in \mathbf{B}$ fulfilling $f(M_0) < f(M_1)$ it holds $f(h(M_0, M_2)) < f(h(M_1, M_2))$. Then one can introduce a monoid C being in a bijective correspondence with **B** determined by the monoid operation h and the linear ordering f.

This induces also an injective well-ordered valuation $\nu : A \setminus \{0\} \rightarrow C$ defined by the leading monomial from **B** in the k-linear expansion. Then **B** is an adapted basis of ν .

One can reorder the monomials in a_1, \ldots, a_m as follows to make the new ordering \triangleleft similar to Theorem 3.39 ii) and to Remark 3.46. We say that for a pair of monomials it holds $a_1^{i_1} \cdots a_m^{i_m} \triangleleft a_1^{j_1} \cdots a_m^{j_m}$ if either $f(\nu(a_1^{i_1} \cdots a_m^{i_m})) < f(\nu(a_1^{j_1} \cdots a_m^{j_m}))$ or $f(\nu(a_1^{i_1} \cdots a_m^{i_m})) = f(\nu(a_1^{j_1} \cdots a_m^{j_m}))$ and $f(a_1^{i_1} \cdots a_m^{i_m}) < f(a_1^{j_1} \cdots a_m^{j_m})$. Then the construction from Theorem 3.39 ii) applied to \lhd produces the same basis **B** which now satisfies the properties from Remark 3.46.

Observe that the valuation produced in Theorem 3.21 fulfills the conditions of Theorem 3.39 ii) and of Remark 3.46. In particular, ν admits an adapted basis of monomials in X_1, \ldots, X_n . The monomials with equal values of ν lie in the planes parallel to H.

3.7. Injective valuations, filtrations and deformations. Now one can establish an inverse statement to Theorem 3.21.

Theorem 3.50. Let A be a commutative domain of dimension d endowed with an injective well-ordered valuation ν onto a finitely-generated monoid. Then there exist a Khovanskii basis X_1, \ldots, X_n of A such that $A = \Bbbk[X_1, \ldots, X_n]/I$, and ν is obtained as in Theorem 3.21.

In other words, there is a prop subplane $H \subset \mathbb{R}^n$ of dimension n - d, being a common subplane for the tropical variety $Trop(I) \subset \mathbb{R}^n$. Moreover, the ideal I is saturated with respect to H. There exists a hyperplane $Q \in ETrop(I) \subset (\mathbb{R}[\varepsilon])^n$ which contains the subplane $H \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$, and Q is determined by a suitable vector $(w_1, \ldots, w_n) \in (\mathbb{R}_{\geq 0}[\varepsilon])^n$. Then ν is defined by (3.3), the valuation monoid $\nu(A \setminus \{0\}) = \varphi(\mathbb{Z}^n_{\geq 0})$ where $\varphi : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n/H$, and the linear order on $\varphi(\mathbb{Z}^n_{\geq 0}) \ni \varphi(i_1, \ldots, i_n)$ is determined by the value of $w_1i_1 + \cdots + w_ni_n$.

Proof. Applying Theorem 3.39 one can find generators X_1, \ldots, X_n of A such that the valuation monoid $C := \nu(A \setminus \{0\})$ equals the set of values $\nu(M)$ over all the monomials $M = X_1^{i_1} \cdots X_n^{i_n}$ in X_1, \ldots, X_n . Then $A = \Bbbk[X_1, \ldots, X_n]/I$ for an appropriate ideal I.

Due to [31] there exist elements $w_1, \ldots, w_n \in \mathbb{R}_{\geq 0}[\varepsilon]$ such that the linear order in C of $\nu(X_1^{i_1} \cdots X_n^{i_n})$ coincides with the order of the values of $w_1 i_1 + \cdots + w_n i_n$ in the semi-ring $\mathbb{R}_{\geq 0}[\varepsilon]$.

Denote by $H \subset \mathbb{R}^n$ a plane being the linear hull of all the vectors of the form $(l_1, \ldots, l_n) - (j_1, \ldots, j_n) \in \mathbb{Z}^n$ where $w_1 l_1 + \cdots + w_n l_n = w_1 j_1 + \cdots + w_n j_n$, the latter is equivalent to $\nu(X_1^{l_1} \cdots X_n^{l_n}) = \nu(X_1^{j_1} \cdots X_n^{j_n})$. Due to Theorem 3.39 the hyperplane Q contains $H \bigotimes_{\mathbb{R}} \mathbb{R}(\varepsilon)$ and supports the Newton polytope $N(g) \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$ for any $g \in I$. Theorem 3.39 also implies that $\dim(H) = n - d$. Hence H is a common subplane of Trop(I). In addition, H is prop since $w_1, \ldots, w_n \in \mathbb{R}_{\geq 0}[\varepsilon]$.

Take two arbitrary points $(l_1, \ldots, l_n), (j_1, \ldots, j_n) \in \mathbb{Z}_{\geq 0}^n$ such that $(l_1, \ldots, l_n) - (j_1, \ldots, j_n) \in H$. Then $w_1 l_1 + \cdots + w_n l_n = w_1 j_1 + \cdots + w_n j_n$. Therefore $\nu(X_1^{l_1} \cdots X_n^{l_n}) = \nu(X_1^{j_1} \cdots X_n^{j_n})$, and due to injectivity and well-ordering of ν there exist $\beta \in \mathbb{k}^*$ and

 $g_1 = \sum_S \gamma_S X^S \in \mathbb{k}[X_1, \dots, X_n]$ such that $\nu(X^S) < \nu(X_1^{j_1} \cdots X_n^{j_n})$ for every X^S occurring in g_1 , and it holds $X_1^{l_1} \cdots X_n^{l_n} - \beta X_1^{j_1} \cdots X_n^{j_n} - g_1 \in I$. Hence I is saturated with respect to H (cf. Theorem 3.21).

Finally, for any $a \in A^*$ one can uniquely express $a = \sum_{b \in B_0} \alpha_b b$ in a basis $\mathbf{B} \supset B_0 \ni b$ produced in Theorem 3.39, $\alpha_b \in \mathbb{k}^*$. Then $\nu(a) = \max_{b \in B_0} \{\nu(b)\}$, thus ν is defined by (3.3).

We say that the linear order \prec defined by $w_1, \ldots, w_n \in \mathbb{R}_{\geq 0}[\varepsilon]$ is archimedian if among w_1, \ldots, w_n there are no infinitesimals. This is equivalent to that for any pair of monomials $m_1, m_2 \neq 1$ there exists an integer N such that $m_1 \prec m_2^N$. The linear order on $\mathbb{Z}_{\geq 0}^n$ is archimedian iff this order is isomorphic to $\mathbb{Z}_{\geq 0}$. For instance, deglex is archimedian, while lex is not.

More generally, we say that a linear order \prec on a commutative monoid C is archimedian if for any elements $1 \neq c_1, c_2 \in C$ there exists an integer N such that $c_2 \prec Nc_1$. Note that if for any $c \in C$ there is at most a finite number of elements $c_0 \in C$ such that $c_0 \prec c$ then C is archimedian and well-ordered. Conversely, if a commutative monoid C is finitely-generated and \prec is an archimedian linear order on C then for any $c \in C$ there is at most a finite number of elements of C less than c. In particular, in this case C is well-ordered. For a not necessary commutative monoid C we also say that a linear order \prec on it is archimedian if for any $c \in C$ there is at most a finite number of elements of C less than c.

Let $c_1, \ldots, c_k \in C$ be a set of generators of a commutative monoid C. Due to [31] there exist elements $w_1, \ldots, w_k \in \mathbb{R}_{\geq 0}[\varepsilon]$ not being infinitesimals such that

 $\nu(j_1c_1 + \dots + j_kc_k) \preceq \nu(l_1c_1 + \dots + l_kc_k) \Leftrightarrow (w_1j_1 + \dots + w_kj_k \leq w_1l_1 + \dots + w_kl_k)$ for any $j_1, \dots, j_k, l_1, \dots, l_k \in \mathbb{Z}_{\geq 0}$. Define a function $W : C \to \mathbb{R}_{\geq 0}[\varepsilon]$ as follows:

$$W(j_1c_1+\cdots+j_kc_k):=w_1j_1+\cdots+w_kj_k.$$

Remark 3.51. Let A be a commutative domain endowed with a valuation (not necessary injective) onto a finitely-generated monoid C with an archimedian linear order \prec . For each $s \in \mathbb{Z}_{>0}$ consider the set

$$A_s := \{ a \in A^* : W(\nu(a)) \le s \} \cup \{ 0 \}.$$

The sequence $A_0 \subset A_1 \subset \cdots$ constitutes a filtration of A. Observe that dim (A_s) is finite since \prec is archimedian.

Remark 3.52. Now let A be a (not necessary commutative) k-algebra endowed with an injective valuation ν to a (not necessary commutative) monoid C with a linear order \prec . Assume also that there is a function $f: C \to \mathbb{Z}_{\geq 0}$ such that $(c_1 \leq c_2) \Rightarrow$ $(f(c_1) \leq f(c_2)), c_1, c_2 \in C$ and $f(c_1 + c_2) \leq f(c_1) + f(c_2)$ satisfying the property that the set $C_n := \{c \in C : f(c) \leq n\}$ is finite for any $n \in \mathbb{Z}_{\geq 0}$. Note that the latter implies that the order \prec is archimedian. Then the subspaces $A_n := \{a \in A :$ $f(\nu(a)) \leq n\} \cup \{0\}, n \in \mathbb{Z}_{\geq 0}$ provide a filtration of A (cf. Remark 3.51) such that dim $A_n = |C_n|$.

Now we assume that the field \Bbbk is radically closed (i.e. each root of an arbitrary degree of any element of \Bbbk also belongs to \Bbbk). Let A be a d-dimensional \Bbbk -algebra

with an injective valuation $\nu : A \setminus \{0\} \to C$ where C is a finitely-generated monoid endowed with a linear well-ordering. Applying Theorem 3.39 construct a Khovanskii basis $x_1, \ldots, x_n \in A$, then $A = \Bbbk[x_1, \ldots, x_n]/I$ for a suitable ideal $I \subset \Bbbk[x_1, \ldots, x_n]$.

Denote by $S \subset \Bbbk[x_1, \ldots, x_n]$ a binomial ideal generated by elements of the form

(3.8)
$$s := \alpha x_1^{i_1} \cdots x_n^{i_n} - \beta x_1^{j_1} \cdots x_n^{j_n}, \, \alpha, \beta \in \mathbb{K}$$

such that $\nu(x_1^{i_1}\cdots x_n^{i_n}) = \nu(x_1^{j_1}\cdots x_n^{j_n})$, and there exists an element $g \in I$ which is the sum of s and of monomials in x_1, \ldots, x_n having valuation ν less than $\nu(x_1^{i_1}\cdots x_n^{i_n})$. Denote a binomial algebra $M := \Bbbk[x_1, \ldots, x_n]/S$. Observe that for any pair of vectors $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \mathbb{Z}_{\geq 0}^n$ there exist s and g as above iff $\nu(x_1^{i_1}\cdots x_n^{i_n}) = \nu(x_1^{j_1}\cdots x_n^{j_n})$ due to Theorem 3.39. In addition, $\nu(x_1^{i_1}\cdots x_n^{i_n}) = \nu(x_1^{j_1}\cdots x_n^{j_n})$ is equivalent to that $(i_1, \ldots, i_n) - (j_1, \ldots, j_n) \in H$ where an (n - d)-dimensional subplane H was constructed in the proof of Theorem 3.50.

Proposition 3.53. (cf. [18]). Let a k-algebra A have an injective valuation ν : $A^* \to C$ where a finitely-generated monoid C is endowed with a linear well-ordering. Assume that k is radically closed. Then both the associated graded algebra $grA := \bigoplus_{c \in C} A_{\leq c}/A_{< c}$ and the binomial algebra M are isomorphic to the monoidal algebra kC.

Proof. There exist $\gamma_1, \ldots, \gamma_n \in \mathbb{k} \setminus \{0\}$ such that the mapping

 $\mu: x_1^{i_1} \cdots x_n^{i_n} \to \gamma_1^{i_1} \cdots \gamma_n^{i_n} \cdot \nu(x_1^{i_1} \cdots x_n^{i_n})$

provides an isomorphism of M and &C because of Theorem 3.39, taking into account that & is radically closed. In other words, if $g \in I$ equals the sum of a binomial (3.8) and of monomials with the valuation ν less than $\nu(x_1^{i_1} \cdots x_n^{i_n}) = \nu(x_1^{j_1} \cdots x_n^{j_n})$ then $\mu(\alpha x_1^{i_1} \cdots x_n^{i_n}) = \mu(\beta x_1^{j_1} \cdots x_n^{j_n})$, i.e. $\alpha \gamma_1^{i_1} \cdots \gamma_n^{i_n} = \beta \gamma_1^{j_1} \cdots \gamma_n^{j_n}$.

Any element $a \in A \setminus \{0\}$ with the valuation $\nu(a) = c$ can be represented uniquely as a k-linear combination of elements of a basis **B** constructed in Theorem 3.39, among which $\alpha x_1^{i_1} \cdots x_n^{i_n}, \alpha \in \mathbb{k}^*, x_1^{i_1} \cdots x_n^{i_n} \in \mathbf{B}$ has the maximal valuation $\nu(x_1^{i_1} \cdots x_n^{i_n}) = c$. We define a mapping $\sigma(a) := \mu(\alpha x_1^{i_1} \cdots x_n^{i_n}) \in \mathbb{k}C$. Then σ defines a correct mapping on grA.

To verify that σ is a homomorphism on grA take monomials $u := x_1^{i_1} \cdots x_n^{i_n}, v := x_1^{k_1} \cdots x_n^{k_n} \in A$. Due to Theorem 3.39 there exists a unique monomial $x_1^{l_1} \cdots x_n^{l_n} \in \mathbf{B}$ such that $\nu(x_1^{l_1} \cdots x_n^{l_n}) = \nu(uv)$. Hence there exists $\alpha \in \mathbb{k}^*$ such that $\nu(\alpha x_1^{l_1} \cdots x_n^{l_n} - uv) < \nu(uv)$ due to Theorem 3.39. Therefore

$$\sigma(uv) = \mu(\alpha x_1^{l_1} \cdots x_n^{l_n}) = \alpha \gamma_1^{l_1} \cdots \gamma_n^{l_n} \nu(uv) = \gamma_1^{i_1+k_1} \cdots \gamma_n^{i_n+k_n} \nu(u)\nu(v) = \sigma(u)\sigma(v)$$

(we use the product notation for the monoid operation).

Finally, one can check that σ is an isomorphism.

Corollary 3.54. Let A be a k-domain of a dimension d over a radically closed field k, and ν be an injective valuation of A^* onto a well-ordered finitely-generated monoid. Then the variety $Spec(gr(A, \nu))$ is toric of dimension d.

Example 3.55. Consider a domain $A := k[x, y]/(x^6 - 2y^4 - 1)$. Take a common line H from the tropical variety $Trop(x^6 - 2y^4 - 1)$ defined by the equation 2i + 3j = 6 (cf. Theorem 3.21) and a corresponding map $\nu : \{x^i y^j : i, j \ge 0\} \to \mathbb{Z}^2_{>0}$ for which

 $\nu(x) = 2, \nu(y) = 3$. We obtain that the graded algebra $gr(A, \nu)$ has a zero divisor iff the polynomial $x^6 - 2y^4$ is reducible over k. In the latter case ν does not provide a valuation on $A \setminus \{0\}$ (see Theorem 3.1), and the variety $Spec(gr(A, \nu))$ is reducible. Otherwise, if $x^6 - 2y^4$ is irreducible over k then ν provides a valuation on A^* being not injective since $\nu(x^3) = \nu(y^2) = 6$, so $\dim(A_{\nu < 6}/A_{\nu < 6}) = 2$.

Example 3.56. We give an example of a non-commutative algebra admitting an injective valuation onto $\mathbb{Z}_{\geq 0}^2$ such that its graded algebra (with respect to the valuation) is non-commutative (unlike Proposition 3.53). Denote by A_q a quantum $\Bbbk(q)$ -algebra generated by x, y satisfying a relation $xy = q^2yx$. Then $\nu(x) = (0, 1), \nu(y) = (1, 0)$ defines an injective valuation $\nu : A_q \setminus \{0\} \twoheadrightarrow \mathbb{Z}_{\geq 0}^2$. The graded algebra grA_q has an adapted basis $\{b(m, n) := q^{-mn}x^my^n : m, n \geq 0\}$ with a multiplication table

$$b(m, n)b(m_1, n_1) = q^{mn_1 - m_1 n}b(m + m_1, n + n_1).$$

Thus, grA_q is a twisted monoidal algebra.

In contrast to A_q , Weyl algebra generated by x, y satisfying a relation xy = yx + 1admitting an injective valuation defined by $\nu(x) = (1,0), \nu(y) = (0,1)$ onto $\mathbb{Z}_{\geq 0}^2$, has a graded algebra isomorphic to the polynomial ring $\Bbbk[x, y]$.

Let $A = \mathbb{k}[x_1, \ldots, x_n]/J$ be a k-algebra with dim A = 1 and $\nu : A^* \twoheadrightarrow C \subset \mathbb{Z}_{\geq 0}$ be an injective valuation (cf. Corollary 3.47). Then there exist non-negative integers r_1, \ldots, r_n such that $\nu(x_1^{i_n} \cdots x_n^{i_n}) \preceq \nu(x_1^{j_1} \cdots x_n^{j_n})$ iff $i_1r_1 + \cdots + i_nr_n \leq j_1r_1 + \cdots + j_nr_n$ (cf. Theorem 3.50).

Take $t \in \mathbb{k}^*$, and for any polynomial $g \in \mathbb{k}[x_1, \ldots, x_n]$ with a monomial $x_1^{i_n} \cdots x_n^{i_n}$ of the highest valuation $\nu(x_1^{i_n} \cdots x_n^{i_n}) = c$ among the monomials of g, replace x_i by $x_i t^{-r_i}, 1 \leq i \leq n$. The resulting polynomial has the form $(g_0 + tg_1)t^{-i_1r_1 - \cdots - i_nr_n}$ where g_0 coincides with the sum of the monomials of g having their valuation equal c, while every monomial in g_1 has the valuation less than c. The resulting ideal we denote by $J_t \subset \mathbb{k}[x_1, \ldots, x_n]$ and denote $A_t := \mathbb{k}[x_1, \ldots, x_n]/J_t$.

Thus, one can view the family A_t , $t \in \mathbb{k}^*$ as a deformation of A_0 being a binomial algebra isomorphic to $\mathbb{k}C$ (see Proposition 3.53) when the field \mathbb{k} is radically closed. Summarizing, we have established the following proposition.

Proposition 3.57. Let $A = \mathbb{k}[x_1, \ldots, x_n]/J$ be a k-algebra with dim A = 1 and $\nu : A \setminus \{0\} \twoheadrightarrow C \subset \mathbb{Z}_{\geq 0}$ be an injective valuation. Then there exist non-negative integers r_1, \ldots, r_n such that for any $t \in \mathbb{k}^*$ the transformation $x_i \to x_i t^{-r_i}, 1 \leq i \leq n$ provides an ideal $J_t \subset \mathbb{k}[x_1, \ldots, x_n]$ and an algebra $A_t := \mathbb{k}[x_1, \ldots, x_n]/J_t$. Obviously, $A_1 = A$. The associated graded algebra $\operatorname{gr} A_t \simeq \mathbb{k}C, t \in \mathbb{k}^*$ (see Proposition 3.53) when \mathbb{k} is radically closed. One can view the family $A_t, t \in \mathbb{k}^*$ as a deformation of A_0 being a binomial algebra isomorphic to $\mathbb{k}C$ (see Proposition 3.53) when the field \mathbb{k} is radically closed.

Conjecture 3.58. For any domain A with injective well-ordered valuation ν there is a family A_t such that $A_1 = A$ and $A_0 = grA$ with respect to the filtration on A induced by ν (as in Proposition 3.57).

Problem 3.59. Let *E* be a locally nilpotent derivation of a domain *A* (see Lemma 4.9). Classify all subalgebras *B* of *A* such that $\lambda_E(B \setminus \{0\}) \cup \{0\}$ is a subalgebra of *A*.

Problem 3.60. Classify all injective decorated valuations (ν, λ) on $\Bbbk[x_1, \ldots, x_n]$ (see Definition 4.5).

Problem 3.61. Given an injective valuation $\nu : \mathbb{k}[x_1, \ldots, x_m] \setminus \{0\} \to \mathbb{Z}_{\geq 0}^m$, is it true that C_{ν} is a finitely generated submonoid of $\mathbb{Z}_{\geq 0}^m$?

Given a submonoid M of \mathbb{Z}^m , denote $\overline{M} := (\mathbb{R}_{\geq 0} \otimes M) \cap \mathbb{Z}^m$ and refer to it as the *saturation* of M. By definition, \overline{M} is a submonoid of \mathbb{Z}^m and M is a submonoid of \overline{M} . We say that M is saturated if $\overline{M} = M$.

Problem 3.62. Suppose that ν is a saturated injective valuation $A \setminus \{0\} \to \mathbb{Z}_{\geq 0}^m$. Is it true that *Spec* A is smooth or rational?

Problem 3.63. If $A \subset \mathbb{k}[x_1, \ldots, x_m]$ and dim A = d < m. Can A be embedded into $\mathbb{k}[y_1, \ldots, y_d]$?

Problem 3.64. Describe all injective valuation on $\mathbb{k}[x_1, \ldots, x_m]$ into $\mathbb{Z}_{\geq 0}^m$ whose valuation monoid is finitely generated but not saturated.

3.8. Algorithm testing a family of generators of a valuation monoid. Let $(f_1, \ldots, f_m) : \mathbb{k}^n \to \mathbb{k}^m, m \leq n$ be a dominant polynomial map, i.e. the polynomials $f_1, \ldots, f_m \in \mathbb{k}[x_1, \ldots, x_n]$ are algebraically independent (cf. Example 3.90). This provides an injective homomorphism $\mathbb{k}[y_1, \ldots, y_m] \hookrightarrow \mathbb{k}[x_1, \ldots, x_n]$, and thereby an injective valuation $\nu : \mathbb{k}[y_1, \ldots, y_m] \setminus \{0\} \to \mathbb{Z}^n_{\geq 0}$ (we fix some injective valuation on $\mathbb{k}[x_1, \ldots, x_n] \setminus \{0\}$) due to Lemma 3.13.

Problem 3.65. Is the valuation monoid $\nu(\Bbbk[y_1,\ldots,y_m]\setminus\{0\}) \subset \mathbb{Z}_{\geq 0}^n$ finitely-generated?

The goal of this subsection is to prove the following proposition.

Proposition 3.66. Let polynomials $f_1, \ldots, f_m \in \mathbb{k}[x_1, \ldots, x_n], m \leq n$ define an injective homomorphism $\mathbb{k}[y_1, \ldots, y_m] \hookrightarrow \mathbb{k}[x_1, \ldots, x_n]$. Given a computable injective valuation on $\mathbb{k}[x_1, \ldots, x_n] \setminus \{0\}$ to $\mathbb{Z}_{\geq 0}^n$ (e.g., lex or deglex), this provides an injective valuation $\nu : \mathbb{k}[y_1, \ldots, y_m] \setminus \{0\} \to \mathbb{Z}_{\geq 0}^n$.

There is an algorithm which given generators $g_1, \ldots, g_p \in \mathbb{k}[y_1, \ldots, y_m]$ of $\mathbb{k}[y_1, \ldots, y_m]$ tests whether the elements

$$\nu(g_1),\ldots,\nu(g_p)\in C:=\nu(\Bbbk[y_1,\ldots,y_m]\setminus\{0\})\subset\mathbb{Z}^n_{>0}$$

generate the valuation monoid C. If not then the algorithm yields an element $c \in C \setminus \mathbb{Z}_{>0}\{\nu(g_1), \ldots, \nu(g_p)\}.$

Proof. We have $\mathbb{k}[y_1, \ldots, y_m] = \mathbb{k}[g_1, \ldots, g_p]/I$ for a suitable ideal I. Consider a (non-strict) linear ordering \prec on monomials in g_1, \ldots, g_p according to ν . Denote by $H \subset \mathbb{R}^p$ a (rational) (p-m)-dimensional plane such that $\nu(g_1^{i_1} \cdots g_p^{i_p}) = \nu(g_1^{j_1} \cdots g_p^{j_p})$ iff $(i_1 - j_1, \ldots, i_p - j_p) \in H$ (cf. the proof of Theorem 3.39), in other words, $(i_1, \ldots, i_p) \preceq (j_1, \ldots, j_p) \preceq (i_1, \ldots, i_p)$ (slightly abusing the notations we identify a monomial $g_1^{i_1} \cdots g_p^{i_p}$ with the vector (i_1, \ldots, i_p)). Define a linear ordering \triangleleft on monomials in g_1, \ldots, g_p as follows: $g_1^{i_1} \cdots g_p^{i_p} \triangleleft g_1^{j_1} \cdots g_p^{j_p}$ iff either $g_1^{i_1} \cdots g_p^{i_p} \prec g_1^{j_1} \cdots g_p^{j_p}$ or $\nu(g_1^{i_1} \cdots g_p^{i_p}) = \nu(g_1^{j_1} \cdots g_p^{j_p})$ and the vector (i_1, \ldots, i_p) is less than (j_1, \ldots, j_p) in deglex (again cf. the proof of Theorem 3.39).

The algorithm constructs a Gröbner basis of I with respect to \triangleleft . Denote by $G \subset \mathbb{Z}_{\geq 0}^p$ the complement to the monomial ideal of leading monomials of the Gröbner basis. Then G is a finite (disjoint) union of sets of the form

$$u + \{(u_1, \ldots, u_p) : u_i \vdash_i 0\}$$

where each $\vdash_i, 1 \leq i \leq p$ is either = or \geq .

Consider a plane $H_0 \subset \mathbb{R}^p$ parallel to H which has a common point with $\mathbb{Z}_{\geq 0}^p$. Then $H_0 \cap G \neq \emptyset$. Indeed, otherwise for any point $g_1^{i_1} \cdots g_p^{i_p} \in H_0$ we get (taking into account properties of Gröbner bases) that

$$g_1^{i_1} \cdots g_p^{i_p} = \sum_L \alpha_L g^L, \alpha_L \in \mathbb{k}^*, \nu(g^L) \prec \nu(g_1^{i_1} \cdots g_p^{i_p}) \text{ for each } L,$$

which contradicts to the subadditivity of the valuation.

First, assume that for every H_0 parallel to H such that $H_0 \cap \mathbb{Z}_{\geq 0}^p \neq \emptyset$ it holds $|G \cap H_0| = 1$. Then we fall in the conditions of Theorem 3.21, and thus $\nu(g_1), \ldots, \nu(g_p)$ constitute a family of generators of $\nu(\Bbbk[y_1, \ldots, y_m] \setminus \{0\}) = \mathbb{Z}_{\geq 0}^p / H_{\mathbb{Z}}$.

Now on the contrary, assume that $|G \cap H_0| \geq 2$ for some H_0 parallel to H. In this case the algorithm can find a pair of different monomials $g_1^{i_1} \cdots g_p^{i_p}, g_1^{j_1} \cdots g_p^{j_p} \in G \cap H_0$ for some H_0 invoking integer linear programming. If the algorithm fails, it means that $|G \cap H_0| = 1$ for all H_0 (see the previous case). Since the valuation ν is injective, there exists $\alpha \in \Bbbk^*$ such that

$$c_0 := \nu(g_1^{i_1} \cdots g_p^{i_p} - \alpha g_1^{j_1} \cdots g_p^{j_p}) \prec \nu(g_1^{i_1} \cdots g_p^{i_p}) = \nu(g_1^{j_1} \cdots g_p^{j_p}).$$

If $\mathbb{Z}_{\geq 0}^p \ni c_0 \notin \mathbb{Z}_{\geq 0}\{\nu(g_1), \ldots, \nu(g_p)\}$ then $c := c_0$ meets the requirements of the Proposition.

Otherwise, if $c_0 \in \mathbb{Z}_{\geq 0}\{\nu(g_1), \ldots, \nu(g_p)\}$ there exists a monomial $g_1^{l_1} \cdots g_p^{l_p}$ such that $\nu(g_1^{l_1} \cdots g_p^{l_p}) = c_0$. Again due to the injectivity there exists $\beta \in \mathbb{k}^*$ for which it holds

$$\nu(g_1^{i_1}\cdots g_p^{i_p}-\alpha g_1^{j_1}\cdots g_p^{j_p}-\beta g_1^{l_1}\cdots g_p^{l_p})\prec c_0.$$

Observe that $g_1^{i_1} \cdots g_p^{i_p} - \alpha g_1^{j_1} \cdots g_p^{j_p} - \beta g_1^{l_1} \cdots g_p^{l_p} \neq 0$, because otherwise this would contradict to that $g_1^{i_1} \cdots g_p^{i_p}, g_1^{j_1} \cdots g_p^{j_p} \in G$. Continuing in a similar way, the algorithm eventually arrives at a required element $c \in C \setminus \mathbb{Z}_{\geq 0}\{\nu(g_1), \ldots, \nu(g_p)\}$ since C is well-ordered.

Remark 3.67. In the proof of the latter Proposition we used f_1, \ldots, f_m only to be able to compute $\nu(g)$ for $g \in \mathbb{k}[y_1, \ldots, y_m]$. In fact, one could stick with an arbitrary computable injective valuation.

3.9. The space of injective valuations on a domain. For a k-domain A we consider the space V := V(A) of all injective valuations $\nu : A \setminus \{0\} \to \mathbb{R}_{\geq 0}$. Given a basis **B** of A endowed with a linear order \prec one can consider a set $V_{\mathbf{B},\prec}$ of mappings $\nu : \mathbf{B} \to \mathbb{R}_{\geq 0}$ such that for any $b, b_0 \in \mathbf{B}$ a relation $b \prec b_0$ implies $\nu(b) < \nu(b_0)$, and for any $b_1, b_2 \in \mathbf{B}$ for which

$$b_1b_2 = \alpha_{b_0}b_0 + \sum_{b \in \mathbf{B}} \alpha_b b, \ \alpha_{b_0}, \alpha_b \in \mathbb{K}, \ b \prec b_0$$

it holds $\nu(b_0) = \nu(b_1) + \nu(b_2)$. Then ν induces an injective valuation on A with an adapted basis **B**: namely, for any $a = \alpha_{b_0}b_0 + \sum_{b \in \mathbf{B}} \alpha_b b, \ b \prec b_0$ we define $\nu(a) := \nu(b_0)$. One can define a topology on V with open basic sets $V_{\mathbf{B},\prec}$ for all \mathbf{B},\prec .

Question. Is V connected?

Recall (see the proof of Theorem 3.50) that for any valuation $\nu : A \setminus \{0\} \twoheadrightarrow C$ onto a well-ordered finitely-generated monoid C one can find generators X_1, \ldots, X_m of A and elements $w_1, \ldots, w_m \in \mathbb{R}_{\geq 0}[\varepsilon]$ such that $\nu(X_1), \ldots, \nu(X_m)$ generate C, and the order on monomials $i_1\nu(X_1) + \cdots + i_m\nu(X_m)$ is determined by $w_1i_1 + \cdots + w_mi_m$. Choosing a basis **B** among monomials of the form $X_1^{i_1} \cdots X_m^{i_m}$ and defining $\nu(X_1^{i_1} \cdots X_m^{i_m}) := w_1i_1 + \cdots + w_mi_m$, one obtains that $\nu \in V$, provided that $w_1, \ldots, w_m \in \mathbb{R}_{\geq 0}$.

3.10. Injective well-ordered valuations of 2-dimensional algebras. Let $A := \mathbb{k}[x, y, z]/(f)$ be a 2-dimensional algebra where $f \in \mathbb{k}[x, y, z]$. Consider a valuation $\nu : A \setminus \{0\} \to \mathbb{Z}_{\geq 0}^2$ studied in Theorem 3.21. Then ν is induced by an edge e of the Newton polytope $N(f) \subset \mathbb{R}_{\geq 0}^3$ and a 2-dimensional plane $Q \subset \mathbb{R}^3$ containing e. Note that in the notations of Theorem 3.21 H is the line passing through e, and $Q \in Trop(f)$. Moreover, the proof of Theorem 3.21 and Remark 3.20 imply that if ν is injective then the endpoints of e (up to a permutation of the coordinates x, y, z) are (p, 0, 0), (0, q, r) where $p, q, r \in \mathbb{Z}_{\geq 0}$ have no common divisor.

Lemma 3.68. If two edges e_1, e_2 of the Newton polytope $N(f) \subset \mathbb{R}^3_{\geq 0}$ induce injective valuations then e_1, e_2 have a common vertex located on a coordinate line.

Proof. We say that a point $v \in N(f)$ belongs to a *roof* of N(f) if on a ray emanating from the origin (0, 0, 0) and passing through v, the latter is the last point from N(f) on the ray. Then e_1, e_2 lie on the roof.

Consider a 2-dimensional plane $Q_1 \subset \mathbb{R}^3$ which contains e_1 and supports N(f). The projection of the roof to Q_1 by means of the rays contains the projections of e_1 and of e_2 . If e_1, e_2 had no common vertex located on a coordinate line then the projections of e_1, e_2 would have a common point being internal in the projection of either e_1 or e_2 . The obtained contradiction completes the proof. \Box

Corollary 3.69. All the edges of the Newton polytope N(f) inducing injective valuations either

i) have a common vertex located on a coordinate line or

ii) form a triangle with its vertices on the coordinate lines, and in this case the roof of N(f) consists of this triangle.

Remark 3.70. i) Observe that in case i) of Corollary 3.69 when all the edges have a common vertex (p, 0, 0), a common adapted basis of all the injective valuations induced by the edges, is $\{x^i y^j z^k : 0 \le i < p, 0 \le j, k < \infty\}$ (cf. the proof of Theorem 3.21, Remark 3.20 and Theorem 3.39).

ii) One can verify that in case ii) of Corollary 3.69 three injective valuations do not possess a common adapted basis.

For example, let $f := z + x^2 + y^3$. Then $A := \mathbb{k}[x, y, z]/(f) \simeq \mathbb{k}[x, y]$. Denote by ν_x the injective valuation induced by the edge (z, y^3) with respect to lex ordering in which y > x, and an adapted basis $\{y^i x^j : 0 \le i, j < \infty\}$. Denote by ν_y the

injective valuation induced by the edge (z, x^2) with respect to lex ordering in which x > y, and an adapted basis $\{x^i y^j : 0 \le i, j < \infty\}$. Finally, denote by ν_z the injective valuation induced by the edge (y^3, x^2) with respect to lex ordering in which x, y > z, and $\nu_z(x) = (3, 0), \nu_z(y) = (2, 0), \nu_z(z) = (0, 1)$. An adapted basis of ν_z is $\{y^i z^j, x y^i z^j : 0 \le i, j < \infty\}$ (cf. Examples 3.6, 3.38).

One can compute all three JHb (see Theorem 4.24):

$$\mathbf{K}_{\nu_{z},\nu_{y}}(2j,i) = (2i,j), \mathbf{K}_{\nu_{z},\nu_{y}}(2j+1,i) = (2i+3,j);$$

$$\mathbf{K}_{\nu_{z},\nu_{x}}(3j,i) = (3i,j), \mathbf{K}_{\nu_{z},\nu_{x}}(3j+1,i) = (3i+2,j), \mathbf{K}_{\nu_{z},\nu_{x}}(3j+2,i) = (3i+4,j);$$

$$\mathbf{K}_{\nu_x,\nu_y}(j,i) = (i,j).$$

One can generalize Corollary 3.69 (in one direction) to hypersurfaces of arbitrary dimensions.

Remark 3.71. Let $A := \mathbb{k}[x_1, \ldots, x_n]/(f)$ where $f \in \mathbb{k}[x_1, \ldots, x_n]$. An edge e of the Newton polytope $N(f) \subset \mathbb{R}^n_{\geq 0}$ induces an injective valuation ν on $A \setminus \{0\}$ iff the endpoints of e are $v = (v_1, \ldots, v_n), u = (u_1, \ldots, u_n) \in \mathbb{Z}^n_{\geq 0}$ such that $\min\{v_i, u_i\} = 0, 1 \leq i \leq n$, and the integers $\max\{v_i, u_i\}, 1 \leq i \leq n$ have no common divisor (cf. Theorem 3.21 and Remark 3.20). Denote by $H \subset \mathbb{R}^n$ the line containing e.

Denote by \prec a linear ordering of the valuation cone (being a subset of \mathbb{Z}^{n-1}) of ν . Following the proof of Theorem 3.39 ii) one can extend \prec to a linear ordering q on $\mathbb{Z}_{\geq 0}^n$ in two different ways according to an ordering (direction) in H. Then according to one of these two choices of q either u or v becomes a leading monomial of f with respect to q. Any of these two choices of q provides an adapted basis of A with respect to ν being a complement of a principal monomial ideal in $\Bbbk[x_1, \ldots, x_n]$ (see Theorem 3.39).

If edges e_1, \ldots, e_s of N(f) induce injective valuations ν_1, \ldots, ν_s , respectively, of A^* and have a common vertex in N(f) then ν_1, \ldots, ν_s possess a common adapted basis being a complement of a principal monomial ideal in $\Bbbk[x_1, \ldots, x_n]$.

Example 3.72. Now we give an example of a pair of injective valuations on a 3dimensional algebra $A := k[x, y, z, t]/(f := x^2 + y^3 + z^5 + t^7)$ to $\mathbb{Z}_{\geq 0}^3$ endowed with the lexicographical ordering. Each pair of monomials of f provides an injective valuation of $A \setminus \{0\}$ (see Theorem 3.21 and Proposition 3.25). In particular, a pair of monomials x^2, y^3 provides an injective valuation ν_1 for which it holds

$$\nu_1(x) = (3, 0, 0), \nu_1(y) = (2, 0, 0), \nu_1(z) = (0, 1, 0), \nu_1(t) = (0, 0, 1), \nu_1($$

In its turn, a pair of monomials z^5, t^7 provides an injective valuation ν_2 for which it holds

$$\nu_2(z) = (7, 0, 0), \nu_2(t) = (5, 0, 0), \nu_2(x) = (0, 1, 0), \nu_2(y) = (0, 0, 1).$$

Denote $w := x^2 + y^3$. Then ν_1, ν_2 have a common adapted basis

$$\{y^{i}t^{j}w^{l}x^{k}z^{m} : i, j, l \ge 0, 0 \le k \le 1, 0 \le m \le 4\},\$$

and $\nu_1(w) = \nu_1(-z^5 - t^7) = (0, 5, 0), \ \nu_2(w) = (0, 2, 0).$

One can generalize this construction to a polynomial of the form $f := \sum_{1 \le i \le n} x_i^{q_i}$ where $q_i, 1 \le i \le n$ are pairwise relatively prime. Each pair of monomials of fprovides an injective valuation of $k[x_1 \ldots, x_n]/(f)^*$ into $\mathbb{Z}_{\ge 0}^{n-1}$.

3.11. Enumerating injective well-ordered valuations of a hypersurface of a prime degree at a main variable.

Remark 3.73. Let a polynomial $f = y^d + f_1 \in \mathbb{k}[y, x_1, \dots, x_n]$ be normalized with respect to y, i.e. $deg_y(f_1) < d$ (one can reduce to this situation invoking Noether normalization). Denote $A := \mathbb{k}[y, x_1, \dots, x_n]/(f)$. We say that an edge of Newton polytope $N_f \subset \mathbb{R}^{n+1}$ is long if its endpoints are $(d, 0, \dots, 0)$, $(0, i_1, \dots, i_n)$ and $GCD(d, i_1, \dots, i_n) = 1$.

Then the domain $A \setminus \{0\}$ admits an injective well-ordered valuation into $\mathbb{Q}_{\geq 0}^n$ with an adapted basis

$$\{y^k x_1^{j_1} \cdots x_n^{j_n} : 0 \le k < d, 0 \le j_1, \dots, j_n < \infty\}$$

and

$$\nu(x_l) = e_l, 1 \le l \le n, \ \nu(y) = \frac{i_1 e_1 + \dots + i_n e_n}{d}$$

where $e_l = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n, 1 \le l \le n$ is an ort vector (cf. Proposition 3.8).

Observe that for any valuation (not necessary injective or well-ordered) on $A \setminus \{0\}$ for which $\nu(x_l) = e_l, 1 \leq l \leq n$ there exists an edge of N_f with endpoints $(d_1, j_1, \ldots, j_n), (d_2, k_1, \ldots, k_n)$ such that

(3.9)
$$\nu(y) = \frac{(k_1 - j_1)e_1 + \dots + (k_n - j_n)e_n}{d_1 - d_2}.$$

As a more general setting than in Remark 3.73 we assume that a domain A is a finite integral extension of $\mathbb{k}[x_1, \ldots, x_n]$ of a rank d. We study injective well-ordered valuations ν on $A \setminus \{0\}$ into $\mathbb{Q}_{>0}^n$ for which $\nu(x_l) = e_l, 1 \leq l \leq n$.

The following construction is valid for a valuation ν not necessary injective or wellordered. Denote by $G(\nu) \subset \mathbb{Q}^n/\mathbb{Z}^n$ the image $\nu(A \setminus \{0\})/\mathbb{Z}^n$. Note that $G(\nu)$ is an abelian semigroup. Moreover, for every element $a \in A \setminus \{0\}$ there exists a polynomial $h \in \mathbb{K}[z, x_1, \ldots, x_n]$ such that h(a) = 0 and $deg_z(h) \leq d$. As in Remark 3.73 there is an edge of Newton polytope $N_h \subset \mathbb{R}^{n+1}$ with endpoints $(d_1, j_1, \ldots, j_n), (d_2, k_1, \ldots, k_n)$ such that

$$\nu(a) = \frac{(k_1 - j_1)e_1 + \dots + (k_n - j_n)e_n}{d_1 - d_2}$$

(see (3.9)). Hence $(d_1 - d_2)\nu(a)$ is the unit element in $G(\nu)$, thus $G(\nu)$ is a group. Moreover,

$$G(\nu) \subset \frac{\mathbb{Z}^n}{LCM\{1,\ldots,d\}}/\mathbb{Z}^n$$

in particular, $G(\nu)$ is finite. We call $G(\nu)$ the group of the valuation.

Lemma 3.74. Let a domain A be a free $\Bbbk[x_1, \ldots, x_n]$ -module of a rank d. Let ν be a well-ordered valuation of $A \setminus \{0\}$ being an extension into $\mathbb{Q}_{\geq 0}^n$ of a valuation on $\Bbbk[x_1, \ldots, x_n] \setminus \{0\}$ with $\nu(x_l) = e_l, 1 \leq l \leq n$. Then for the size s of the group of the valuation $G(\nu)$ holds

- i) $s \leq d;$
- ii) if ν is injective and archimedian then s = d.

Proof. i) Let $b_1, \ldots, b_d \in A$ be a free $\Bbbk[x_1, \ldots, x_n]$ -basis of A. Pick $t_1, \ldots, t_s \in A \setminus \{0\}$ such that $\nu(t_l) - \nu(t_j) \notin \mathbb{Z}^n$ for $1 \leq l \neq j \leq s$. Express $t_j = \sum_{1 \leq i \leq d} h_{j,i}b_i$ for appropriate polynomials $h_{j,i} \in \Bbbk[x_1, \ldots, x_n]$. Denote

$$M_N := \{x_1^{l_1} \cdots x_n^{l_n} : 0 \le l_1 + \cdots + l_n \le N\}, W_N := t_1 M_N + \cdots + t_s M_N$$

for an integer N. Then $\dim_{\mathbb{K}} W_N = s|M_N|$ due to the valuation property. On the other hand,

$$W_N \subset b_1 M_{N+c} + \dots + b_d M_{N+c}$$

for a suitable constant $c \in \mathbb{Z}_{\geq 0}$. Therefore, considering sufficiently big N, we obtain that $s \leq d$.

ii) Denote

$$V_N := \{ (i_1, \dots, i_n) \in \mathbb{Z}_{>0}^n : \nu(i_1, \dots, i_n) \le N \}$$

(we identify the archimedian valuation with defining it linear form). Then $|V_N| \sim c_0 \cdot N^n$ for an appropriate $0 < c_0 \in \mathbb{R}$. Observe that ν on $\mathbb{Q}^n_{\geq 0}$ is defined by the same linear form as ν is.

Denote by g_1, \ldots, g_s the unique representatives of the elements of $G(\nu)$ in the cube $[0,1)^n$. Then

$$\nu(b_1 \cdot V_N + \dots + b_d \cdot V_N) \subset g_1 \cdot V_{N+c} \bigsqcup \cdots \bigsqcup g_s \cdot V_{N+c}$$

for a suitable constant c. Since ν is injective this implies that $s \geq d$ taking into account that $\dim(b_1 \cdot V_N + \cdots + b_d \cdot V_N) = d|V_N|$.

Remark 3.75. One can literally extend Lemma 3.74 to a domain $A \supset \Bbbk[x_1, \ldots, x_n]$ such that $\Bbbk(x_1, \ldots, x_n)$ -dimension of $A \otimes_{\Bbbk[x_1, \ldots, x_n]} \Bbbk(x_1, \ldots, x_n)$ equals d.

It would be interesting to clarify whether Lemma 3.74 and Remark 3.75 are true for not necessary archimedian valuations.

Corollary 3.76. Under the conditions of Lemma 3.74 or Remark 3.75 in case of a square-free d the group $G(\nu)$ is cyclic of size d and every its generating element has a form $\frac{i_1e_1+\dots+i_ne_n}{d}$ where $GCD(d, i_1, \dots, i_n) = 1$.

Now let $A := k[y, x_1, \dots, x_n]/(f)$ be as in Remark 3.73 and d be a prime. Our goal is to design an algorithm which enumerates all injective well-ordered archimedian valuations on $A \setminus \{0\}$ into $\mathbb{Q}_{\geq 0}^n$ (with $\nu(x_l) = e_l, 1 \leq l \leq n$, cf. Corollary 3.76).

The algorithm produces a finite tree T by recursion. Some leaves of T correspond to injective well-ordered valuations on $A \setminus \{0\}$. As a base of recursion a root of T is produced.

As a recursive hypothesis assume that at a vertex v of a depth s of T a constructible set $U_v \subset \mathbb{k}^s$ and a set of monomials $m_1, \ldots, m_s \in \mathbb{Z}_{\geq 0}^n$ are produced (we identify monomials in the variables x_1, \ldots, x_n with $\mathbb{Z}_{\geq 0}^n$). We suppose that m_s does not belong to the monomial ideal generated by m_1, \ldots, m_{s-1} . In addition, the algorithm produces a polynomial $f_v = y^d + f_{v,1} \in \mathbb{k}[y, x_1, \ldots, x_n, z_1, \ldots, z_s]$ (where $deg_y(f_{v,1}) < d$) such that for any point $(\alpha_1, \ldots, \alpha_s) \in U_v$ it holds

$$(3.10) f_v(y - \alpha_1 m_1 - \dots - \alpha_s m_s, x_1, \dots, x_n, \alpha_1, \dots, \alpha_s) = 0$$

and that Newton polytopes $N_{f_v} \subset \mathbb{R}^{n+1}$ are the same for all the points $(\alpha_1, \ldots, \alpha_s) \in U_v$.

Now we proceed to the description of the recursive step of the algorithm. First assume that Newton polytope N_{f_v} has a long edge with endpoints $(d, 0, \ldots, 0), (0, i_1, \ldots, i_n),$ denote by $g(z_1, \ldots, z_n) \in \mathbb{k}[z_1, \ldots, z_n]$ the coefficient of f_v at the monomial $x_1^{i_1} \cdots x_n^{i_n}$. The algorithm verifies (invoking linear programming) whether there exists a linear archimedian ordering \succ (compatible with addition) on $\mathbb{Q}_{\geq 0}^n$ such that $m_1 \succ \cdots \succ$ $m_s \succ \frac{i_1e_1+\cdots+i_ne_n}{d}$ (otherwise, the algorithm ignores the long edge under consideration). If such an ordering does exist then Remark 3.73 provides an injective well-ordered valuation ν on A^* such that

$$\nu(y - \alpha_1 m_1 - \dots - \alpha_s m_s) = \frac{i_1 e_1 + \dots + i_n e_n}{d}.$$

Thus, as an adapted basis of A with respect to ν one can take

$$\{(y - \alpha_1 m_1 - \dots - \alpha_s m_s)^k \cdot x_1^{j_1} \cdots x_n^{j_n}\}, 0 \le k < d, 0 \le j_1, \dots, j_n < \infty.$$

The algorithm produces a vertex being a son of v and a leaf in T which corresponds to ν .

Now we consider a not long edge of N_{f_v} with endpoints (d_1, j_1, \ldots, j_n) , (d_2, k_1, \ldots, k_n) (obviously, $0 \le d_1 \ne d_2 \le d$). Denote

$$m_{s+1} := \frac{(k_1 - j_1)e_1 + \dots + (k_n - j_n)e_n}{d_1 - d_2}$$

(cf. (3.9), (3.10)), provided that $m_{s+1} \in \mathbb{Z}_{\geq 0}^n$. Observe that if $m_{s+1} \notin \mathbb{Z}_{\geq 0}^n$ then for no element $a \in A$ it holds $\nu(a) = m_{s+1}$ for an injective well-ordered archimedian valuation ν because of Corollary 3.76.

The algorithm verifies (invoking linear programming) whether there exists a linear archimedian ordering \succ on $\mathbb{Z}_{\geq 0}^n$ such that $m_1 \succ \cdots \succ m_s \succ m_{s+1}$ (otherwise, the algorithm ignores the edge of N_{f_v} under consideration). The latter is necessary because the algorithm looks for an injective well-ordered valuation ν such that $\nu(y - \alpha_1 m_1 - \cdots - \alpha_s m_s) = \nu(m_{s+1})(= m_{s+1})$. Note that in particular, the existence of a suitable linear ordering \succ implies that m_{s+1} does not belong to the monomial ideal generated by m_1, \ldots, m_s .

The algorithm calculates a polynomial $g \in \mathbb{k}[y, x_1, \ldots, x_n, z_1, \ldots, z_s, z_{s+1}]$ such that

$$g(y - \alpha_1 m_1 - \dots - \alpha_s m_s - z_{s+1} m_{s+1}, x_1, \dots, x_n, \alpha_1, \dots, \alpha_s, z_{s+1}) = 0$$

for any point $(\alpha_1, \ldots, \alpha_s) \in U_v$ (cf. (3.10)). Note that it still holds $g = y^d + g_1$ where $deg_y(g_1) < d$. For different values of z_{s+1} in k there is a finite number of possible shapes of Newton polytopes $N_g \subset \mathbb{R}^{n+1}$ (which are determined by their vertices). For each fixed shape the algorithm produces a vertex w being a son of v in T together with a constructible set $U_w \subset k^{s+1}$ assuring the fixed shape. We put a polynomial $f_w := g$. This completes the description of the recursive step of the algorithm.

Observe that the tree T is finite since along every its path the monomial ideal generated by m_1, \ldots, m_s strictly increases and therefore, the path is finite due to Hilbert's Idealbasissatz (also we make use of König's Lemma). Summarizing, we have established the following proposition.

Proposition 3.77. There is an algorithm which for a polynomial $f = y^d + f_1 \in \mathbb{k}[y, x_1, \ldots, x_n]$ with a prime d where $deg_y(f_1) < d$, enumerates all injective well-ordered archimedian valuations on $(\mathbb{k}[y, x_1, \ldots, x_n]/(f)) \setminus \{0\}$.

It would be interesting to generalize the latter proposition to arbitrary affine algebras. The next example demonstrates that it is not possible to generalize it directly even to domains being free $k[x_1, \ldots, x_n]$ -modules of composite ranks.

Example 3.78. A domain $A = \mathbb{k}[x^{1/2}, y^{1/2}]$ is 4-dimensional free $\mathbb{k}[x, y]$ -module. Then $\nu(x^{1/2}) = e_1/2, \nu(y^{1/2}) = e_2/2$ and the group $G(\nu)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that an element $z := x^{1/2} + y^{1/2}$ has a minimal polynomial of degree 4, namely $z^4 - 2(x+y)z^2 + x^2 + y^2 - 4xy = 0$ whose Newton polytope has no long edge (cf. Remark 3.73).

Remark 3.79. Let a domain A be a $\Bbbk[x_1, \ldots, x_n]$ -module, admitting an injective well-ordered valuation ν satisfying the conditions of Remark 3.75 such that d is square-free. Due to Corollary 3.76 the group $G(\nu)$ is cyclic of the size d. Pick $y \in A$ such that $\nu(y)/\mathbb{Z}^n$ is a generator of $G(\nu)$. Denote by $f \in \Bbbk[x_1, \ldots, x_n, y]$ the minimal polynomial of y over $\Bbbk[x_1, \ldots, x_n]$. Then $deg_y f = d$ since Newton polytope of f contains an edge with a denominator equal to d, and $f = qy^d + \cdots, q \in$ $\Bbbk[x_1, \ldots, x_n]$. The domain $A_0 := \Bbbk[x_1, \ldots, x_n, qy]$ is a free $\Bbbk[x_1, \ldots, x_n]$ -module with a basis $1, qy, \ldots, (qy)^{d-1}$, and Newton polytope of the minimal polynomial $q^{d-1}f$ of qy contains a long edge. According to Remark 3.73 this long edge provides on $A_0 \setminus \{0\}$ the valuation coinciding with the restriction of ν .

Furthermore, this restriction is extended uniquely to $A \setminus \{0\}$. Namely, for any $a \in A \setminus \{0\}$ there exists $p \in \Bbbk[x_1, \ldots, x_n]$ such that $pa \in A_0 \setminus \{0\}$, therefore $\nu(a) = \nu(pa) - \nu(p)$.

3.12. Convexity of the extended Jordan-Hölder bijection for valuations in an archimedian monoid. Let $\nu : A \setminus \{0\} \to \mathbb{Z}_{\geq 0}^n$ be an injective valuation in a monoid with archimedian linear order (cf. Remark 3.51). We say that ν is finitary if the ordering \prec on vectors $v = (v_1, \ldots, v_n) \in \mathbb{Z}_{\geq 0}^n$ is determined by a linear function $\alpha(v) := \alpha_1 v_1 + \cdots + \alpha_n v_n$ where positive reals $\alpha_1, \ldots, \alpha_n$ are Q-linearly independent. Then the ordering \prec is isomorphic to $\mathbb{Z}_{\geq 0}$, i.e. the monoid $\mathbb{Z}_{\geq 0}^n$ is archimedian. For instance, lex is not finitary. In particular, for any $c \in \nu(A \setminus \{0\})$ the k-linear space $A_{\leq c} := \{a \in A \setminus \{0\} : \nu(a) \leq c\} \cup \{0\}$ is finite-dimensional.

Let $\nu_0 : A \setminus \{0\} \to \mathbb{Z}^n_{\geq 0}$ be another injective valuation (not necessary archimedain). Denote the valuation cones $C := \nu(A \setminus \{0\}), C_0 := \nu_0(A \setminus \{0\})$. Consider a convex cone $C_0^{(\mathbb{Q})} := C_0 \otimes \mathbb{Q}_{\geq 0}$ and denote

$$S_0 := C_0^{(\mathbb{Q})} \cap \{ (v_1, \dots, v_n) \in \mathbb{Q}_{\geq 0}^n : v_1 + \dots + v_n = 1 \}.$$

The following map

$$\mathbf{K}(c_0) := \min_{\prec} \{ \nu(\nu_0^{-1}(c_0)) \} \in C$$

is a bijection $\mathbf{K} : C_0 \to C$ called generalized JHb (see Theorem 4.24). Denote a function $\overline{\mathbf{K}} := \alpha \circ \mathbf{K} : C_0 \to \mathbb{R}_{\geq 0}$. One can define $\overline{\mathbf{K}}$ on S_0 (so, on rational points) as follows. For a point $u := (u_1, \ldots, u_n) \in S_0$ we have $\overline{\mathbf{K}}((p+q)u) \leq \overline{\mathbf{K}}(pu) + \overline{\mathbf{K}}(qu)$, provided that $pu, qu \in C_0$, since the subadditivity $\overline{\mathbf{K}}(w_1 + w_2) \leq \overline{\mathbf{K}}(w_1) + \overline{\mathbf{K}}(w_2)$ holds for any elements $w_1, w_2 \in C_0$. Therefore, due to Fekete's subadditivity lemma [33] there exists the limit

$$\underline{\mathbf{K}}(u) := \lim_{p \to \infty, p u \in C_0} \frac{\mathbf{K}(pu)}{p}$$

Proposition 3.80. Function $\underline{\mathbf{K}} : S_0 \to \mathbb{R}_{\geq 0}$ is convex.

Proof. Consider a convex combination $w = \sum_i \lambda_i w_i$ of points $w, w_i \in S_0$ where $0 < \lambda_i \in \mathbb{Q}, \sum_i \lambda_i = 1$ for all *i*. Then for a suitable $0 < s \in \mathbb{Z}$ it holds $sw, sw_i \in C_0$ for all *i*. Denote by *q* the common denominator of all λ_i . Hence $\overline{\mathbf{K}}(pqsw) \leq \sum_i \overline{\mathbf{K}}(pqs\lambda_iw_i)$ for any $p \in \mathbb{Z}_{\geq 0}$ because of the subadditivity of $\overline{\mathbf{K}}$. Dividing the both sides of the latter inequality by pqs and tending *p* to infinity, we conclude that $\underline{\mathbf{K}}(w) \leq \sum_i \lambda_i \underline{\mathbf{K}}(w_i)$, which completes the proof. \Box

Remark 3.81. We have taken an ordering \prec to be archimedian since otherwise one can't assure an inequality after tending to a limit. For example, in *lex* ordering each element of a sequence $(1 - 1/p, 1) \in \mathbb{Q}^2_{\geq 0}$ is less than (1, 0), while their limit against p is not.

Corollary 3.82. One can extend $\underline{\mathbf{K}}$ (from rational points) to real points in the interior $int(S_0 \otimes \mathbb{R}_{>0})$ being a (continuous) convex function.

Proof. <u>K</u> is locally Lipschitz in $int(S_0)$ (cf. e. g. [28]), hence it can be (uniquely) extended to a continuous function (moreover, locally Lipschitz) on $int(S_0 \otimes \mathbb{R}_{\geq 0})$ which is also convex. \Box

3.13. Jordan-Hölder bijections for valuations of an algebra. We say that a map $\mathbf{K} : P \to Q$ of ordered partial semigroups is *sub-multiplicative* if it satisfies the following:

 $\mathbf{K}(u \circ v) \preceq \mathbf{K}(u) \circ \mathbf{K}(v)$ whenever $u \circ v$ and $\mathbf{K}(u) \circ \mathbf{K}(v)$ are defined in P and in Q, respectively.

Proposition 3.83. Let $\nu : A \setminus \{0\} \to P, \nu_1 : A \setminus \{0\} \to P'$ be a pair of injective valuations of an algebra A to partial semigroups P and P', respectively. Then JHb $\mathbf{K}_{\nu',\nu}$ from $\nu(A \setminus \{0\})$ to $\nu'(A \setminus \{0\})$ is sub-multiplicative.

Proof. For any elements $u, v \in P$ for which $u \circ v$ and $\mathbf{K}(u) \circ \mathbf{K}(v)$ are defined take $a, b \in A \setminus \{0\}$ such that $\nu(a) = u, \nu(b) = v$ and $\nu'(a) = \mathbf{K}(u), \nu'(b) = \mathbf{K}(v)$. Then $\nu(ab) = u \circ v$ and $\mathbf{K}(u \circ v) \preceq \nu'(ab) = \mathbf{K}(u) \circ \mathbf{K}(v)$.

Remark 3.84. We expect that the converse also holds. If **K** is a sub-multiplicative bijection $P \rightarrow Q$ of partial semigroups such that \mathbf{K}^{-1} is sub-multiplicative as well, then there exist injective valuations $\nu : A \setminus \{0\} \rightarrow P, \nu' : A \setminus \{0\} \rightarrow P'$ of an appropriate algebra A such that $\mathbf{K} = \mathbf{K}_{\nu,\nu'}$.

Example 3.85. Consider a partial semigroup P endowed with two different orders. They provide two injective valuations $\nu_1, \nu_2 : \mathbb{k}P \setminus \{0\} \twoheadrightarrow P$. Then $\{[u] : u \in P\} \subset \mathbb{k}P$ (Definition 2.28) is a common adapted basis for ν_1, ν_2 , and the JH bijection \mathbf{K}_{ν_1,ν_2} is identity map Id_P . **Proposition 3.86.** Let P be a set with two (partial) operations $(a, b) \mapsto a \circ b$ and $(a, b) \mapsto a \bullet b$ so that P has partial semigroup structures P_{\circ} and P_{\bullet} respectively. Define a new operation

$$ab := a \circ b + a \bullet b$$

on the vector space $\mathbb{k}P$ (with the convention if a summand is not defined it is replaced by zero) and denote this algebra by $A_{\circ,\bullet}$.

(a) $A_{\circ,\bullet}$ is associative iff \circ and \bullet are mutually associative:

$$(a \circ b) \bullet c = a \bullet (b \circ c), \ (a \bullet b) \circ c = a \circ (b \bullet c)$$

for all $a, b, c \in P$. We say that $(a \circ b) \bullet c$ is defined if both $a \circ b$ and $(a \circ b) \bullet c$ are defined (here we assume that $(a \circ b) \bullet c$ and $a \bullet (b \circ c)$ are defined or not defined simultaneously (as well as $(a \bullet b) \circ c$ and $a \circ (b \bullet c)$).

(b) Suppose additionally that both P_{\circ} and P_{\bullet} are ordered with \preceq° and \preceq^{\bullet} , respectively, and it holds

$$a \circ b \preceq^{\bullet} a \bullet b \preceq^{\circ} a \circ b$$

provided that $a \circ b$ and $a \bullet b$ are defined (cf. Proposition 3.83). Then the assignment $[a] \mapsto a$ define injective valuations ν_{\circ} and ν_{\bullet} on $A^*_{\circ,\bullet}$ to P_{\circ} and to P_{\bullet} , respectively Moreover, identity map $P \mapsto P$ is the corresponding JH bijection, and $[P] \subset A_{\circ,\bullet}$ is a common adapted basis of valuations ν_{\circ} and ν_{\bullet} .

Proof. Prove (a). Indeed,

$$(ab)c = (a \circ b + a \bullet b)c = (a \circ b)c + (a \bullet b)c$$
$$= (a \circ b) \circ c + (a \circ b) \bullet c + (a \bullet b) \circ c + (a \bullet b) \bullet c$$

On the other hand,

$$a(bc) = a(b \circ c + b \bullet c)$$

= $a \circ (b \circ c) + a \bullet (b \circ c) + a \circ (b \bullet c) + a \bullet (b \bullet c)$

This gives associativity because $(a \circ b) \bullet c = a \bullet (b \circ c)$ and $(a \bullet b) \circ c = a \circ (b \bullet c)$.

(b) We claim that ν_{\bullet} is a valuation. Take $a, b \in P \subset A_{\circ,\bullet}$ such that $a \bullet b$ is defined. If $a \circ b \prec^{\bullet} a \bullet b$ (or $a \circ b$ is not defined) then $\nu_{\bullet}(ab) = a \bullet b = \nu_{\bullet}(a) \bullet \nu_{\bullet}(b)$. Otherwise, if $a \circ b = a \bullet b$ then $ab = 2a \bullet b$, and again we get that $\nu_{\bullet}(ab) = a \bullet b = \nu_{\bullet}(a) \bullet \nu_{\bullet}(b)$. The claim is proved.

In a similar manner we establish that ν_{\circ} is a valuation as well.

We mention that an issue of whether a sum of two associative products form again an associative product similar to (a) is widely studied (see, e.g. [29]), while not in the context of semigroup algebras.

Remark 3.87. Let A be a k-algebra with a basis B equipped with a linear order \prec . For $b_1, b_2 \in B$ define $b_1 \circ b_2 \in B$ to be the highest (with respect to \prec) element in the decomposition in B of b_1b_2 , provided that $b_1b_2 \neq 0$, otherwise $b_1 \circ b_2$ is not defined. Assume that the following properties are fulfilled:

i) if $b_1 \prec b_2$ then $b_1 \circ b_0 \preceq b_2 \circ b_0$, provided that $b_1 \circ b_0, b_2 \circ b_0$ are defined (respectively, $b_0 \circ b_1 \preceq b_0 \circ b_2$, provided that $b_0 \circ b_1, b_0 \circ b_2$ are defined);

ii) $(b_1 \circ b_2) \circ b_3 = b_1 \circ (b_2 \circ b_3)$, moreover, $b_1 \circ b_2, (b_1 \circ b_2) \circ b_3$ are defined iff $b_2 \circ b_3, b_1 \circ (b_2 \circ b_3)$ are defined, $b_1, b_2, b_3 \in B$.

Then (B, \circ) is a partial semigroup. For any $a \in A \setminus \{0\}$ consider its decomposition $a = \lambda b + \cdots, \lambda \in \mathbb{k}^*$ in B where $b \in B$ is the highest element of B in this decomposition, we define $\nu(a) := b$. Then $\nu : A \setminus \{0\} \to B$ is an injective valuation.

Conversely, having an injective valuation $\nu : A \setminus \{0\} \rightarrow P$ and an adapted basis B one defines (as above) the (partial) operation \circ on B such that the partial semigroup (B, \circ) is isomorphic to P.

Note that for two injective valuations ν, ν' on $A \setminus \{0\}$ the images $\nu(A \setminus \{0\})$ and $\nu'(A \setminus \{0\})$ are not necessarily isomorphic as (partial) semigroups. It was demonstrated in Remark 3.70, we give here more examples.

Example 3.88. Let φ and ψ are injective homomorphisms $\mathbb{k}[z_1, z_2] \to \mathbb{k}[t_1^{\pm 1}, t_2^{\pm 1}]$ given respectively by: $\varphi(z_1) = t_1$, $\varphi(z_2) = t_2$, $\psi(z_1) = t_1$, $\psi(z_2) = t_1^2 + t_2$. Clearly, $C_{\varphi} = C_{\psi} = \mathbb{Z}^2_{>0}$. One can easily see that the set

$$\mathbf{B} = \{ b_{\mathbf{d}}^{\varepsilon} = z_1^{\varepsilon} z_2^{d_1} (z_2 - z_1^2)^{d_2} \, | \, \mathbf{d} = (d_1, d_2) \in \mathbb{Z}_{\geq 0}^2, \, \varepsilon \in \{0, 1\} \}$$

is a basis of $k[z_1, z_2]$ adapted to both ν_{φ} and ν_{ψ} (i.e., is an JH-basis in $k[z_1, z_2]$) and

$$\nu_{\varphi}(b_{\mathbf{d}}^{\varepsilon}) = (\varepsilon + 2d_2, d_1), \nu_{\psi}(b_{\mathbf{d}}^{\varepsilon}) = (\varepsilon + 2d_1, d_2)$$

for all $\mathbf{d} = (d_1, d_2) \in \mathbb{Z}^2_{>0}, \ \varepsilon \in \{0, 1\}$. Therefore,

$$\mathbf{K}_{\varphi,\psi}(a_1, a_2) = (2a_2 + (a_1)_2, \left\lfloor \frac{a_1}{2} \right\rfloor)$$

for all $a_1, a_2 \in \mathbb{Z}_{\geq 0}$. where $(a)_2 = a - 2 \lfloor \frac{a}{2} \rfloor$ is the parity of a. Moreover, $\mathbf{K}_{\varphi,\psi}$ is an involution on $\mathbb{Z}^2_{>0}$.

Example 3.89. Let φ be an endomorphism of $A = \mathbb{C}[t_{11}, t_{12}, t_{21}, t_{22}]$ given by $\varphi\begin{pmatrix}t_{11} & t_{12}\\t_{21} & t_{22}\end{pmatrix} = \begin{pmatrix}t_{11}t_{12}t_{21} & t_{11}t_{12}\\t_{11}t_{21} & t_{11}+t_{22}\end{pmatrix}$. One can show that φ is injective, therefore, $\nu_0 \circ \varphi$ is a well-defined injective valuation $A^* \to \mathbb{Z}^4_{\geq 0}$ (here ν_0 denotes the tautological valuation, see subsection 3.4). A common adapted basis of injective valuations ν_0 and $\nu_0 \circ \varphi$ is

$$\mathbf{B} = \{x_{11}^{m_1} x_{12}^{m_2} x_{21}^{m_3} x_{22}^{m_4} (x_{11} x_{22} - x_{12} x_{21})^{m_5} : \text{all } m_i \in \mathbb{Z}_{\ge 0}, \min(m_1, m_4) = 0\}$$

Then $(\nu_0 \circ \varphi)(A^*)$ is a submonoid of $\nu_0(A^*) = \mathbb{Z}_{\geq 0}^4$ given by $\left\{ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} : c_{11} \geq \max(c_{12}, c_{21}), \min(c_{12}, c_{21}) \geq c_{2,2} \geq 0 \right\}$, and $\mathbf{K}_{\nu_0 \circ \varphi, \nu_0}$ and $\mathbf{K}_{\nu_0, \nu_0 \circ \varphi}$ are given, respectively, by

$$\mathbf{K}_{\nu_{0}\circ\varphi,\nu_{0}}(\mathbf{d}) = \begin{pmatrix} \max(d_{11}, d_{22}) + d_{12} + d_{21} & d_{11} + d_{12} \\ d_{11} + d_{21} & \min(d_{11}, d_{22}) \end{pmatrix}, \mathbf{d} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \in \nu_{0}(A^{*}),$$
$$\mathbf{K}_{\nu_{0},\nu_{0}\circ\varphi}(\mathbf{c}) = \begin{pmatrix} \max(c_{22}, c_{12} + c_{21} - c_{11}) & \min(c_{11} - c_{21}, c_{12} - c_{22}) \\ \min(c_{11} - c_{12}, c_{21} - c_{22}) & \max(c_{22}, c_{11} + 2c_{22} - c_{12} - c_{21}) \end{pmatrix}, \mathbf{c} \in \nu_{0} \circ \varphi(A^{*})$$

Example 3.90. (String valuations and JH bijections on polynomials in 3 variables) Let E_1, E_2 be (locally nilpotent) derivations of $\Bbbk[x_1, x_2, x_3]$ given by

$$E_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \ E_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}.$$

Clearly, the Lie algebra generated by E_1, E_2 is isomorphic to the 3×3 nilpotent matrices because of the defining Serre relations $[E_i, [E_i, E_{3-i}]] = 0$ for i = 1, 2.

Let $\mathbf{E}_i = (E_i, E_{3-i}, E_i)$ for i = 1, 2, in the notation of Section 4.2 and abbreviate $\nu_i := \nu_{\mathbf{E}_i}, i = 1, 2.$ One can show that $\nu_i = \nu_0 \circ \varphi_i$, where $\nu_0 : \mathbb{k}[t_1, t_2, t_3] \setminus \{0\} =$ $\mathbb{RZ}^3_{>0} \setminus \{0\} \to \mathbb{Z}^3_{>0}$ is the standard (tautological) valuation and $\varphi_1, \varphi_2 : \mathbb{R}[x_1, x_2, x_3] \hookrightarrow \mathbb{R}[x_1, x_2, x_3]$ $k[t_1, t_2, t_3]$ are injective homomorphisms given respectively by

$$\varphi_1(x_1, x_2, x_3) = (t_1 + t_3, t_2, t_1 t_2), \ \varphi_2(x_1, x_2, x_3) = (t_2, t_1 + t_3, t_2 t_3)$$

It is easy to see that the basis $x^d = x_1^{d_1} x_2^{d_2} x_3^{d_3}$ $d = (d_1, d_2, d_3) \in \mathbb{Z}^3_{\geq 0}$ is adapted to ν_2 and the basis $\tilde{x}^d = x_1^{d_1} x_2^{d_2} x_4^{d_3}, d = (d_1, d_2, d_3) \in \mathbb{Z}_{\geq 0}^3$, where $x_4 := x_1 x_2 - x_3$, is adapted to ν_1 .

Actually, they have a common adapted basis. Let $b_{\mathbf{m}} = x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}$ for $\mathbf{m} =$ $(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4_{>0}$, and let

$$\mathbf{M} = \{ (m_1, m_2, m_3, m_4) \in \mathbb{Z}_{\geq 0}^4 : \min(m_1, m_2) = 0 \} .$$

Clearly, the relation $x_1x_2 = x_3 + x_4$ implies that the set $\mathbf{B} := \{b_{\mathbf{m}}, \mathbf{m} \in \mathbf{M}\}$ is a basis of $k[x_1, x_2, x_3]$. We claim that **B** is adapted to both ν_1 and ν_2 .

- Indeed, $E_i(x_j) = \delta_{i,j}$ and $E_i(x_{j+2}) = \delta_{i,j}x_{3-i}$ for i, j = 1, 2 and $\nu_{E_i}(b_{\mathbf{m}}) = m_i + m_{i+2}, \ \lambda_{E_i}(b_{\mathbf{m}}) = x_{3-i}^{m_{3-i}+m_{2+i}}x_{5-i}^{m_{5-i}}.$
- $\nu_{E_{3-i}}(\lambda_{E_i}(b_{\mathbf{m}})) = m_{3-i} + m_{2+i} + m_{5-i}, \ \lambda_{(E_i, E_{3-i})}(b_{\mathbf{m}}) = \lambda_{E_{3-i}}(\lambda_{E_i}(b_{\mathbf{m}})) = x_i^{m_{5-i}}.$ $\nu_{E_i}(\lambda_{(E_i, E_{3-i})}(b_{\mathbf{m}})) = m_{5-i}, \ \lambda_{\mathbf{E}_i}(b_{\mathbf{m}}) = \lambda_{E_i}(x_i^{m_{5-i}}) = 1.$

Therefore,

$$\nu_i(b_{\mathbf{m}}) = (m_i + m_{i+2}, m_{3-i} + m_{i+2} + m_{5-i}, m_{5-i})$$

and $C_{\varphi_i} = C_{\nu_{\mathbf{E}_i}} = C$ for i = 1, 2, where

$$C = \{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 0} : a_2 \geq a_3\}$$
.

Taking into account that $\varphi_i^*(b_{\mathbf{m}}) = (t_1 + t_3)^{m_i} t_2^{m_{3-i}} (t_1 t_2)^{m_{2+i}} (t_2 t_3)^{m_{5-i}}$, we obtain

$$\nu_i(b_{\mathbf{m}}) = \nu_0(\varphi_i^*(b_{\mathbf{m}})) = (m_i + m_{i+2}, m_{3-i} + m_{i+2} + m_{5-i}, m_{5-i}) = \nu_i(b_{\mathbf{m}}) .$$

The JH bijection $K_{\nu_{\mathbf{E}_2},\nu_{\mathbf{E}_1}}: C \to C$ is given by

$$(a_1, a_2, a_3) \mapsto (\max(a_3, a_2 - a_1), a_1 + a_3, \min(a_1, a_2 - a_3))$$
.

Finally, define injective homomorphisms $\psi_i : \mathbb{k}[x_1, x_2, x_3] \hookrightarrow \mathbb{k}[t_1, t_2, t_3], i = 1, 2$ by

$$\psi_1(x_1, x_2, x_3) = (t_1, t_3, t_2), \ \psi_2(x_1, x_2, x_3) = (t_2, t_1t_2 - t_3, t_1)$$

and abbreviate $\nu'_i := \nu \circ \psi_i$

It is easy to see that $\nu'_{b_{\mathbf{m}}} = (m_i + m_{5-i}, m_{i+2}, m_{3-i} + m_{5-i})$ for i = 1, 2 and all $\mathbf{m} \in \mathbf{M}$. Therefore, **B** is adapted to both ψ_1 and ψ_2 as well and $C_{\nu'_1} = C_{\nu'_2} = \mathbb{Z}^3_{>0}$. The JH bijection $K_{\nu'_2,\nu'_1}: \mathbb{Z}^3_{\geq 0} \xrightarrow{\sim} \mathbb{Z}^3_{\geq 0}$ is given by

$$(d_1, d_2, d_3) \mapsto (d_2 + \max(0, d_3 - d_1), \min(d_1, d_3), d_2 + \max(0, d_1 - d_3))$$
.

Also, $K_{\nu_i,\nu'_i}: \mathbb{Z}^3_{>0} \to C$ is given by

$$(d_1, d_2, d_3) \mapsto (d_2 + \max(0, d_1 - d_3), d_2 + d_3, \min(d_1, d_3))$$

and $K_{\nu_{3-i},\nu'_i}: \mathbb{Z}^3_{\geq 0} \xrightarrow{\sim} C$ is given by $(d_1, d_2, d_3) \mapsto (d_3, d_1 + d_2, d_2).$

Example 3.91. Now we study a non-commutative analog of Example 3.90. Denote by $\Bbbk \langle t_1, t_2, t_3 \rangle$ the free algebra endowed with a well-ordering \prec on monomials defined as follows. If a monomial m_1 is shorter than a monomial m_2 then $m_1 \prec m_2$. Otherwise, if their lengths coincide then \prec is determined by lex with respect to $t_3 \prec t_2 \prec t_1$.

Consider homomorphisms

$$\varphi_i: A := \Bbbk \langle x_1, x_2, x_3 \rangle \to \Bbbk \langle t_1, t_2, t_3 \rangle, i = 1, 2;$$

 $\varphi_1(x_1) = t_1 + t_3, \\ \varphi_1(x_2) = t_2, \\ \varphi_1(x_3) = t_1 \\ t_2; \\ \varphi_2(x_1) = t_2, \\ \varphi_2(x_2) = t_1 + t_3, \\ \varphi_2(x_3) = t_2 \\ t_3.$

Denote $m := x_1x_2 - x_3$. We claim that a set $\mathbf{B} \subset A$ of monomials in x_1, x_2, x_3, m without submonomials x_1x_2 constitutes a common adapted basis of A with respect to valuations $\nu_{\varphi_1}, \nu_{\varphi_2}$.

First, **B** spans A since in any monomial in x_1, x_2, x_3 one can replace each occurrence of submonomial x_1x_2 by $m + x_3$.

Now we observe that given a monomial $b \in \mathbf{B}$ in order to compute the leading monomial of $\varphi_1(b) \in \mathbb{k}\langle t_1, t_2, t_3 \rangle$ one has to replace each occurrence of x_1, x_2, x_3 and of m as follows:

$$x_1 \rightarrow t_1, x_2 \rightarrow t_2, x_3 \rightarrow t_1 t_2, m \rightarrow t_3 t_2.$$

Note that these leading monomials are pairwise distinct for different monomials from **B** since each such monomial T in t_1, t_2, t_3 can be uniquely represented in the following way. Between any pair of adjacent occurences in T of submonomials of the form either t_1t_2 or t_3t_2 the submonomial of T has a form $t_2 \ldots t_2t_1 \ldots t_1$. In other words, one can describe the set of leading monomials of $\varphi_1(\mathbf{B})$ as the monoid $C_{\varphi_1} \subset \langle t_1, t_2, t_3 \rangle$ generated by t_1, t_2, t_3t_2 .

This implies that the elements of **B** are linearly independent, so **B** is a basis of A, in addition that φ_1 is a monomorphism, and **B** is an adapted basis with respect to the valuation ν_{φ_1} .

In a similar manner, the leading monomial of $\varphi_2(b)$ is obtained by means of the following replacements:

$$x_1 \rightarrow t_2, x_2 \rightarrow t_1, x_3 \rightarrow t_2 t_3, m \rightarrow t_2 t_1.$$

Again, these leading monomials are distinct for different elements $b \in \mathbf{B}$. Any leading monomial T_2 can be uniquely represented as follows. Between an adjacent occurences of a pair of submonomials of the form either t_2t_3 or t_2t_1 the submonomial of T_2 coincides with $t_1 \ldots t_1 t_2 \ldots t_2$. Hence φ_2 is also a monomorphism. Thus, **B** is a common adapted basis with respect to both valuations $\nu_{\varphi_1}, \nu_{\varphi_2}$. The set of leading monomials of $\varphi_2(\mathbf{B})$ equals the monoid $C_{\varphi_2} \subset \langle t_1, t_2, t_3 \rangle$ generated by $t_1, t_2, t_2 t_3$.

Note that unlike the commutative case (Example 3.90) it holds $C_{\varphi_1} \neq C_{\varphi_2}$: for instance, $t_3t_2 \in C_{\varphi_1} \setminus C_{\varphi_2}$, while $t_2t_3 \in C_{\varphi_2} \setminus C_{\varphi_1}$. One obtains JHb $\mathbf{K}_{\nu_{\varphi_2},\nu_{\varphi_1}}$ as follows. In a monomial $T \in C_{\varphi_1}$ (see the notations above) we replace each occurrence of t_1t_2 by t_2t_3 , respectively, each occurrence of t_3t_2 by t_2t_1 , in addition, in a submonomial of the form $t_2 \ldots t_2t_1 \ldots t_1$ between a pair of adjacent occurrences of either t_1t_2 or t_3t_2 , we replace t_2 by t_1 and t_1 by t_2 , thereby we get a submonomial $t_1 \ldots t_1t_2 \ldots t_2$. The resulting monomial is $\mathbf{K}_{\nu_{\varphi_2},\nu_{\varphi_1}}(T) \in C_{\varphi_2}$. **Definition 3.92.** Let ν_{\bullet} and ν_{\circ} be injective valuations on an algebra A. Suppose that a basis **B** is adapted to both valuations. This turns **B** into ordered partial semigroups $(\mathbf{B}, \circ, \preceq^{\circ})$ and $(\mathbf{B}, \bullet, \preceq^{\bullet})$, see Remark 3.87.

We say that ν_{\bullet} and ν_{\circ} are polar with respect to **B** if any b'' occurring in $bb' \neq 0$ satisfies

$$b \bullet b' \preceq^{\circ} b'', b \circ b' \preceq^{\bullet} b''.$$

Remark 3.93. i) For the algebra $\mathbb{k}[x, y, z]/(z + x^2 + y^3)$ constructed in Remark 3.70, each pair among its injective valuations ν_x, ν_y, ν_z is polar with respect to the produced common adapted basis. For instance, for the common basis $\{y^i z^j x^k : i, j \ge 0, 0 \le k \le 1\}$ of ν_y, ν_z it holds $x \cdot x = -y^3 - z$, hence $\nu_y(x^2) = \nu_y(z) \succ \nu_y(y^3), \nu_z(x^2) = \nu_z(y^3) \succ \nu_z(z)$.

In a similar way, for the algebra $\mathbb{k}[x, y, z, t]/(x^2 + y^3 + z^5 + t^7)$ constructed in Example 3.72, its valuations ν_1, ν_2 are polar with respect to the produced their common adapted basis $\{y^i t^j w^l x^k z^m : i, j, l \ge 0, 0 \le k \le 1, 0 \le m \le 4\}$. Indeed, $x \cdot x = w - y^3, z^2 \cdot z^3 = z \cdot z^4 = w - t^7$, and $\nu_1(x^2) = \nu_1(w) \succ \nu_1(y^3), \nu_2(x^2) =$ $\nu_2(y^3) \succ \nu_2(w), \nu_1(z^5) = \nu_1(t^7) \succ \nu_1(w), \nu_2(z^5) = \nu_2(w) \succ \nu_2(t^7)$, therefore the polar condition holds also for the decomposition in the basis of the products $xz^2 \cdot xz^3 =$ $xz \cdot xz^4 = -wt^7 + w^2 + y^3t^7 - y^3w$.

ii) Observe that the injective valuations produced in Examples 3.88, 3.89 are polar with respect to the basis **B**. The same is true for the injective valuations $\nu_0 \circ \varphi_1, \nu_0 \circ \varphi_2$ with respect to the basis **B** produced in Example 3.90.

4. Appendix: Valuations of vector spaces and Jordan-Hölder Bijections

4.1. Valuations on vector spaces. Given a vector space S over a field \Bbbk and a totally ordered set (C, <), following [19, 22, 23], we say that a map $\nu : S \setminus \{0\} \to C$ is a valuation if $\nu(\Bbbk^{\times} \cdot x) = \nu(x)$ for all nonzero $x \in S$ and $\nu(x + y) \leq \max(\nu(x), \nu(y))$ for $x + y \neq 0$ (this implies that $\nu(x + y) = \max(\nu(x), \nu(y))$ whenever $\nu(x) \neq \nu(y)$). Denote by C_{ν} the image $\nu(S \setminus \{0\})$.

One can construct valuations on another vector space (resp. integral domain) S' by importing a given valuation ν on S via any injective k-linear map $f: S' \hookrightarrow S$ (resp. an injective homomorphism of k-algebras). Namely, the composition $\nu \circ f$ is a valuation on S'.

Each valuation $\nu: S \setminus \{0\} \to C$ defines a filtration S_{\leq} of subspaces on S via

$$S_{\leq a} := \{0\} \cup \{x \in S \setminus \{0\} : \nu(x) \le a\}$$

for $a \in C_{\nu}$ (if S is an integral domain, this is a filtration on a k-algebra). We also abbreviate $S_{\langle a \rangle} := \sum_{a' \langle a \rangle} S_{\langle a' \rangle}$ and denote $S_a := S_{\langle a \rangle} S_{\langle a \rangle}$ for $a \in C$ (S_a is called in [19] the *leaf at a*).

Conversely, if C is a well-order, then any increasing filtration $S_{\leq a}$, $a \in C$ of S defines a (well-ordered) valuation $\nu : S \setminus \{0\} \to C$ via

$$\nu(x) = \min\{a \in C : x \in S_{\le a}\}$$

for all $x \in S \setminus \{0\}$.

Following [4, 20], we say that $\mathbf{B} \subset S$ is *adapted* to a valuation $\nu : S \setminus \{0\} \to C$ if for each $a \in C$ the restriction to $\mathbf{B}_a = \{b \in \mathbf{B} \mid \nu(b) = a\}$ of the canonical projection $\pi_a : S_{\leq a} \to S_a$ is injective and the image $\pi_a(\mathbf{B}_a)$ is a basis of S_a .

If the filtration on S is induced by a valuation $\nu : S \setminus \{0\} \to C$, we refer to adapted subsets of S as ν -adapted.

Remark 4.1. Such a (necessarily independent) subset **B** of *S* is called *valuation-independent* in [22], [23], but we prefer terminology of [20, Definition 2]. If we denote $gr S := \bigoplus_{a \in C} S_a$, then clearly any adapted subset of *S* defines a basis of gr S.

We say that ν is *locally finite* iff there exists an isomorphism $f: S \rightarrow gr S = \bigoplus_{a \in C} S_a$

of k-vector spaces such that $f(S_{\leq a}) = \bigoplus_{a' \leq a} S_{a'}$ for each $a \in C$.

The following is immediate.

Lemma 4.2. Let S be a k-vector space and $\nu : S \setminus \{0\} \to C$ be any valuation. Then: (a) For any basis $\underline{\mathbf{B}}$ of gr S such that any $\underline{\mathbf{B}}_a := S_a \cap \underline{\mathbf{B}}$ a basis of S_a one can construct an adapted set \mathbf{B} as follows.

$$\mathbf{B}_a = \iota_a(\underline{\mathbf{B}}_a) \; ,$$

where and $\iota_a : S_a \hookrightarrow S_{\leq a}$ is any simultaneous splitting of canonical projections $\pi_a : S_{\leq a} \twoheadrightarrow S_a$.

(b) ν is locally finite iff S admits an adapted basis.

(c) If C is a well-order, then ν is locally finite, moreover, any adapted subset of S is a basis.

Example 4.3. Let $S = \mathbb{k}((t^{-1}))$ be the algebra of all formal Laurent series in t^{-1} over \mathbb{k} . Then setting $\nu(f) = n$ for each $f = \sum_{m=-\infty}^{n} a_m t^m$ with $a_n \neq 0$ defines a valuation $S \setminus \{0\} \to \mathbb{Z}$. This valuation is **not** locally finite, in particular, the subset $\mathbf{B} = \{t^m, m \in \mathbb{Z}\}$ is adapted to ν , however, it is not a basis of S. In fact, there is **no** adapted bases in S. At the same time, the restriction of ν to the subalgebra $S_0 = \mathbb{k}[t, t^{-1}]$ of Laurent polynomials is a locally-finite valuation on S_0 .

It turns out that we can always propagate valuations to tensor products without assuming that they are well-ordered.

Proposition 4.4. Let S be k-vector space and $\nu : S \setminus \{0\} \to C$ be a valuation. Then for any k-vector space S' we have:

(a) There exists a unique a valuation $\nu^{S'}: S \otimes S' \setminus \{0\} \to C$ such that

(4.1)
$$\nu^{S'}(x \otimes y) = \nu(x)$$

for all $x \in S \setminus \{0\}$, $y \in S' \setminus \{0\}$ so that the associated filtration on $S \otimes S'$ is

$$(S \otimes S')_{\leq a} = S_{\leq a} \otimes S'$$

for $a \in C$.

(b) For any valuation $\nu' : S' \setminus \{0\} \to C'$ there exists a unique valuation $\nu \otimes \nu' : S \otimes S' \setminus \{0\} \to C \times C'$ such that

(4.2)
$$(\nu \otimes \nu')(x \otimes y) = (\nu(x), \nu'(y))$$

for $x \in S \setminus 0$, $y \in S' \setminus 0$ (where we equip $C \times C'$ with the lexicographic ordering, i.e., $(a, a') < (\tilde{a}, \tilde{a}')$ whenever either $a < \tilde{a}$ or $\tilde{a} = a$ and $a' < \tilde{a}'$) so that the associated filtration on $S \otimes S'$ is

$$(S \otimes S')_{\leq (a,a')} = S_{\leq a} \otimes S'_{\leq a'} + S_{$$

for $(a, a') \in C \times C'$.

We prove Proposition 4.4 in Section 4.8.

By definition, $C_{\nu S'} = C_{\nu}$, $\nu^{S'}(s \otimes (S' \setminus \{0\})) = \nu(s)$ for any $s \in S \setminus \{0\}$ and $\nu^{S'}((S \setminus \{0\}) \otimes s') = C_{\nu}$ for any $s' \in S' \setminus \{0\}$ in Proposition 4.4(a).

4.2. **Decorated valuations.** In this section we generalize valuations by taking into account their leading coefficients.

Definition 4.5. Given k-vector spaces S, S' and a valuation $\nu : S \setminus \{0\} \to C$, we say a map $\lambda : S \setminus \{0\} \to S' \setminus \{0\}$ is a *leading coefficient* of ν if $\lambda(cx) = c\lambda(x)$ for all $c \in \mathbb{k}, x \in S \setminus \{0\}$ and $\lambda(x+y) = \begin{cases} \lambda(x) + \lambda(y) & \text{if } \nu(x) = \nu(y) = \nu(x+y) \\ \lambda(x) & \text{if } \nu(x) > \nu(y) \\ \lambda(y) & \text{if } \nu(x) < \nu(y) \end{cases}$ for

any $x, y \in S$ such that $x + y \neq 0$. Sometimes we will refer to the pair (ν, λ) as a *decorated valuation*.

If both S and S' are k-algebras, we require additionally any leading coefficient λ of ν to satisfy

(4.3)
$$\lambda(xy) = \lambda(x)\lambda(y)$$

for all $x, y \in S \setminus \{0\}$ (i.e., λ is a homomorphism of multiplicative semigroups).

We will sometimes refer to a leading coefficient λ satisfying (4.3) as *multiplicative*. The following are immediate.

Lemma 4.6. For any decorated valuation (ν, λ) on a vector space S one has

(a) $\lambda(x_1 + \dots + x_r) = \sum_{\substack{j \in [1,r]:\\\nu(x_j) = \max(\nu(x_1),\dots,\nu(x_r))}} \lambda(x_j)$ whenever $x_1 + \dots + x_r \neq 0$ and

 $\nu(x_1 + \dots + x_r) = \max(\nu(x_1), \dots, \nu(x_r)).$

(b) For any subspace S_0 of S the restriction of (ν, λ) to $S_0 \setminus \{0\}$ is a decorated valuation on S_0 .

(c) For any injective linear map $f: S' \hookrightarrow S''$ the pair $(\nu, f \circ \lambda)$ is a decorated valuation on S.

Lemma 4.7. Let $\nu : S \setminus \{0\} \to C$ and $\nu' : S' \setminus \{0\} \to C'$ be valuations and $\lambda : S \setminus \{0\} \to S' \setminus \{0\}$ be a leading coefficient of ν . Then the assignments $x \mapsto (\nu(x), \nu'(\lambda(x)))$ define a valuation $\nu \times_{\lambda} \nu' : S \setminus \{0\} \to C \times C'$, with the lexicographic order on $C \times C'$.

The following immediate result gives various characterizations of decorated valuations in terms of adapted bases.

Lemma 4.8. Let S and S' be k-vector spaces, $\nu : S \setminus \{0\} \to C$ be a valuation, and **B** be a basis of S adapted to ν . Then

(a) Any map $f : \mathbf{B} \to S' \setminus \{0\}$ uniquely extends to a leading coefficient $\lambda = \lambda_{\mathbf{B},f} : S \setminus \{0\} \to S' \setminus \{0\}.$

(b) The assignments $b \otimes s' \mapsto s'$, $b \in \mathbf{B}$, $s' \in S'$ define a leading coefficient $\lambda^{\mathbf{B}} : S \otimes S' \setminus \{0\} \to S'$ of the valuation $\nu^{S'}$ (in the notation of Proposition 4.4(a)).

(c) For any leading coefficient $\lambda : S \setminus \{0\} \to S' \setminus \{0\}$ of ν there exists a unique injective linear map $\delta = \delta_{\mathbf{B},\lambda} : S \hookrightarrow S \otimes S'$ such that $\lambda = \lambda^{\mathbf{B}} \circ \delta$ (in fact, δ is given by $\delta(b) = b \otimes \lambda(b)$ for all $b \in \mathbf{B}$).

The following provides an example of decorated valuations "in nature."

Lemma 4.9. Let k be of characteristic 0, S be a k-vector space, and E be a locally nilpotent linear map $S \to S$, i.e., for each nonzero $x \neq 0$ there is a unique number $\nu_E(x) \in \mathbb{Z}_{\geq 0}$ such that $E^{\nu_E(x)}(x) \neq 0$ and $E^{\nu_E(x)+1}(x) = 0$. Then

(a) The assignments $x \mapsto \nu_E(x)$ define a valuation $\nu_E : S \setminus \{0\} \to \mathbb{Z}_{\geq 0}$.

(b) The assignments $x \mapsto E^{(\nu_E(x))}(x)$ define the leading coefficient $\lambda_E : S \setminus \{0\} \to S \setminus \{0\}$, where we abbreviate $E^{(n)} := \frac{1}{n!}E^n$, the n-th divided power.

(c) If S is an integral domain over \Bbbk and E is a locally nilpotent derivation of S, then ν_E is an additive valuation on S and λ_E is its multiplicative leading coefficient.

More generally, let $\mathbf{E} = (E_1, \ldots, E_m)$ be a family of locally nilpotent linear maps $S \to S$. Define the map $\lambda_{\mathbf{E}} : S \setminus \{0\} \to S \setminus \{0\}$ by

$$\lambda_{\mathbf{E}} := \lambda_{E_m} \circ \cdots \circ \lambda_{E_1} ,$$

where $\lambda_E : S \setminus \{0\} \to S \setminus \{0\}$ is as in Lemma 4.9 (with the convention $\lambda_{\emptyset} = Id_{S \setminus \{0\}}$). Then define the map $\nu_{\mathbf{E}} : S \setminus \{0\} \to \mathbb{Z}_{>0}^m$ by

$$\nu_{\mathbf{E}}(x) = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m ,$$

where $a_k = \nu_{E_k}(\lambda_{(E_1,\dots,E_{k-1})}(x))$ for $k \in [m]$ (actually, $a_1 = \nu_{E_1}(x)$). The following is a generalization of Lemma 4.9.

Corollary 4.10. Let \Bbbk be of characteristic 0, S be a \Bbbk -vector space. Then for any family $\mathbf{E} = (E_1, \ldots, E_m)$ of locally nilpotent linear maps $S \to S$ one has:

(a) $\nu_{\mathbf{E}} : S \setminus \{0\} \to \mathbb{Z}_{\geq 0}^m$ is a valuation and $\lambda_{\mathbf{E}} : S \setminus \{0\} \to S \setminus \{0\}$ is its leading coefficient.

(b) If S is an integral domain over \Bbbk and each E_k is a locally nilpotent derivation of S, then $\nu_{\mathbf{E}}$ is additive and $\lambda_{\mathbf{E}}$ is multiplicative.

Remark 4.11. The decorated valuations $(\nu_{\mathbf{E}}, \lambda_{\mathbf{E}})$ generalize *string* valuations and their leading coefficients introduced by Andrei Zelevinsky and the first author in [5].

Remark 4.12. In fact, all valuations $\nu_{\mathbf{E}}$ factor (and thus can be defined recursively) as in Lemma 4.7: $\nu_{(E_1,\ldots,E_m)} = \nu_{(E_1,\ldots,E_k)} \times_{\lambda_{(E_1,\ldots,E_k)}} \nu_{(E_{k+1},\ldots,E_m)}$ for any $k \in [1, m-1]$.

4.3. Injective valuations. Our main focus is on the class of what we call *injective* valuations, i.e., locally finite valuations such that $S_a = S_{\leq a}/S_{\langle a}$ is one-dimensional for each $a \in C_{\nu}$ (such valuations were called valuations with one-dimensional leaves in [18]). Note, however, that the valuation on $\mathbb{k}((t^{-1}))$ in Example 4.3 has one-dimensional leaves, but is not locally finite, hence not injective.

The following is an immediate consequence of Lemma 4.2(c).

Lemma 4.13. A well-ordered valuation $\nu : S \setminus \{0\} \to C$ is injective iff there exists a basis **B** of S such that the restriction of ν to **B** is an injective map $\mathbf{B} \hookrightarrow C$.

As in Section 4.1, we refer to any basis **B** satisfying Lemma 4.13 as *adapted* to ν and denote by \mathbf{A}_{ν} the set of all bases of *S* adapted to ν (in [22], [23] each $\mathbf{B} \in \mathbf{A}_{\nu}$ is referred to as a *valuation basis*).

One can easily show that for any basis **B** adapted to (an injective valuation) ν one has $\nu(\mathbf{B}) = C_{\nu}$ and $S_{\leq a} = \bigoplus_{b \in \mathbf{B}: \nu(b) \leq a} \mathbb{k} \cdot b$, $S_{< a} = \bigoplus_{b \in \mathbf{B}: \nu(b) < a} \mathbb{k} \cdot b$ for all $a \in C_{\nu}$.

The following result establishes a convenient criterion of injectivity of a valuation.

Proposition 4.14 (Euclidean property). The following are equivalent for a given well-ordered valuation $\nu : S \setminus \{0\} \to C$.

(a) ν is injective.

(b) For any non-zero $x, y \in S$ such that $\nu(x) = \nu(y)$ and $x \notin \mathbb{k} \cdot y$ there exists (a unique) $c \in \mathbb{k}^{\times}$ such that $\nu(x - cy) < \nu(x)$.

We prove Proposition 4.14 in Section 4.8.

Proposition 4.14 is well-known for finite-dimensional S (see e.g., [19]), for infinite-dimensional S we could not find it in the literature.

Corollary 4.15. For a given well-ordered injective valuation $\nu : S \setminus \{0\} \to C$ any ν -adapted set is an (adapted) basis of S.

Remark 4.16. We demonstrate that the conclusion of Corollary 4.15 is not valid without the assumption of well-orderness. Consider a space S with a basis $\{e_i : 0 \leq i < \infty\}$ and an injective valuation $\nu : S \setminus \{0\} \to \mathbb{Z}_{\leq 0}$ such that $\nu(e_i) = -i$. Then a set $R := \{e_i + e_{i+1} : 0 \leq i < \infty\}$ is adapted, while it is not a basis of S: for instance, e_1 does not belong to the span of R.

We can build new injective valuations out of existing ones by the following immediate consequence of the injectivity criterion in Proposition 4.14(b).

Corollary 4.17. Let $\nu : S \setminus \{0\} \to C$ be a well-ordered injective valuation. Then for any subspace <u>S</u> of S the restriction $\underline{\nu} := \nu|_{S \setminus \{0\}}$ is an injective valuation on <u>S</u>.

Remark 4.18. It is interesting whether an analog of Corollary 4.17 holds without assumption of well-orderness of C.

Given a valuation $\nu : S \setminus \{0\} \to C$, we say that a family $S_i, I \in I$ of subspaces of S is ν -compatible if $\nu(\bigcap_{i \in I} S_i \setminus \{0\}) = \bigcap_{i \in I} \nu(S_i \setminus \{0\})$ (clearly, for any ν the left hand side is always a subset of the right hand side).

Proposition 4.19. Suppose that a valuation ν on a space S is well-ordered injective. Then a family of subspaces $\{S_i, i \in I\}$ of S is ν -compatible iff there exists an adapted with respect to ν basis \mathbf{B} in S such that $\mathbf{B} \cap S_i$ is a basis in S_i for each $i \in I$. In addition, in this case $\mathbf{B} \cap \bigcap_{j \in J} S_j$ is a basis in $\bigcap_{j \in J} S_j$ for each $J \subseteq I$ and $\nu((\sum_{j \in J} S_j) \setminus \{0\}) = \bigcup_{j \in J} \nu(S_j \setminus \{0\})$ for every subset $J \subseteq I$.

We prove Proposition 4.19 in Section 4.8.

Remark 4.20. If $\{S_i, i \in I\}$ is a ν -compatible family, $|I| < \infty$, dim $(S_i) < \infty$, $i \in I$ then for any subset $J \subseteq I$ it holds

$$\dim(\sum_{j\in J} S_j) = \sum_{L\subseteq J} (-1)^{|L|+1} \dim(\bigcap_{l\in L} S_l).$$

We can also construct injective valuations on the quotients as follows.

Proposition 4.21. Let S be a k-vector space and let $\nu : S \setminus \{0\} \to C$ is an (injective) valuation for some well-order C. Then for any subspace $J \subset S$ the assignments

$$\nu'(v+J) := \min\{\nu(v+J)\}\$$

for all non-zero $v + J \in S/J$ define an (injective) valuation $\nu' : S/J \setminus \{0\} \to C$.

Remark 4.22. Note, however, that if S is a commutative integral domain, J a prime ideal, C a monoid and $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in S \setminus \{0\}$ in Proposition 4.21, then $\nu'(ab+J) \leq \nu'(a+J) + \nu'(b+J)$ for all $a, b \in S \setminus J$ because of the inequality $\min\{\nu(X \cdot Y)\} \leq \min\{\nu(X)\} + \min\{\nu(Y)\}$ for any subsets $X, Y \subset S \setminus \{0\}$ (here $X \cdot Y$ is the k-linear span of $\{xy \mid x \in X, y \in Y\}$).

It turns out that any valuation can be assembled out of injective ones as follows.

Proposition 4.23. Let S be a k-vector space and $\nu : S \setminus \{0\} \to C$ be a locally finite valuation (see Section 4.1). Then there are k-vector spaces <u>S</u> and S', an injective valuation $\underline{\nu} : \underline{S} \setminus \{0\} \to C$ and a k-linear embedding $\mathbf{j} : S \to \underline{S} \otimes S'$ such that $C_{\underline{\nu}} = C_{\nu}$ and

$$\nu = \underline{\nu}^{S'} \circ \mathbf{j}$$

in the notation (4.1).

We prove Proposition 4.23 in Section 4.8.

4.4. Jordan-Hölder bijections. For any valuations $\nu, \nu' : S \setminus \{0\} \to C$ such that ν' is well-ordered define a map $\mathbf{K}_{\nu',\nu} : C_{\nu} \to C_{\nu'}$ by

(4.4)
$$\mathbf{K}_{\nu',\nu}(a) = \min\{\nu'(\nu^{-1}(a))\}$$

for all $a \in C_{\nu}$.

Our first result provides an "industry" for establishing combinatorial bijections.

Theorem 4.24. For any well-ordered injective valuations ν and ν' on S the maps $\mathbf{K}_{\nu',\nu}: C_{\nu} \to C_{\nu'}$ and $\mathbf{K}_{\nu,\nu'}: C_{\nu'} \to C_{\nu}$ are well-defined and mutually inverse bijections. Moreover, there exists a basis $\mathbf{B}_{\nu,\nu'}$ of S adapted to both ν and ν' such that $\mathbf{K}_{\nu',\nu}(\nu(b)) = \nu'(b)$ for all $b \in \mathbf{B}_{\nu,\nu'}$.

We prove Theorem 4.24 in Section 4.8.

We refer to $\mathbf{K}_{\nu',\nu}$ as Jordan-Hölder bijection (JHb) and call any basis $\mathbf{B}_{\nu,\nu'}$ as an JH-basis.

Remark 4.25. In fact, Theorem 4.24 generalizes well-known facts that any two complete flags in \mathbb{k}^n have a canonical relative position w, which is a permutation of $\{1, \ldots, n\}$, and admit a common basis. Namely, an injective valuation $\nu : S \setminus \{0\} \to C$ defines a complete flag \mathcal{F}_{ν} indexed by C_{ν} via $(\mathcal{F}_{\nu})_{\leq a} = \{v \in S \setminus \{0\} : \nu(v) \leq a\},$ $a \in C_{\nu}$ (see Sections 4.1 and 4.3 for details). Conversely, any complete flag \mathcal{F} on Sis of the form \mathcal{F}_{ν} . If the indexing sets for flags \mathcal{F}_{ν} and $\mathcal{F}_{\nu'}$ are well-ordered, then Theorem 4.24 asserts that there exist a canonical relative position $\mathbf{K}_{\nu',\nu}$ of \mathcal{F}_{ν} and $\mathcal{F}_{\nu'}$ and a common (JH) basis. This can be also reformulated in terms of generalized Jordan-Hölder correspondence developed by Abels in 1991, see, e.g., Section 2.3 of 8.

The following result is an immediate consequence of Theorem 4.24.

Corollary 4.26. In the assumptions of Theorem 4.24 the set $A_{\nu} \cap A_{\nu'}$ is nonempty.

The following result is a reverse of Theorem 4.24, however, we do not assume that valuations are well-ordered.

Proposition 4.27. Let ν and ν' be (not necessarily well-ordered) injective valuations on S such that $\mathbf{A}_{\nu} \cap \mathbf{A}_{\nu'}$ is nonempty. Then the assignments (4.4) define a bijection $\mathbf{K}_{\nu',\nu}: C_{\nu} \xrightarrow{\sim} C_{\nu'}$ so that $\nu'(b) = \mathbf{K}_{\nu',\nu}(\nu(b))$ for any $\mathbf{B} \in \mathbf{A}_{\nu} \cap \mathbf{A}_{\nu'}$ and all $b \in \mathbf{B}$.

We prove Proposition 4.27 in Section 4.8.

Example 4.28. Let $S = \mathbb{k}[t]$ and let $\nu, \nu' : S \setminus \{0\} \to -\mathbb{Z}_{\geq 0}$ be valuations given by $u(t^k) - u'(t^k + 1) - -k$

$$\nu(c) = \nu(c + 1) = -\pi$$

for $k \in \mathbb{Z}_{\geq 0}$. These valuations are obviously injective, and are adapted respectively to the bases $\mathbf{B} = \{t^k, k \in \mathbb{Z}_{\geq 0}\}, \mathbf{B}' = \{1 + t^k, k \in \mathbb{Z}_{\geq 0}\}$, however, $\nu(\mathbf{B}') = \nu'(\mathbf{B}) = \{0\}$. Denote $\mathbf{B}'' = \{t^k - t^{k+1}, k \in \mathbb{Z}_{\geq 0}\}$. Clearly, $\nu(\mathbf{B}'') = \nu'(\mathbf{B}'') = -\mathbb{Z}_{\geq 0}$ because

$$\nu(t^{k} - t^{k+1}) = -k, \ \nu'(t^{k} - t^{k+1}) = \max(\nu'(1+t^{k}), \nu'(1+t^{k+1})) = -k$$

for $k \in \mathbb{Z}_{>0}$. However **B**'' is not a basis of S, moreover, $\mathbf{A}_{\nu} \cap \mathbf{A}_{\nu'} = \emptyset$ In particular, Proposition 4.27 and Theorem 4.24 are not applicable to (ν, ν') (note that $-\mathbb{Z}_{\geq 0}$ endowed with the natural order is not well-ordered) and thus $\mathbf{K}_{\nu',\nu}$ is undefined.

In some cases, we can obtain injective valuations by utilizing leading coefficients of valuations on their ambient spaces (see Section 4.2).

Proposition 4.29. Let $\nu : S \setminus \{0\} \to C$ be a well-ordered valuation and $\lambda : S \setminus \{0\} \to C$ $S' \setminus \{0\}$ be its leading coefficient. Let S_0 be a subspace of S such that $\lambda(S_0) = \mathbb{k}^{\times} \cdot s'$ for some $s' \in S'$. Then the restriction of ν to S_0 is an injective valuation on S_0 .

Proof. Without loss of generality, we consider the case when $S_0 = S$, S' = k, s' = 1. It suffices to verify the condition (b) of Proposition 4.14. Indeed, let $x, y \in S \setminus \{0\}$ be such that $y \notin kx$ and $\nu(x) = \nu(y)$. Denote $c := \frac{\lambda(x)}{\lambda(y)}$. Suppose, by contradiction, that $\nu(x - cy) = \nu(x)$. Then $\lambda(x - cy) = \lambda(x) + \lambda(-cy) = \lambda(x) - c\lambda(y) = 0$, which is impossible.

The contradiction finishes the proof.

We can apply this result to integral domains as follows. Given a commutative integral domain \mathcal{B} over k and a subalgebra \mathcal{A} , denote by $\mathcal{B}_{\mathcal{A}}$ the set of all $x \in \mathcal{B}$ such that $\mathcal{A} \cdot x \cap (\mathcal{A} \setminus \{0\})$ is nonempty. Clearly, $\mathcal{B}_{\mathcal{A}}$ is a subalgebra of \mathcal{B} (we will sometimes refer to it as the *localization* of \mathcal{A} in \mathcal{B}).

Theorem 4.30. Let \mathcal{B} be an integral domain over \Bbbk , M be a well-ordered monoid, ν be an additive valuation $\mathcal{B} \setminus \{0\} \to M$, and $\lambda : \mathcal{B} \to \mathcal{C}$ be its multiplicative leading

coefficient (here C is an integral domain over \Bbbk). Suppose that A is a subalgebra of \mathcal{B} such that $\lambda(A \setminus \{0\}) = \Bbbk^{\times}$. Then

(a) $\lambda(\mathcal{B}_{\mathcal{A}} \setminus \{0\}) = \mathbb{k}^{\times}$. (b) The restriction of ν to $\mathcal{B}_{\mathcal{A}}$ is an injective additive valuation $\mathcal{B}_{\mathcal{A}} \setminus \{0\} \to M$.

Proof. Indeed, let $b \in \mathcal{B}_{\mathcal{A}} \setminus \{0\}$. That is, xb = y for some $x, y \in \mathcal{A} \setminus \{0\}$. Therefore,

$$\lambda(y) = \lambda(xb) = \lambda(x)\lambda(b)$$

since λ is multiplicative. Hence $\lambda(y) \in \mathbb{k}^{\times}$ because $\lambda(x), \lambda(y) \in \mathbb{k}^{\times}$ by the assumption. This proves (a).

Part (b) follows from (a) and Proposition 4.29.

The theorem is proved.

4.5. Well-ordered submonoids of \mathbb{Z}^m . For $M \subset \mathbb{Z}^m$ and $k \in [m-1]$ denote by M_k the image of M under the standard projection $\mathbb{Z}^m \to \mathbb{Z}^k (a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_k)$.

Proposition 4.31. Let $m \ge 1$ and $M \subset \mathbb{Z}^m$. Then the following are equivalent:

(a) M is well-ordered with respect to the lexicographic order on \mathbb{Z}^m .

(b) For k = 0, ..., m - 1 there exist functions $f_k : M_k \to \mathbb{Z}$ such that:

$$a_1 + f_0 \ge 0, a_2 + f_1(a_1) \ge 0, a_3 + f_2(a_1, a_2) \ge 0, \dots, a_m + f_{m-1}(a_1, \dots, a_{m-1}) \ge 0$$

for all $a = (a_1, \ldots, a_m) \in M$.

If M is a monoid one can additionally require in (b) that $f_0 = f_1(0) = \cdots = f_{m-1}(0, \ldots, 0) = 0$.

Proof. First assume (a). For any $0 \le k < m$ fix a point $(a_1 \ldots, a_k) \in M_k$. There exists an integer N such that $a_{k+1} \ge N$ for any a_{k+1} such that $(a_1, \ldots, a_k, a_{k+1}) \in M_{k+1}$. We put $f_k(a_1, \ldots, a_k) := -N$.

Conversely, assume (b) and that (a) is false. Then there exists an infinite decreasing sequence of elements of M. Therefore for a suitable maximal possible $0 \le k < m$ all elements of the sequence starting with some point have the same prefix a_1, \ldots, a_k for appropriate $(a_1, \ldots, a_k) \in M_k$. Since $a_{k+1} + f_k(a_1, \ldots, a_k) \ge 0$ we get a contradiction with the maximality of k.

When M is a monoid and an element $a := (0, \ldots, 0, a_k, \ldots, a_m) \in M$ where $a_k \neq 0$, it holds $a_k > 0$ because otherwise $a > 2a > 3a > \ldots$. This implies the last statement of the proposition.

Example 4.32. Given $r \in \mathbb{Q}_{>0}$, then $M_r = \{(a_1, a_2) \in \mathbb{Z}^2 : a_1 \ge 0, a_2 + ra_1^2 \ge 0\}$ is a well-ordered submonoid of \mathbb{Z}^2 .

We say that $g \in GL_m(\mathbb{Q})$ is *tame* if $g(e_j) \in e_j + \sum_{i=1}^{j-1} \mathbb{Q}_{\geq 0} \cdot e_i$ for $j \in [m]$, where $\{e_1, \ldots, e_m\}$ is the standard basis of \mathbb{Z}^m .

Corollary 4.33. A finitely generated submonoid $M \subset \mathbb{Z}^m$ is well-ordered (with respect to the lexicographic order on \mathbb{Z}^m) iff $M \subset g^{-1}(\mathbb{Z}^m_{\geq 0})$ for some tame $g \in GL_m(\mathbb{Z})$.

4.6. Tame valuations on the Laurent polynomial ring. In this section we will view each \mathbb{R}^n as a totally ordered set with respect to the lexicographic ordering.

We say that a valuation $\nu : \mathbb{k}[x_1^{\pm 1}, \dots, x_m^{\pm 1}] \setminus \{0\} \to \mathbb{R}^n$ is *tame* if it is completely determined by its values $\nu(x_i) = v_i \in \mathbb{R}^n$, $i = 1, \dots, n$. Clearly, ν is tame iff it is of the form $\nu_{\mathbf{v}}, \mathbf{v} = (v_1, \dots, v_m) \in (\mathbb{R}^n)^m$:

$$\nu_{\mathbf{v}}\left(\sum_{d\in\mathbb{Z}^m}c_dx^d\right)=\max_{d\in\mathbb{Z}^m:c_d\neq 0}\left\{d_1v_1+\cdots d_mv_m\right\}\ .$$

The following is obvious.

Lemma 4.34. A tame valuation $\nu = \nu_{\mathbf{v}}$ is injective iff the vectors v_1, \ldots, v_m are linearly independent (in particular, $n \ge m$).

Since the monomials form an adapted basis to a tame valuation one can apply Proposition 4.27 to any pair of tame valuations on the Laurent polynomial algebra $\mathbb{k}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ and get

Corollary 4.35. Any pair of injective tame valuations on $\mathbb{k}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ has an adapted basis.

A tame valuation provides a total well ordering of the monomials. [31] and Theorem 9 from [11] state that all the total well orderings of the monomials are exhausted by the tame ones.

One can view **v** as $n \times m$ matrix. Then lex ordering corresponds to the unit matrix, and deglex ordering corresponds to $m \times m$ matrix with ones on the diagonal and in the first row with zeroes at the rest entries.

4.7. Algorithms computing Jordan-Hölder bijections. Consider a pair ν, ν' : $S \setminus \{0\} \to (C, <)$ of injective well-ordered valuations. Assume that there are given algorithms mapping C_{ν} (respectively, $C_{\nu'}$) to an adapted basis $\mathbf{B} \in \mathbf{A}_{\nu}$ (respectively, $\mathbf{B}' \in \mathbf{A}_{\nu'}$). Also assume that $(C_{\nu}, <)$ is isomorphic to $\mathbb{Z}_{\geq 0}$ and there is given an algorithm exhibiting this isomorphism. Note that deglex on the polynomial ring fulfills the latter feature. Then one can compute JHb $\mathbf{K}_{\nu',\nu}$: $C_{\nu} \to C'_{\nu'}$ and a common adapted basis from $\mathbf{A}_{\nu} \cap \mathbf{A}_{\nu'}$.

Indeed, for any $a \in C_{\nu}$ the algorithm produces $b_a \in \mathbf{B}$ with $\nu(b_a) = a$ and all $b_i \in \mathbf{B}, i \in I$ such that $\nu(b_i) < a$. The algorithm expands each $b_a, b_i, i \in I$ in basis \mathbf{B}' and an element $b_a + \sum_{i \in I} c_i \cdot b_i$ with indeterminate coefficients $c_i, i \in I$

$$b_a + \sum_{i \in I} c_i \cdot b_i = \sum_{1 \le j \le p} A_j \cdot b'_j$$

for suitable linear functions A_j , $1 \leq j \leq p$ in c_i , $i \in I$ and $b'_j \in \mathbf{B}'$, $1 \leq j \leq p$. Let $\nu'(b'_1) >' \nu'(b'_2) >' \cdots >' \nu'(b'_p)$.

Consecutively, for l = 1, 2, ..., p the algorithm tests whether a linear in $c_i, i \in I$ system $A_1 = A_2 = \cdots = A_{l-1} = 0$ has a solution. Consider maximal l satisfying the latter property. Then JHb $\mathbf{K}_{\nu',\nu}(a) = \nu'(b'_l)$. Pick any solution $c_i, i \in I$ of the linear system $A_1 = A_2 = \cdots = A_{l-1} = 0$. then

$$\{b_a + \sum_{i \in I} c_i \cdot b_i\}_{a \in C_{\nu}}$$

constitute a common adapted basis for $\mathbf{A}_{\nu} \cap \mathbf{A}_{\nu'}$.

Just described algorithm computes JHb $\mathbf{K}_{\nu',\nu}(a)$ for an arbitrary input $a \in C_{\nu}$ in a general case of a vector space. Since in case of a polynomial algebra JHb is more rigid than in general, one is able to design a partial algorithm for computing JHb and a common adapted basis for both ν, ν' in a finite form (we call this form a piece-wise monoidal representation), provided that the partial algorithm terminates. Moreover, the partial algorithm terminates iff JHb admits a piece-wise monoidal representation. Below we assume that $C_{\nu} = \mathbb{Z}_{\geq 0}^m$.

We accomplish the algorithm from the beginning of this subsection for computing JHb $\mathbf{K}_{\nu',\nu}(a)$ step by step for increasing $a \in C_{\nu}$ by recursion. Thus, we assume as a recursive hypothesis that $\mathbf{K}_{\nu',\nu}(a)$ is already computed for all $a < a_0$ for some a_0 . After each step the result of the algorithm can be given as the following *piecewise monoidal representation*. Polynomials $f_1, \ldots, f_n \in \mathbb{k}[x_1, \ldots, x_m]$ are given together with a partition of $\mathbb{R}_{\geq 0}^m$ into simplicial cones generated by vectors $a_1 := \nu(f_1), \ldots, a_n := \nu(f_n) \in \mathbb{Z}_{\geq 0}^m$. Consider one (with some dimension $p \leq m$) of these cones generated by vectors a_{i_0}, \ldots, a_{i_p} and denote by $M \subset \mathbb{Z}_{\geq 0}^m$ the monoid generated by vectors a_{i_0}, \ldots, a_{i_p} . In addition, to each integer point a from the parallelotop $P = \{\alpha_0 \cdot a_{i_0} + \cdots + \alpha_p \cdot a_{i_p} : 0 \leq \alpha_0, \ldots, \alpha_p < 1\} \subset \mathbb{R}_{\geq 0}^m$ generated by a_{i_0}, \ldots, a_{i_p} is attached a polynomial $f_a \in \mathbb{k}[x_1, \ldots, x_m]$ with $\nu(f_a) = a$. Then the monoid of all integer points from $(M \otimes \mathbb{R}_{\geq 0}) \cap \mathbb{Z}_{\geq 0}^m$ is a disjoint union of shifted monoids M + a for all $a \in P \cap \mathbb{Z}_{\geq 0}^m$.

These data determine a basis **B** of $\mathbb{k}[x_1, \ldots, x_m]$ adapted for ν . Namely, for any point $v = c_0 \cdot a_{i_0} + \cdots + c_p \cdot a_{i_p} + a \in M + a$ where $c_0, \ldots, c_p \in \mathbb{Z}_{\geq 0}$ put $b_v := f_{i_0}^{c_0} \cdots + f_{i_p}^{c_p} \cdot f_a \in \mathbf{B}$, hence $\nu(b_v) = v$. Also we define map $\mathbf{K} : C_{\nu} \to C_{\nu'}$ by $\mathbf{K}(v) := \nu'(b_v) = c_0 \cdot \nu'(f_{i_0}) + \cdots + c_p \cdot \nu'(f_{i_p}) + \nu'(f_a)$. Thereby, **K** is linear on each shifted monoid M + a. Clearly, $\mathbf{K}_{\nu',\nu} \leq '\mathbf{K}$ holds point-wise.

Now we produce an algorithmic criterion whether the partial algorithm terminates at the current step of recursion. It terminates iff for every pair of distinct points $v, v_0 \in \mathbb{Z}_{\geq 0}^m$ it holds $\nu'(b_v) \neq \nu'(b_{v_0})$. The latter condition is equivalent to non-solvability of a suitable integer programming problem. If the partial algorithm terminates then **B** is a common adapted basis for both ν, ν' and $\mathbf{K} = \mathbf{K}_{\nu',\nu'}$ (see Proposition 4.27).

Otherwise, if the partial algorithm does not terminate at the current recursive step, the algorithm described at the beginning of this subsection accomplishes the next step for computing JHb at a greater (wrt the ordering < on C_{ν}) point. Assume (for the sake of simplicity) that the algorithm at this step computes just $\mathbf{K}_{\nu',\nu}(a_0)$ and $f_0 \in \mathbb{K}[x_1, \ldots, x_m]$ satisfying $\mathbf{K}_{\nu',\nu}(c_0) <' \mathbf{K}(a_0)$ such that $\nu(f_0) = a_0$ and $\nu'(f_0) = \mathbf{K}_{\nu',\nu}(a_0)$. Then at the current recursive step the partial algorithm adds f_0 to f_1, \ldots, f_n .

Let a_0 belong to a *p*-dimensional simplicial cone *T* generated by vectors a_{i_0}, \ldots, a_{i_p} for some $p \leq m$. The partition of *T* into simplicial cones T_j , $0 \leq j \leq p$ generated by $a_{i_0}, \ldots, a_{i_{j-1}}, a_0, a_{i_{j+1}}, \ldots, a_{i_p}$ induces the partition of $\mathbb{Z}_{\geq 0}^m$ into the union of shifted monoids (we keep from the previous recursive step the partitions of all the cones not containing a_0), and thereby, we get a piecewise monoidal representation after the current recursive step. To define (the modified after the current recursive step) $\mathbf{K}' : C_{\nu} \to C_{\nu'}$ on a shifted monoid $M'_j + a$ where monoid M'_j is generated by $a_{i_0}, \ldots, a_{i_{j-1}}, a_0, a_{i_{j+1}}, \ldots, a_{i_p}$, and an integer point s belongs to the parallelotope generated by the same vectors $a_{i_0}, \ldots, a_{i_{j-1}}, a_0, a_{i_{j+1}}, \ldots, a_{i_p}$, we take polynomial $f_a := b_a \in \mathbf{B}$ constructed at the previous recursive step.

This completes the description of a piecewise monoidal representation of \mathbf{K}' at the current recursive step and the design of the partial algorithm.

Proposition 4.36. The designed partial algorithm terminates and in this case yields a piece-wise monoidal representation of $\mathbf{K}_{\nu,\nu'}$ (provided that $C_{\nu} = \mathbb{Z}_{\geq 0}^m$) together with a common adapted basis for both ν, ν' iff $\mathbf{K}_{\nu',\nu}$ admits a piece-wise monoidal representation (in particular, $\mathbf{K}_{\nu',\nu}$ is linear on each of the shifted monoids from the representation).

Proof. We have already shown that if the designed partial algorithm terminates then $\mathbf{K} = \mathbf{K}_{\nu',\nu}$.

Conversely, suppose that $\mathbf{K}_{\nu',\nu}$ admits a piece-wise monoidal representation with vectors $a_1, \ldots, a_n \in C_{\nu}$. After that the partial algorithm computes $\mathbf{K}_{\nu',\nu}(a)$ for $a \in \{a_1, \ldots, a_n\}$ and for all integer points a belonging to the parallelotopes from the latter piece-wise monoidal representation generated by vectors a_1, \ldots, a_n , the resulting $\mathbf{K} \leq' \mathbf{K}_{\nu',\nu}$ since \mathbf{K} is determined by these values $\mathbf{K}_{\nu',\nu}(a)$. On the other hand, always holds $\mathbf{K} \geq' \mathbf{K}_{\nu',\nu}$, therefore $\mathbf{K} = \mathbf{K}_{\nu',\nu}$ and the Proposition is proved. \Box

It would be interesting to understand, whether the designed partial algorithm always terminates when say, ν is deglex valuation and $\nu' = \nu_{\varphi}$ for any injective homomorphism. $\tau : \mathbb{k}[x_1, \ldots, x_m] \to \mathbb{k}[x_1, \ldots, x_m]$?

4.8. Proofs of results of Section 4.

Proof of Proposition 4.4. Prove (a). Let S and S' be k-vector spaces, for any nonzero $z \in S \otimes S'$ denote by $V(z) \subset S$ the smallest (by inclusion) subspace of S such that $z \in V(z) \otimes S'$.

The following is obvious.

Lemma 4.37. For each nonzero $z \in S \otimes S'$ one has:

(a) $V(z) \neq 0$ and $V(\mathbb{k}^{\times} \cdot z) = V(z)$,

(b) $V(z+z') \subseteq V(z) + V(z')$ for any nonzero $z' \in S \otimes S' \setminus \{0, -z\}$.

(c) V(z) is finite dimensional, moreover, it is the k-linear span of $\{x_1, \ldots, x_m\}$ for any expansion

$$(4.5) z = x_1 \otimes y_1 + \dots + x_m \otimes y_m$$

with minimal possible m (such an m was called rank of z in [15]).

Furthermore, given a valuation $\nu : S \setminus \{0\} \to C$. Then, clearly, for any finitedimensional subspace $S_0 \subset S$ the set

$$\{\nu(x) \mid x \in S_0 \setminus \{0\}\}$$

is a finite subset of C; denote by $\nu(S_0)$ its maximal element.

Furthermore, in the notation of Lemma 4.37, for each nonzero $z \in S \otimes S'$, denote

(4.6)
$$\nu^{S'}(z) := \nu(V(z))$$

Clearly, $V(x \otimes y) = \mathbb{k} \cdot x$ for any nonzero $x \in S$, $y \in S'$, hence $\nu^{S'}(x \otimes y) = \nu(x)$, as in (4.1). This and Lemma 4.37 imply that the assignment $z \mapsto \nu^{S'}(z)$ is the desired valuation on $S \otimes S'$. This finishes the proof of Proposition 4.4(a).

Prove (b). For any nonzero $z \in S \otimes S'$ denote by $\underline{V}(z)$ the smallest (by inclusion) subspace of $S_{\leq a}/S_{\leq a}$, $a = \nu^{S'}(z)$ such that

$$z + S_{$$

in $(S_{\leq a}/S_{< a}) \otimes S' = (S_{\leq a} \otimes S')/(S_{< a} \otimes S')$ (that is, $\underline{V}(z)$ is the image of V(z) under the canonical projection $S_{\leq a} \twoheadrightarrow S_{\leq a}/S_{< a}$). Furthermore, denote by V'(z) the smallest (by inclusion) subspace of S' such that

$$z + S_{$$

in $(S_{\leq a}/S_{< a}) \otimes S'$, where $a = \nu^{S'}(z)$.

The following is obvious.

Lemma 4.38. For each nonzero $z \in S \otimes S'$, the subspace V'(z) is finite-dimensional. Moreover, dim $\underline{V}(z) = \dim V'(z)$ and for any expansion (4.5) with smallest possible m, one has

$$\underline{V}(z) = \bigoplus_{i \in [1,m]: \nu(x_i) = a} \mathbb{k} \cdot (x_i + V(z)_{< a}), \ V'(z) = \bigoplus_{i \in [1,m]: \nu(x_i) = a} \mathbb{k} \cdot y_i \ .$$

Furthermore for each nonzero $z \in S \otimes S'$, denote

(4.7)
$$(\nu \otimes \nu')(z) := (\nu^{S'}(z), \nu'(V'(z)))$$

Clearly, $V'(x \otimes y) = \mathbb{k} \cdot y$ for any nonzero $x \in S$, $y \in S'$. Since $\nu^{S'}(x \otimes y)$, we obtain $(\nu \otimes \nu')(x \otimes y) = (\nu(x), \nu'(y))$, as in (4.2). This and Lemma 4.38 imply that the assignment $z \mapsto (\nu \otimes \nu')(z)$ is the desired valuation on $S \otimes S'$. This finishes the proof of Proposition 4.4(b).

The proposition is proved.

Proof of Proposition 4.14. Prove (a)=>(b). Indeed, let $x, y \in S \setminus \{0\}$ with $a = \nu(x) = \nu(y)$. Then $S_{\leq a} = \Bbbk x + S_{< a} = \Bbbk y + S_{< a}$ which implies that $x - cy \in S_{< a}$ for some (unique) nonzero scalar $c \in \Bbbk$. This proves the implication (a)=>(b).

Prove (b)=>(a). Choose $\mathbf{B} \in \mathbf{A}_{\nu}$ in the notation of Section 4.1. By Lemma 4.2(c), this is a basis of S such that the restriction of ν to \mathbf{B} is a surjective map $\mathbf{B} \to C_{\nu}$ and $\mathbf{B}_{\langle a} = S_{\langle a} \cap \mathbf{B}$ is a basis of $S_{\langle a}$ for all $a \in C_{\nu}$. It remains to establish injectivity of $\nu|_{\mathbf{B}}$, which we will do by contradiction. Suppose $b, b' \in \mathbf{B}$ be such that $b \neq b'$, $\nu(b) = \nu(b')$. Then there exists $c \in \mathbb{k}^{\times}$ such that $b' - cb \in S_{\langle a}$, where $a = \nu(b)$, which implies that b' is a linear combination of elements of \mathbf{B} . The resulting contradiction proves that $\nu|_{\mathbf{B}} : \mathbf{B} \to C_{\nu}$ is a bijection. In view of Lemma 4.13, this proves the implication (b)=>(a).

The proposition is proved.

Proof of Proposition 4.19. Let **B** be an adapted basis of *S* such that $\mathbf{B} \cap S_i$ is a basis in S_i for each $i \in I$. Then $\mathbf{B} \cap S_i = \{b \in \mathbf{B} : \nu(b) \in \nu(S_i \setminus \{0\})\}$ (cf. Corollary 4.17). Therefore, if for $b \in \mathbf{B}$ it holds $\nu(b) \in \bigcap_{j \in J} \nu(S_j \setminus \{0\})$ for some $J \subseteq I$ then $b \in \bigcap_{i \in J} S_i$, hence $\nu(b) \in \nu(\bigcap_{i \in J} S_j \setminus \{0\})$, this justifies that the family

 $\{S_i, i \in I\}$ is ν -compatible. Moreover, this implies that $\mathbf{B} \cap \bigcap_{j \in J} S_j = \{b \in \mathbf{B} : \nu(b) \in \nu(\bigcap_{j \in J} S_j \setminus \{0\})\}$ is a basis of $\bigcap_{j \in J} S_j$. In addition, $\sum_{j \in J} S_j$ is contained in the linear hull of the vectors $\{b \in \mathbf{B} : \nu(b) \in \mathcal{B}\}$

In addition, $\sum_{j \in J} S_j$ is contained in the linear hull of the vectors $\{b \in \mathbf{B} : \nu(b) \in \bigcup_{j \in J} \nu(S_j \setminus \{0\})\}$. Thus, $\nu((\sum_{j \in J} S_j) \setminus \{0\}) \subseteq \bigcup_{j \in J} \nu(S_j \setminus \{0\})$. The opposite inclusion is obvious.

Now conversely, assume that the family $\{S_i, i \in I\}$ is ν -compatible. For each $c \in \nu(S \setminus \{0\})$ there exists a unique subset $J \subseteq I$ such that

$$c \in \bigcap_{j \in J} \nu(S_j \setminus \{0\}) \setminus \bigcup_{l \notin J} \nu(S_l \setminus \{0\}).$$

The case $J = \emptyset$ means that $c \notin \bigcup_{i \in I} \nu(S_i \setminus \{0\})$. Due to ν -compatibility there exists a vector $b_c \in \bigcap_{j \in J} S_j$ such that $\nu(b_c) = c$ (the case $J = \emptyset$ means that $b_c \in S$). Observe that for any $c \in \bigcap_{j \in J} \nu(S_j \setminus \{0\})$ it holds that the vector $b_c \in \bigcap_{j \in J} S_j$. Hence $\{b_c : c \in \bigcap_{j \in J} \nu(S_j \setminus \{0\})\}$ is a basis of $\bigcap_{j \in J} S_j$ since ν is injective (cf. Corollary 4.17). Hence the basis $\mathbf{B} := \{b_c : c \in \nu(S \setminus \{0\})\}$ is required.

Proof of Proposition 4.23. By Lemma 4.2(b), \mathbf{A}_{ν} is non-empty, so fix $\mathbf{B} \in \mathbf{A}_{\nu}$. Then $\mathbf{B}_{\leq a} := S_{\leq a} \cap \mathbf{B} = \bigsqcup_{a' \leq a} \mathbf{B}_{a'}$ is a basis in $S_{\leq a}$, where $\mathbf{B}_{a'} = \{b \in \mathbf{B} : \nu(b) = a'\}$.

Furthermore, choose a well-ordering of each \mathbf{B}_a . Denote by $\mathbf{B}^0 \subset \mathbf{B}$ the set which consists of all minimal (with respect to the chosen well-ordering) elements of all \mathbf{B}_a . By the construction, $|\mathbf{B}^0 \cap \mathbf{B}_a| = 1$ for all $a \in C_{\nu}$ and the restriction of ν to \mathbf{B}^0 is a bijection $\mathbf{B}^0 \xrightarrow{\sim} C_{\nu}$.

Using transfinite induction, we repeat this procedure and obtain the following.

Lemma 4.39. For each $\mathbf{B} \in \mathbf{A}_{\nu}$ there is a well-ordered set \mathbf{I} with the minimal element $\mathbf{0}$ such that

- $\mathbf{B} = \bigsqcup_{\mathbf{i} \in \mathbf{I}} \mathbf{B}^{\mathbf{i}}$.
- The restriction of ν to $\mathbf{B}^{\mathbf{i}}$ is an injective map $\mathbf{B}^{0} \hookrightarrow C_{\nu}$.
- $\nu(\mathbf{B}^{\mathbf{i}}) \subset \nu(\mathbf{B}^{\mathbf{i}'})$ if $\mathbf{i}' \leq \mathbf{i}$ and $\nu(\mathbf{B}^0) = C_{\nu}$

Using this, we obtain an injective map $\underline{\mathbf{j}}:\mathbf{B}\hookrightarrow\mathbf{B}^0\times\mathbf{I}$ given by

$$\mathbf{j}(b) = (b^0, \mathbf{i})$$

for each $b \in \mathbf{B}^{\mathbf{i}}$, where b^0 is the only element of \mathbf{B}^0 such that $\nu(b) = \nu(b^0)$.

Linearizing, we obtain an injective k-linear map $\mathbf{j} = \mathbf{k}\mathbf{j} : S \to \underline{S} \otimes S'$, where $S = \mathbf{k}\mathbf{B}^0 \subset S$ and $S' = \mathbf{k}\mathbf{I}$.

By the very construction, the restriction of ν to \underline{S} is an injective valuation $\underline{\nu}$: $\underline{S} \setminus \{0\}$. Also, for each nonzero $x \in S$ written as $x = \sum_{i \in \mathbf{I}, b \in \mathbf{B}^i} c_b^i b^i$ one has

$$\mathbf{j}(x) = \sum_{\mathbf{i} \in \mathbf{I}, b \in \mathbf{B}^{\mathbf{i}}} c^{\mathbf{i}}_b(b^0, \mathbf{i})$$

in the notation (4.8).

In particular,
$$\nu(x) = \max_{\mathbf{i} \in I, b \in \mathbf{B}^{\mathbf{i}}: c_b^{\mathbf{i}} \neq 0} \{\nu(b)\} = \max_{\mathbf{i} \in I, b \in \mathbf{B}^{\mathbf{i}}: c_b^{\mathbf{i}} \neq 0} \{\nu(b^0)\} = \underline{\nu}^{S'}(x).$$

The proposition is proved.

Proof of Theorem 4.24. Let ν and ν' be injective valuations on S. Fix any $a \in C_{\nu}$. Then choose $x \in S \setminus \{0\}$ such that $\nu(x) = a$ and $\nu'(x) = \min\{\nu'(\nu^{-1}(a))\}$ and $y \in S \setminus \{0\}$ such that $\nu(y) = \min\{\nu(\nu'^{-1}(\nu'(x)))\}$, hence $\nu'(y) = \nu'(x)$. By definition, $\nu'(x) = \mathbf{K}_{\nu',\nu}(a)$ and $\nu(y) = \mathbf{K}_{\nu,\nu'}(\nu'(x)) = \mathbf{K}_{\nu,\nu'}(\mathbf{K}_{\nu',\nu}(a))$. Note that $x \in \nu'^{-1}(\nu'(x))$ hence $\nu(y) \leq \nu(x)$.

Using Proposition 4.14(b), choose $c \in \mathbb{k}$ such that $\nu'(x-cy) < \nu'(x)$. Thus

$$\mathbf{K}_{\nu',\nu}(a) = \nu'(x) > \nu'(x - cy) \; .$$

Since $\nu'(x) \leq \nu'(z)$ for all $z \in S \setminus \{0\}$ with $\nu(z) = \nu(x)$, this implies that $\nu(x - cy) \neq 0$ $\nu(x)$. In turn, this and the inequality $\nu(y) \leq \nu(x)$ imply that $\nu(y) = \nu(x)$.

This proves that $\mathbf{K}_{\nu,\nu'}(\mathbf{K}_{\nu',\nu}(a)) = a$ for all $a \in C_{\nu}$, i.e., $\mathbf{K}_{\nu,\nu'} \circ \mathbf{K}_{\nu',\nu} = Id_{C_{\nu}}$. Switching ν and ν' in the above argument, we also obtain $\mathbf{K}_{\nu',\nu} \circ \mathbf{K}_{\nu,\nu'} = Id_{C_{\nu'}}$.

This proves the first assertion of the theorem.

Prove the second assertion now. For each $a \in C_{\nu}$ denote by S_a the set of all $b \in S \setminus \{0\}$ such that $\nu(b) = a$. Furthermore, let S_a^{min} be the set of all $b \in S_a$ such that $\nu'(b) = \min\{\nu'(b') : b' \in S_a\}$. Then, well-ordering of ν implies that S_a^{min} is nonempty and then $\nu'(b) = \mathbf{K}_{\nu',\nu}(a)$ for each $b \in S_a^{min}$ by (4.4). Finally, for each $a \in C_{\nu}$ choose a single element $b_a \in S_a^{min}$. Clearly, $\mathbf{B} = \{b_a : a \in C_{\nu}\}$ is adapted to ν because the restriction of ν to **B** is a bijection $\mathbf{B} \rightarrow C_{\nu}$ $(b_a \rightarrow \nu(b_a) = a)$. Hence $\mathbf{B} \in \mathbf{A}_{\nu}$ is a basis of S by Lemma 4.2(c). Finally, injectivity of $\mathbf{K}_{\nu',\nu}$ implies that **B** is adapted to ν' because for any distinct $a, a' \in C_{\nu}$ one has $\nu'(b_a) = \mathbf{K}_{\nu',\nu}(a) \neq \mathbf{K}_{\nu',\nu}(a') = \nu'(b_{a'})$.

The theorem is proved.

Proof of Proposition 4.27. Let $\mathbf{B} \in \mathbf{A}_{\nu} \cap \mathbf{A}_{\nu'}$ and for each $d \in C_{\nu}$ denote by b_d the only element of **B** with $\nu(b_d) = d$.

Clearly, for any $a \in C_{\nu}$ each $s \in \nu^{-1}(a)$ can be uniquely written as:

$$s = c_a \cdot b_a + \sum_{\tilde{a} < a} c_{\tilde{a}} b_{\tilde{a}}$$

where $c_a \neq 0$.

Therefore, $\nu'(s) \ge \nu'(b_a)$ for all $s \in \nu^{-1}(a)$, i.e., $\nu'(\nu^{-1}(a)) \ge \nu'(b_a)$ in C'.

On the other hand, $b_a \in v^{-1}(a)$, therefore the minimum in $\min\{\nu'(\nu^{-1}(a))\}$ (see (4.4)) is attained and equals $\nu'(b_a) \in \nu'(\nu^{-1}(a))$, i. e. $\mathbf{K}_{\nu',\nu}(a) = \nu'(b_a)$ is defined. The proposition is proved.

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