# VALUATIONS, BIJECTIONS, AND BASES 

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#### Abstract

The aim of this paper is to build a theory of commutative and noncommutative injective valuations of various algebras. The targets of our valuations are (well-)ordered commutative and noncommutative (partial or entire) semigroups including any sub-semigroups of the free monoid $F_{n}$ on $n$ generators and various quotients. In the case when the (partial) valuation semigroup is finitely generated, we construct a generalization of the standard monomial bases for the so-valued algebra, which seems to be new in noncommutative case. Quite remarkably, for any pair of well-ordered valuations one has canonical bijections between the valuation semigroups, which serve as analogs of the celebrated Jordan-Hölder correspondences and these bijections are "almost" homomorphisms of the involved (partial and entire) semigroups.


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## 1. Introduction

The aim of this paper is twofold:

- To initiate and systematically study injective valuations of algebras into (partial) semigroups.
- For pairs of such injective valuations of a given algebra establish and study a bijection between their images, which we refer to as Jordan-Hölder (JH) bijection.

Recall (cf. [35] Ch. 18) that a valuation $\nu$ on a $\mathbb{k}$-vector space $V$ is a map $V \backslash\{0\} \rightarrow P$ where $P$ is a totally ordered set with an order $\preceq$ such that $\nu(a v)=\nu(v)$ for all $v \in V, a \in \mathbb{k}^{\times}$and

$$
\begin{equation*}
\nu(u+v) \preceq \max (\nu(u), \nu(v)) \tag{1.1}
\end{equation*}
$$

whenever $u+v \neq 0$ (in some papers max is replaced by min and $\preceq$ by $\succeq$ ). It is immediate that (1.1) becomes an equality if $\nu(u) \neq \nu(v)$.

In addition, if $V$ is a $\mathbb{k}$-algebra, we require $P$ to be a (partial) semigroup (see e.g. [1]) with the operation $\circ$ (see Section 2 for details) and

- If $\nu(u) \circ \nu(v)$ is defined in $P$ for some $u, v \in V \backslash\{0\}$ then $u v \neq 0$ and

$$
\nu(u v)=\nu(u) \circ \nu(v) .
$$

In particular, if $(P, \circ)$ is an entire (rather than partial) semigroup then the algebra $V$ is necessarily a domain, and conversely if $V$ is not a domain, then $(P, \circ)$ is necessarily a partial semigroup.

Also we impose the following condition on the order in $P$ : if $c, c^{\prime}, d, d^{\prime} \in P$ satisfy inequalities $c \preceq d, c^{\prime} \preceq d^{\prime}$ then

$$
\begin{equation*}
c \circ c^{\prime} \preceq d \circ d^{\prime} \tag{1.2}
\end{equation*}
$$

provided that $c \circ c^{\prime}, d \circ d^{\prime} \in P$.
If $P$ was an entire semigroup then the axiom (1.2) would follow from weaker ones: $a \preceq b$ implies $c \circ a \preceq c \circ b$ and $a \circ b \preceq b \circ c$ (for ordered entire semigroups one can look in [7]). However, in partial semigroups, (1.2) is not always derived (see, e.g., Example 2.47).

Note that this axiomatic is rather strong: if $P$ is an (entire) ordered monoid then for any non-unital invertible element $c$, the unit is strictly between $c$ and $c^{-1}$.

Additionally, we say that the order $\prec$ on a partial semigroup $P$ satisfies the strict property if $c \prec d$ or $c^{\prime} \prec d^{\prime}$ implies that $c \circ c^{\prime} \prec d \circ d^{\prime}$ in (1.2).

One can show (see e.g., Remark 2.25) that for any valuation $\nu$ of $V$ the image $P_{\nu}:=\nu(V \backslash\{0\})$ is always a (partial) subsemigroup of $P$.

Following [19, 22] we say that a valuation $\nu$ on (a vector space or an algebra) $V$ is injective if there exists a basis $\mathbf{B}$ of $V$ such that $\left.\nu\right|_{\mathbf{B}}$ is an injective map $\mathbf{B} \hookrightarrow P$ (we refer to such a basis as adapted to $\nu$ ).

Note also that for some finitely generated commutative domains $A$ there is no injective valuation to an entire semigroup (see Lemma 3.32, Theorem 3.33).

For contrast, we claim that injective valuations to reasonable partial semigroups always exist (in Section 2 we construct a class of coideal partial semigroups, see Definition 2.1, which will provide such reasonable valuations).

Theorem 1.1 (Theorem 2.38). Any finitely generated commutative algebra $\mathcal{A}$ admits an injective valuation onto a coideal partial subsemigroup of $\mathbb{Z}_{\geq 0}^{m}$. Thereby, the order in the coideal partial subsemigroup satisfies the strict property.

In view of above, this means that even finite-dimensional algebras are "domains." Our next result extends Theorem 1.1 to the noncommutative case.

Theorem 1.2 (Theorem 2.38). Any finitely generated algebra $\mathcal{A}$ admits an injective valuation onto a coideal partial subsemigroup of $F_{m}$ (the free monoid on $m$ generators) with respect to (strict) deglex order (Lemma 2.13 and Remark 2.14).

We can refine Theorem 1.1 by employing Gröbner basis-like approach (cf. [20]). Nameley, consider a commutative algebra $\mathcal{A}$ generated by a finite set $X$ and a finite set $\mathbf{S}$ of monomials in $X$. We say that a monomial $b$ in $X$ is standard if $b$ does not contain elements of $\mathbf{S}$ as submonomials. If the set $\mathbf{B}$ of all standard monomials is a basis of $\mathcal{A}$, it is referred as a standard monomial one.

The following is a restatement of a well-known result (asserting that any finitely generated commutative algebra admits a standard monomial basis) in terms of injective valuation into partial semigroups.

Theorem 1.3. Let $\mathcal{A}$ be a finitely generated commutative algebra. Then
(a) (Proposition 2.45) Any standard monomial basis $\mathbf{B}$ defines a structure of an ordered partial semigroup on $\mathbb{Z}_{\geq 0}^{X}$ and an injective valuation $\nu: \mathcal{A} \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{X}$ such that $\mathbf{B}$ is adapted to $\nu$.
(b) (Theorem 2.36). Conversely, let $\nu: \mathcal{A} \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{m}$ be an injective valuation such that $\nu(\mathcal{A} \backslash\{0\})=\mathbb{Z}_{\geq 0}^{m} \backslash[\mathbf{S}]$, where $[\mathbf{S}]:=\bigcup_{s \in \mathbf{S}}\left(s+\mathbb{Z}_{\geq 0}^{m}\right)$ for some finite subset $\mathbf{S}$ of $\mathbb{Z}_{\geq 0}^{m}$. Then there exists a standard monomial basis $\mathbf{B}$ of $\mathcal{A}$ adapted to $\nu$.

A notable example is the (generalized) Stanley-Reisner algebra $\mathcal{A}_{\mathbf{S}}$ defined for any finite subset $\mathbf{S} \subset \mathbb{Z}_{>0}^{m}$ as follows. $\mathcal{A}_{\mathbf{S}}$ is generated by $X=\left\{x_{1}, \ldots, x_{m}\right\}$ and has a presentation $x^{s}=0$ for all $s \in \mathbf{S}$. Clearly, the set $x^{c}, c \in \mathbb{Z}_{\geq 0}^{n} \backslash[\mathbf{S}]$ form a standard monomial basis $\mathbf{B}$.

The complement $\mathbb{Z}_{\geq 0}^{n} \backslash[\mathbf{S}]$ has a structure of a partial semigroup via: for $c, c^{\prime} \in$ $\mathbb{Z}_{\geq 0}^{n} \backslash[\mathbf{S}]$ their sum $c+c^{\prime}$ is defined unless it belongs to $[\mathbf{S}]$, which is an ideal of $\mathbb{Z}_{\geq 0}^{n}$. Then the lexicographic (or deglex) order on $\mathbb{Z}_{\geq 0}^{n}$ gives rise to a (unique) injective
valuation $\nu_{\mathbf{S}}$ on $\mathcal{A}_{\mathbf{S}}$ via

$$
\nu_{\mathbf{S}}\left(x^{a}\right)=a
$$

for all $a \in \mathbb{Z}_{>0}^{n} \backslash[\mathbf{S}]$. Clearly, $\mathbf{B}$ is adapted to $\nu_{\mathbf{S}}$.
A theory generalizing Buchberger's algorithm producing Gröbner bases of ideals was developed for certain classes of noncommutative algebras (see e.g. [24, 27]). In our version we construct bases for injectively valued (noncommutative) algebras as follows (some difference in approach is that we focus mainly on bases of quotient algebras rather than of ideals).

First, we will replace $\mathbb{Z}_{\geq 0}^{n} \backslash[\mathbf{S}]$ by a (not necessarily commutative) partial semigroup $(P, \circ)$, second, fix an injective valuation $\nu: \mathcal{A} \backslash\{0\} \rightarrow P$ (and denote by $P_{\nu}$ the image of $\nu$, which is automatically a partial sub-semigroup of $P$ ), and, third, construct a linearly independent set $\mathbf{B}:=\left\{b_{c}, c \in P\right\}$ of certain monomials in $\mathcal{A}$ which is adapted to $\nu$, i.e., $\nu(\mathbf{B})=P_{\nu}$. It is critical for the construction that $\mathbf{B}$ is a basis of $\mathcal{A}$ which is guaranteed for any adapted set whenever $P$ is well-ordered, see Corollary 4.15 (the assumption of well-orderness seems to be indispensible, see Remark 4.16).

More precisely, let $P$ be a partial semigroup and $P_{0}$ be a generating set of $P$. A factorization of $c \in P$ is any sequence $\vec{c}=\left(c_{1}, \ldots, c_{\ell}\right) \in P_{0}^{\ell}$ such that

$$
c=c_{1} \circ \cdots \circ c_{\ell}
$$

in $P$; denote by $F(c)$ the set of all factorizations of $c$ of the shortest length (to be denoted $\ell(c)$ ). Now suppose that $P$ is well-ordered. We refer to $\vec{c} \in R(c)$ as standard if it is the smallest in the lexicographic order on $R(c)$ and denote it by $\vec{c}^{s t}$.

Then fix an injective valuation $\nu: \mathcal{A} \backslash\{0\} \rightarrow P$, a generating set of $\left(P_{\nu}\right)_{0}$, and choose $x_{c_{0}} \in \mathcal{A} \backslash\{0\}$ for all $c_{0} \in\left(P_{\nu}\right)_{0}$ such that $\nu\left(x_{c_{0}}\right)=c_{0}$ for all $c \in\left(P_{\nu}\right)_{0}$. For any $c \in P_{\nu}$ and any $\vec{c}=\left(c_{1}, \ldots, c_{\ell}\right) \in D(c)$ denote $x_{\vec{c}}:=x_{c_{1}} \cdots x_{c_{\ell}}$.

Theorem 1.4 (Theorem 2.36). In assumptions as above, the set of all $x_{c}:=x_{\tilde{c}^{s t}}$, $c \in P$ is a basis of $\mathcal{A}$ adapted to $\nu$.

This is a generalization of Theorem 1.3(b). Note that unlike in the commutative case, $P$ can be any small category, in particular, with countably many arrows. In that case, its generating set $P_{0}$ is a quiver (possibly with relations) so that $P$ is the set of all paths in $P_{0}$ (cf. Examples 2.16, 2.17).

A large class of injective valuations on $\mathcal{A}$ can be constructed by restriction of a given injective valuation $\hat{\nu}$ on a larger algebra $\mathcal{B}$ for various injective homomorphisms $\mathbf{j}: \mathcal{A} \hookrightarrow \mathcal{B}$ by $\nu:=\hat{\nu} \circ \mathbf{j}$ with some care. We say that a valuation $\nu: V \backslash\{0\} \rightarrow P$ is well-ordered if its range $P_{\nu}$ is a well-ordered set.

Lemma 1.5. (cf. Lemma 3.13.) Suppose that $\hat{\nu}$ is an injective well-ordered valuation of an algebra $\mathcal{B}$. Then the restriction $\hat{\nu}$ to any subalgebra $\mathcal{A}$ is also a well-ordered injective valuation.

In fact, for any ordered partial semigroup $P$ we can construct a "default" invective valuation onto $P$ as follows.

First, recall that for any partial semigroup $P$ and any field $\mathbb{k}$ we equip the linear span $\mathbb{k} P=\bigoplus_{c \in P} \mathbb{k} \cdot[c]$ with an associative algebra structure via

$$
[c]\left[c^{\prime}\right]= \begin{cases}{\left[c \circ c^{\prime}\right]} & \text { if } c \circ c^{\prime} \text { is defined in } P \\ 0 & \text { otherwise }\end{cases}
$$

for all $c, c^{\prime} \in P$. In particular, if $P$ is a group then $\mathbb{k} P$ is the group algebra of $P$. In the case when $P=\mathbb{Z}_{\geq 0} \backslash[\mathbf{S}], \mathbb{k} P$ is the Stanley-Reisner algebra $\mathcal{A}_{\mathbf{S}}$.

Second, suppose that $P$ is ordered and $\mathbb{k}$ is of characteristic 0 . Clearly, the assignments $[c] \mapsto c, c \in P$ define an injective valuation $\nu_{P}: \mathbb{k} P \backslash\{0\} \rightarrow P$ (Definition 2.28). We refer to $\nu_{P}$ as the tautological valuation of $\mathbb{k} P$. For instance, if $P=\mathbb{Z}_{\geq 0}^{n}$, i.e., $\mathbb{k} P=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ and the order is lexicographic, then $\nu_{P}$ is just the usual leading degree valuation of $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$.

Using this, we extend Theorems 1.1, 1.2 as follows.
Theorem 1.6 (Theorem 2.38). Let $P$ be a well-ordered partial semigroup and I be a two-sided ideal of $\mathbb{k} P$.
i) Then $J:=\nu_{P}(I \backslash\{0\})$ is an ideal in $P$ (i.e., it is invariant under left and right compositions), thus $P \backslash J$ is a well-ordered (coideal) partial semigroup. Moreover, the assignments $\nu(a+I):=\min _{j \in I} \nu_{P}(a+j)$ define an injective well-ordered valuation $\nu: A / I \backslash\{0\} \rightarrow P \backslash J$.
ii) Then $\mathbf{B}:=\{[c]+I: c \in P \backslash J\}$ is a standard monomial basis of $\mathbb{k} P / I$ with respect to $\nu$.

Thus, combining constructions of Theorem 1.6 with Lemmas 1.5 and Definition 2.28 , we obtain a large class of injective valuations of commutative and noncommutative algebras into various entire and partial semigroups.

The following are examples of partial semigroups involved in some valuations.
Example 1.7. For any Coxeter group $W=<s_{i}, i \in I>$ its Nil-Coxeter monoid $W_{\mathbf{0}}$ is a partial monoid generated by $s_{i}, i \in I$ and has a presentation $s_{i} \circ s_{i}$ is undefined and $\underbrace{s_{i} \circ s_{j} \circ s_{i} \ldots}_{m_{i j}}=\underbrace{s_{j} \circ s_{i} \circ s_{j} \ldots}_{m_{i j}}$. It is well-known that $W_{\mathbf{0}}=W$ as a set and its multiplication table is given by $w \circ w^{\prime}=w w^{\prime}$ iff $\ell\left(w w^{\prime}\right)=\ell(w)+\ell\left(w^{\prime}\right)$, where $\ell$ denotes the length of words in generators, otherwise, $w \circ w^{\prime}$ is not defined. For example, if $W=S_{3}$, then $W_{0}=\left\{1, s_{1}, s_{2}, s_{1} \circ s_{2}, s_{2} \circ s_{1}, s_{1} \circ s_{2} \circ s_{1}=s_{2} \circ s_{1} \circ s_{2}\right\}$.

More generally, let $C$ be a monoid, $(D, \bullet)$ is an ordered monoid, and $\ell: C \rightarrow D$ be a map such that $\ell\left(c c^{\prime}\right) \preceq \ell(c) \bullet \ell\left(c^{\prime}\right)$. This defines a partial monoid structure on $C$ via $c \circ c^{\prime}=c c^{\prime}$ iff $\ell\left(c c^{\prime}\right)=\ell(c) \bullet \ell\left(c^{\prime}\right)$.

In Example 2.18 we construct an order on $W_{0}$ when $|I|=2$ (and expect that such an order does not exist if $|I|>2$ ).
Example 1.8. It turns out that one can construct (injective) valuations from any finite-dimensional algebra $\mathcal{A}$ to the groupoid (e.g., a partial semigroup) $M_{n}$, which is the set of all pairs $(i, j), i, j \in\{1, \ldots, n\}$ with the composition $(i j) \circ(j k)=(i k)$ (see Example 2.47 for details). Fix a total ordering on $M_{n}$ (compatible with the composition):

$$
(1, n)<\ldots<(1,1)<(2, n)<\ldots<(2,1)<\cdots<(n, n)<\ldots<(n, 1)
$$

(i.e., the lexicographic ordering on pairs $(i, n-j)$ ) and define a valuation $\nu_{0}: \mathbb{k} M_{n}=$ $\operatorname{Mat}_{n}(\mathbb{k}) \backslash\{0\} \rightarrow M_{n}$, that is, $\nu_{0}\left(e_{i j}\right)=(i j)$ where $e_{i j}$ is the (ij)-th matrix unit.

Given a finite-dimensional algebra $\mathbb{k}$-algebra $\mathcal{A}$, any faithful $n$-dimensional representation $\rho$ of $\mathcal{A}$ defines an injective valuation $\nu_{0} \circ \rho: A \backslash\{0\} \rightarrow M_{n}$ (This, in particular, applies to $\mathbb{k} G$ for any finite group $G$, even though $G$ has no compatible total ordering, see Section 2 for details).

By varying total orderings on $M_{n}$ compatible with the operation, will give a large class of new injective valuations on $A$. In Section 2 we construct some such orderings (the symmetric group $S_{n}$ permutes them) and pose a problem of their classification.

One can show that any finitely generated partial semigroup $P$ can be covered by a coideal of an entire semigroup $\widehat{P}$ (Proposition 2.11). For instance, we can take $\widehat{M}_{n}$ to be generated by all $\widehat{(i j)}, i, j=1, \ldots, n$ subject to $\widehat{(i j)} \circ \widehat{(j k)}=\widehat{(i k)}$. However, in a contrast with free coideal semigroup in Proposition 2.11, we do not know whether (an appropriate coideal of) $\widehat{M}_{n}$ is ordered in a compatible way. It would be interesting to classify all ordered partial semigroups $P$ which admit such a lifting to (coideals of) ordered entire semigroups $\widehat{P}$.

On the other hand, one can apply Theorem 1.6 to the free semigroup $\widehat{P}$ freely generated by $(i j), 1 \leq i, j \leq n$, tautological valuation $\nu_{0}$ on $\mathbb{k} \widehat{P}$, and the ideal

$$
I:=\left\langle\left\{e_{i j} e_{p q}: p \neq q\right\} \cup\left\{e_{i j} e_{j l}-e_{i l}\right\}\right\rangle \subset \mathbb{k} \widehat{P} .
$$

Then one obtains an injective valuation $\nu:(\mathbb{k} \widehat{P} / I) \backslash\{0\}=A \backslash\{0\} \rightarrow \widehat{P} \backslash J$, where $J=\nu_{0}(I \backslash\{0\})$ is the corresponding ideal of $\widehat{P}$. By definition, $\widehat{P} \backslash J$ is a partial semigroup consisting of $n^{2}$ elements $\{(i j): 1 \leq i, j \leq n\}$ such that no composition of them is defined. Thus, $\widehat{P} \backslash J$ differs from $M_{n}$.

Returning to general partial semigroups, we can construct new valuations from $P$-filtered algebras and vice versa. We say that $\mathcal{A}$ is filtered by an ordered (partial) semigroup $P$ if $\mathcal{A}=\sum_{c \in P} \mathcal{A}_{\preceq c}, \mathcal{A}_{\preceq c^{\prime}} \subset \mathcal{A}_{\preceq c}$ whenever $c^{\prime} \preceq c$, and $\mathcal{A}_{\preceq c} A_{\preceq c^{\prime}} \subset \mathcal{A}_{\preceq c o c^{\prime}}$ whenever $c \circ c^{\prime}$ is defined in $P$.

Proposition 1.9 (Proposition 2.46). (a) For any valuation $\nu: \mathcal{A} \backslash\{0\} \rightarrow P$ the subsets $\mathcal{A}_{\preceq c}:=\{x \in \mathcal{A} \backslash\{0\}: \nu(x) \preceq c\}$ define a $P_{\nu}$-filtration of $\mathcal{A}$.
(b) Conversely, for a well-ordered $P$, given a $P$-filtration $\mathcal{A} \preceq \bullet$ of $\mathcal{A}$, setting $\nu(x):=$ $\min \left\{c: x \in \mathcal{A}_{\preceq c}\right\}$ for any nonzero $x \in \mathcal{A}$, defines a $P$-valuation of $\mathcal{A}$.

In particular, if $P$ is well-ordered, by the standard procedure $g r$ this defines a $P$-graded algebra (recall that $\mathcal{A}$ is graded by a (partial) semigroup $P$ if $\mathcal{A}=\bigoplus_{c \in P} \mathcal{A}_{c}$ so that $\mathcal{A}_{c} A_{c^{\prime}} \subset \mathcal{A}_{c \circ c^{\prime}}$ whenever $c \circ c^{\prime}$ is defined in $P$ ). Thus, Proposition 1.9 applied to $\mathcal{A}:=\mathbb{k} P$ recovers the tautological valuation $\nu_{P}: \mathbb{k} P \backslash\{0\} \rightarrow P$.

It turns out that having a pair of injective valuations $\nu, \nu^{\prime}$ of an algebra (or even a vector space) $\mathcal{A}$ to partial semigroups $P$ and $P^{\prime}$ gives an interesting information about both the algebra and the pair $P_{\nu}=\nu(\mathcal{A} \backslash\{0\}), P_{\nu^{\prime}}^{\prime}=\nu^{\prime}(\mathcal{A} \backslash\{0\})$.

Theorem 1.10. [Theorem 4.24] Suppose that $\nu: \mathcal{A} \backslash\{0\} \rightarrow P$ and $\nu^{\prime}: \mathcal{A} \backslash\{0\} \rightarrow P^{\prime}$ are injective valuations and $P$ and $P^{\prime}$ are well-ordered. Then the assignments $a \mapsto$ $\min \left\{\nu^{\prime}\left(\nu^{-1}(a)\right)\right\}$ define a bijection $\mathbf{K}_{\nu^{\prime}, \nu}: P_{\nu} \rightrightarrows P_{\nu^{\prime}}^{\prime}$. Moreover, $\mathbf{K}_{\nu^{\prime}, \nu}^{-1}=\mathbf{K}_{\nu, \nu^{\prime}}$.

We call $\mathbf{K}_{\nu^{\prime}, \nu}$ a Jordan-Hölder bijection (JH bijection). It can be reformulated in terms of the generalized Jordan-Hölder correspondence on matroids developed by Abels in 1991 [8], cf. also Remark 4.25.

In addition, under the same assumptions there is a common adapted basis for both valuations.

Theorem 1.11. [Theorem 4.24] Under assumptions of Theorem 1.10, there exists a basis $\mathbf{B}_{\nu, \nu^{\prime}}$ of $\mathcal{A}$ adapted to both $\nu$ and $\nu^{\prime}$ and such that $\mathbf{K}_{\nu^{\prime}, \nu}(\nu(b))=\nu^{\prime}(b)$ for all $b \in \mathbf{B}_{\nu, \nu^{\prime}}$.

We sometimes refer to such a basis as an JH-basis of $\mathcal{A}$.
The following result asserts that any JH-bijection is almost a homomorphism of partial semigroups.

Theorem 1.12. [Proposition 3.83] Under assumptions of Theorem 1.10 the JHbijection $\mathbf{K}:=\mathbf{K}_{\nu^{\prime}, \nu}$ is sub-multiplicative in the following sense:

$$
\mathbf{K}\left(c \circ c^{\prime}\right) \preceq \mathbf{K}(c) \circ \mathbf{K}\left(c^{\prime}\right)
$$

whenever $c \circ c^{\prime}$ and $\mathbf{K}(c) \circ \mathbf{K}\left(c^{\prime}\right)$ are defined in $P$ and $P^{\prime}$, respectively.
This implies that $\mathbf{K}_{\nu, \nu^{\prime}}=\mathbf{K}_{\nu^{\prime}, \nu}^{-1}$ is also sub-multiplicative, which will, in particular, allow to stratify both $P_{\nu}$ and $P_{\nu^{\prime}}^{\prime}$ into "multiplicativity domains" (see Examples 3.88, 3.89, 3.90 and 3.91).

In fact, JH bijection as well as any sub-multiplicative maps $P \rightarrow Q$ can be viewed as "homomorphisms" of partial semigroups in the following sense.

We say that a map $f: P \rightarrow Q$ is a partial homomorphism of partial semigroups if the operation $\circ_{f}$ determined by the requirement: $c \circ_{f} c^{\prime}=c \circ c^{\prime}$ whenever the latter one is defined and $f\left(c \circ c^{\prime}\right)=f(c) \circ f\left(c^{\prime}\right)$, gives a structure of partial semigroup on $P$ (In particular, $f$ becomes a homomorphism of partial semigroups $\left(P, \circ_{f}\right) \rightarrow Q$ ).

Partial homomorphisms are abound in "nature", for instance, Proposition 2.35 asserts that for any valuation $\nu: \mathbb{k} P \backslash\{0\} \rightarrow Q$, the assignments $c \mapsto \nu([c])$ define a partial homomorphism $P \rightarrow Q$.

Proposition 1.13 (Proposition 2.10). Let $P, Q$ be ordered partial semigroups and $f: P \rightarrow Q$ be sub-multiplicative. Suppose that the order on $Q$ has a strict property (see Definition 2.9). Then $f$ is a partial homomorphism.

Thus, the JH bijection $\mathrm{K}: P_{\nu} \leftrightarrows P_{\nu^{\prime}}^{\prime}$ is a partial isomorphism (whenever the orders on $P_{\nu}$ and $P_{\nu^{\prime}}^{\prime}$ have strict property) and is an "honest," rather than partial, isomorphism $\left(P_{\nu}, \circ_{\mathbf{K}}\right) \widetilde{\rightarrow}\left(P_{\nu^{\prime}}^{\prime},{ }_{\mathbf{K}^{-1}}\right)$.

In fact, if $\nu^{\prime}=f \circ \nu$ for some isomorphism $f: P \rightrightarrows P^{\prime}$ of ordered partial semigroups, then $\mathbf{K}=\left.f\right|_{P_{\nu}}$ is also an isomorphism of ordered partial semigroups $P_{\nu} \underset{\rightarrow}{\leftrightarrows} P_{\nu^{\prime}}^{\prime}$.

## 2. Injective valuations on algebras with zero divisors

In this section we extend the concept of valuations to algebras with (possibly) zero divisors.

### 2.1. Partial semigroups.

Definition 2.1. We say that ( $P, \circ$ ) is a (not necessary commutative) partial semigroup if for some elements $c, d \in P$ their composition $c \circ d \in P$ is defined (in this case we say that $c, d$ are composable) satisfying the following property (of associativity). If two elements $c \circ c^{\prime},\left(c \circ c^{\prime}\right) \circ c^{\prime \prime} \in P$ are defined then $c^{\prime} \circ c^{\prime \prime}, c \circ\left(c^{\prime} \circ c^{\prime \prime}\right) \in P$ are also defined and $\left(c \circ c^{\prime}\right) \circ c^{\prime \prime}=c \circ\left(c^{\prime} \circ c^{\prime \prime}\right)$. Vice versa also holds (this is equivalent to that $\hat{P}=P \sqcup\{\mathbf{0})$ is an entire semigroup with the requirement that $\mathbf{0} \cdot P=P \cdot \mathbf{0}=\{\mathbf{0}\})$.

We say that a subset $J \subset P$ is an ideal in $P$ if for any elements $c \in P, d \in J$ it holds $c \circ d \in J$, provided that $c \circ d \in P$, and similarly, $d \circ c \in J$, provided that $d \circ c \in P$. Then $P \backslash J$ is a partial semigroup. If $M$ is a semigroup and $J \subset M$ is an ideal we call $M \backslash J$ a coideal partial semigroup.

We assume that $P$ is endowed with a linear order $\prec$ satisfying the following property. For $c, c^{\prime}, d, d^{\prime} \in P$ the inequalities $c \preceq d, c^{\prime} \preceq d$ imply that $c \circ c^{\prime} \preceq d \circ d^{\prime}$ (sometimes we consider partial semigroups without apriori linear order which we introduce afterwards).

A mapping $f: P \rightarrow Q$ of partial semigroups $P, Q$ preserving the orders is called a homomorphism if for any $c, d \in P$ it holds that $c \circ d$ is defined iff $f(c) \circ f(d)$ is defined as well, and in this case the equality $f(c \circ d)=f(c) \circ f(d)$ is true.

Remark 2.2. i) For a partial semigroup $P$ we call $P_{0} \subset P$ a subsemigroup of $P$ if for any $c, d \in P_{0}$ it holds $c \circ d \in P_{0}$ whenever $c \circ d \in P$. Any subset $R \subset P$ generates the uniquely defined minimal subsemigroup $\bar{R} \subset P$ such that $R \subset \bar{R}$.
ii) If $f: P \rightarrow Q$ is a homomorphism of partial semigroups then the image $f(P)$ is a subsemigroup of $Q$.

For any partial semigroup $P$ we say that a subset $S \subset P \times P$ is admissible if it defines a partial semigroup on $P$. The following is immediate.

Lemma 2.3. The intersection of any family of admissible subsets of $P$ is admissible
This defines an admissible closure of any $X \subset P \times P$ to be the intersection of all admissible subsets containing $X$. This means that any $X \subset P \times P$ defines a canonical partial semigroup on $P$ so that pairs $(a, b) \in X$ are composable which we denote by $P_{X}$.

Definition 2.4. We say that a mapping $f: P \rightarrow Q$ of partial semigroups $P, Q$ is a partial homomorphism if the set $S_{f}$ of all pairs $(a, b) \in P \times P$ such that $a, b$ are composable in $P$ and $f(a), f(b)$ are composable in $Q$ is admissible, in addition we require that $f(a \circ b)=f(a) \circ f(b)$ for $(a, b) \in S_{f}$.

Remark 2.5. If $f: P \rightarrow Q$ is a partial homomorphism of partial semigroups then $f: P_{S_{f}} \rightarrow Q$ is a homomorphism (see Definition 2.1).

Remark 2.6. If $\prec$ is a linear order on a semigroup $M$ then it induces a linear order on a coideal partial semigroup $P \subset M$.

The following is immediate.
Lemma 2.7. Let $f: P \rightarrow Q$ be an (ordered) epimorphism of (ordered) partial semigroups. Suppose that the fibers of $f$ are well-ordered. Then the assignments
$x \mapsto \min \left\{f^{-1}(x)\right\}$ define a section $f^{*}: Q \hookrightarrow P$ of $f$. In turn, this defines a vector space decomposition $\mathbb{k} P=\mathbb{k} f^{*}(Q) \oplus I$ where $I$ is the kernel of the canonical homomorphism $\mathbb{k} P \rightarrow \mathbb{k} Q$. Also, all elements $y-f^{*}(x)$, $y \in f^{-1}(x)$, $y \neq f^{*}(x)$ form a basis $\mathbf{B}_{I}$ of $I$.

One can easily verify the following proposition.
Proposition 2.8. Let $P$ be an (ordered) partial semigroup and $Q$ be a partial semigroup (without apriori an order). Suppose that there is an order on $Q$ viewed as a set and let $f$ be a surjective order-preserving map $P \rightarrow Q$. Suppose that $f$ is also $a$ homomorphism of partial semigroups. Then $Q$ is an ordered partial semigroup.

Definition 2.9. We say that the order $\prec$ in a partial semigroup $Q$ fulfills a strict property if for any elements $a, b, c, d \in Q$ such that $a \prec b, c \preceq d$ it holds $a \circ c \prec b \circ d$ (respectively, $c \circ a \prec d \circ b$ ), provided that $a \circ c, b \circ d \in Q$ (respectively, provided that $c \circ a, d \circ b \in Q)$, cf. Definition 2.1.

Given ordered partial semigroups $P$ and $Q$, we say that a map $f: P \rightarrow Q$ is sub-multiplicative if $f\left(c \circ c^{\prime}\right) \preceq f(c) \circ f\left(c^{\prime}\right)$ whenever $c, c^{\prime}$ are composable in $P$ and $f(c), f\left(c^{\prime}\right)$ are composable in $Q$.

Proposition 2.10. Let $P, Q$ be ordered partial semigroups and $f: P \rightarrow Q$ be submultiplicative. Suppose that the order on $Q$ has a strict property. Then $f$ is a partial homomorphism.

Proof. It suffices to show that that $S_{f}$ is admissible. Indeed, suppose that $\left(c, c^{\prime}\right) \in S_{f}$ and $\left(c \circ c^{\prime}, c^{\prime \prime}\right) \in S_{f}$. Then

$$
f\left(c \circ c^{\prime}\right)=f(c) \circ f\left(c^{\prime}\right), f\left(\left(c \circ c^{\prime}\right) \circ c^{\prime \prime}\right)=f\left(c \circ c^{\prime}\right) \circ f\left(c^{\prime \prime}\right)=f(c) \circ f\left(c^{\prime}\right) \circ f\left(c^{\prime \prime}\right)
$$

Thus, $c^{\prime}, c^{\prime \prime}$ and $c, c^{\prime} \circ c^{\prime \prime}$ are composable, as well as $f\left(c^{\prime}\right), f\left(c^{\prime \prime}\right)$ and $f(c), f\left(c^{\prime}\right) \circ f\left(c^{\prime \prime}\right)$ are composable. Therefore

$$
f\left(\left(c \circ c^{\prime}\right) \circ c^{\prime \prime}\right)=f\left(c \circ\left(c^{\prime} \circ c^{\prime \prime}\right)\right) \preceq f(c) \circ f\left(c^{\prime} \circ c^{\prime \prime}\right) \preceq f(c) \circ f\left(c^{\prime}\right) \circ f\left(c^{\prime \prime}\right) .
$$

Since both the latter inequalities are actually, equalities, we get that $\left(c, c^{\prime} \circ c^{\prime \prime}\right) \in S_{f}$, finally the strict property of $Q$ implies that $f\left(c^{\prime} \circ c^{\prime \prime}\right)=f\left(c^{\prime}\right) \circ f\left(c^{\prime \prime}\right)$, thus $\left(c^{\prime}, c^{\prime \prime}\right) \in S_{f}$. The admissibility of $S_{f}$ is established.

Denote by $F_{k}$ the free semigroup generated freely by $k$ elements.
The following result provides a converse statement to Proposition 2.8 under the strict property of the order.

Theorem 2.11. Let $Q$ be a partial semigroup generated by $k$ elements $u_{1}, \ldots, u_{k}$ and let $\prec$ be an order on $Q$ satisfying the strict property. Then there exists a coideal partial semigroup $F \subset F_{k}$ and an epimorphism $f: F \rightarrow Q$ of (ordered) partial semigroups.

Proof. By definition, $\mathbb{k} F_{k}=\mathbb{k}<u_{1}, \ldots, u_{k}>$ and one has a epimorphism of algebras $\bar{f}: \mathbb{k} F_{k} \rightarrow \mathbb{k} Q$ which sends any monomial to element of $Q$ or 0 . Denote by $F$ the set of all elements of $F_{k}$ whose image is not 0 . Clearly, $F$ is a coideal of $F_{k}$.

This induces a natural epimorphism of partial semigroup $f: F \rightarrow Q$.

Introduce an order $\triangleleft$ on $F$ as follows. We say that $F \ni v:=u_{i_{1}} \circ \cdots \circ u_{i_{m}} \triangleleft w:=$ $u_{j_{1}} \circ \cdots \circ u_{j_{p}} \in F$ iff either

- $f(v) \prec f(w)$, either
- $f(v)=f(w)$ and $m<p$, or
- $f(v)=f(w), m=p$ and the word $v$ is less than $w$ in the lexicographical order (defined on $u_{1}, \ldots, u_{k}$ in an arbitrary way). Denote the length $l(v):=m$.

We claim that the epimorphism $f$ fulfills Definition 2.1. Indeed, let $v, w, v_{1}, w_{1} \in$ $F ; v \unlhd w, v_{1} \unlhd w_{1} ; v \circ v_{1}, w \circ w_{1} \in F$. If either $f(v) \prec f(w)$ or $f\left(v_{1}\right) \prec f\left(w_{1}\right)$ then $f\left(v \circ v_{1}\right)=f(v) \circ f\left(v_{1}\right) \prec f(w) \circ f\left(w_{1}\right)=f\left(w \circ w_{1}\right)$ due to the assumption in the theorem. If $f(v)=f(w), f\left(v_{1}\right)=f\left(w_{1}\right)$ and either $l(v)<l(w)$ or $l\left(v_{1}\right)<l\left(w_{1}\right)$ then $l\left(v \circ v_{1}\right)<l\left(w \circ w_{1}\right)$. Finally, if $l(v)=l(w), l\left(v_{1}\right)=l\left(w_{1}\right)$ then the word $v \circ v_{1}$ is less than $w \circ w_{1}$ in the lexicographical order, unless $v=w, v_{1}=w_{1}$.

Using an argument similar to that of the proof of Theorem 2.11, we establish the following.
Lemma 2.12. Let $P_{1}$ and $P_{2}$ be any partial semigroups. Then
(a) their direct product $P_{1} \times P_{2}$ is also a (partial) semigroup. Moreover, if $P_{1}$ and $P_{2}$ are ordered so that the ordering on $P_{1}$ fulfills the strict property then $P_{1} \times P_{2}$ is ordered as well via $\left(p_{1}, p_{2}\right) \preceq\left(p_{1}^{\prime}, p_{2}^{\prime}\right)$ iff either $p_{1} \prec p_{1}^{\prime}$ or $p_{1}=p_{1}^{\prime}$ and $p_{2} \preceq p_{2}^{\prime}$.
(b) Moreover, if $P_{2}$ also fulfills the strict property then $P_{1} \times P_{2}$ fulfills the strict property as well.

We say that a function $\ell: P \rightarrow \mathbb{Z}_{>0}$ is length if $\ell\left(c \circ c^{\prime}\right)=\ell(c)+\ell\left(c^{\prime}\right)$ for all composable $c, c^{\prime} \in P$. We say that $(P, \ell)$ is a graded partial semigroup if $\ell$ is a length on $P$ (sometimes we omit $\ell$ ).

We say that an order $\prec$ on a graded partial semigroup is length compatible if $\ell(c)<\ell\left(c^{\prime}\right)$ implies that $c \prec c^{\prime}$.

The following is immediate
Lemma 2.13. (Generalized deglex) Let $P$ be a free semigroup freely generated by a set $X$. Then
(a) Any function $f: X \rightarrow \mathbb{Z}_{>0}$ defines (unique) length function on $P$ and vice versa.
(b) For any length function $\ell: P \rightarrow \mathbb{Z}_{>0}$ any total order $\prec$ of $X$ such that $\ell(x)<$ $\ell(y)$ implies $x \prec y$, determines a unique length compatible order (fulfilling the strict property) on $P$ such that $x a \prec y b$ whenever $x, y \in X, \ell(x a)=\ell(y b)$ and $x \prec y$ or $x=y$ and $a \prec b$.
(c) If for any $m \in \mathbb{Z}_{>0}$ the preimage $f^{-1}(m) \subset X$ is finite then $\prec$ is a well ordering.
Remark 2.14. If $\ell(x)=1$ for any generator of $P$ (e.g., when $P=F_{+}^{n}$ or $P=\mathbb{Z}_{\geq 0}^{n}$ ) this becomes an ordinary deglex on $P$.

Denote by $P_{1} * P_{2}=P_{2} * P_{1}$ the free product of (partial) semigroups $P_{1}$ and $P_{2}$.
Suppose that $P_{1}$ and $P_{2}$ are ordered. We say that that an order on the partial semigroup $P_{1} * P_{2}$ is compatible if $p \prec p^{\prime}$ implies $p * c \prec p^{\prime} * c$ and $c * p \prec c * p^{\prime}$ for any $c \in P_{1} * P_{2}$ and any $p, p^{\prime} \in P_{i}, i=1,2$.

If $P_{1}$ and $P_{2}$ are entire then there are several constructions of the order on $P_{1} * P_{2}$ (see e.g. [6]). In particular, for an ideal $J_{1}$ in $P_{1}$ and an ideal $J_{2}$ of $P_{2}$, any such
order restricts to an order on the free product $\left(P_{1} \backslash J_{1}\right) *\left(P_{2} \backslash J_{2}\right)$ of coideal partial semigroups.

The following immediate fact gives another construction of an order on free products of graded partial semigroups (making use of Lemma 2.13).

Lemma 2.15. Let $P_{1}$ and $P_{2}$ be any graded partial semigroups. Then their free product $P_{1} * P_{2}=P_{2} * P_{1}$ is also a graded (partial) semigroup.

Example 2.16. i) For a monoid $M:=\mathbb{Z}_{\geq 0}^{n}=\left\{x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}: i_{1}, \ldots, i_{n} \geq 0\right\}$ and a family of monomials $u_{1}, \ldots, u_{s}$ in the variables $x_{1}, \ldots, x_{n}$, the set of monomials not dividing any of $u_{1}, \ldots, u_{s}$, forms a coideal partial monoid $P\left(u_{1}, \ldots, u_{s}\right)$. Then $P\left(u_{1}, \ldots, u_{s}\right)$ coincides with the complement to the monomial ideal $J\left(u_{1}, \ldots, u_{s}\right):=$ $\bigcup_{1 \leq j \leq s}\left(u_{j}+\mathbb{Z}_{\geq 0}^{n}\right)$.
ii) For a free monoid $M_{n}:=\left\langle x_{1}, \ldots, x_{n}\right\rangle$ consider an ideal $J:=\left\langle x_{i} x_{j} x_{k}: 1 \leq\right.$ $i, j, k \leq n\rangle$. We define a well-ordering $\prec$ on $M$ as follows. For a pair of words $u, v \in M$ we say that $u \prec v$ if either $u$ is shorter than $v$ or they have the same length and $u$ is lower than $v$ with respect to the lexicographical order in which $x_{n} \prec \cdots \prec x_{1}$ (see Lemma 2.13). Then $P:=M \backslash J$ is a finite coideal partial monoid.
iii) For a free monoid $M_{n}:=\left\langle x_{1}, \ldots, x_{n}\right\rangle$ consider an ideal $J_{n}:=\left\langle x_{j} x_{i}: j \geq i\right\rangle$. Then $P_{n}:=M_{n} \backslash J_{n}$ is a finite coideal partial monoid consisting of $2^{n}$ elements of the form $u:=x_{i_{1}} \cdots x_{i_{k}}, 1 \leq i_{1}<\cdots<i_{k} \leq n$. For an element $v:=x_{j_{1}} \cdots x_{j_{l}}, 1 \leq$ $j_{1}<\cdots<j_{l} \leq n$ the composition $u \circ v \in P_{n}$ iff $i_{k}<j_{1}$.

Example 2.17. i) Now we modify the construction of Example 2.16 iii) and produce a partial monoid $Q_{n}$ coinciding as a set with $P_{n}$ and equipped with the same ordering $\prec$. The composition law in $Q_{n}$ differs from the one in $P_{n}$ : let $u:=x_{i_{1}} \cdots x_{i_{k}}, 1 \leq$ $i_{1}<\cdots<i_{k} \leq n, v:=x_{j_{1}} \cdots x_{j_{l}}, 1 \leq j_{1}<\cdots<j_{l} \leq n$ be two elements of $Q_{n}$, then

$$
u \circ v=x_{i_{1}} \cdots x_{i_{k}} \circ x_{j_{2}} \cdots x_{j_{l}}
$$

iff $i_{k}=j_{1}$; otherwise the composition is not defined.
The partial monoid $Q_{n}$ is isomorphic to the following partial monoid $R_{n}$. The generators of $R_{n}$ are $\left\{y_{i, j}: 1 \leq i<j \leq n\right\}$. The composition $y_{i, j} \circ y_{k, l}$ is defined iff $j=k$. The isomorphism of $R_{n}$ and $Q_{n}$ is established by mapping of $y_{i_{1}, i_{2}} \circ y_{i_{2}, i_{3}} \circ$ $\cdots \circ y_{i_{k-1}, i_{k}}$ to $x_{i_{1}} \circ \cdots \circ x_{i_{k}}$.
ii) One can yield a family of partial submonoids of $R_{n}$ as follows. Consider a directed acyclic graph $G$ with $n$ vertices numbered by $\{1, \ldots, n\}$ in such a way that for any arrow $(i, j)$ of $G$ it holds $i<j$. Then one can consider a partial submonoid $R_{G}$ of $R_{n}$ generated by the elements $\left\{y_{k, l}\right\}$ for which there is a path from a vertex $k$ to a vertex $l$ in $G$.
iii) Alternatively, one can consider a partial submonoid $T_{G}$ of $R_{G}$ of paths in $G$. A partial monoid $T_{G}$ is generated by the elements $\left\{z_{k, l}\right\}$ where $(k, l)$ is an arrow in $G$ (cf. [30]).

More generally, one can consider a partial monoid $T_{G}$ of paths in an arbitrary directed graph $G$ (when $G$ contains cycles, $T_{G}$ is infinite). One can treat $T_{G}$ as a coideal partial submonoid of the free monoid $M_{G}$ generated by $\left\{z_{k, l}\right\}$ where $(k, l)$ is an arrow in $G$. Then $T_{G}=M_{G} \backslash J_{G}$ where the ideal $J_{G}$ is generated by all compositions of the form $z_{k, l} \circ z_{i, j}$ where $l \neq i$.

Example 2.18. Denote by $W_{0}(m), m \geq 1$ the nil-Coxeter semigroup generated by $s_{1}, s_{2}$ satisfying the following relations:

$$
s_{1} \circ s_{1}, s_{2} \circ s_{2} \notin W_{\mathbf{0}}(m), \underbrace{s_{1} \circ s_{2} \circ s_{1} \circ s_{2} \ldots}_{m}=\underbrace{s_{2} \circ s_{1} \circ s_{2} \circ s_{1} \ldots}_{m} .
$$

Then $W_{\mathbf{0}}(m)$ consists of $2 m-1$ elements: for each $1 \leq k<m$ it contains two elements

$$
c_{k}:=\underbrace{s_{1} \circ s_{2} \circ s_{1} \circ s_{2} \ldots}_{k}, d_{k}:=\underbrace{s_{2} \circ s_{1} \circ s_{2} \circ s_{1} \ldots}_{k}
$$

of length $k$, and in addition the element $c_{m}=d_{m}$.
The following compositions are defined in $W_{\mathbf{0}}(m)$ :

$$
c_{2 k} \circ c_{l}=c_{2 k+l}, c_{2 k+1} \circ d_{l}=d_{2 k+l}, d_{2 k} \circ d_{l}=d_{2 k+l}, d_{2 k+1} \circ c_{l}=d_{2 k+l+1},
$$

provided that $2 k+l \leq m$ or respectively, $2 k+l+1 \leq m$. All other compositions are not defined. One can verify that $W_{\mathbf{0}}(m)$ is a partial semigroup with an order defined by $d_{k} \prec c_{k} \prec d_{k+1}, 1 \leq k<m$.

Observe that in case of $W_{\mathbf{0}}(m)$ one cannot replace the axiom from Definition 2.1 by weaker axioms that $c \preceq d$ implies that $b \circ c \preceq b \circ d, c \circ b \preceq d \circ b$, provided that $b \circ c, b \circ d, c \circ b, d \circ b$ are defined.

If one applies Theorem 2.11 to the partial semigroup $W_{0}(m)$, then one obtains the partial semigroup $\overline{W_{\mathbf{0}}(m)}$ generated by two elements $\overline{s_{1}}, \overline{s_{2}}$ such that $\overline{s_{1}} \circ \overline{s_{1}}, \overline{s_{2}} \circ \overline{s_{2}}$ and $\underbrace{\overline{s_{1}} \circ \overline{s_{2}} \circ \cdots}_{m+1}, \underbrace{\overline{s_{2}} \circ \overline{s_{1}} \circ \cdots}_{m+1}$ are not defined; thus consisting of $2 m$ elements of the form either $\overline{c_{k}}=\underbrace{s_{1} \circ s_{2} \circ \cdots}_{k}$ or $\overline{d_{k}}=\underbrace{s_{2} \circ s_{1} \circ \cdots}_{k}, 1 \leq k \leq m$. The epimorphism $f: \overline{W_{\mathbf{0}}(m)} \rightarrow W_{\mathbf{0}}(m)$ sends $f\left(\overline{c_{k}}\right)=c_{k}, f\left(\overline{d_{k}}\right)=d_{k}, 1 \leq k \leq m$. Thus, $f$ is not injective just on two elements: $f\left(\overline{c_{m}}\right)=f\left(\overline{d_{m}}\right)=c_{m}=d_{m}$. The order on $\overline{W_{\mathbf{0}}(m)}$ is defined by $\overline{c_{k}} \triangleleft \overline{d_{k}} \triangleleft \overline{c_{k+1}}, 1 \leq k<m$ and in addition, $\overline{c_{m}} \triangleleft \overline{d_{m}}$.

The following two propositions provide constructions of extending partial semigroups.
Proposition 2.19. Let $P$ and $Q$ be partial semigroups. Then $P^{\prime}=P \sqcup Q$ is a partial semigroup with the composition inherited from $P$ and $Q$ and $p q=q p=q$ for all $p \in P, q \in Q$. Suppose that $P$ and $Q$ are ordered such that
i) $q \preceq q q^{\prime}$ and $q \preceq q^{\prime} q$ for all $q, q^{\prime} \in Q$ (this property is called positive ordering, see, e.g. [32], [30]). Then the assignments $p \prec q$ for $p \in P$ and $q \in Q$ turn $P^{\prime}$ into an ordered partial semigroup;
ii) $q \succeq q q^{\prime}$ and $q \succeq q^{\prime} q$ for all $q, q^{\prime} \in Q$. Then the assignments $p \succ q$ for $p \in P$ and $q \in Q$ turn $P^{\prime}$ into an ordered partial semigroup.

Note that when $P, Q$ are commutative partial semigroups, the resulting $P^{\prime}$ is commutative as well. The next proposition allows one to construct non-commutative partial semigroups from arbitrary (in particular, commutative) ones.

Proposition 2.20. Let $P, Q$ be partial semigroups. Consider a partial semigroup $R:=Q \sqcup\{x\} \sqcup\{y\} \sqcup P$ defined as follows:

$$
x z=x, y z=y, z \in\{P, Q, x, y\} ; P z_{1}=y, z_{1} \in\{x, y, Q\} ; Q z_{2}=x, z_{2} \in\{x, y, P\}
$$

Then $R$ is a partial semigroup with an ordering $Q \prec x \prec y \prec P$.
When in Propositions 2.19, 2.20 $P, Q$ are entire semigroups, the results of constructions are entire semigroups as well. In contrast, there are no entire finite semigroups satisfying the strong property.

Proposition 2.21. There are no entire finite semigroups (with more than one element) satisfying the strong property.

Proof. Suppose the contrary. If for some element $a$ of the semigroup it holds $a \prec a^{2}$ then $a^{i} \prec a^{i+1}, i \geq 1$, which leads to a contradiction. By the same token an assumption $a \succ a^{2}$ leads to a contradiction as well. Thus, $a=a^{2}$ for any element $a$.

For any pair of elements $a \prec b$ it holds $a=a^{2} \prec a b$, hence $a b \prec a b^{2}=a b$. The obtained contradiction completes the proof.

Now we concoct a construction for extending partial monoids satisfying the strict property.

Proposition 2.22. Let $P$ be a partial monoid with an order $\prec^{0}$ satisfying the strict property. For $x \notin P$ construct a partial monoid

$$
Q:=P \sqcup P \circ x \sqcup \cdots \sqcup P \circ x^{\circ k}
$$

such that $x \circ P, x^{\circ(k+1)}$ are not defined. We set the order $\prec$ on $Q$ as follows:

$$
\begin{gathered}
P \circ x^{\circ i} \prec P \circ x^{\circ(i+1)}, 0 \leq i<k \text { and } \\
c \circ x^{\circ j} \prec d \circ x^{\circ j} i f f c \prec^{0} d, c, d \in P, 0 \leq j \leq k .
\end{gathered}
$$

Then $Q$ is a partial monoid satisfying the strict property.
Alternatively, one could set the order as $P \circ x^{\circ i} \succ P \circ x^{\circ(i+1)}$.
Proof. To verify the strict property consider elements $u, v, w, t \in Q$ such that $u \preceq v, w \preceq t$ and at least one of two latter inequalities is strict. We assume that $u \circ w, v \circ t \in Q$. When $u, v, w, t \in P$, the strict property follows from the strict property for $P$. Otherwise, $v \in P, t=t_{0} \circ x^{\circ i}$ for some $1 \leq i \leq k$. Therefore, $u \in P, w=w_{0} \circ x^{\circ j}$ for suitable $0 \leq j \leq i$. If $j<i$ then $u \circ w \prec v \circ t$. Otherwise, if $j=i$ then it holds $w_{0} \preceq^{0} t_{0}$. Since one of two inequalities $u \preceq^{0} v, w_{0} \preceq^{0} t_{0}$ is strict, we deduce from the strict property for $P$ that $u \circ w_{0} \prec^{0} v \circ t_{0}$, which implies the strict property for $Q$ :

$$
u \circ w=u \circ w_{0} \circ x^{\circ i} \prec v \circ t_{0} \circ x^{\circ i}=v \circ t .
$$

By the same token one considers an alternative order $P \circ x^{\circ i} \succ P \circ x^{\circ(i+1)}$.

Remark 2.23. One can generalize Proposition 2.22 to partial semigroups (rather than monoids). For a partial semigroup $P$ satisfying the strict property consider a partial semigroup $Q:=\bigsqcup_{0 \leq i \leq k}(i, P)$ (where $(i, P)$ is a copy of $P$ ), in which the product is defined as $\left(0, p_{0}\right) \circ(i, p):=\left(i, p_{0} \circ p\right), p_{0}, p \in P, 0 \leq i \leq k$, and $(j, P) \circ(i, P)$ is not defined when $j>0$. The order in $Q$ is lexicographical with respect to $i$ and to the order in $P$. As in the proof of Proposition 2.22 one can verify that $Q$ fulfills the strict property.

### 2.2. Valuations of algebras in partial semigroups.

Definition 2.24. For a $\mathbb{k}$-algebra $\mathcal{A}$ a mapping $\nu: A \backslash\{0\} \rightarrow P$ onto a partial semigroup $P$ is a valuation if for any $a, b \in A \backslash\{0\}$ it holds the following:
i) $\nu\left(\mathbb{k}^{*} a\right)=\nu(a)$;
ii) $\nu(a+b) \preceq \max \{\nu(a), \nu(b)\}$, provided that $a+b \neq 0$;
iii) $\nu(a b)=\nu(a) \circ \nu(b)$, provided that $\nu(a) \circ \nu(b) \in P$ (in particular, in this case it holds $a b \neq 0$ ).
Remark 2.25. Alternatively, one could consider a mapping $\nu: A \backslash\{0\} \rightarrow P_{0}$ satisfying the properties similar to i), ii), iii) where $P_{0}$ is a partial semigroup. Then $P_{\nu}:=\nu(A \backslash\{0\}) \subset P_{0}$ is also a partial semigroup.
Remark 2.26. In this section we do not suppose that $A$ is unital or $P$ contains 1 .
We say that a valuation $\nu: A \backslash\{0\} \rightarrow P$ onto a partial semigroup $P$ is injective if there exists a $\mathbb{k}$-basis $\mathbf{B} \subset A$ of $A$ such that $\nu: \mathbf{B} \rightarrow P$ is a bijection. A basis fulfilling the latter property is called adapted with respect to $\nu$.

The proof of the following proposition is straightforward.
Proposition 2.27. Let $\nu: A \backslash\{0\} \rightarrow P$ be an injective valuation and let $f: P \rightarrow Q$ be an ordered homomorphism of partial semigroups. Then $f \circ \nu: A \backslash\{0\} \rightarrow Q$ is also an injective valuation.

Definition 2.28. For a partial semigroup $P$ define a semigroup algebra $\mathbb{k} P$ as having a basis $\{[u]: u \in P\}$. We define $[u][v]=[u \circ v]$, provided that $u \circ v \in P$, otherwise $[u][v]=0$.

Then a tautological valuation $\nu:=\nu_{P}: \mathbb{k} P \backslash\{0\} \rightarrow P$ is defined by

$$
\nu\left(\sum_{u \in P} \alpha_{u}[u]\right):=\max \{u\}, \alpha_{u} \in \mathbb{k}^{*} .
$$

Observe that the order on $P$ is not necessary to be a well order: still, we get an injective valuation with an adapted basis $\{[u]: u \in P\}$.

It is immediate that a homomorphism $f: P \rightarrow Q$ of partial semigroups induces a natural homomorphism of algebras $\mathbb{k} P \rightarrow \mathbb{k} Q$. In addition, it induces a (not necessary injective) valuation $f \circ \nu: \mathbb{k} P \backslash\{0\} \rightarrow Q$.

Note however that if $P$ is a partial semigroup and $Q$ is a coideal in $P$, there is a homomorphism of algebras $\mathbb{k} P \rightarrow \mathbb{k} Q$ given by $[c] \rightarrow\left\{\begin{array}{ll}{[c]} & \text { if } c \in Q \\ 0 & \text { otherwise }\end{array}\right.$ but, in general, there is no corresponding partial homomorphism from $P$ to $Q$.
Remark 2.29. In the conditions of Lemma $2.7 \mathbf{B}=f^{*}(Q) \sqcup \mathbf{B}_{I}$ is a basis of $\mathbb{k} P$ adapted to the tautological valuation of $\nu_{P}: \mathbb{k} P \backslash\{0\} \rightarrow P$ (produced in Definition 2.28).
Example 2.30. Following Example 2.16 i) one can consider the monoidal algebra $\mathbb{k} P\left(u_{1} \ldots, u_{s}\right)$. It is called a Stanley-Reisner algebra in case when all $u_{1}, \ldots, u_{s}$ are square-free.

The proof of the following proposition is similar to the proof of Theorem 3.1 ii), iv).

Proposition 2.31. Let $\nu: A \backslash\{0\} \rightarrow P$ be a valuation onto a partial semigroup $P$. When $P$ is well-ordered and $\operatorname{dim}\left(\mathcal{A}_{u}\right)=1$ for any $u \in P$, the valuation $\nu$ is injective. Every set $\mathbf{B} \subset A$ such that the mapping $\nu: \mathbf{B} \rightarrow P$ is a bijection, is an adapted basis of $A$ (with respect to $\nu$ ).

Vice versa, if $\nu$ is injective then $\operatorname{dim}\left(\mathcal{A}_{u}\right)=1$ for any $u \in P$.
Proposition 2.32. Let $\nu: A \backslash\{0\} \rightarrow P$ be an injective valuation of an algebra $A$ into a well-ordered partial semigroup $P$, and $B$ be a subalgebra of $A$. Then the restriction of $\nu$ on $B \backslash\{0\}$ is also an injective valuation.
Remark 2.33. In view of Remark 4.18, it is interesting whether an analog of Proposition 2.32 holds without assumption of well-orderness of $P$.

The following proposition for vector spaces is established in Proposition 4.4 (b). The extension to algebras is straight-forward.
Proposition 2.34. Let $\mathcal{A}_{i}, i=1,2$ be algebras and $\nu_{i}: \mathcal{A}_{i} \backslash\{0\} \rightarrow P_{i}$ be their valuations to respective partial semigroups. Then the assignments $a_{1} \otimes a_{2} \mapsto\left(\nu_{1}\left(a_{1}\right), \nu_{2}\left(a_{2}\right)\right)$ extend to a valuation $\nu: \mathcal{A}_{1} \otimes \mathcal{A}_{2} \backslash\{0\} \rightarrow P_{1} \times P_{2}$ (an order in $P_{1} \times P_{2}$ is defined in Lemma 2.12). If both valuations $\nu_{1}, \nu_{2}$ are injective then $\nu$ is injective as well.
Proposition 2.35. Consider a partial semigroup $P$ and an ordered partial semigroup $Q$. Let $\nu: \mathbb{k} P \backslash\{0\} \rightarrow Q$ be a valuation. Then the mapping $c \mapsto \nu([c])$ is a partial homomorphism $P \rightarrow Q$. In particular, $P$ acquires a new structure of a partial semigroup $P_{S_{f}}$ in notation of Remark 2.5.
Proof. Take $c, c^{\prime}, c^{\prime \prime} \in P$ such that $\nu([c]), \nu\left(\left[c^{\prime}\right]\right)$ are composable and that $\nu([c]) \circ$ $\nu\left(\left[c^{\prime}\right]\right), \nu\left(\left[c^{\prime \prime}\right]\right)$ are also composable. Let $S_{\nu}$ be the set of all $\left(c, c^{\prime}\right) \in P \times P$ such that $\nu([c]), \nu\left(\left[c^{\prime}\right]\right)$ are composable. Then due to Definition 2.1 it holds that $\nu\left(\left[c^{\prime}\right]\right), \nu\left(\left[c^{\prime \prime}\right]\right)$ are composable and that $\nu([c]), \nu\left(\left[c^{\prime}\right]\right) \circ \nu\left(\left[c^{\prime \prime}\right]\right)$ are composable as well. Thus, $S_{\nu}$ is admissible due to Definition 2.24.

If $\left(c, c^{\prime}\right) \in S_{\nu}$ then $c, c^{\prime}$ are composable in $P_{S_{\nu}}$ and $\nu\left(\left[c \circ c^{\prime}\right]\right)=\nu([c]) \circ \nu\left(\left[c^{\prime}\right]\right)$ again due to Definition 2.24.

In the following theorem we consider different words in generators of a partial semigroup representing the same element of the partial semigroup, among them we choose the minimal with respect to deglex (also for non-commutative partial semigroups), cf. Lemma 2.13, and call this word canonical. The following theorem can be easily deduced from Corollary 4.15.
Theorem 2.36. Let $\nu: A \backslash\{0\} \rightarrow P_{\nu} \subset P$ be an injective well-ordered valuation into a partial semigroup $P$, generated by $P_{0}$, and let $X_{0}$ be a generating set of $A$ such that $\left.\nu\right|_{X_{0}}$ is a bijection $X_{0} \underset{\rightarrow}{\rightarrow} P_{0}$. Then the set of all monomials $x_{u}:=\prod_{x \in X_{0}} x$ corresponding to the canonical factorization of $u \in P_{\nu}$ is an adapted to $\nu$ basis in $A$, and $\nu\left(x_{u}\right)=u$ (we will refer to the elements $x_{u}$ as standard monomials).
Remark 2.37. $X_{0}$ is not always a minimal generating set for $\mathcal{A}$. The same applies to $P_{0}$. In principle we can require that $P_{0}$ is minimal by inclusion. In some cases, including submonoids of $\mathbb{Z}_{\geq 0}^{m}$, $P^{\text {ind }}$ of indecomposable elements of $P$ generate $P$, in which case we can choose $\bar{P}_{0}:=P^{\text {ind }}$.

In the following theorem we show that given an injective valuation on an algebra, how one can define it on its quotient algebra.

Theorem 2.38. Let $A$ be $a \mathbb{k}$-algebra and $\nu_{0}: A \backslash\{0\} \rightarrow P$ be an injective valuation onto a well-ordered partial semigroup $P$. Let $I \subset A$ be an ideal.
i) Then $\nu_{0}(I \backslash\{0\})$ is an ideal in $P$. For $a \in(A / I) \backslash\{0\}$ the formula from Proposition 4.21, i.e.,

$$
\begin{equation*}
\nu(a):=\min \nu_{0}(a+I) \tag{2.1}
\end{equation*}
$$

defines an injective valuation $\nu:(A / I) \backslash\{0\} \rightarrow\left(P \backslash \nu_{0}(I \backslash\{0\})\right)$. If $\nu_{0}(a) \in$ $P \backslash \nu_{0}(I \backslash\{0\})$ then $\nu(a)=\nu_{0}(a)$.
ii) If $u \in P \backslash \nu_{0}(I \backslash\{0\})$ is indecomposable then $u$ is also indecomposable in $P$.
iii) Let $\left\{x_{u}: u \in P\right\}$ be a standard monomial basis of $A$ with respect to $\nu_{0}$ (cf. Theorem 2.36). Then $\mathbf{B}:=\left\{q\left(x_{u}\right): u \in P \backslash \nu_{0}(I \backslash\{0\})\right\}$ is a standard monomial basis of $A / I$ with respect to $\nu$ where $q: A \rightarrow A / I$ is the natural projection.

Proof. i) First, we note that if $\nu_{0}(a) \in P \backslash \nu_{0}(I \backslash\{0\})$ then $\nu(a)=\nu_{0}(a)$. Indeed, suppose that on the contrary it holds $\nu_{0}(a+f) \prec \nu_{0}(a)(c f .(2.1))$. Then $\nu_{0}(f)=\nu_{0}(a)$ which contradicts the supposition.

Observe that for any $a \in(A / I) \backslash\{0\}$ it holds $\nu(a) \notin \nu_{0}(I \backslash\{0\})$. Indeed, otherwise $\nu(a)=\nu_{0}(a+f) \in \nu_{0}(I \backslash\{0\})$ for suitable $f \in I \backslash\{0\}$. Then there exists $g \in I \backslash\{0\}$ such that $\nu_{0}(g)=\nu_{0}(a+f)$. Due to the injectivity of $\nu_{0}$ there exists $\alpha \in \mathbb{k}^{*}$ for which holds $\nu_{0}(a+f+\alpha g) \prec \nu_{0}(a+f)$, this contradicts to the equality $\nu(a)=\nu_{0}(a+f)$ and to (2.1).

Now let $a, b \in(A / I) \backslash\{0\}$ and $f, g \in I$ be such that $\nu(a)=\nu_{0}(a+f), \nu(b)=\nu_{0}(b+g)$ according to (2.1). Then
$\nu(a+b) \preceq \nu_{0}(a+f+b+g) \preceq \max \left\{\nu_{0}(a+f), \nu_{0}(b+g)\right\}=\max \{\nu(a+f), \nu(b+g)\}$
which justifies Definition 2.24 ii) for $\nu$.
To verify Definition 2.24 iii) for $\nu$ assume that $\nu(a) \circ \nu(b) \in P \backslash \nu_{0}(I \backslash\{0\})$. Since

$$
\nu(a b) \preceq \nu_{0}(a b+a g+f b+f g)=\nu_{0}(a+f) \circ \nu_{0}(b+g)
$$

due to (2.1) and to Definition 2.24 iii) for $\nu_{0}$, we get $\nu(a b) \preceq \nu(a) \circ \nu(b)$. Suppose that $\nu(a b) \prec \nu(a) \circ \nu(b)$. Let $\nu(a b)=\nu_{0}\left((a+f)(b+g)+f_{0}\right)$ for appropriate $f_{0} \in I \backslash\{0\}$ (see (2.1)). Hence

$$
\nu_{0}\left((a+f)(b+g)+f_{0}\right) \prec \nu(a) \circ \nu(b)=\nu_{0}((a+f)(b+g))
$$

and thereby, $\nu_{0}((a+f)(b+g))=\nu_{0}\left(f_{0}\right) \in \nu_{0}\left(I^{*}\right)$. The obtained contradiction shows that $\nu(a b)=\nu(a) \circ \nu(b)$.

Finally, we prove that $\nu$ is injective. Let $a, b \in(A / I) \backslash\{0\}$ and $f, g \in I$ be such that $\nu_{0}(a+f)=\nu(a)=\nu(b)=\nu_{0}(b+g)$ (see (2.1)). Since $\nu_{0}$ is injective there exists $\alpha \in \mathbb{k}^{*}$ such that either $\nu_{0}((a+f)+\alpha(b+g)) \prec \nu_{0}(a+f)$ or $a+f+\alpha(b+g)=0$. In the former case $\nu(a+\alpha b) \preceq \nu_{0}((a+f)+\alpha(b+g)) \prec \nu(a)$, while in the latter case $(A / I) \ni a+\alpha b=0$.
ii) Suppose the contrary, then $u=u_{1} u_{2}$ for suitable $u_{1}, u_{2} \in P$. It holds $u_{1}, u_{2} \notin$ $\nu_{0}(I \backslash\{0\})$, this contradicts to that $u \in P \backslash \nu_{0}(I \backslash\{0\})$ is indecomposable.
iii) Due to i) it holds $\nu\left(q\left(x_{u}\right)\right)=\nu_{0}\left(x_{u}\right)=u$ for $x_{u} \in \mathbf{B}$ (cf. Theorem 2.36) and $\nu((A / I) \backslash\{0\})=P \backslash \nu_{0}(I \backslash\{0\})=\nu(\mathbf{B})$. Therefore, Proposition 2.31 implies that $\mathbf{B}$
is an adapted basis of $A / I$ with respect to $\nu$. Finally, ii) entails that $\mathbf{B}$ is a standard monomial basis.

Example 2.39. Let an algebra $A:=\mathbb{k}[x, y] /\left(x^{2}-y^{3}\right)$. Following Theorem 3.21 one produces an injective valuation $\nu: A \backslash\{0\} \rightarrow C$ onto a semigroup $C:=\{(i, j): 0 \leq$ $i<\infty, j=0,1\}$ where $(0,1) \circ(0,1)=(3,0)$, and $\nu\left(y^{i}\right)=(i, 0), \nu\left(x y^{i}\right)=(i, 1)(c f$. Example 3.38).

On the other hand, applying Theorem 2.38 one obtains an injective valuation $\underline{\nu}: A \backslash\{0\} \rightarrow P \subset \mathbb{Z}_{\geq 0}^{2}$ onto a partial semigroup $P$ which coincides with $C$ as a set, while $(0,1) \circ(0, \overline{1})$ is not defined in $P$. The values of $\underline{\nu}$ coincide with the corresponding values of $\nu$, i.e. $\underline{\nu}\left(y^{i}\right)=(i, 0), \underline{\nu}\left(x y^{i}\right)=(i, 1)$.

Remark 2.40. One can study the following inverse issue to Theorem 2.38. Let $J \subset A$ be an ideal in a commutative algebra $A$, and let $\nu: J \backslash\{0\} \rightarrow P$ be a valuation (not necessary injective) in a partial semigroup $P$ whose ordering $\prec$ fulfills the strict property. Assume in addition that for any element $a \in A$ it holds $a J \neq\{0\}$ and that for any $c \in P$ there exists $d \in P$ such that $c, d$ are composable.

When one can extend the valuation $\bar{\nu}: A \backslash\{0\} \rightarrow Q$ for a suitable partial semigroup $Q \supset P$ such that $\left.\bar{\nu}\right|_{J \backslash\{0\}}=\nu$ ? To define $Q$ consider a set $P \times P$ with the componentwise composition $\left(c_{1}, c_{2}\right) \circ Q\left(d_{1}, d_{2}\right):=\left(c_{1} \circ d_{1}, c_{2} \circ d_{2}\right)$ (provided that both $c_{1}, d_{1}$ and $c_{2}, d_{2}$ are composable) and impose the following relations (the idea is to treat $Q$ as a set of "fractions" with numerators and denominators from $P$ ). Firstly, we identify pairs $\left(c \circ d_{1}, d_{1}\right),\left(c \circ d_{2}, d_{2}\right) \in P \times P$ (provided that both $c, d_{1}$ and $c, d_{2}$ are composable). Secondly, we identify in $Q$ pairs $\left(c_{1}, c_{2}\right)$, $\left(d_{1}, d_{2}\right)$ if $c_{1} \circ d_{2}=c_{2} \circ d_{1}$ (provided that both $c_{1}, d_{2}$ and $c_{2}, d_{1}$ are composable in $P$ ). Thirdly, for any $a \in A$ if $a b_{1}=b_{2}, a b_{3}=b_{4}$ for non-zero $b_{1}, b_{2}, b_{3}, b_{4} \in J$, we identify in $Q$ the pairs $\left(\nu\left(b_{2}\right), \nu\left(b_{1}\right)\right)$ and $\left(\nu\left(b_{4}\right), \nu\left(b_{3}\right)\right)$. We define an order on $Q$ as follows: $\left(c_{1}, c_{2}\right) \prec_{Q}$ $\left(d_{1}, d_{2}\right)$ iff $c_{1} \circ d_{2} \prec c_{2} \circ d_{1}$ (provided that both $c_{1}, d_{2}$ and $c_{2}, d_{1}$ are composable).

If the resulting semigroup $Q$ is ordered and contains $P$ embedded for $c \in P$ by $(c \circ d, d) \in Q$ such that $c, d$ are composable, then one can define an extension $\bar{\nu}(a):=\left(\nu\left(b_{2}\right), \nu\left(b_{1}\right)\right)$. Moreover, in this case the order $\prec_{Q}$ fulfills the strict property.

Denote by $A_{1} * A_{2}=A_{2} * A_{1}$ the free product of algebras $A_{1}$ and $A_{2}$.
The following is immediate.
Lemma 2.41. Let $A_{i}, i=1,2$ be algebras and let $\nu_{i}: A_{i} \backslash\{0\} \rightarrow P_{i}, i=1,2$ be a valuation of $A_{i}$ to a (partial) semigroup $P_{i}$. Suppose that $P_{1} * P_{2}$ has a compatible order (see the definition after Remark 2.14). Then the free product $A_{1} * A_{2}$ has a natural valuation $\nu_{1} * \nu_{2}: A_{1} * A_{2} \backslash\{0\} \rightarrow P_{1} * P_{2}$.

Example 2.42. Let $W$ be a finite reflection group on the space $V$, recall that its coinvariant algebra $A_{W}=S(V) /<S(V)_{+}^{W}>$ has dimension $|W|$. Also if $W$ is a Weyl group of a complex semisimple group $G$, then $A_{W} \cong H^{*}(G / B)$, where $B$ is the Borel subgroup of $G$. In this case, $A_{W}$ has a canonical Schubert basis $X_{w}, w \in W$.

If $W=<s_{1}, s_{2} \mid s_{1}^{2}=s_{2}^{2}=1,\left(s_{1} s_{2}\right)^{n}=1>$ is dihedral of order $2 n$, then $A_{W}=$ $\mathbb{C}[z, \bar{z}] /<z \bar{z}, z^{n}+\bar{z}^{n}>$ because $W$ acts on $V=\mathbb{C} \cdot z \oplus \mathbb{C} \cdot \bar{z}$ by $s_{1}(z)=\bar{z}, s_{1} s_{2}(z)=$ $\zeta z, s_{1} s_{2}(\bar{z})=\zeta^{-1} \bar{z}$, where $\zeta=e^{\frac{2 \pi i}{n}}$, therefore $z \bar{z}$ and $z^{n}+\bar{z}^{n}$ are basic $W$-invariants. Writing $z=x+i y$ we expect that the Schubert basis is $\left\{R e z^{k}=\frac{z^{k}+\bar{z}^{k}}{2}, \operatorname{Im} z^{k}=\right.$
$\left.\frac{z^{k}-\bar{z}^{k}}{2 i}, k=0, \ldots, n\right\} \backslash\{0\}$. Note that in this case $A_{W} \cong \mathbb{C}[z, \bar{z}] /<z \bar{z}, z^{n}-\bar{z}^{n}>=\mathbb{C} P$, where $P$ is a partial additive monoid on $M_{n} \sqcup_{0, n} M_{n}$ where $M_{n}$ is the partial monoid on $[0, n]$ with $a \circ b$ defined iff $a+b \leq n$, in which case the composition is $a+b$ and $\sqcup_{0, n}$ stands for disjoint union with identified unit 0 and identified $n$. Namely, the first copy of $M_{n}$ consists of $z^{k}, 0 \leq k \leq n$, while the second copy consists of $\bar{z}^{k}, 0 \leq k \leq n$, note that $z^{n+1}=\bar{z}^{n+1}=0$.

Note also that $A_{S_{3}} \cong \mathbb{C}\left[x_{1}, x_{2}, x_{3}\right] /<e_{1}, e_{2}, e_{3}>=\mathbb{C}\left[x_{1}, x_{2}\right] /<x_{1}^{2}+x_{1} x_{2}+$ $x_{2}^{2}, x_{1} x_{2}\left(x_{1}+x_{2}\right)>$ where $e_{1}=x_{1}+x_{2}+x_{3}, e_{2}=x_{1} x_{2}+x_{1} x_{3}+x_{2} x_{3}, e_{3}=x_{1} x_{2} x_{3}$. An $S_{3}$-equivariant isomorphism is given by $z=x_{1}-\zeta x_{2}, \bar{z}=x_{2}-\zeta x_{1}$. The latter algebra has a Schubert basis $\left\{1, x_{1}, x_{1}+x_{2}, x_{1}^{2}, x_{1} x_{2}, x_{1}^{2} x_{2}\right\}$.

Note also that $A_{I_{2}(4)} \cong \mathbb{C}\left[x_{1}, x_{2}\right] /<x_{1}^{2}+x_{2}^{2}, x_{1}^{2} x_{2}^{2}>$ with the action given by $s_{1}\left(x_{1}\right)=x_{2}, s_{2}\left(x_{2}\right)=-x_{2}, s_{2}\left(x_{1}\right)=x_{1}$. An $I_{2}(4)$-equivariant isomorphism is given by $z=x_{1}-i x_{2}, \bar{z}=x_{1}+i x_{2}$. The latter algebra has a Schubert basis $\left\{1, x_{1}, x_{1}+\right.$ $\left.x_{2}, x_{1}^{2}, x_{1} x_{2}, x_{1}^{2} x_{2}+x_{1} x_{2}^{2}, x_{1}^{3}, x_{1}^{3} x_{2}\right\}$.

When $n$ is odd $P$ admits the following ordering fulfilling the strict property (see Definition 2.9):

$$
1 \prec z \prec \cdots \prec z^{(n-1) / 2} \prec \bar{z} \prec \cdots \prec \bar{z}^{n-1} \prec z^{(n+1) / 2} \prec \cdots \prec z^{n-1} \prec z^{n}\left(=\bar{z}^{n}\right)
$$

In contrast, when $n$ is even there is no ordering of $P$ fulfilling the strict property since if $z^{n / 2} \prec \bar{z}^{n / 2}$ (or, respectively $z^{n / 2} \succ \bar{z}^{n / 2}$ ) then $z^{n} \prec \bar{z}^{n}$ (respectively, $z^{n} \succ \bar{z}^{n}$ ). On the other hand, any ordering on $P$ merging the orderings $1 \prec z \prec \cdots \prec z^{n}$ and $1 \prec \bar{z} \prec \cdots \prec \bar{z}^{n}$ satisfies Definition 2.1.

One can consider another representation $A_{W}=\mathbb{C} Q$ where $Q$ is a partial monoid

$$
Q:=\left\{c_{k}:=z^{k}+\bar{z}^{k}: 0 \leq k \leq n\right\} \sqcup\left\{d_{k}:=z^{k}-\bar{z}^{k}: 0<k<n\right\}
$$

with the following composition rules:

- $c_{k} \circ c_{l}=d_{k} \circ d_{l}=c_{k+l}$ iff $k+l \leq n$;
- $c_{k} \circ d_{l}=d_{k+l}$ iff $k+l<n$.

Then $Q$ satisfies Definition 2.1 with an ordering

$$
1 \prec c_{1} \prec d_{1} \prec \cdots \prec c_{n-1} \prec d_{n-1} \prec c_{n}
$$

while $Q$ does not admit an ordering fulfilling the strict property.
Now we construct a common adapted basis of $A_{W}$ for a pair of injective valuations $\nu_{P}: A_{W} \backslash\{0\}(=\mathbb{C} P \backslash\{0\}) \rightarrow P$ and $\nu_{Q}: A_{W} \backslash\{0\}(=\mathbb{C} Q \backslash\{0\}) \rightarrow Q$ (see Theorem 1.11). When $n$ is odd, the common basis consists of

$$
\left\{1, z^{n}\right\} \sqcup\left\{z^{k}, z^{k}+\bar{z}^{k}: 1 \leq k<n / 2\right\} \sqcup\left\{\bar{z}^{k}, z^{k}+\bar{z}^{k}: n / 2<k<n\right\} .
$$

When $n$ is even, fix the following ordering in $P$ :

$$
1 \prec z \prec \bar{z} \prec \cdots \prec z^{k} \prec \bar{z}^{k} \prec \cdots \prec z^{n-1} \prec \bar{z}^{n-1} \prec z^{n}\left(=\bar{z}^{n}\right) .
$$

Then the common basis consists of

$$
\left\{1, z^{n}\right\} \sqcup\left\{z^{k}, z^{k}+\bar{z}^{k}: 1 \leq k<n\right\}
$$

Consider the tautological injective valuation $\nu_{0}: \mathbb{C}[z, \bar{z}] \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{2}$ where $\mathbb{Z}_{\geq 0}^{2}$ is endowed with lex ordering in which $\bar{z} \prec z$. If we apply Theorem $2 . \overline{3}$ to an ideal
$I:=\left\langle z \bar{z}, z^{n}-\bar{z}^{n}\right\rangle \subset \mathbb{C}[z, \bar{z}]$ then we obtain an injective valuation $\nu: A_{W} \backslash\{0\} \rightarrow P^{\prime}$, where $P^{\prime}=\left\{1, z, \ldots, z^{n-1}, \bar{z}, \ldots, \bar{z}^{n}\right\}$. Thus, $P^{\prime}$ contains also $2 n$ elements as $P$, but differs from $P$ as a partial monoid since $z^{n}$ is not defined in $P^{\prime}$ unlike $P$. Nevertheless, $\nu$ coincides with $\nu_{P}$ element-wise.
Example 2.43. Recall that the nil Hecke algebra $\mathcal{H}_{S_{2}}$ of $S_{2}$ is generated by $\alpha, x$ subject to

$$
x^{2}=0, x \alpha+\alpha x=-2
$$

In particular, $s=\alpha x+1=-x \alpha-1$ is an involution.
More generally, let $W=<s_{i}, i \in I \mid s_{i}^{2}=1,\left(s_{i} s_{j}\right)^{m_{i j}}=1>$ be a Coxeter group and $V=\oplus \mathbb{k} \alpha_{i}$ be its reflection representation with a basis $\alpha_{i}$ so that the action is given by

$$
s_{i}\left(\alpha_{j}\right)=\alpha_{j}-a_{i j} \alpha_{i}
$$

where $A$ is the corresponding Cartan matrix.
Then $\mathcal{H}_{W}$ is generated by $x_{i}, \alpha_{i}$ subject to

$$
x_{i}^{2}=0, x_{i} \alpha_{j}=s_{i}\left(\alpha_{j}\right) x_{i}-a_{i j}
$$

and the braid relation

$$
\underbrace{x_{i} x_{j} x_{i} \cdots}_{m_{i j}}=\underbrace{x_{j} x_{i} x_{j} \cdots}_{m_{i j}}
$$

It is easy to see that $\mathbb{k} W$ embeds into $\mathcal{H}_{W}$ via $s_{i} \mapsto \alpha_{i} x_{i}+1$.
This embedding extends to an embeddings $S(V) \rtimes \mathbb{k} W \hookrightarrow \mathcal{H}_{W}$ and $\mathcal{H}_{W} \hookrightarrow$ $\operatorname{Frac}(S(V)) \rtimes \mathbb{k} W$ and

$$
\mathcal{H}_{W} \cong(\mathbb{k} W)_{0} \otimes S(V)
$$

as a vector space where $(\mathbb{k} W)_{0}=<x_{i}>$ is the nil-Coxeter algebra.
$\mathcal{H}_{W}$ admits a quotient $\underline{\mathcal{H}}_{W}$ by the ideal in $S(V)$ generated by $W$-invariants so that

$$
\underline{\mathcal{H}}_{W} \cong(\mathbb{k} W)_{0} \otimes A_{W}
$$

It is proved in [14], [9] that if $|W|=N$, then the algebra $\mathcal{H}_{W}$ is isomorphic to $\operatorname{Mat}_{N}\left(S(V)^{W}\right)$, hence $\underline{\mathcal{H}}_{W} \cong \operatorname{Mat}_{N}(\mathbb{k})$. Therefore, one can apply to the algebra $\mathcal{H}_{W}$ Proposition 2.34 and Example 2.47, and thereby produce an injective valuation on $\mathcal{H}_{W}$.

Example 2.44. [Galois extensions] Let $\mathbb{K}$ be a finite Galois extension of $\mathbb{k}$ and let $G=G a l(\mathbb{K} / \mathbb{k})$. Then the assignments $g \otimes a \mapsto g \circ L_{a}$ for all $g \in G, a \in \mathbb{K}$ define an isomorphism of algebras $\mathbb{K} \rtimes \mathbb{k} G \leadsto \rightarrow \operatorname{End}_{\mathbb{k}}(\mathbb{K})$ (where $L_{a}: \mathbb{K} \rightarrow \mathbb{K}$ is the multiplication by $a \in \mathbb{K}$ ), this again follows from [14], [9]. In particular, any choice of basis $\mathbb{k}$-basis $\left\{b_{1}, \ldots, b_{n}\right\}$ of $\mathbb{K}$ canonically identifies the algebra $\mathbb{K} \rtimes \mathbb{k} G$ with $M a t_{n}(\mathbb{k})$. Similar to Example 2.43 one can produce an injective valuation on $\mathbb{K} \rtimes \mathbb{k} G$.

One can verify the following proposition.
Proposition 2.45. For a commutative $\mathbb{k}$-algebra $A$ let a set $P$ of monomials in elements $a_{1}, \ldots, a_{n} \in A$ form $a \mathbb{k}$-basis in $A$, and $P$ be a partial semigroup (in particular, $P$ is endowed with a linear order). For an element

$$
a=\sum_{u \in P} \alpha_{u} u \in A \backslash\{0\}, \alpha_{u} \in \mathbb{k}^{*}
$$

define $\nu(a):=\max \{u\}$ where max ranges over $u$ from the latter sum. Then $\nu$ : $A \backslash\{0\} \rightarrow P$ is an injective valuation onto $P$. Moreover, $P$ is adapted with respect to $\nu$.

### 2.3. Injective valuations onto coideal partial semigroups via tropical geometry and adapted bases.

Proposition 2.46. i) Let $\nu: A \backslash\{0\} \rightarrow P$ be a valuation in a partial semigroup $P$. For $u \in P$ denote $a \mathbb{k}$-linear space $A_{\preceq u}:=\{a \in A \backslash\{0\}: \nu(a) \preceq u\} \cup\{0\}$ (see Definition 2.24 i), ii)). Then $A_{\preceq u} A_{\preceq v} \subset A_{\preceq u \circ v}$, provided that $u \circ v \in P$. Thus the family $\left\{A_{\preceq u}: u \in P\right\}$ forms a filtration of $A$.
ii) Let $\bar{P}$ be a well-ordered partial semigroup and $\left\{A_{u}: u \in P\right\}$ be a filtration of an algebra $A$. For $a \in A \backslash\{0\}$ setting $\nu(a)$ to be the minimal $u \in P$ such that $a \in A_{u}$, defines a valuation $\nu: A \backslash\{0\} \rightarrow P$.
Example 2.47. Consider a partial semigroup $P_{k}:=\{(i, j): 1 \leq i, j \leq k\}$ where $(i, j) \circ(j, l):=(i, l)$ and $(i, j) \circ(m, l)$ is not defined when $m \neq j$. We define a linear order $\prec$ on $(i, j) \in P_{k}$ being lexicographical with respect to a vector
i) $(j-i,-i)$ or
ii) $(-i, j)$.

In both cases $P_{k}$ endowed with $\prec$ satisfies Definition 2.1. Moreover, $\prec$ satisfies the strict property (see Definition 2.9).

Note that the axiom of the order from Definition 2.1 for $P_{k}$ cannot be deduced from weaker axioms: $c \preceq d$ implies that $a \circ c \preceq a \circ d$ and $c \circ a \preceq d \circ a$, provided that $a \circ c, a \circ d, c \circ a, d \circ a \in P_{k}$.

It would be interesting to clarify whether $P_{k}$ can be represented as a coideal semigroups (with a compatible ordering).

Clearly, $\operatorname{Mat}_{k}(\mathbb{k})=\mathbb{k} P_{k}$ and the tautological valuation $\nu: \operatorname{Mat}_{k}(\mathbb{k}) \backslash\{0\} \rightarrow P_{k}$ is injective and given by $\nu\left(e_{i j}\right)=(i, j)$ from Definition 2.28.

Observe that the valuation of the unit of the algebra $\operatorname{Mat}_{k}(\mathbb{k})$ equals $\nu\left(e_{1,1}+\cdots+\right.$ $\left.e_{k, k}\right)=(1,1)$ in both cases i), ii).

Note that one cannot take a vector ( $i, j$ ) in place of vectors from either i) or ii) since the induced ordering does not satisfy Definition 2.1.

Consider a partial semigroup $P^{\prime}$ with an ordering $\triangleleft$. One can construct (see Lemma 2.12) a partial semigroup $P_{k} \times P^{\prime}$ in which the ordering is given by the lexicographical pair $(\prec, \triangleleft)$, where $\prec$ is one of the described above orderings on $P_{k}$. If $\triangleleft$ fulfills the strict property then the resulting ordering fulfills the strict property as well (cf. Lemma 2.12). Thus, if an algebra $A$ admits an injective valuation onto $P^{\prime}$ then the matrix algebra $M a t_{k}(\mathbb{k}) \otimes A$ admits an injective valuation onto $P_{k} \times P^{\prime}$, see Proposition 2.34.

Denote by $T_{k}$ the partial monoid of paths in the complete directed graph having $k$ vertices and loops (cf. Example 2.17 iii$)$ ). So, $T_{k}$ is generated by the set $\left\{z_{i, j}: 1 \leq\right.$ $i, j \leq k\}$. Following the construction in the proof of Theorem 2.11 we produce an epimorphism $f: T_{k} \rightarrow P_{k}$ such that $f\left(z_{i, j}\right)=(i, j)$, thus $f\left(z_{i_{1}, i_{2}} \circ z_{i_{2}, i_{3}} \circ \cdots \circ z_{i_{s-1}, i_{s}}\right)=$ $\left(i_{1}, i_{s}\right)$. Denote by $\prec$ one of the introduced above orders on $P_{k}$ (say, i) or ii)). Now define an order $\triangleleft$ on $T_{k}$ as follows. We say that $z_{i_{1}, i_{2}} \circ \cdots \circ z_{i_{s-1}, i_{s}} \triangleleft z_{j_{1}, j_{2}} \circ \cdots \circ z_{j_{l-1}, j_{l}}$ if either

- $f\left(z_{i_{1}, i_{2}} \circ \cdots \circ z_{i_{s-1}, i_{s}}\right) \prec f\left(z_{j_{1}, j_{2}} \circ \cdots \circ z_{j_{l-1}, j_{l}}\right)$, either
- $f\left(z_{i_{1}, i_{2}} \circ \cdots \circ z_{i_{s-1}, i_{s}}\right)=f\left(z_{j_{1}, j_{2}} \circ \cdots \circ z_{j_{l-1}, j_{l}}\right)$ and $s<l$, or
- $f\left(z_{i_{1}, i_{2}} \circ \cdots \circ z_{i_{s-1}, i_{s}}\right)=f\left(z_{j_{1}, j_{2}} \circ \cdots \circ z_{j_{l-1}, j_{l}}\right), s=l$ and the vector $\left(i_{1}, \ldots, i_{s}\right)$ is less than $\left(j_{1}, \ldots, j_{l}\right)$ in the lexicographical order (in which, say, $1<\cdots<n$ ). One can verify that $f$ satisfies Proposition 2.8.

Note that $f$ induces a natural epimorphism of semigroup algebras $\mathbb{k} T_{k} \rightarrow \mathbb{k} P_{k}=$ $M a t_{k}(\mathbb{k})$.

Consider $P_{\infty}:=\{(i, j): 1 \leq i, j<\infty\}$, this is naturally an ordered partial semigroup of infinite matrices, and inclusions $P_{k} \subset P_{\infty}$ are ordered, moreover $P_{\infty}$ is their injective limit. Despite $\prec$ is not a well ordering, the semigroup algebra $\mathbb{k} P_{\infty}$ is the algebra $M a t_{\infty}(\mathbb{k})$ of infinite matrices with finite support, and the injective valuation onto $P_{\infty}$ provides the tautological valuation $M a t_{\infty}(\mathbb{k}) \backslash\{0\} \rightarrow P_{\infty}$ with an adapted basis $\left\{e_{i, j}: 1 \leq i, j<\infty\right\}$.

Denote by $F:=\mathbb{k}\left\langle\left\{e_{i, j}: 1 \leq i, j \leq k\right\}\right\rangle$ the free algebra with the natural injective valuation $\nu_{0}$ onto the free semigroup $P:=<(i, j), 1 \leq i, j \leq k>$. We assume that $P$ is equipped with the well ordering produced in Lemma 2.13. Denote by

$$
I:=\left\langle\left\{e_{i, j} e_{p, q}: j \neq p, 1 \leq i, j, p, q \leq k\right\} \cup\left\{e_{i, l}-e_{i, j} e_{j, l}: 1 \leq i, j, l \leq k\right\}\right\rangle
$$

an ideal in $F$. When we apply Theorem 2.38 we obtain an injective valuation $\nu$ : $(F / I) \backslash\{0\}=\operatorname{Mat}_{k}(\mathbb{k}) \backslash\{0\} \rightarrow P \backslash \nu_{0}(I \backslash\{0\})$. Observe that $P \backslash \nu_{0}(I \backslash\{0\})$ is a partial semigroup consisting of $k^{2}$ elements $\{(i, j): 1 \leq i, j \leq k\}$ such that no composition of them is defined since $\nu_{0}\left(e_{i, j} e_{p, q}\right)=(i, j) \circ(p, q)$ and $\nu_{0}\left(e_{i, l}-e_{i, j} e_{j, l}\right)=(i, j) \circ(j, l)$. Thus, $P \backslash \nu_{0}(I \backslash\{0\})$ differs from $P_{k}$.

Problem 2.48. Describe all possible orderings on $P_{k}$.
Example 2.49. Let us apply the construction from Proposition 2.35 to the symmetric group $P:=S_{k}$ and $Q:=P_{k}$ taking as $\nu$ the valuation from Example 2.47 ii . We consider the standard representation of $S_{k}$ in $G L_{k}$. This provides a partial homomorphism $f: S_{k} \rightarrow P_{k}$. Then following Remark 2.5 one obtains a partial semigroup $R_{k}:=\left(S_{k}\right)_{S_{f}}$ and a homomorphism from $R_{k}$ to $P_{k}$. One can explicitly describe $R_{k}$ as follows. Two permutations $p, q \in P_{k}$ are composable in $R_{k}$ iff $p(1)=1$ taking into account that $\nu(p)=\left(1, p^{-1}(1)\right)$.

In case of a coideal partial semigroup $P$ the following construction allows one to obtain a stronger property of filtrations.
Definition 2.50. For an algebra $A$ we say that $\nu: A \backslash\{0\} \rightarrow P$ is a valuation onto a coideal partial semigroup $P \subset M$ if in addition to Definition 2.24 for any elements $a, b \in A \backslash\{0\}$ an inequality $\nu(a b) \preceq \nu(a) \circ \nu(b) \in M$ holds, provided that $a b \neq 0$.

Recall (cf. the definition prior to Remark 3.51) that an order $\prec$ on a semigroup $M$ is archimedian if for any $u \in M$ the set all elements of $M$ less than $u$ is finite.

Proposition 2.51. Let $\nu: A \backslash\{0\} \rightarrow P$ be a valuation on $a \mathbb{k}$-algebra onto a coideal partial semigroup $P \subset M$. We assume that $M$ is endowed with an archimedian order $\prec$. In this case for the filtration: $A_{\preceq u}$ it holds

$$
A_{\preceq u} A_{\preceq v} \subset A_{\max \{P \ni w \preceq u \circ v\}}
$$

owing to Definition 2.50. Observe that the latter maximum exists since $\prec$ is archimedian.

One can also denote $\mathcal{A}_{u}:=A_{\preceq u} / A_{\prec u}$ and the natural projection $p_{u}: A_{\preceq u} \rightarrow \mathcal{A}_{u}$. The following remark extends Remark 3.52 to coideal partial semigroups.

Remark 2.52. Let $A$ be a (not necessary commutative) $\mathbb{k}$-algebra and $\nu: A \rightarrow P \subset$ $M$ be an injective valuation onto a coideal partial semigroup $P$. We assume that $M$ is endowed with a linear order $\prec$ and a function $f: M \rightarrow \mathbb{Z}_{\geq 0}$ such that $c_{1} \prec c_{2}$ implies that $f\left(c_{1}\right) \leq f\left(c_{2}\right)$, and $f\left(c_{1}+c_{2}\right) \leq f\left(c_{1}\right)+f\left(c_{2}\right)$ for $c_{1}, c_{2} \in M$, moreover the set $C_{n}:=\{c \in M: f(c) \leq n\}$ is finite for any $n \in \mathbb{Z}_{\geq 0}$. Note that the latter implies that the order $\prec$ is archimedian. Then the $\mathbb{k}$-subspaces $A_{n}:=\{a \in A \backslash\{0\}$ : $f(\nu(a)) \leq n\} \cup\{0\}, n \in \mathbb{Z}_{\geq 0}$ provide a filtration of $A$ such that $\operatorname{dim}\left(A_{n}\right)=\left|C_{n}\right|$.

Proposition 2.53. When $\nu: A \backslash\{0\} \rightarrow P$ is a valuation onto a coideal partial semigroup $P \subset M$, one can define a graded associated algebra $\mathcal{A}:=\bigoplus_{u \in P} \mathcal{A}_{u}$ as follows. Let $u, v \in P, c \in \mathcal{A}_{u}, d \in \mathcal{A}_{v}$. If $u+v \in P, a \in A_{\preceq u}, b \in A_{\preceq v}$ such that $p_{u}(a)=c, p_{v}(b)=d$ then we define the product $c d:=p_{u+v}(\bar{a} b) \in \mathcal{A}_{u+v}$. It holds $c d \neq 0$. Otherwise, if $u+v \notin P$ then we define $c d:=0$.

Proof. The correctness of the definition of the product $c d$ and that $c d \neq 0$ in case when $u+v \in P$ follows from Definitions 2.24 iii), 2.50. The associativity of $\mathcal{A}$ can be verified taking into account Definition 2.1.

The following proposition generalizes Theorem 3.21 to partial semigroups. We utilize the notations from Theorem 3.21.

Proposition 2.54. Let $A=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] / I$ be an algebra and $I_{\text {trop }} \subset I$ be $a(n-d)$ dimensional subideal satisfying the following properties. Assume that there exists a common $(n-d)$-dimensional rational plane $H \subset \mathbb{R}^{n}$ of the tropical variety $\operatorname{Trop}\left(I_{\text {trop }}\right)$ such that $H$ is prop and $I_{\text {trop }}$ is saturated with respect to $H$. Then there exists a coideal partial monoid $P \subset \mathbb{Z}_{\geq 0}^{n} / H_{\mathbb{Z}}$ and an injective valuation $\nu: A \backslash\{0\} \rightarrow P$. A linear order on $P$ is induced by a linear order on $\mathbb{Z}_{\geq 0}^{n} / H_{\mathbb{Z}}$ which in its turn, is determined by a hyperplane from $\operatorname{ETrop}\left(I_{\text {trop }}\right)$.

Proof. Apply Theorem 3.21 to the ideal $I_{\text {trop }}$ and obtain an injective valuation

$$
\nu_{0}:\left(\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] / I_{\text {trop }}\right) \rightarrow \mathbb{Z}_{\geq 0}^{n} / H_{\mathbb{Z}} .
$$

Then apply Theorem 2.38 to $\nu_{0}$ which results in $\nu$. Observe that $P=\left(\mathbb{Z}_{\geq 0}^{n} / H_{\mathbb{Z}}\right) \backslash$ $\nu_{0}\left(\left(I / I_{\text {trop }}\right) \backslash\{0\}\right)$.

The following proposition is inverse to Proposition 2.45 and extends Theorem 3.39 to valuations onto coideal partial monoids.

Proposition 2.55. Let $A$ be a commutative $\mathbb{k}$-algebra and $\nu: A \backslash\{0\} \rightarrow P \subset$ $M$ be an injective valuation onto a finitely-generated coideal partial commutative monoid $P$, where the monoid $M$ is endowed with a linear well-ordering $\prec . ~ A s-$ sume that $c_{1}, \ldots, c_{s} \in P$ is a family of generators of $P$. Take $a_{1}, \ldots, a_{s} \in A$ such that $\nu\left(a_{i}\right)=c_{i}, 1 \leq i \leq s$. Then $A=\mathbb{k}\left[a_{1}, \ldots, a_{s}\right] / I$ for a suitable ideal $I \subset \mathbb{k}\left[a_{1}, \ldots, a_{s}\right]$. Consider a linear ordering $\triangleleft$ on monomials in $a_{1}, \ldots, a_{s}$ such that $a_{1}^{i_{1}} \cdots a_{s}^{i_{s}} \triangleleft a_{1}^{j_{1}} \cdots a_{s}^{j_{s}}$ if either $M \ni i_{1} \nu\left(a_{1}\right) \circ \cdots \circ i_{s} \nu\left(a_{s}\right) \prec j_{1} \nu\left(a_{1}\right) \circ \cdots \circ j_{s} \nu\left(a_{s}\right) \in M$
or $i_{1} \nu\left(a_{1}\right) \circ \cdots \circ i_{s} \nu\left(a_{s}\right)=j_{1} \nu\left(a_{1}\right) \circ \cdots \circ j_{s} \nu\left(a_{s}\right)$ and the monomial $a_{1}^{i_{1}} \cdots a_{s}^{i_{s}}$ is less than $a_{1}^{j_{1}} \cdots a_{s}^{j_{s}}$ in deglex. Then the monomials in $a_{1}, \ldots, a_{s}$ belonging to the complement of the monomials ideal $J$ of leading monomials of the Gröbner basis of I (relatively to $\triangleleft$ ), constitute an adapted basis of $A$ with respect to $\nu$.

Proof. For a monomial $a:=a_{1}^{i_{1}} \cdots a_{s}^{i_{s}}$ such that $i_{1} \nu\left(a_{1}\right) \circ \cdots \circ i_{s} \nu\left(a_{s}\right) \in M \backslash P$ it holds $\nu(a) \prec i_{1} \nu\left(a_{1}\right) \circ \cdots \circ i_{s} \nu\left(a_{s}\right)$ due to Definitions 2.24 iii), 2.50. Therefore $J$ contains all monomials $a_{1}^{i_{1}} \cdots a_{s}^{i_{s}}$ for which $i_{1} \nu\left(a_{1}\right) \circ \cdots \circ i_{s} \nu\left(a_{s}\right) \in M \backslash P$.

On the other hand, among all monomials $a_{1}^{j_{1}} \cdots a_{s}^{j_{s}}$ with a fixed valuation $v:=$ $j_{1} \nu\left(a_{1}\right) \circ \cdots \circ j_{s} \nu\left(a_{s}\right) \in P$ all these monomials belong to $J$ except of a single one being minimal in deglex since $\nu$ is injective (cf. the proof of Theorem 3.39 ii) and remark 3.46). Denote the latter monomial by $a_{v}$. Then the set $\left\{a_{v}: v \in P\right\}$ constitutes an adapted basis of $A$ with respect to $\nu$.

For a commutative partial monoid $P$ we define its rank $r k(P)$ to be the maximal number of elements $c_{1}, \ldots, c_{r} \in P$ such that all the elements $i_{1} c_{1} \circ \cdots \circ i_{r} c_{r} \in$ $P, i_{1}, \ldots, i_{r} \geq 0$ are pairwise distinct. In this case we call elements $c_{1}, \ldots, c_{r}$ independent. The following corollary extends Corollary 3.47 to coideal partial monoids.
Corollary 2.56. Let $A$ be a commutative $\mathbb{k}$-algebra and $\nu: A \backslash\{0\} \rightarrow P \subset M$ be an injective valuation onto a finitely-generated coideal partial commutative monoid, where monoid $M$ is endowed with a linear well-ordering. Then $\operatorname{dim}(A)=r k(P)$.

Proof. Denote $r:=r k(P)$ and let $c_{1}, \ldots, c_{r} \in P$ be independent. Take $a_{1}, \ldots, a_{r} \in$ $A$ for which $\nu\left(a_{i}\right)=c_{i}, 1 \leq i \leq r$. Then monomials $a_{1}^{i_{1}} \cdots a_{r}^{i_{r}}, i_{1}, \ldots, i_{r} \geq 0$ are $\mathbb{k}$ linearly independent, hence $d:=\operatorname{dim}(A) \geq r$.

Conversely, among monomials belonging to the complement of $J$ (see Proposition 2.55) there are monomials $b_{1}, \ldots, b_{d}$ such that all monomials in $b_{1}, \ldots, b_{d}$ belong to the complement of $J$ taking into account the property of the Gröbner basis (cf. the proof of Corollary 3.47). Then $\nu\left(b_{1}\right), \ldots, \nu\left(b_{d}\right) \in P$ are independent due to Proposition 2.55, hence $r \geq d$.

## 3. InJective valuations on domains

In this section we consider (more familiar) valuations of algebras in semigroups (rather than in partial semigroups as in section 2).
3.1. Valuations of domains into semigroups. Let $C$ be a semigroup endowed with a linear ordering < compatible with the semigroup operation + (not necessary commutative). For a $\mathbb{k}$-algebra $A$ its valuation we define as a mapping $\nu: A \backslash\{0\} \rightarrow C$ such that

$$
\begin{gathered}
\nu(\alpha a)=\nu(a), \nu\left(a+a_{0}\right) \leq \max \left\{\nu(a), \nu\left(a_{0}\right)\right\}, \nu\left(a a_{0}\right)=\nu(a)+\nu\left(a_{0}\right), \\
a, a_{0}, a+a_{0} \in A \backslash\{0\}, \alpha \in \mathbb{k}^{*} .
\end{gathered}
$$

Denote $C_{\nu}:=\nu(A \backslash\{0\})$. An example is provided by a semigroup algebra $\mathbb{k} C$ with a valuation (see Proposition 2.45)

$$
\nu\left(\alpha_{1} c_{1}+\cdots+\alpha_{k} c_{k}\right):=\max \left\{c_{1}, \ldots, c_{k}\right\}, c_{1}, \ldots, c_{k} \in C, \alpha_{1}, \ldots, \alpha_{k} \in \mathbb{R}^{*}
$$

Let $C:=\left\langle c_{1}, \ldots, c_{n}\right\rangle$ be a free semigroup generated by $c_{1}, \ldots, c_{n}$. One can define a linear ordering on $C$ as follows (see Lemma 2.13): $c_{i_{1}} \cdots c_{i_{m}}<c_{j_{1}} \cdots c_{j_{s}}, 1 \leq$
$i_{1}, \ldots, i_{m}, j_{1}, \ldots, j_{s} \leq n$ iff either $m<s$ or $m=s$ and the vector $\left(i_{1}, \ldots, i_{m}\right)$ is less than the vector $\left(j_{1}, \ldots, j_{m}\right)$ with respect to lex. Note that this provides a wellordering on $C$.

We say that a valuation $\nu$ is injective if there exists a $\mathbb{k}$-basis $\left\{a_{c}: c \in C_{\nu}\right\}$ of $A$, where $\nu\left(a_{c}\right)=c, c \in C_{\nu}$ (such a basis we call adapted with respect to $\nu$ ). Then $\nu$ has one-dimensional leaves ([18]). Observe that

$$
a_{c_{1}} a_{c_{2}}=\alpha\left(c_{1}, c_{2}\right) a_{c_{1}+c_{2}}+\sum_{c<c_{1}+c_{2}} \alpha_{c} c
$$

for suitable $\alpha\left(c_{1}, c_{2}\right) \in \mathbb{k}^{*}, \alpha_{c} \in \mathbb{k}$. Observe that due to the associativity in $A$ the following relations are fulfilled:

$$
\alpha\left(c_{1}, c_{2}\right) \alpha\left(c_{1}+c_{2}, c_{3}\right)=\alpha\left(c_{2}, c_{3}\right) \alpha\left(c_{1}, c_{2}+c_{3}\right)
$$

For example, $\{c \in C\}$ is an adapted basis of $\mathbb{k} C$ with respect to the tautological valuation (see Definition 2.28).

More generally, for an arbitrary valuation $\nu$ on $A$ we say that $\left\{a_{i} \in A\right\}_{i}$ is an adapted basis [20] (with respect to $\nu$ ) if for any $a=\sum_{j} \alpha_{j} a_{j} \in A \backslash\{0\}, \alpha_{j} \in \mathbb{k}^{*}$ it holds $\nu(a)=\max _{j}\left\{\nu\left(a_{j}\right)\right\}$. In particular, if $A$ has a Khovanskii basis [20] then one can produce relying on it an adapted basis.

Theorem 3.1. Let $A$ be $a \mathbb{k}$-algebra, $\nu: A \backslash\{0\} \rightarrow C_{\nu}$ be a mapping onto a linearly ordered semigroup $C_{\nu}$ such that $\nu(\alpha a)=\nu(a), \nu\left(a+a_{0}\right) \leq \max \left\{\nu(a), \nu\left(a_{0}\right)\right\}, a, a_{0}, a+$ $a_{0} \in A \backslash\{0\}, \alpha \in \mathbb{R}^{*}$. Denote

$$
A_{c}:=\{a \in A \backslash\{0\}: \nu(a) \leq c\} \cup\{0\}
$$

and $G_{c}:=A_{c} / A_{<c}$. Consider an associated graded algebra $G:=\bigoplus_{c \in C_{\nu}} G_{c}$.
i) $\nu$ is a valuation iff $G$ is a domain. In this case $\nu_{0}(g)=c$ for $g \in G_{c}^{*}$ defines a valuation on $G^{*}$.
ii) Let $C_{\nu}$ be well-ordered and $\nu$ be a valuation. Then $\nu$ is an injective valuation iff $\operatorname{dim}_{\mathbb{k}}\left(G_{c}\right)=1$ for any $c \in C_{\nu}$.
iii) Let $\nu$ be an injective valuation and $\mathcal{C}=\left\{c_{i}\right\} \subset C_{\nu}$ be a set of generators of a well-ordered $C_{\nu}$. Then $A$ has an adapted basis of the form $\left\{a_{i_{1}} \cdots a_{i_{k}}:\left(i_{1}, \ldots, i_{k}\right) \in\right.$ $S\}$ for an appropriate set $S$, where $\nu\left(a_{i}\right)=c_{i} \in \mathcal{C}$ and $C_{\nu}=\left\{c_{i_{1}} \cdots c_{i_{k}}:\left(i_{1}, \ldots, i_{k}\right) \in\right.$ $S\}$. Note that $\mathcal{C}$ can be infinite.
iv) For a valuation $\nu$ on $A$ and a well-ordered $C$ there is an adapted basis of $A$.

Proof. i) Let $\nu$ be a valuation. Denote by $p_{c}: A_{c} \rightarrow G_{c}$ the projection. For any $g \in G_{c} \backslash\{0\}, g_{0} \in G_{c_{0}} \backslash\{0\}$ take $a \in p_{c}^{-1}(g), a_{0} \in p_{c_{0}}^{-1}\left(g_{0}\right)$. Then $\nu(a)=c, \nu\left(a_{0}\right)=c_{0}$. Since $\nu\left(a a_{0}\right)=\nu(a)+\nu\left(a_{0}\right)$, it holds $a a_{0} \notin A_{<\left(c+c_{0}\right)}$. Therefore $g g_{0}=p_{c+c_{0}}\left(a a_{0}\right) \neq 0$, i.e. $G$ is a domain.

In a similar manner one can verify the inverse statement.
ii) Let $\operatorname{dim}\left(G_{c}\right)=1$ for any $c \in C_{\nu}$. For each $c \in C_{\nu}$ pick $a_{c} \in A_{c}$ such that $\nu\left(a_{c}\right)=c$. We claim that the elements $\left\{a_{c}: c \in C_{\nu}\right\}$ constitute an adapted basis of $A$ with respect to $\nu$ (this implies the injectivity of $\nu$ ).

Clearly, the elements $\left\{a_{c}: c \in C_{\nu}\right\}$ are linearly independent. For any element $a \in A \backslash\{0\}$ with $\nu(a)=c_{0}$ there exists (and unique) $\alpha \in \mathbb{k}^{*}$ such that $\nu\left(a-\alpha a_{c_{0}}\right)<c_{0}$ since $\operatorname{dim}\left(G_{c_{0}}\right)=1$. Applying a similar argument to $a-\alpha a_{c_{0}}$ (in place of $a$ ), unless
$a-\alpha a_{c_{0}}=0$, and continuing in this way, we arrive eventually at a decomposition of $a$ in a linear combination of the elements from $\left\{a_{c}: c \in C_{\nu}\right\}$, taking into account that $C_{\nu}$ is well-ordered. The claim is proved.

In a similar manner one can verify the inverse statement.
iii) follows from ii).
iv) Choose a basis $\left\{b_{c, i}: i \in I_{c}\right\}$ of $G_{c}$ and elements $a_{c, i} \in A_{c}$ such that $p_{c}\left(a_{c, i}\right)=$ $b_{c, i}$. We claim that $\mathcal{A}:=\left\{a_{c, i}: c \in C, i \in I_{c}\right\}$ constitute an adapted basis of $A$.

Indeed, consider $a=\sum_{c, j} \alpha_{c, j} a_{c, j} \in A \backslash\{0\}, \alpha_{c, j} \in \mathbb{k}^{*}$. Denote a subsum $e:=$ $\sum_{l} \alpha_{c_{0}, l} a_{c_{0}, l}$ which ranges over all $l$ such that $c_{0}:=\nu\left(a_{c_{0}, l}\right)=\max _{c, j}\left\{\nu\left(a_{c, j}\right)\right\}$. Then $\nu(e)=c_{0}$ owing to the choice of $a_{c, i}$. Hence $\nu(a)=c_{0}$. In particular, the elements of $\mathcal{A}$ are independent.

Similar to the proof above of ii) one can express any element of $A$ as a linear combination of elements from $\mathcal{A}$ which proves the claim.
Remark 3.2. If $\nu$ is an injective valuation on an algebra $A$ over an radically closed field onto a well-ordered finitely-generated monoid $C$ then one can treat $A$ as a deformation of $\mathbb{k} C$ (see Proposition 3.53 below).

Now we describe a construction which starting with a valuation on an algebra, produces a valuation on its quotient (cf. Theorem 2.38). Let $A=\bigoplus_{c \in C} A_{c}$ be a domain over a field $\mathbb{k}$ graded by an ordered monoid $C$. For $a \in A \backslash\{0\}$ denote by $l t(a) \in A_{c_{0}}$ the leading term of $a$ for suitable $c_{0} \in C$, i.e. $a-l t(a) \in \bigoplus_{c<c_{0}} A_{c}$. Note that $\nu_{0}(a):=c_{0}$ defines a valuation on $A \backslash\{0\}$ (not necessary injective). For an ideal $J \subset A$ denote by $l t(J) \subset A$ the homogeneous ideal generated by $l t(f)$ for $f \in J$.

Theorem 3.3. Let $A=\bigoplus_{c \in C} A_{c}$ be a domain over a field $\mathbb{k}$ graded by an ordered monoid $C$. For an ideal $J \subset A$ one can define a mapping $\nu$ on the algebra $(A / J) \backslash\{0\}$ filtered by $C$ as follows. For $g \in(A / J) \backslash\{0\}$ denote

$$
\nu(g):=\min \left\{\nu_{0}(g+J)\right\} \in C .
$$

i) $\nu\left(\alpha g_{1}\right)=\nu\left(g_{1}\right), \nu\left(g_{1}+g_{2}\right) \leq \max \left\{\nu\left(g_{1}\right), \nu\left(g_{2}\right)\right\}$;
ii) $\nu\left(g_{1} g_{2}\right)=\nu\left(g_{1}\right)+\nu\left(g_{2}\right)$ for any $g_{1}, g_{2} \in(A / J) \backslash\{0\}$ iff the ideal $l t(J) \subset A$ is prime;
iii) $\nu$ is injective iff $C$ is well-ordered and $\operatorname{dim}_{\mathbb{k}}\left(A_{c} /\left(l t(J) \cap A_{c}\right)\right)=1$ for each $c \in C$.

Thus, when the conditions in ii), iii) are satisfied, $\nu$ is an injective well-ordered valuation of $(A / J) \backslash\{0\}$.

Proof. i) is straight-forward.
One can verify the following lemma.
Lemma 3.4. Assume that for $g \in(A / J) \backslash\{0\}$ it holds $\nu(g)=\nu_{0}\left(g+f_{0}\right)=c_{0}, f_{0} \in J$. Then $l t\left(g+f_{0}\right) \notin l t(J)$. In addition, for any $c>c_{0}$ and $l t(g+f) \in A_{c}, f \in J$ it holds $l t(g+f) \in l t(J)$.
ii) Let $l t(J)$ be prime, and $\nu\left(g_{1}\right)=\nu_{0}\left(g_{1}+f_{1}\right), \nu\left(g_{2}\right)=\nu_{0}\left(g_{2}+f_{2}\right)$ for appropriate $f_{1}, f_{2} \in J$. It holds $l t\left(g_{1}+f_{1}\right) l t\left(g_{2}+f_{2}\right) \notin l t(J)$ since $l t(J)$ is prime and employing Lemma 3.4. Therefore $\nu\left(g_{1} g_{2}\right)=\nu\left(g_{1}\right)+\nu\left(g_{2}\right)$ again due to Lemma 3.4.

One can prove ii) in the opposite direction in a similar way.
iii) Let $\operatorname{dim}\left(A_{c} /\left(l t(J) \cap A_{c}\right)\right)=1$ for any $c \in C$. Then for every $g_{1}, g_{2} \in(A / J) \backslash\{0\}$ such that $\nu\left(g_{1}\right)=\nu\left(g_{2}\right)=c, l t\left(g_{1}+f_{1}\right), l t\left(g_{2}+f_{2}\right) \in A_{c}$, we have $l t\left(g_{1}+f_{1}\right)-$ $\alpha \cdot l t\left(g_{2}+f_{2}\right) \in l t(J)$ for a suitable $\alpha \in \mathbb{k}$. Hence there exists $f \in J, l t(f)=$ $l t\left(g_{1}+f_{1}\right)-\alpha \cdot l t\left(g_{2}+f_{2}\right)$ for which $\nu_{0}\left(\left(g_{1}+f_{1}\right)-\alpha \cdot\left(g_{2}+f_{2}\right)-f\right)<c$ that establishes the injectivity of $\nu$, taking into account that $C$ is well-ordered (cf. the proof of Theorem 3.1 ii ).

One can prove iii) in the opposite direction in a similar way.
Remark 3.5. Assume that $A=\mathbb{k}\left\langle x_{1}, \ldots, x_{n}\right\rangle / J$ for a prime ideal $J \subset \mathbb{k}\left\langle x_{1}, \ldots, x_{n}\right\rangle$ such that $x_{i} \notin J, 1 \leq i \leq n$, and $\nu$ is a valuation (not necessary injective) on $A \backslash\{0\}$. Then one can define $\nu_{0}\left(x_{i}\right):=\nu\left(x_{i}\right), 1 \leq i \leq n$ which provides a grading on $\mathbb{k}\left\langle x_{1}, \ldots, x_{n}\right\rangle$. Now if we apply the construction from Theorem 3.3 to the latter graded algebra $\mathbb{k}\left\langle x_{1}, \ldots, x_{n}\right\rangle$ and to the ideal $J$, we arrive at the initial valuation $\nu$ on $A \backslash\{0\}$.

Let $\nu: A \backslash\{0\} \rightarrow C$ be a well-ordered injective valuation of an algebra $A$.
Given an ideal $J$ in $A$, we say that a generating set $B$ of $J$ is a $\nu$-Gröbner basis of $J$ if $(A b A, b \in B)$ is a $\nu$-ensemble, that is,

$$
\nu(J \backslash\{0\})=\bigcup_{b \in B} C_{\nu} \cdot \nu(b) \cdot C_{\nu}
$$

### 3.2. Examples of injective valuations on algebras of dimension 2.

Example 3.6. Consider the following injective valuation on the ring $A:=\mathbb{k}[x, y] \backslash$ $\{0\}$. One can uniquely represent an arbitrary polynomial $f \in A$ as $f=g\left(y, y^{3}-\right.$ $\left.x^{2}\right)+x h\left(y, y^{3}-x^{2}\right)$ for some polynomials $g, h$. Define $\nu$ and its adapted basis as follows:

$$
\nu\left(y^{k}\left(y^{3}-x^{2}\right)^{l}\right):=(2 k, l), \nu\left(x y^{k}\left(y^{3}-x^{2}\right)^{l}\right):=(2 k+3, l), k, l \geq 0 .
$$

Therefore, $\nu(f)=\max \left\{\nu\left(g\left(y, y^{3}-x^{2}\right)\right), \nu\left(x h\left(y, y^{3}-x^{2}\right)\right)\right\}$. The valuation monoid is $\left\{(u, v) \in \mathbb{Z}_{\geq 0}^{2}: u \neq 1\right\}$. We consider its linear ordering with respect to deglex, say, with $u$ being higher than $v$. Thus, $\nu$ is not induced by a minimal generating set of $\mathbb{k}[x, y]$.

One can straightforwardly verify the following proposition.
Proposition 3.7. Let $\nu$ be an injective valuation $\nu$ on an algebra $A \backslash\{0\}$ with $a$ finitely generated valuation in a well-ordered semigroup C. Consider a partition of $C$ according to [21]. Namely, each element of the partition has a form $c+D$ where $c \in C$ and a semigroup $D \subseteq C$ is isomorphic to $\mathbb{Z}_{\geq 0}^{k}$ for some $k$ with basis vectors $c_{1}, \ldots, c_{k} \in D$. Let $a, a_{1}, \ldots, a_{k} \in A$ be such that $\bar{\nu}(a)=c, \nu\left(a_{1}\right)=c_{1}, \ldots, \nu\left(a_{k}\right)=$ $c_{k}$. Then the elements aa $a_{1}^{i_{1}} \cdots a_{k}^{i_{k}}, i_{1}, \ldots, i_{k} \in \mathbb{Z}_{\geq 0}$ for all the elements of the partition of $C$ form an adapted basis of $A$ with respect to $\nu$.

Let $A$ be a finitely generated $\mathbb{k}$-algebra of dimension $d$. Let $\nu: A \backslash\{0\} \rightarrow C$ be a valuation on $A \backslash\{0\}$ and $C$ be a well-ordered semigroup of a rank $r$.

For each $c \in C$ pick an arbitrary element $a_{c} \in A$ such that $\nu\left(a_{c}\right)=c$. Then the elements $\left\{a_{c}: c \in C\right\}$ are $\mathbb{k}$-linearly independent. Therefore, $r \leq d$. Indeed, otherwise take linearly independent $c_{0}, \ldots, c_{d} \in C$ (in Grothéndieck group of $C$ ), then
all the monomials in the elements $a_{c_{0}}, \ldots, a_{c_{d}}$ are linearly independent. The obtained contradiction justifies the inequality $r \leq d$. Note that for the latter inequality we did not use the injectivity of $\nu$.

Obviously, one can yield a well-ordered injective valuation on $\mathbb{k}\left[x_{1}, \ldots, x_{d}\right] \backslash\{0\}$ (in a unique manner) by means of assigning linearly independent vectors $\nu\left(x_{1}\right), \ldots, \nu\left(x_{d}\right) \in$ $\mathbb{Z}_{\geq 0}^{d}$ and defining a well-ordering on $\mathbb{Z}_{\geq 0}^{d}$.

Below we produce a different family of well-ordered injective valuations of rank 2 on the polynomial ring $\mathbb{k}[x, y] \backslash\{0\}$ generalizing Example 3.6 in which $\nu(x), \nu(y)$ are linearly dependent.
Proposition 3.8. Let $f=x^{n}+\sum_{0 \leq i<n} f_{i} x^{i}, f_{i} \in \mathbb{k}[y]$ be a polynomial such that $m:=\operatorname{deg}_{y}\left(f_{0}\right)$ is relatively prime with $n$, and $m i+\operatorname{ndeg}_{y}\left(f_{i}\right) \leq m n$ for $0 \leq i \leq n$. Then there is a well-ordered injective valuation $\nu:(\mathbb{k}[x, y] \backslash\{0\}) \rightarrow \mathbb{Z}_{\geq 0}^{2}$ defined as follows on its adapted basis:

$$
\begin{equation*}
\nu\left(x^{i} y^{k} f^{l}\right):=(m i+n k, l), 0 \leq i<n, 0 \leq k, l . \tag{3.1}
\end{equation*}
$$

Proof. We observe that $\mathbb{k}[x, y]$ is a finite $\mathbb{k}[f, y]$-module with a basis $1, x, \ldots, x^{n-1}$ with an irreducible monic polynomial $f(x, y)-f$ defining $x^{n}$. This justifies that in (3.1) we have a basis of $\mathbb{k}[x, y]$. The right-hand sides of (3.1) are pairwise distinct due to relative primality of $m, n$.

To verify the multiplicativity of $\nu$ note that $m i+n d e g_{y}\left(f_{i}\right)<m n$ for $0<i<n$, hence

$$
\nu\left(x^{j}\right)+\nu\left(x^{n-j}\right)=(m j+m(n-j), 0)=\nu\left(y^{m}\right)=\nu\left(\sum_{0 \leq i<n} f_{i}(y) x^{i}-f\right)=\nu\left(x^{n}\right) .
$$

One can extend this construction.
Corollary 3.9. Let a ring $B$ be a finite $A$-module with an integral basis $1, x, \ldots, x^{n-1}$. Let $\nu$ be a well-ordered injective valuation on $A \backslash\{0\}$ with a valuation semigroup $C \subseteq \mathbb{Z}_{\geq 0}^{d}$. Assume that $x^{n}$ satisfies a polynomial $f=x^{n}+\sum_{0 \leq i<n} f_{i} x^{i}$ where $f_{i} \in$ $A, 0 \leq i<n$ such that

$$
\frac{i \nu\left(f_{0}\right)}{n} \notin G(C), 0<i<n, n \nu\left(f_{i}\right)<(n-i) \nu\left(f_{0}\right), 0 \leq i \leq n
$$

where $G(C)$ denotes Grothéndieck group of $C$. Then one can uniquely extend $\nu$ to a well-ordered injective valuation $\nu_{1}$ on $B \backslash\{0\}$ such that $\nu_{1}(x)=\nu\left(f_{0}\right) / n$. Clearly, the valuation semigroup of $\nu_{1}$ has the same rank as of $\nu$.
Remark 3.10. Let $\nu: \mathbb{k}[x, y] \backslash\{0\} \rightarrow C$ be a well-ordered injective valuation. When the values $\nu(x), \nu(y)$ are independent, the semigroup $C$ is isomorphic to $\mathbb{Z}_{\geq 0}^{2}$, while in Proposition 3.8 the semigroup of the produced valuation consists of $n$ copies of (shifted) $\mathbb{Z}_{\geq 0}^{2}$.
Example 3.11. In Corollary 3.9 we have provided a construction of an extension of a domain with an injective valuation. In the course of this construction the Grothéndieck group of the valuation monoid is also extended. Now we give an example of an extension of a domain with an injective valuation when the Grothéndieck group of monoids does not change.

Let a domain $A_{0}:=\mathbb{k}[x, y]$ and $\nu$ be its valuation onto $\mathbb{Z}_{\geq 0}^{2}$ such that $\nu(x)=$ $(1,0), \nu(y)=(0,1)$ (one can take an arbitrary linear well-ordering on $\mathbb{Z}_{>0}^{2}$ ). Consider polynomials $a, b \in A_{0}$ such that the leading monomial (with respect to $\bar{\nu}$ ) of $a$ equals $x^{k}$ for some $k \geq 1$, while the leading monomial of $b$ equals $y^{l}$ for some $l \geq 1$. Denote $A:=A_{0}[b / a] \subset \mathbb{k}(x, y)$. Therefore, the extension of $\nu$ on $A \backslash\{0\}$ is inherited uniquely from $\nu$. Observe that $\nu(A \backslash\{0\}) \subset \mathbb{Z}^{2}$ is well-ordered since $l \geq 1$. The following set forms an adapted basis of $A$ :

$$
\left\{x^{i} y^{j}: i, j \geq 0\right\} \bigsqcup\left\{(b / a)^{s} x^{i} y^{j}: s \geq 1,0 \leq i<k, 0 \leq j\right\} .
$$

Indeed, this set spans $A$. On the other hand, $\nu\left(x^{i} y^{j}\right)=(i, j), \nu(b / a)^{s} x^{i} y^{j}=(-s k+$ $i, s l+j)$, and these values are pairwise distinct for different $i, j, s$.

Example 3.12. Consider an injective homomorphism $\mathbb{k}[x, y] \hookrightarrow \mathbb{k}\left[x-y^{3 / 2}, y^{1 / 2}\right]$ and an injective well-ordered valuation $\nu_{1}$ on the latter algebra defined by $\nu_{1}\left(x-y^{3 / 2}\right):=$ $(-3,1), \nu_{1}\left(y^{1 / 2}\right):=(1,0)$. Then $\nu_{1}\left(x-y^{3 / 2}\right), \nu_{1}\left(y^{1 / 2}\right)$ are linearly independent (cf. Remark 3.10). One can verify that the restriction of $\nu_{1}$ to $\mathbb{k}[x, y] \backslash\{0\}$ coincides with $\nu$.
3.3. Valuations on polynomial algebras. The following is a particular case of Corollary 4.17.

Lemma 3.13. Let $\varphi: \mathbb{k}\left[x_{1}, \ldots, x_{n}\right] \rightarrow \mathbb{k}\left[t_{1}, \ldots, t_{m}\right]$ be an injective homomorphism of algebras. Then the composition $\nu_{0} \circ \varphi$ is an injective valuation $\nu_{\varphi}: \mathbb{k}\left[x_{1}, \ldots, x_{n}\right] \backslash$ $\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{m}$ (here $\nu_{0}$ denotes the tautological injective valuation $\nu_{0}: \mathbb{k}\left[t_{1}, \ldots, t_{m}\right] \backslash$ $\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{m}$ given by $\nu_{0}\left(t_{1}^{i_{1}} \cdots t_{m}^{i_{n}}\right)=\left(i_{1}, \ldots, i_{m}\right) \in \mathbb{Z}_{\geq 0}^{m}$ with respect to the lexicographical ordering on $\mathbb{Z}_{\geq 0}^{n}$ ).

Problem 3.14. Classify all injective valuations $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{m}$.
Problem 3.15. Given $N \geq m$ and a valuation $\nu: \mathbb{k}\left[x_{1}, \ldots, x_{N}\right] \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{m}$, describe all subalgebras $\mathcal{A}$ of $\mathbb{k}\left[x_{1}, \ldots, x_{N}\right]$ such that

- $\mathcal{A} \cong \mathbb{k}\left[t_{1}, \ldots, t_{m}\right]$
- The restriction of $\nu$ to $\mathcal{A} \backslash\{0\}$ is injective.

For instance, if $N=2, m=1$, and $\nu$ is given by a locally nilpotent derivation $E$ (see Lemma 4.9), then $\mathcal{A}=\mathbb{k}[t]$, where $E(t)=1$. More generally, if a nilpotent group $U$ acts on a variety $X$, and $\nu$ is a string valuation based on the action of $\operatorname{Lie}(U)$ on $\mathbb{k}[X]$, then we search for subgroups $U^{\prime}$ of $U$ such that $\mathcal{A}=\mathbb{k}[X]^{\operatorname{Lie}\left(U^{\prime}\right)}$.

This problem is related to the following linear algebra problem.
Problem 3.16. Let $F=\left(V_{1} \subset \cdots \subset V_{m}=\mathbb{C}^{N}\right)$ be a partial flag in $\mathbb{C}^{N}$. Describe the set $\operatorname{Gr}(m, N)_{F}$ of all $U \in G r(m, N)$ such that $\operatorname{dim}\left(V_{i} \cap U\right)=i$ for $i=1, \ldots, m$.

It is obvious that any $A \in \operatorname{Mat}_{m \times N}(\mathbb{Z})$ such that $A \cdot \mathbb{Z}_{\geq 0}^{N} \subset \mathbb{Z}_{\geq 0}^{m}$ must belong to $M a t_{m \times N}(\mathbb{Z})$

Problem 3.17. Given a finite subset $S$ of $\mathbb{Z}_{\geq 0}^{N}$, classify all $A \in M a t_{m \times N}\left(\mathbb{Z}_{\geq 0}\right)$ such that the restriction of the map $x \rightarrow A x$ to the complement $\mathbb{Z}_{\geq 0}^{N} \backslash\left(\bigcup_{v \in S}\left(v+\mathbb{Z}_{\geq 0}^{N}\right)\right)$ is injective.

Problem 3.18. Classify Zariski closed subsets $X \subset \mathbb{A}^{N}$ by injective valuations on $\mathbb{k}[X]$ and vice versa.
3.4. Injective well-ordered valuations on varieties based on tropical geometry. In the sequel we provide a realization of the construction from Theorem 3.3. Let $I \subseteq \mathbb{k}\left[X_{1}, \ldots, X_{n}\right]$ be a prime ideal where $\mathbb{k}$ is a field of zero characteristic. Our purpose is to construct injective well-ordered valuations on the quotient ring $A \backslash\{0\}=\left(\mathbb{k}\left[X_{1}, \ldots, X_{n}\right] / I\right) \backslash\{0\}$ which are induced from the tautological valuation $\nu_{0}\left(X_{1}^{j_{1}} \cdots X_{n}^{j_{n}}\right):=\left(j_{1}, \ldots, j_{n}\right) \in \mathbb{Z}_{\geq 0}^{n}$ on $\mathbb{k}\left[X_{1}, \ldots, X_{n}\right] \backslash\{0\}$. Note that to determine $\nu_{0}$ completely, one has to fix also a linear ordering on $\mathbb{Z}_{\geq 0}^{n}$.

For the sake of convenience we need to describe linear orders $\prec$ on monomials $X^{J}=$ $X_{1}^{j_{1}} \cdots X_{n}^{j_{n}}$ compatible with the product, i.e. $X^{J} \prec X^{K}$ implies $X^{J+L} \prec X^{K+L}$, in a different language than in section 4.6. To this end, we introduce an infinitesimal $\varepsilon$, i.e. $0<\varepsilon<y$ for any $0<y \in \mathbb{R}$. Then $\mathbb{R}[\varepsilon]$ is an ordered ring. Assign weights $w_{i}=w\left(X_{i}\right) \in \mathbb{R}_{\geq 0}[\varepsilon], 1 \leq i \leq n$. This induces a linear (non-strict) order on monomials $X^{J}$ according to the value of $w_{1} j_{1}+\cdots+w_{n} j_{n} \in \mathbb{R}_{\geq 0}[\varepsilon]$. This determines a well-ordering, in other words, there does not exist a strictly decreasing sequence of monomials. It is proved in [31] and in Theorem 9 [11] (in a different language) that any linear order on monomials can be obtained in the described manner. For instance, for two variables lex corresponds to the vector of weights $(1, \varepsilon)$, and deglex corresponds to $(1+\varepsilon, 1)$.

Definition 3.19. Denote $d:=\operatorname{dim} A$. Consider the tropical variety $T:=\operatorname{Trop}(I) \subseteq$ $\mathbb{R}^{n}[25]$. One can view each element of $T$ as a hyperplane in $\mathbb{R}^{n}$ which supports from above Newton polytope $N(f) \subset \mathbb{R}^{n}$ at least at two points (thus, at least at an edge) for every $f \in I$. In such a case we say that this edge is located on the roof of $N(f)$. Then $T$ is equidimensional of dimension $d$ [25] being a finite union of polyhedra each of dimension $d$. Every polyhedron corresponds to a union of hyperplanes containing a (unique) common subplane of dimension $n-d$ which is dual to the polyhedron (we call these subplanes common for the tropical variety $T$ ). Every such common subplane $H \subset \mathbb{R}^{n}$ is supporting to $N(f)$ for any $f \in I$ and is definable by linear equations with rational coefficients.

We extend $\operatorname{Trop}(I)$ considering

$$
E \operatorname{Trop}(I):=\operatorname{Trop}(I) \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon] \subset(\mathbb{R}[\varepsilon])^{n}
$$

where $\operatorname{ETrop}(I)$ satisfies the same linear inequalities as $\operatorname{Trop}(I)$. Thus, one can view $\operatorname{Etrop}(I)$ still as a finite union of polyhedra. Each hyperplane from $\operatorname{ETrop}(I)$ contains $H \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$ for some common subplane $H$ of $\operatorname{Trop}(I)$ and supports $N(f) \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$ at least at two points.

We call a subplane $H$ prop if $H_{0} \cap \mathbb{R}_{\geq 0}^{n}=\left\{X_{l_{1}}=\cdots=X_{l_{m}}=0\right\} \cap \mathbb{R}_{\geq 0}^{n}$ for suitable $1 \leq l_{1}, \ldots, l_{m} \leq n$ where $H_{0}$ is parallel to $H$ and contains the origin $(0, \ldots, 0)$.

Let us fix a common subplane $H$ for the time being. We say that the ideal $I$ is saturated (with respect to $H$ ) if for any pair of integer points $u, v \in \mathbb{Z}_{\geq 0}^{n}$ such that $v-u \in H$ there exists a polynomial $f \in I$ whose Newton polytope $N(f)$ possesses an edge $(u, v)$ on its roof. Below (see Theorem 3.21) under the condition of saturation we obtain an injective valuation, so this condition is stronger than the property that
an initial ideal corresponding to $H$ is prime in $\mathbb{k}\left[X_{1}, \ldots, X_{n}\right]$ (cf. Theorem 3.3 ii) and [20]).

Remark 3.20. In fact, one can reduce the condition of saturation to a finite number of conditions. Indeed, consider a semigroup

$$
G:=\left\{(u, v): u, v \in \mathbb{Z}_{\geq 0}^{n}, u-v \in H\right\} \subset \mathbb{Z}_{\geq 0}^{2 n}
$$

Due to Gordan's lemma [13] $G$ is finitely generated. Among its generators select all $(u, v)$ such that $u \neq v$. Denote a vector $\left(w_{1}^{\prime}, \ldots, w_{n}^{\prime}\right)=: u-v$ and a vector $w:=\left(w_{1}, \ldots, w_{n}\right)=:\left(w_{1}^{\prime}, \ldots, w_{n}^{\prime}\right) / G C D\left(w_{1}^{\prime}, \ldots, w_{n}^{\prime}\right)$. Introduce points
$u_{0}:=\left(\max \left\{w_{1}, 0\right\}, \ldots, \max \left\{w_{n}, 0\right\}\right), v_{0}:=\left(\max \left\{-w_{1}, 0\right\}, \ldots, \max \left\{-w_{n}, 0\right\}\right) \in \mathbb{Z}_{\geq 0}^{n}$.
Then $u_{0}-v_{0}=w$.
One can verify that it suffices for the saturation to impose for all the constructed pairs of points $u_{0}, v_{0}(3.2)$ the existence of a polynomial $f \in I$ such that $N(f)$ has an edge $\left(u_{0}, v_{0}\right)$ on its roof.

From now on we assume that the subplane $H$ is prop and $I$ is saturated (with respect to $H$ ). Our aim is to produce a valuation $\nu:=\nu_{H}$ on $A \backslash\{0\}$. Denote $H_{\mathbb{Z}}:=H \cap \mathbb{Z}^{n}$. The following construction of a valuation is similar to [20].

Consider an epimorphism $\varphi: \mathbb{Z}^{n} \rightarrow \mathbb{Z}^{n} / H_{\mathbb{Z}}$. Then the image $C:=\varphi\left(\mathbb{Z}_{\geq 0}^{n}\right)$ is a semigroup cone (since $H$ is prop). The valuation $\nu$ under production will have $C$ as its valuation cone. Choose some linear ordering $<$ on $C$ for definiteness by fixing a prop hyperplane determined by a vector $\left(w_{1}, \ldots, w_{n}\right) \in\left(\mathbb{R}_{\geq 0}[\varepsilon]\right)^{n}$ from $\operatorname{ETrop}(I)$ which contains $H \otimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$.

Take $0 \neq a \in A$. Assume that there exists $f \in a+I$ such that its Newton polytope $N(f)$ contains no edge in $H$. Then there is a unique vertex $v$ of $N(f)$ with the maximal value of the ordering of $\varphi(v) \in C$. Put $\nu(a):=\varphi(v)$.

Let us establish the correctness of this definition. If otherwise, for some $f_{1} \in a+I$ its Newton polytope $N\left(f_{1}\right)$ has a unique vertex $v_{1}$ with the maximal value of the ordering of $\varphi\left(v_{1}\right)$, then $\varphi(v)=\varphi\left(v_{1}\right)$ taking into account that $f-f_{1} \in I$.

Next we show that for any $0 \neq a \in A$ there exists $f \in a+I$ for which $N(f)$ contains no edge in $H$. Indeed, take $f \in a+I$ such that the vertices $v$ of $N(f)$ with the maximal value of the ordering of $\varphi(v) \in C$ are minimal among all $f \in a+I$. If $u$ is another vertex of $N(f)$ for which $\varphi(v)=\varphi(u)$, i. e. an interval $(u, v)$ lies in $H$, then due to the saturation condition there exists $g \in I$ whose Newton polytope $N(g)$ contains an edge $(u, v)$ on its roof. Therefore, for a suitable $\alpha \in \mathbb{k}$ the support of the polynomial $f+\alpha g$ does not contain $u$. Continuing in this way, we arrive eventually to a polynomial $f_{1} \in a+I$ such that its Newton polytope $N\left(f_{1}\right)$ contains a single vertex $w_{0}$ with the maximal ordering of $\varphi\left(w_{0}\right) \in C$ greater than the orderings of $\varphi(w)$ for all other vertices $w$ of $N\left(f_{1}\right)$. Clearly, $\varphi\left(w_{0}\right)=\varphi(v)$ due to the choice of $f$ satisfying the minimality property.

Observe that we have proved at the same time that one can equivalently define

$$
\begin{equation*}
\nu(a)=\min _{f \in a+I} \max _{v \in N(f)}\{\varphi(v)\} \tag{3.3}
\end{equation*}
$$

where $v \in N(f)$ means that $v$ is a vertex of $N(f)$.

Thus, the valuation $\nu$ on $A \backslash\{0\}$ is defined correctly. If $0 \neq a_{1}, a_{2} \in A$ then take polynomials $f_{1} \in a_{1}+I, f_{2} \in a_{2}+I$ such that Newton polytope $N\left(f_{1}\right)$ (respectively, $N\left(f_{2}\right)$ ) contains a unique vertex $v_{1}$ (respectively, $v_{2}$ ) such that $\varphi\left(v_{1}\right)$ (respectively, $\varphi\left(v_{2}\right)$ ) has a greater ordering than $\varphi(w)$ for all other vertices $w$ of $N\left(f_{1}\right)$ (respectively, $\left.N\left(f_{2}\right)\right)$. Then $\nu\left(a_{1}+a_{2}\right) \leq \max \left\{\varphi\left(v_{1}\right), \varphi\left(v_{2}\right)\right\}=\max \left\{\nu\left(a_{1}\right), \nu\left(a_{2}\right)\right\}$ because of (3.3). In addition, for a polynomial $f_{1} f_{2} \in a_{1} a_{2}+I$ its Newton polytope $N\left(f_{1} f_{2}\right)$ contains a unique vertex $v_{1}+v_{2}$ such that $\varphi\left(v_{1}+v_{2}\right) \in C$ has a greater ordering than all other vertices of $N\left(f_{1} f_{2}\right)$, hence $\nu\left(a_{1} a_{2}\right)=\varphi\left(v_{1}+v_{2}\right)=\varphi\left(v_{1}\right)+\varphi\left(v_{2}\right)$.

Now we verify the injectivity of $\nu$. Let $\nu\left(a_{1}\right)=\nu\left(a_{2}\right)$ for $0 \neq a_{1}, a_{2} \in A$. Take $f_{1} \in a_{1}+I, f_{2} \in a_{2}+I$ with vertices $v_{1} \in N\left(f_{1}\right), v_{2} \in N\left(f_{2}\right)$ as above. Thus, $\varphi\left(v_{1}\right)=\varphi\left(v_{2}\right)$. Therefore, there exists $g \in I$ such that its Newton polytope $N(g)$ contains an edge $\left(v_{1}, v_{2}\right)$ on its roof due to the saturation condition. Hence, for any vertex $w \in N\left(f_{1}+\alpha f_{2}+\beta g\right)$ we have $\varphi(w)<\varphi\left(v_{1}\right)=\nu\left(a_{1}\right)$ for appropriate $\alpha, \beta \in \mathbb{k}$. Thus, $\nu\left(a_{1}+\alpha a_{2}\right)<\nu\left(a_{1}\right)$, see (3.3), which justifies the injectivity of $\nu$.

We summarize the proved above in the following theorem.
Theorem 3.21. Let $A=\mathbb{k}\left[X_{1}, \ldots, X_{n}\right] / I$ be a domain of dimension d. Let $H \subset \mathbb{R}^{n}$ be one of a finite number of (rationally definable) common subplanes of dimension $n-d$ dual to a (highest dimensional) polyhedron of dimension $d$ of the tropical variety Trop $(I) \subset \mathbb{R}^{n}$ (see Definition 3.19). Assume that $H$ is prop and $I$ is saturated with respect to $H$. Consider a natural epimorphism $\varphi:(\mathbb{R}[\varepsilon])^{n} \rightarrow(\mathbb{R}[\varepsilon])^{n} /\left(H \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]\right)$. Fix a prop hyperplane from ETrop $(I)$ which contains $H \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$, it determines a linear order on $\varphi\left(\mathbb{Z}_{\geq 0}^{n}\right)$. Then (3.3) defines a well-ordered injective valuation $\nu$ on $A \backslash\{0\}$ having a valuation cone $\varphi\left(\mathbb{Z}_{\geq 0}^{n}\right)$.
Remark 3.22. The constructions of injective valuations on $\mathbb{k}[x, y] \backslash\{0\}$ from Example 3.6 and Proposition 3.8 are particular cases of Theorem 3.21 when one represents $\mathbb{k}[x, y] \simeq \mathbb{k}\left[x_{1}, \ldots, x_{n}\right] / I$ for suitable ideals $I \subset \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$.

Let $(M, \cdot)$ be a (not necessary commutative) monoid. We say that an equivalence relation $\sim$ is admissible if $u \sim v$ implies $w u \sim w v, u w \sim v w$ for any $u, v, w \in M$. Then one can define a quotient monoid $M / \sim$ on equivalence classes. A linear order $\prec$ on equivalence classes $U \prec V$ (or on $M / \sim$ ) is defined as $u \prec v$ for any $u \in U, v \in V$, we require that this linear order on $M / \sim$ is correct. The latter linear order is admissible if $U \prec V$ implies $U W \prec V W, W U \prec W V$ (cf. Definition 2.1). Below we consider only admissible equivalence relations and linear orders.

Denote by $M:=\left\langle a_{1}, \ldots, a_{s}\right\rangle$ the free monoid generated by $a_{1}, \ldots, a_{s}$. Let $A:=$ $\mathbb{k}\left\langle a_{1}, \ldots, a_{s}\right\rangle / I$ be a (not necessary commutative) algebra where $I \subset \mathbb{k}\left\langle a_{1}, \ldots, a_{s}\right\rangle$ is an ideal. We say that an equivalence relation $\sim$ on $M$ and a linear order $\prec$ on $M / \sim$ are compatible with $I$ if for any element

$$
\begin{equation*}
f=\sum_{u \in \operatorname{supp}(f) \subset M} \alpha_{u} u \in I, \alpha_{u} \in \mathbb{k}^{*} \tag{3.4}
\end{equation*}
$$

there are elements $u_{1}, u_{2} \in \operatorname{supp}(f)$ such that $u_{1} \sim u_{2}$ and for every $u \in \operatorname{supp}(f)$ it holds $u \preceq u_{1}$. One can treat this concept as a generalization of the tropical variety of $I$ to the non-commutative case.

We say that $I$ is saturated with respect to $\sim, \prec$ if for any pair $u_{1} \sim u_{2}, u_{1} \neq$ $u_{2}$ there exists $f$ of the form (3.4) such that $u_{1}, u_{2} \in \operatorname{supp}(f)$ and for every $u \in$
$\operatorname{supp}(f), u \neq u_{1}, u_{2}$ it holds $u \prec u_{1}$. Similarly to the proof of Theorem 3.21 one can verify the following proposition.

Proposition 3.23. Let $A:=\mathbb{k}\left\langle a_{1}, \ldots, a_{s}\right\rangle / I$ be an algebra, $\sim$ be an admissible equivalence relation on the free monoid $M:=\left\langle a_{1}, \ldots, a_{s}\right\rangle$, and $\prec$ be an admissible well order on $M / \sim$. Assume that $\sim, \prec$ are compatible with $I$, and $I$ is saturated with respect to $\sim, \prec$. Then there is an injective valuation $\nu: A \backslash\{0\} \rightarrow M / \sim$ defined as follows: for $f \in A \backslash\{0\}$ put $\nu(f)$ as the minimal equivalence class $U_{0}$ such that

$$
f=\alpha_{u_{0}} u_{0}+\sum_{u \in M} \alpha_{u} u, \alpha_{u_{0}}, \alpha_{u} \in \mathbb{k}^{*},
$$

where $u_{0} \in U_{0}$ and $u \prec u_{0}$ for all $u$ (cf. (3.3)). In addition, one can pick an adapted basis among monomials from $M$.

### 3.5. Injective well-ordered valuations on algebraic curves.

Remark 3.24. Assume that the field $\mathbb{k}$ is algebraically closed and $A=\mathbb{k}\left[X_{1}, \ldots, X_{n}\right] / I$ is a domain. Let $\nu: A \backslash\{0\} \rightarrow C$ be a valuation and let $\psi: C \rightarrow \mathbb{Q}$ be a homomorphism preserving the order. There exist Puiseux series $\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{k}\left(\left(\varepsilon^{1 / \infty}\right)\right)^{n}$ satisfying $I$ of the form $\alpha_{0} \varepsilon^{s_{0} / q}+\alpha_{1} \varepsilon^{s_{1} / q}+\cdots \in \mathbb{k}\left(\left(\varepsilon^{1 / \infty}\right)\right)^{n}$ where integers $s_{0}>s_{1}>\cdots$ decrease, such that $\operatorname{ord}\left(x_{i}\right)=\psi \circ \nu\left(X_{i}\right), 1 \leq i \leq n$ [25].
Proposition 3.25. Let a valuation $\nu$ on an irreducible curve $A \backslash\{0\}:=\mathbb{k}[x, y] /(f) \backslash$ $\{0\}, f \in \mathbb{k}[x, y]$ fulfill Theorem 3.21, i.e. $\nu$ is injective and $\nu(a) \geq 0$ for any $a \in$ $A \backslash\{0\}$, where $\mathbb{k}$ is algebraically closed. Then there exists an injective homomorphism $\eta: A \hookrightarrow \mathbb{k}\left(\left(\varepsilon^{1 / \infty}\right)\right)$ such that for every $a \in A \backslash\{0\}$ it holds $\nu(a)=\operatorname{ord}(\eta(a))$, provided that $\nu(x)=1$ for normalization.

Proof. According to Theorem 3.21 Newton polygon $N_{f}$ has an edge with endpoints $(p, 0),(0, q)$ for relatively prime $p, q \geq 1$. Let $p \leq q$ for definiteness. Then the equation $f(x, y)=0$ has a Puiseux series solution of the form $y(x)=\alpha x^{p / q}+\ldots$ where $\alpha \in \mathbb{k}^{*}$ and the terms in dots contain powers of $x$ less than $p / q$.

One can define $\eta(x):=\varepsilon, \eta(y):=y(\varepsilon)$. Then $\eta$ is injective since $f$ is irreducible. The monomials $\mathbf{B}:=\left\{x^{i} y^{j}: 0 \leq i<\infty, 0 \leq j<q\right\}$ constitute a basis of $A$. The orders $\operatorname{ord}\left(\eta\left(x^{i} y^{j}\right)\right)=i+j p / q=\nu\left(x^{i} y^{j}\right)$ are pairwise distinct for the monomials from B.

One can prove a certain converse statement to Proposition 3.25.
Remark 3.26. Let for a polynomial $f \in \mathbb{k}[x, y]$ where $\mathbb{k}$ is algebraically closed, its Newton polygon $N_{f} \subset \mathbb{R}^{2}$ is not of the shape from Proposition 3.25, i.e. $N_{f}$ does not contain an edge with vertices $(p, 0),(0, q)$ with relatively prime $p, q$. Then the imbedding $\eta: A:=\mathbb{k}[x, y] /(f) \hookrightarrow \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ into the field of Puiseux series induces a valuation $\nu: A \backslash\{0\} \rightarrow \mathbb{Q}$ by a formula $\nu(a):=\operatorname{ord}(\eta(a))$, being not an injective well-ordered.

Any automorphism $\varphi$ of $\mathbb{k}[x, y]$ produces an injective valuation on the algebra $\mathbb{k}[x, y] /(f \circ \varphi) \backslash\{0\}$.

Consider an algebra $A:=\mathbb{k}[x, y] /(f)$ of a curve where $f$ is irreducible. Let $A \hookrightarrow$ $\mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ be an injective homomorphism into the field of Newton-Puiseux series. We investigate when this induces an injective well-ordered valuation $\nu(=$ ord $)$ on
$A \backslash\{0\}$. W.l.o.g. one can suppose that $\operatorname{ord}(x)=1$ and $f=y^{d}+f_{1}$ is normalized, i.e. $\operatorname{deg}_{y}\left(f_{1}\right)<d$.

Lemma 3.27. Let $M \subset \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ be a free $\mathbb{k}[x]$-module of a rank $d$. Then $M \backslash\{0\}$ admits $a \mathbb{k}[x]$-basis $s_{1}, \ldots, s_{d}$ such that $\operatorname{ord}\left(s_{1}\right), \ldots, \operatorname{ord}\left(s_{d}\right)$ are non-negative and $\operatorname{ord}\left(s_{i}\right)-\operatorname{ord}\left(s_{j}\right) \notin \mathbb{Z}$ for each pair $1 \leq i \neq j \leq d$ iff for any $s \in M \backslash\{0\}$ it holds $\operatorname{ord}(s) \geq 0$.

Proof. In one direction the lemma is evident, so assume that $\operatorname{ord}(s) \geq 0$ for any $s \in M \backslash\{0\}$. Let $p_{1}, \ldots, p_{d} \in M$ be a $\mathbb{k}[x]$-basis of $M$. If $\operatorname{ord}\left(p_{i}\right)-\operatorname{ord}\left(p_{j}\right) \in \mathbb{Z}_{\geq 0}$ for some $1 \leq i \neq j \leq d$ and $\operatorname{ord}\left(p_{i}-\alpha x^{\operatorname{ord}\left(p_{i}\right)}\right)<\operatorname{ord}\left(p_{i}\right), \operatorname{ord}\left(p_{j}-\beta x^{\operatorname{ord}\left(p_{j}\right)}<\operatorname{ord}\left(p_{j}\right)\right.$ for suitable $\alpha, \beta \in \mathbb{k}^{*}$, one can replace $p_{j}$ by $p_{j}^{\prime}:=p_{j}-(\beta / \alpha) x^{\operatorname{ord}\left(p_{i}\right)-\operatorname{ord}\left(p_{j}\right)} p_{i}$. Clearly, $\operatorname{ord}\left(p_{j}^{\prime}\right)<\operatorname{ord}\left(p_{j}\right)$. Continuing in this way, we arrive to a required basis $s_{1}, \ldots, s_{d}$.

Remark 3.28. Let $f=y^{d}+f_{1} \in \mathbb{Z}[x, y]$ be normalized. Assume that the bit-sizes of the integer coefficients of $f$ do not exceed $L$. Here we agree that the field $\mathbb{k}=\overline{\mathbb{Q}}$.

For a root $Y \in \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ of $f$ consider a free $\mathbb{k}[x]$-module $M \subset \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ with a basis $1, Y, \ldots, Y^{d-1}$. Then the algorithm designed in the proof of Lemma 3.27 either yields a basis $s_{1}, \ldots, s_{d}$ of $M$ such that $\operatorname{ord}\left(s_{i}\right) \geq 0$ and $\operatorname{ord}\left(s_{i}\right)-\operatorname{ord}\left(s_{j}\right) \notin \mathbb{Z}$ for every pair $1 \leq i \neq j \leq d$ or the algorithm discovers an element $s \in M$ such that $\operatorname{ord}(s)<0$.

The complexity of the algorithm is polynomial in $d, \operatorname{deg}(f), L$. It follows from the polynomial complexity bound for developing Newton-Puiseux series [10].

Now we are able to summarize the obtained above in the following corollary.
Corollary 3.29. Let $A=\mathbb{k}[x, y] /(f)$ be an algebra of an irreducible curve. Let $Y \in \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ be a root of $f$ in the field of Newton-Puiseux series. Denote by $M \subset \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ the $\mathbb{k}[x]$-module generated by $1, Y, \ldots, Y^{d-1}$. The valuation ord on $A \backslash\{0\}$ induced by means of an injective homomorphism $A \hookrightarrow \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ where $y \rightarrow Y$, is injective and well-ordered iff for any $s \in M \backslash\{0\}$ it holds ord $(s) \geq 0$ (agreeing ord $(x)=1$ ).

In the case of $f \in \mathbb{Z}[x, y]$ and $\mathbb{k}=\overline{\mathbb{Q}}$ there is an algorithm which either yields an adapted (with respect to ord) $\mathbb{k}[x]$-basis of $A$ or discovers an element $s \in M \backslash\{0\}$ such that ord $(s)<0$.

Remark 3.30. Due to Lemma 3.27 an adapted basis yielded in Corollary 3.29 has a form $\left\{s_{i} x^{j}: 1 \leq i \leq d, 0 \leq j\right\}$ for appropriate elements $s_{i} \in M, 1 \leq i \leq d$.
Example 3.31. Let $f:=\left(y^{2}-x\right)^{3}-8 x^{2}$. The Newton-Puiseux expansion of its root is $Y=x^{1 / 2}+x^{1 / 6}+\cdots$. Denote $a:=y^{2}-x$. Newton polygon $N_{f}$ has an edge with the endpoints $(3,0),(0,6)$. Therefore, it does not fulfill the conditions of Theorem 3.21. Nevertheless, the algebra $A:=\overline{\mathbb{Q}}[x, a] \backslash\{0\}$ admits an injective well-ordered valuation ord with an adapted basis of a form

$$
x^{j}, y x^{j}, a x^{j}, a y x^{j}, a^{2} x^{j}, a^{2} y x^{j}, j \geq 0
$$

due to Corollary 3.29. It holds $\operatorname{ord}(y)=1 / 2, \operatorname{ord}(a)=2 / 3, \operatorname{ord}(\operatorname{ay})=7 / 6, \operatorname{ord}\left(a^{2}\right)=$ $4 / 3, \operatorname{ord}\left(a^{2} y\right)=11 / 6$.

Now we proceed to a proof of a converse statement to Corollary 3.29: if an algebra $A:=\mathbb{k}[x, y] /(f) \backslash\{0\}$ of an irreducible curve admits an injective well-ordered valuation $\nu$, then $\nu$ is inherited from an injective homomorphism $A \hookrightarrow \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ under which $y$ is mapped to a root of $f$ (and in addition, $\nu$ does not depend on a choice of a root). We agree that $\nu(x)=1$. Denote $f=y^{d}+f_{1}, \operatorname{deg}_{y}\left(f_{1}\right)<d$.

We will repeatedly make use of the following easy observation. Let $a=\sum_{0 \leq i<d} \alpha_{i} y^{i} \in$ $A$ and $g(a)=0$ for a suitable polynomial $g \in \mathbb{k}[x, z], \operatorname{deg}_{z}(g) \leq d$. Then the value $\nu(a)$ is among the slopes of the edges of Newton polygon $N_{g}$.

First, we recall some properties of Newton-Puiseux expansions of the roots in $\mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ of $f$ (see e.g. [36]). There is a partition of the roots of $f$ into classes of cardinalities $d_{1}, \ldots, d_{k}$ where $d_{1}+\cdots d_{k}=d$. For each class of a cardinality $d_{i}$ every root from this class has a form

$$
\begin{equation*}
Y=\sum_{j \geq 0} \beta_{j} x^{p_{j} / d_{i}} \in \mathbb{k}\left(\left(x^{1 / \infty}\right)\right) \tag{3.5}
\end{equation*}
$$

where integers $p_{0}>p_{1}>\cdots$ decrease. Moreover, all the roots from this class are exhausted by Newton-Puiseux series

$$
\begin{equation*}
\sum_{j \geq 0} \beta_{j} \omega^{p_{j}} x^{p_{j} / d_{i}} \tag{3.6}
\end{equation*}
$$

where $\omega$ ranges over the roots of unity of the degree $d_{i}$. In the process of NewtonPuiseux expanding of $Y$ for any intermediate current polynomial $h \in \mathbb{k}[x, y]$ for the slope $p / q \in \mathbb{Q}$ of each edge of Newton polygon $N_{h}$ it holds $q \mid d_{i}$.

Lemma 3.32. If an algebra $A=\mathbb{k}[x, y] /(f) \backslash\{0\}$ of a curve admits an injective well-ordered valuation $\nu$ then the roots of $f$ in the field of Newton-Puiseux series constitute a single class.

Proof. Denote by $d_{1}, \ldots, d_{k}$ the cardinalities of the classes of the roots of $f$. Consider an element $a=\sum_{0 \leq i<d} \alpha_{i} y^{i} \in A \backslash\{0\}$. Let $g(a)=0$ for an appropriate polynomial $g \in \mathbb{k}[x, y]$, $\operatorname{deg}_{y}(\bar{h}) \leq d$. Then $g\left(\sum_{0 \leq i<d} \alpha_{i} Y^{i}\right)=0$ for any root $Y \in$ $\mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ of $f$. Therefore, for the slope $p / q$ of every edge of Newton polygon $N_{g}$ it holds $q \mid d_{l}$ for suitable $1 \leq l \leq k$.

Hence the values of $\nu$ on $A \backslash\{0\}$ are contained in a set

$$
\mathbb{Z}_{\geq 0} / d_{1} \cup \cdots \cup \mathbb{Z}_{\geq 0} / d_{k}
$$

Here we use that $\nu$ is well-ordered, so non-negative on $A \backslash\{0\}$. Denote by $L_{N} \subset A$ for an integer $N \geq 0$ the $\mathbb{k}$-linear space with a basis $y^{i} x^{j}: 0 \leq i<d, 0 \leq j<N$. Then $\operatorname{dim}\left(L_{N}\right)=N d$. On the other hand, $\nu$ attains on $L_{N}$ the values from a set

$$
\{0, \ldots, N+\text { const }\} \cup \bigcup_{1 \leq l \leq k, 1 \leq p<d_{l}}\left(\{0, \ldots, N+\text { const }\}+p / d_{l}\right)
$$

The cardinality of the latter set does not exceed ( $N+$ const $)(d-k+1)$. Thus, if $k \geq 2$ then the valuation $\nu$ attains on $L_{N}$ less than $\operatorname{dim}\left(L_{N}\right)$ values, which contradicts to the injectivity of $\nu$. This completes the proof of the lemma.

For any $a=\sum_{0 \leq i<d} \alpha_{i} y^{i} \in A \backslash\{0\}$ due to Lemma 3.32 we have

$$
\sum_{0 \leq i<d} \alpha_{i} Y^{i}=\gamma x^{p / q}+\cdots \in \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)
$$

where $Y$ is a root of $f(3.5), p / q$ is the leading exponent of Newton-Puiseux expansion, and $\gamma \in \mathbb{R}^{*}, p \in \mathbb{Z}$. Let $g(a)=0$ for a polynomial $g \in \mathbb{k}[x, y]$, $\operatorname{deg}_{y}(g) \leq d$. All the roots of $g$ have an expansion of the form $\gamma \omega^{p} x^{p / d}+\cdots$, where $\omega$ ranges over the roots of unity of the degree $d$. Hence Newton polygon $N_{g}$ has a unique edge with the slope $p / d$, thus $\nu(a)=p / d$.

Summarizing, we have established the following theorem.
Theorem 3.33. If an algebra $A=\mathbb{k}[x, y] /(f) \backslash\{0\}$ of an irreducible curve admits an injective well-ordered valuation $\nu$ then $\nu$ is inherited from the valuation ord on $\mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ by means of an injective homomorphism $A \hookrightarrow \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ where y is mapped to $a \operatorname{root} Y \in \mathbb{k}\left(\left(x^{1 / \infty}\right)\right)$ of $f$. The value of $\nu$ does not depend on a choice of a root.

Remark 3.34. Corollary 3.29 and Theorem 3.33 together describe all the injective well-ordered valuations on a curve, and moreover, provide an algorithm to yield all such valuations.

Example 3.35. We provide a complete description when an algebra $A=\mathbb{k}[x, y] /(g)$ where $g$ is a quadratic polynomial, admits an injective well-ordered valuation $\nu$.

First, if $g=x y+p x+q y+t$ then either $\nu(x)=0$ or $\nu(y)=0$ (cf. Theorem 3.21). In both cases we get a contradiction with the injectivity of $\nu$.

Now we assume that $g=x^{2}+e x y+b y^{2}+p x+q y+t$ and either $b \neq 0$ or $e \neq 0$. Then $\nu(x)=\nu(y)$ (unless $b=0, e \neq 0$ when one should consider in addition, another possibility $\nu(x)=0$, which contradicts to the injectivity, cf. above). Therefore, due to the injectivity, there exists $\alpha \in \mathbb{k}$ such that for $u:=x+\alpha y$ it holds $\nu(u)<\nu(y)$. Substituting $u-\alpha y$ for $x$ in $g$, we deduce that $\alpha^{2}-\alpha e+b=0$ (being the coefficient at the highest monomial $y^{2}$ in $g$ ) and $2 \alpha-e=0$ (being the coefficient at the next highest monomial $u y$ in $g$ ). Hence $\alpha=e / 2$ and $e^{2}-4 b=0$ (being the discriminant of the highest form of $g$ ). Thus, $g=u^{2}+p u+(q-e p / 2) y+t$.

If $q-e p / 2 \neq 0$, we fall in the conditions of Theorem 3.21, therefore $A$ admits an injective well-ordered valuation $\nu$, and the monomials in $u$ constitute an adapted basis of $A$ with respect to $\nu$. By the same token this arguments covers also the case $b=e=0$.

Else if $q-e p / 2=0$, we have $g=u^{2}+p u+t$, hence $\nu(u)=0$ which contradicts to the injectivity of $\nu$ (cf. above).

Thus, $A$ admits an injective well-ordered valuation iff (the discriminant of the highest form of $g) e^{2}-4 b=0$, while $q-e p / 2 \neq 0$.

Consider a domain $A=\mathbb{k}[x, y] /(g)$ where $g \in \mathbb{k}[x, y]$. We study necessary conditions when $A \backslash\{0\}$ admits an injective well-ordered valuation $\nu$ (cf. the sufficient conditions from Theorem 3.21). There exists an edge $e$ of the roof of Newton polygon $\mathcal{N}(g)$ such that for any points $(i, j),(k, l)$ from the edge $e$ it holds $\nu\left(x^{i} y^{j}\right)=\nu\left(x^{k} y^{l}\right)$. In this case we say that $\nu$ goes along the edge $e$.

Proposition 3.36. Let $\nu$ be an injective well-ordered valuation on $\mathbb{k}[x, y] /(g) \backslash\{0\}$ which goes along an edge of $\mathcal{N}(g)$ being parallel to the line $\{x=-y\}$, and $\operatorname{deg}(g)>1$. Then the discriminant of the leading homogeneous form of $g$ vanishes.

Proof. We have $\nu(x)=\nu(y)$ because $\nu$ goes along the edge parallel to the line $\{x=-y\}$. The injectivity implies the existence of $0 \neq \alpha \in \mathbb{k}$ such that for $z:=x-\alpha y \in A$ it holds $\nu(z)<\nu(x)$. Since $d:=\operatorname{deg}(g)>1$ the element $z \notin \mathbb{k}$, hence $\nu(z)>0$.

Denote by $h(x, y):=b_{0} x^{d}+b_{1} x^{d-1} y+\cdots+b_{d} y^{d}$ the leading homogeneous form of $g$, where $b_{0}, \ldots, b_{d} \in \mathbb{k}$. Replace $x$ in $g$ by $z+\alpha y$ and the resulting polynomial denote by $\tilde{g} \in \mathbb{k}[z, y]$. In $\tilde{g}$ the monomial $y^{d}$ has the higher valuation than the other monomials. Therefore, the coefficient in $\tilde{g}$ at this monomial, which equals $h(\alpha, 1)$, vanishes. The monomial $z y^{d-1}$ has the higher valuation than the other monomials in $\tilde{g}$ (except of the monomial $y^{d}$ ). Therefore, the coefficient in $\tilde{g}$ at the monomial $z y^{d-1}$ which equals the derivative $h_{x}(\alpha, 1)$, vanishes as well. Since $h$ and its derivative have a common root, its discriminant vanishes.

### 3.6. Adapted bases in domains with injective well-ordered valuations.

Remark 3.37. In case when $A=\mathbb{k}\left[X_{1}, \ldots, X_{n}\right] /(g)$ is a ring of regular functions on an irreducible hypersurface, we consider an edge of Newton polytope $N(g)$ with the endpoints $u=\left(u_{1}, \ldots, u_{n}\right), v=\left(v_{1}, \ldots, v_{n}\right) \in \mathbb{Z}_{>0}^{n}$. Denote by $H$ the line passing through $u, v$. The principal ideal $(g)$ is saturated with respect to $H$ iff $\min \left\{u_{i}, v_{i}\right\}=0,1 \leq i \leq n$ and in addition, $u_{1}, \ldots, u_{n}, v_{1}, \ldots, v_{n}$ have no nontrivial common divisor, cf. Remark 3.20 and (3.2). Moreover, $H$ is prop iff either $0 \neq u, v$ or one of vectors $u, v$ equals 0 and the other one has a single non-zero coordinate equal 1. When $H$ is prop and $I$ is saturated with respect to $H$, there exists a well-ordered injective valuation $\nu$ on $A \backslash\{0\}$ with a valuation cone $\varphi\left(\mathbb{Z}_{\geq 0}^{n}\right) \subset \mathbb{Z}^{n} / H_{\mathbb{Z}}$ according to Theorem 3.21.

Observe that in this way one can obtain a well-ordered valuation $\nu$ on $\mathbb{k}[x, y] \backslash\{0\} \simeq$ $\left(\mathbb{k}[x, y, z] /\left(z-y^{3}+x^{2}\right)\right) \backslash\{0\}$ produced in Example 3.6 (see also Proposition 3.8). Indeed, Newton polytope of the polynomial $f:=z-y^{3}+x^{2}$ is a triangle. As $H$ we take the line passing through the edge $(2,0,0),(0,3,0)$. The principal ideal $(f)$ is saturated with respect to $H$ (cf. Proposition 3.25 and Example 3.38). Therefore, Theorem 3.21 provides just the valuation $\nu$ as in Example 3.6.

Example 3.38. Let $g \in X^{3}+Y^{2}+\mathcal{L}\left\{1, Y, X, X Y, X^{2}\right\}$ where $\mathcal{L}$ denote the linear hull. The domain $A:=\mathbb{k}[X, Y] /(g)$ defines a curve. Then the line $H=\{2 X+3 Y=$ $0\}$ and $\varphi: \mathbb{Z}^{2} \rightarrow \mathbb{Z}$ is given by $\varphi(i, j)=2 i+3 j$, the valuation cone $\varphi\left(\mathbb{Z}_{\geq 0}^{2}\right)=\mathbb{Z}_{\geq 0} \backslash\{1\}$. The valuation $\nu\left(X^{i_{0}} Y^{j_{0}}+\mathcal{L}\left\{X^{i} Y^{j}: 2 i+3 j<2 i_{0}+3 j_{0}\right\}\right)=2 i_{0}+3 \bar{j}_{0}$ on $A \backslash\{0\}$ is well-ordered and injective.

Theorem 3.39. Let $A$ be a $\mathbb{k}$-algebra.
i) Then for any finite set of its generators $x_{1}, \ldots, x_{m}$ there is a finite set of vectors $S \subset \mathbb{Z}_{\geq 0}^{m}$ such that all monomials $x^{w}, w \in \mathbb{Z}_{\geq 0}^{m}$ for which holds $(w-S) \cap \mathbb{Z}_{\geq 0}^{m}=\emptyset$ form a basis $\mathbf{B}$ of $A$;
ii) let $\nu: A \backslash\{0\} \rightarrow C$ be an injective valuation onto a monoid $C$ generated by $c_{1}, \ldots, c_{m}$ endowed with a linear well ordering $\prec$. Let $a_{1}, \ldots, a_{m} \in A$ be such that
$\nu\left(a_{i}\right)=c_{i}, 1 \leq i \leq m$. Similar to $\left.i\right)$ there exists a finite set $S$ of monomials in $a_{1}, \ldots, a_{m}$ such that $\mathbf{B}$ consisting of monomials off the monomial ideal generated by $S$, form an adapted basis of $A$ with respect to $\nu$.

Proof. i) Choose a finite presentation $A=\mathbb{k}\left[x_{1}, \ldots, x_{m}\right] / J$ and fix an injective linear weight function $q: \mathbb{Z}_{\geq 0}^{m} \rightarrow \mathbb{R}$ inducing a well-ordering on $\mathbb{Z}_{\geq 0}^{m}$ and being compatible with the addition: if $q\left(v_{1}\right)<q\left(v_{2}\right)$ then $q\left(v_{1}+v\right)<q\left(v_{2}^{-}+v\right)$ for any $v, v_{1}, v_{2} \in \mathbb{Z}_{\geq 0}^{m}$. In particular, one can take $q\left(u_{1}, \ldots, u_{m}\right)=\alpha_{1} u_{1}+\cdots+\alpha_{m} u_{m}$ for $0<\alpha_{1}, \ldots, \bar{\alpha}_{m} \in \mathbb{R}$ being $\mathbb{Q}$-linearly independent.

Take a Gröbner basis of $J$ (with respect to the ordering $q$ ). In each element $a=\sum_{i} \beta_{i} x^{v_{i}}, \beta_{i} \in \mathbb{k}$ of the basis choose $v_{i_{0}}$ with the biggest value of $q\left(v_{i_{0}}\right)$ among $q\left(v_{i}\right)$. We call $v_{i_{0}}:=\operatorname{lev}(a)$ the leading exponent vector of $a$. Put $S$ to consist of the leading exponent vectors of all the elements of the basis.

First, we verify that the elements of $\mathbf{B}:=\left\{x^{w}: w \notin S+\mathbb{Z}_{\geq 0}^{m}\right\}$ are $\mathbb{k}$-linearly independent in $A$. Indeed, otherwise let

$$
\sum_{j} \gamma_{j} x^{w_{j}} \in J, \gamma_{j} \in \mathbb{k}, w_{j} \notin S+\mathbb{Z}_{\geq 0}^{m}
$$

This contradicts to the property of Gröbner bases that the monomial ideal $S+\mathbb{Z}_{\geq 0}^{m}$ coincides with the ideal of the leading monomials of all the elements of $J$.

Now we show that any element of the form $x^{v}, v \in \mathbb{Z}_{\geq 0}^{m}$ is a $\mathbb{k}$-linear combination of the elements of $\mathbf{B}$. If $x^{v} \notin \mathbf{B}$ then, again due to the property of Gröbner bases, there exists an element $a_{0} \in J$ such that its leading monomial coincides with $x^{v}$. Consider a linear combination $x^{v}+\alpha a_{0}$ for an appropriate (unique) $\alpha \in \mathbb{k}$ for which $q\left(\operatorname{lev}\left(x^{v}+\alpha a_{0}\right)\right)<q(v)$. Then we continue in a similar way, taking the biggest monomial in $x^{v}+\alpha a_{0}$ which does not belong to $\mathbf{B}$, provided that it does exist. This process terminates due to the well-ordering with respect to $q$. i) is proved.
ii) Again pick positive reals $\alpha_{1}, \ldots, \alpha_{m}$ being $\mathbb{Q}$-linearly independent. Introduce a well-ordering $q$ on the monomials in $a_{1}, \ldots, a_{m}$ as follows. We say that $q\left(a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}\right)<$ $q\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}\right)$ iff either $\nu\left(a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}\right) \prec \nu\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}\right)$ or $\nu\left(a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}\right)=\nu\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}\right)$ and $\alpha_{1} j_{1}+\cdots+\alpha_{m} j_{m}<\alpha_{1} i_{1}+\cdots+\alpha_{m} i_{m}$.

The elements $a_{1}, \ldots, a_{m}$ are generators of $A$ since $\nu$ is injective and $\prec$ is wellordered. Therefore, $A=\mathbb{k}\left[a_{1}, \ldots, a_{m}\right] / J$ for certain ideal $J$. Consider a Gröbner basis of $J$ with respect to $q$.

We claim that the basis $\mathbf{B}$ of $A$ (consisting of some monomials in $a_{1}, \ldots, a_{m}$ ) produced in i), is adapted with respect to $\nu$. Suppose the contrary. Let $\nu\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}\right)=$ $\nu\left(a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}\right)$ for two different monomials from the basis $\mathbf{B}$. Let $\alpha_{1} i_{1}+\cdots+\alpha_{m} i_{m}>$ $\alpha_{1} j_{1}+\cdots+\alpha_{m} j_{m}$ for definiteness. There exists (and unique) $\beta \in \mathbb{k}$ for which $\nu\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}+\beta a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}\right) \prec \nu\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}\right)$ holds, because $\nu$ is injective. There exists an element $a_{1}^{l_{1}} \cdots a_{m}^{l_{m}} \in \mathbf{B}$ such that $\nu\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}+\beta a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}\right)=\nu\left(a_{1}^{l_{1}} \cdots a_{m}^{l_{m}}\right)$. We continue the process this way. Due to well-ordering of $\nu$ the process terminates, and we arrive at an element of the form

$$
\begin{equation*}
a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}+\beta a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}+\sum_{K} \beta_{K} a^{K} \in J \tag{3.7}
\end{equation*}
$$

for appropriate $\beta_{K} \in \mathbb{k}$, where for all the monomials from the latter sum in (3.7) it holds $\nu\left(a^{K}\right) \prec \nu\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}\right)$. Thus, $a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}$ is the highest (with respect to $q$ ) monomial in (3.7), hence $a_{1}^{i_{1}} \cdots a_{m}^{i_{m}} \notin \mathbf{B}$ due to the construction of $\mathbf{B}$ in i). The obtained contradiction proves the claim and ii).

Remark 3.40. The elements $a_{1}, \ldots, a_{m}$ produced in the proof of Theorem 3.39 ii) constitute a Khovanskii basis of $A$ [20].
Remark 3.41. i) The proof of Theorem 3.39 provides an inverse to the construction from Theorem 3.3. Namely, let $\nu$ be an injective well-ordered valuation on $A \backslash\{0\}$ with a valuation semigroup $C$, and a set of generators $a_{1}, \ldots, a_{m}$ of $A$ be produced as in the proof of Theorem 3.39. One can represent the polynomial algebra $\mathbb{k}\left[a_{1}, \ldots, a_{m}\right]=\bigoplus_{c \in C} D_{c}$ as a graded domain where a $\mathbb{k}$-basis of $D_{c}$ consists of all the monomials $p=a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}$ such that $\nu(p)=c$. Then we fall in the conditions of Theorem 3.3.
ii) Theorem 3.39 implies that one can view $A$ as a deformation of $\mathbb{k} C$.

Definition 3.42. Given a basis $\mathbf{B}$ of an algebra $\mathcal{A}$ we say that a map $\nu: \mathbf{B} \rightarrow \mathbb{Z}^{m}$ is a $\mathbf{B}$-prevaluation of $\mathcal{A}$ if

- $\nu\left(b b^{\prime}\right)=\nu(b)+\nu\left(b^{\prime}\right)$ for any $b, b^{\prime} \in \mathbf{B}$ such that $b b^{\prime} \in \mathbf{B}$.
- $\nu(\mathbf{B})$ is a submonoid in $\mathbb{Z}^{m}$.

If $\nu$ is injective, then, clearly, the basis $\mathbf{B}$ is naturally labeled by the monoid $\nu(\mathbf{B})$. It is also clear that if $\nu: \mathcal{A} \backslash\{0\} \rightarrow \mathbb{Z}^{m}$ is a valuation, then $\left.\nu\right|_{\mathbf{B}}$ is a $\mathbf{B}$-prevaluation of $\mathcal{A}$.

The following are immediate
Lemma 3.43. If $\nu, \nu^{\prime}$ are injective $\mathbf{B}$-prevaluations, then the assignments $a \rightarrow$ $\nu\left(\nu^{-1}(b)\right)$ define a bijection $\mathbf{K}_{\nu, \nu^{\prime}}: \nu(\mathbf{B}) \widetilde{\rightarrow} \nu^{\prime}(\mathbf{B})$ (we refer to it as a generalized JB bijection).

Problem 3.44. Suppose that $\mathbb{k}$ is a ring and $A$ is a finitely generated and finitely presented commutative algebra over $\mathbb{k}$. If $A$ is a free $\mathbb{k}$-module, does it admit a standard basis $\mathbf{B}$ (i.e., as in Theorem 3.39 i)?

Problem 3.45. Using an adapted basis B of Theorem 3.39 i), we can define a multivariate Hilbert series of $A$ by

$$
\operatorname{Hilb}(A)=\sum_{b \in \mathbf{B}} b
$$

By definition, this is a rational function with denominator being the product of $\left(1-x_{i}\right)$.

Therefore, we can define a multivariate Hilbert polynomial of $A$ as the "numerator" of $\operatorname{Hilb}(A)$. The question is whether this definition gives more information about $G r A$ and $A$ than the ordinary Hilbert series $\operatorname{Hilb}(A, t)$.

Remark 3.46. The adapted basis $\mathbf{B}$ produced in Theorem 3.39 ii) consists of the following elements: for each $c \in C$ take the monomial $M$ in $a_{1}, \ldots, a_{m}$ being minimal (with respect to $f$ ) among the monomials for which $\nu(M)=c$ holds.

Another description is that $\mathbf{B}$ consists of all the monomials being $\mathbb{k}$-linearly independent (in $A$ ) from less (with respect to $f$ ) monomials. For any monomial $M_{0} \in \mathbf{B}$ consider the next (with respect to $f$ ) monomial $M_{1} \in \mathbf{B}$. Then for any monomial $M$ such that $f\left(M_{0}\right) \leq f(M)<f\left(M_{1}\right)$ it holds $\nu(M)=\nu\left(M_{0}\right)$.

Corollary 3.47. Let $A$ be a commutative $\mathbb{k}$-algebra, $\nu: A \backslash\{0\} \rightarrow C$ be an injective valuation onto a finitely-generated monoid $C$ of rank $r$ endowed with a linear well ordering. Then $r=d:=\operatorname{dim}(A)$.

Proof. First we show that $r \leq d$. Pick independent elements $c_{1}, \ldots, c_{r} \in C$ and $a_{1}, \ldots, a_{r} \in A$ such that $\nu\left(a_{i}\right)=c_{i}, 1 \leq i \leq r$. Then all monomials in $a_{1}, \ldots, a_{r}$ have pairwise distinct valuations $\nu$, therefore $a_{1}, \ldots, a_{r}$ are algebraically independent, thus $r \leq d$. Now we prove the opposite inequality.

Due to Theorem 3.39 B is the complement of a monomial ideal generated by the leading monomials of Gröbner basis of the ideal $J$ in the representation $A=$ $\mathbb{k}\left[a_{1}, \ldots, a_{m}\right] / J$. Therefore, there exist $1 \leq l_{1}<\cdots<l_{d} \leq m$ such that all the monomials in $a_{l_{1}}, \ldots, a_{l_{d}}$ belong to $\mathbf{B}$, see Proposition 3 in Chapter 9.1 and Proposition 4 in Chapter 9.3 [12]. Hence the elements $\nu\left(a_{l_{1}}\right), \ldots, \nu\left(a_{l_{d}}\right)$ are independent in $C$, taking into account that the basis $\mathbf{B}$ is adapted to $\nu$ due to Theorem 3.39 ii). Thus, $r \geq d$.

Remark 3.48. Assume that $C$ is a (not necessary commutative) monoid generated by $c_{1}, \ldots, c_{r}$. We call the length $|c|$ of $c \in C$ the minimal length of words in $c_{1}, \ldots, c_{r}$ equal $c$. Let $C$ be endowed with a linear well-ordering $\prec$ compatible with the length, i.e. $\left|c_{0}\right|<|c|, c_{0}, c \in C$ implies $c_{0} \prec c$. For example, the ordering described prior to Theorem 3.1 of the free monoid is compatible with the length.

Consider an algebra $A$ having an injective valuation $\nu: A \rightarrow C$, and pick elements $a_{1}, \ldots, a_{r} \in A$ such that $\nu\left(a_{i}\right)=c_{i}, 1 \leq i \leq r$. For each $c \in C$ choose a monomial $a_{c}$ in $a_{1}, \ldots, a_{r}$ for which $\nu\left(a_{c}\right)=c$. Then $\left\{a_{c}: c \in C\right\}$ form an adapted basis of $A$ (and $a_{1}, \ldots, a_{r}$ form a Khovanskii basis of $A$ ). Then the linear subspaces $A_{k}:=\{a \in A:|\nu(a)| \leq k\}, k \geq 0$ constitute a filtration of $A$, and $\operatorname{dim} A_{k}$ coincides with the cardinality of the set $C_{k}:=\{c \in C:|c| \leq k\}$, moreover $\nu\left(A_{k}\right)=C_{k}$. We recall that in the commutative case the latter cardinality grows polynomially in $k$ (being a Hilbert polynomial, see e.g. [21]).

The following remark is inverse to Theorem 3.39 ii ) and to Remark 3.46. According to Theorem 3.39 ii) and to Remark 3.46 every injective well-ordered valuation on an algebra can be obtained as described in the remark.

Remark 3.49. Let $a_{1}, \ldots, a_{m}$ be generators of a commutative $\mathbb{k}$-algebra $A$ endowed with a linear order $f$ on monomials in $a_{1}, \ldots, a_{m}$ (compatible with the product).

Consider the family $\mathbf{B}$ of all the monomials being $\mathbb{k}$-linearly independent in $A$ from the less ones (with respect to $f$ ). Then $\mathbf{B}$ forms a basis of $A$. For each $M_{1}, M_{2} \in \mathbf{B}$ denote by $h\left(M_{1}, M_{2}\right)\left(=h\left(M_{2}, M_{1}\right)\right) \in \mathbf{B}$ the leading monomial in the $\mathbb{k}$-linear expansion of the product $M_{1} M_{2}$ in $\mathbf{B}$. Assume that for any pair of monomials $M_{0}, M_{1} \in \mathbf{B}$ fulfilling $f\left(M_{0}\right)<f\left(M_{1}\right)$ it holds $f\left(h\left(M_{0}, M_{2}\right)\right)<f\left(h\left(M_{1}, M_{2}\right)\right)$. Then one can introduce a monoid $C$ being in a bijective correspondence with $\mathbf{B}$ determined by the monoid operation $h$ and the linear ordering $f$.

This induces also an injective well-ordered valuation $\nu: A \backslash\{0\} \rightarrow C$ defined by the leading monomial from $\mathbf{B}$ in the $\mathbb{k}$-linear expansion. Then $\mathbf{B}$ is an adapted basis of $\nu$.

One can reorder the monomials in $a_{1}, \ldots, a_{m}$ as follows to make the new ordering $\triangleleft$ similar to Theorem 3.39 ii ) and to Remark 3.46. We say that for a pair of monomials it holds $a_{1}^{i_{1}} \cdots a_{m}^{i_{m}} \triangleleft a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}$ if either $f\left(\nu\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}\right)\right)<f\left(\nu\left(a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}\right)\right)$ or $f\left(\nu\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}\right)\right)=f\left(\nu\left(a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}\right)\right)$ and $f\left(a_{1}^{i_{1}} \cdots a_{m}^{i_{m}}\right)<f\left(a_{1}^{j_{1}} \cdots a_{m}^{j_{m}}\right)$. Then the construction from Theorem 3.39 ii ) applied to $\triangleleft$ produces the same basis $\mathbf{B}$ which now satisfies the properties from Remark 3.46.

Observe that the valuation produced in Theorem 3.21 fulfills the conditions of Theorem 3.39 ii ) and of Remark 3.46. In particular, $\nu$ admits an adapted basis of monomials in $X_{1}, \ldots, X_{n}$. The monomials with equal values of $\nu$ lie in the planes parallel to $H$.
3.7. Injective valuations, filtrations and deformations. Now one can establish an inverse statement to Theorem 3.21.

Theorem 3.50. Let $A$ be a commutative domain of dimension $d$ endowed with an injective well-ordered valuation $\nu$ onto a finitely-generated monoid. Then there exist a Khovanskii basis $X_{1}, \ldots, X_{n}$ of $A$ such that $A=\mathbb{k}\left[X_{1}, \ldots, X_{n}\right] / I$, and $\nu$ is obtained as in Theorem 3.21.

In other words, there is a prop subplane $H \subset \mathbb{R}^{n}$ of dimension $n-d$, being a common subplane for the tropical variety $\operatorname{Trop}(I) \subset \mathbb{R}^{n}$. Moreover, the ideal I is saturated with respect to $H$. There exists a hyperplane $Q \in \operatorname{ETrop}(I) \subset(\mathbb{R}[\varepsilon])^{n}$ which contains the subplane $H \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$, and $Q$ is determined by a suitable vector $\left(w_{1}, \ldots, w_{n}\right) \in\left(\mathbb{R}_{\geq 0}[\varepsilon]\right)^{n}$. Then $\nu$ is defined by (3.3), the valuation monoid $\nu(A \backslash$ $\{0\})=\varphi\left(\mathbb{Z}_{\geq 0}^{n}\right)$ where $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} / H$, and the linear order on $\varphi\left(\mathbb{Z}_{\geq 0}^{n}\right) \ni \varphi\left(i_{1}, \ldots, i_{n}\right)$ is determined by the value of $w_{1} i_{1}+\cdots+w_{n} i_{n}$.

Proof. Applying Theorem 3.39 one can find generators $X_{1}, \ldots, X_{n}$ of $A$ such that the valuation monoid $C:=\nu(A \backslash\{0\})$ equals the set of values $\nu(M)$ over all the monomials $M=X_{1}^{i_{1}} \cdots X_{n}^{i_{n}}$ in $X_{1}, \ldots, X_{n}$. Then $A=\mathbb{k}\left[X_{1}, \ldots, X_{n}\right] / I$ for an appropriate ideal $I$.

Due to [31] there exist elements $w_{1}, \ldots, w_{n} \in \mathbb{R}_{\geq 0}[\varepsilon]$ such that the linear order in $C$ of $\nu\left(X_{1}^{i_{1}} \cdots X_{n}^{i_{n}}\right)$ coincides with the order of the values of $w_{1} i_{1}+\cdots+w_{n} i_{n}$ in the semi-ring $\mathbb{R}_{\geq 0}[\varepsilon]$.

Denote by $H \subset \mathbb{R}^{n}$ a plane being the linear hull of all the vectors of the form $\left(l_{1}, \ldots, l_{n}\right)-\left(j_{1}, \ldots, j_{n}\right) \in \mathbb{Z}^{n}$ where $w_{1} l_{1}+\cdots+w_{n} l_{n}=w_{1} j_{1}+\cdots+w_{n} j_{n}$, the latter is equivalent to $\nu\left(X_{1}^{l_{1}} \cdots X_{n}^{l_{n}}\right)=\nu\left(X_{1}^{j_{1}} \cdots X_{n}^{j_{n}}\right)$. Due to Theorem 3.39 the hyperplane $Q$ contains $H \bigotimes_{\mathbb{R}} \mathbb{R}(\varepsilon)$ and supports the Newton polytope $N(g) \bigotimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$ for any $g \in I$. Theorem 3.39 also implies that $\operatorname{dim}(H)=n-d$. Hence $H$ is a common subplane of $\operatorname{Trop}(I)$. In addition, $H$ is prop since $w_{1}, \ldots, w_{n} \in \mathbb{R}_{\geq 0}[\varepsilon]$.

Take two arbitrary points $\left(l_{1}, \ldots, l_{n}\right),\left(j_{1}, \ldots, j_{n}\right) \in \mathbb{Z}_{\geq 0}^{n}$ such that $\left(l_{1}, \ldots, l_{n}\right)$ $\left(j_{1}, \ldots, j_{n}\right) \in H$. Then $w_{1} l_{1}+\cdots+w_{n} l_{n}=w_{1} j_{1}+\cdots+w_{n} j_{n}$. Therefore $\nu\left(X_{1}^{l_{1}} \cdots X_{n}^{l_{n}}\right)=$ $\nu\left(X_{1}^{j_{1}} \cdots X_{n}^{j_{n}}\right)$, and due to injectivity and well-ordering of $\nu$ there exist $\beta \in \mathbb{k}^{*}$ and
$g_{1}=\sum_{S} \gamma_{S} X^{S} \in \mathbb{k}\left[X_{1}, \ldots, X_{n}\right]$ such that $\nu\left(X^{S}\right)<\nu\left(X_{1}^{j_{1}} \cdots X_{n}^{j_{n}}\right)$ for every $X^{S}$ occurring in $g_{1}$, and it holds $X_{1}^{l_{1}} \cdots X_{n}^{l_{n}}-\beta X_{1}^{j_{1}} \cdots X_{n}^{j_{n}}-g_{1} \in I$. Hence $I$ is saturated with respect to $H$ (cf. Theorem 3.21).

Finally, for any $a \in A^{*}$ one can uniquely express $a=\sum_{b \in B_{0}} \alpha_{b} b$ in a basis $\mathbf{B} \supset$ $B_{0} \ni b$ produced in Theorem 3.39, $\alpha_{b} \in \mathbb{k}^{*}$. Then $\nu(a)=\max _{b \in B_{0}}\{\nu(b)\}$, thus $\nu$ is defined by (3.3).

We say that the linear order $\prec$ defined by $w_{1}, \ldots, w_{n} \in \mathbb{R}_{\geq 0}[\varepsilon]$ is archimedian if among $w_{1}, \ldots, w_{n}$ there are no infinitesimals. This is equivalent to that for any pair of monomials $m_{1}, m_{2} \neq 1$ there exists an integer $N$ such that $m_{1} \prec m_{2}^{N}$. The linear order on $\mathbb{Z}_{\geq 0}^{n}$ is archimedian iff this order is isomorphic to $\mathbb{Z}_{\geq 0}$. For instance, deglex is archimedian, while lex is not.

More generally, we say that a linear order $\prec$ on a commutative monoid $C$ is archimedian if for any elements $1 \neq c_{1}, c_{2} \in C$ there exists an integer $N$ such that $c_{2} \prec N c_{1}$. Note that if for any $c \in C$ there is at most a finite number of elements $c_{0} \in C$ such that $c_{0} \prec c$ then $C$ is archimedian and well-ordered. Conversely, if a commutative monoid $C$ is finitely-generated and $\prec$ is an archimedian linear order on $C$ then for any $c \in C$ there is at most a finite number of elements of $C$ less than $c$. In particular, in this case $C$ is well-ordered. For a not necessary commutative monoid $C$ we also say that a linear order $\prec$ on it is archimedian if for any $c \in C$ there is at most a finite number of elements of $C$ less than $c$.

Let $c_{1}, \ldots, c_{k} \in C$ be a set of generators of a commutative monoid $C$. Due to [31] there exist elements $w_{1}, \ldots, w_{k} \in \mathbb{R}_{\geq 0}[\varepsilon]$ not being infinitesimals such that
$\nu\left(j_{1} c_{1}+\cdots+j_{k} c_{k}\right) \preceq \nu\left(l_{1} c_{1}+\cdots+l_{k} c_{k}\right) \Leftrightarrow\left(w_{1} j_{1}+\cdots+w_{k} j_{k} \leq w_{1} l_{1}+\cdots+w_{k} l_{k}\right)$ for any $j_{1}, \ldots, j_{k}, l_{1}, \ldots, l_{k} \in \mathbb{Z}_{\geq 0}$. Define a function $W: C \rightarrow \mathbb{R}_{\geq 0}[\varepsilon]$ as follows:

$$
W\left(j_{1} c_{1}+\cdots+j_{k} c_{k}\right):=w_{1} j_{1}+\cdots+w_{k} j_{k} .
$$

Remark 3.51. Let $A$ be a commutative domain endowed with a valuation (not necessary injective) onto a finitely-generated monoid $C$ with an archimedian linear order $\prec$. For each $s \in \mathbb{Z}_{\geq 0}$ consider the set

$$
A_{s}:=\left\{a \in A^{*}: W(\nu(a)) \leq s\right\} \cup\{0\} .
$$

The sequence $A_{0} \subset A_{1} \subset \cdots$ constitutes a filtration of $A$. Observe that $\operatorname{dim}\left(A_{s}\right)$ is finite since $\prec$ is archimedian.

Remark 3.52. Now let $A$ be a (not necessary commutative) $\mathbb{k}$-algebra endowed with an injective valuation $\nu$ to a (not necessary commutative) monoid $C$ with a linear order $\prec$. Assume also that there is a function $f: C \rightarrow \mathbb{Z}_{\geq 0}$ such that $\left(c_{1} \preceq c_{2}\right) \Rightarrow$ $\left(f\left(c_{1}\right) \leq f\left(c_{2}\right)\right), c_{1}, c_{2} \in C$ and $f\left(c_{1}+c_{2}\right) \leq f\left(c_{1}\right)+f\left(c_{2}\right)$ satisfying the property that the set $C_{n}:=\{c \in C: f(c) \leq n\}$ is finite for any $n \in \mathbb{Z}_{\geq 0}$. Note that the latter implies that the order $\prec$ is archimedian. Then the subspaces $A_{n}:=\{a \in A$ : $f(\nu(a)) \leq n\} \cup\{0\}, n \in \mathbb{Z}_{\geq 0}$ provide a filtration of $A$ (cf. Remark 3.51) such that $\operatorname{dim} A_{n}=\left|C_{n}\right|$.

Now we assume that the field $\mathbb{k}$ is radically closed (i.e. each root of an arbitrary degree of any element of $\mathbb{k}$ also belongs to $\mathbb{k}$ ). Let $A$ be a $d$-dimensional $\mathbb{k}$-algebra
with an injective valuation $\nu: A \backslash\{0\} \rightarrow C$ where $C$ is a finitely-generated monoid endowed with a linear well-ordering. Applying Theorem 3.39 construct a Khovanskii basis $x_{1}, \ldots, x_{n} \in A$, then $A=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] / I$ for a suitable ideal $I \subset \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$.

Denote by $S \subset \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ a binomial ideal generated by elements of the form

$$
\begin{equation*}
s:=\alpha x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}-\beta x_{1}^{j_{1}} \cdots x_{n}^{j_{n}}, \alpha, \beta \in \mathbb{k}^{*} \tag{3.8}
\end{equation*}
$$

such that $\nu\left(x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}\right)=\nu\left(x_{1}^{j_{1}} \cdots x_{n}^{j_{n}}\right)$, and there exists an element $g \in I$ which is the sum of $s$ and of monomials in $x_{1}, \ldots, x_{n}$ having valuation $\nu$ less than $\nu\left(x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}\right)$. Denote a binomial algebra $M:=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] / S$. Observe that for any pair of vectors $\left(i_{1}, \ldots, i_{n}\right),\left(j_{1}, \ldots, j_{n}\right) \in \mathbb{Z}_{\geq 0}^{n}$ there exist $s$ and $g$ as above iff $\nu\left(x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}\right)=$ $\nu\left(x_{1}^{j_{1}} \cdots x_{n}^{j_{n}}\right)$ due to Theorem 3.39. In addition, $\nu\left(x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}\right)=\nu\left(x_{1}^{j_{1}} \cdots x_{n}^{j_{n}}\right)$ is equivalent to that $\left(i_{1}, \ldots, i_{n}\right)-\left(j_{1}, \ldots, j_{n}\right) \in H$ where an $(n-d)$-dimensional subplane $H$ was constructed in the proof of Theorem 3.50.

Proposition 3.53. (cf. [18]). Let $a \mathbb{k}$-algebra $A$ have an injective valuation $\nu$ : $A^{*} \rightarrow C$ where a finitely-generated monoid $C$ is endowed with a linear well-ordering. Assume that $\mathbb{k}$ is radically closed. Then both the associated graded algebra gr $A:=$ $\bigoplus_{c \in C} A_{\leq c} / A_{<c}$ and the binomial algebra $M$ are isomorphic to the monoidal algebra $\mathbb{k} C$.

Proof. There exist $\gamma_{1}, \ldots, \gamma_{n} \in \mathbb{k} \backslash\{0\}$ such that the mapping

$$
\mu: x_{1}^{i_{1}} \cdots x_{n}^{i_{n}} \rightarrow \gamma_{1}^{i_{1}} \cdots \gamma_{n}^{i_{n}} \cdot \nu\left(x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}\right)
$$

provides an isomorphism of $M$ and $\mathbb{k} C$ because of Theorem 3.39, taking into account that $\mathbb{k}$ is radically closed. In other words, if $g \in I$ equals the sum of a binomial (3.8) and of monomials with the valuation $\nu$ less than $\nu\left(x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}\right)=\nu\left(x_{1}^{j_{1}} \cdots x_{n}^{j_{n}}\right)$ then $\mu\left(\alpha x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}\right)=\mu\left(\beta x_{1}^{j_{1}} \cdots x_{n}^{j_{n}}\right)$, i.e. $\alpha \gamma_{1}^{i_{1}} \cdots \gamma_{n}^{i_{n}}=\beta \gamma_{1}^{j_{1}} \cdots \gamma_{n}^{j_{n}}$.

Any element $a \in A \backslash\{0\}$ with the valuation $\nu(a)=c$ can be represented uniquely as a $\mathbb{k}$-linear combination of elements of a basis $\mathbf{B}$ constructed in Theorem 3.39, among which $\alpha x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}, \alpha \in \mathbb{k}^{*}, x_{1}^{i_{1}} \cdots x_{n}^{i_{n}} \in \mathbf{B}$ has the maximal valuation $\nu\left(x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}\right)=c$. We define a mapping $\sigma(a):=\mu\left(\alpha x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}\right) \in \mathbb{k} C$. Then $\sigma$ defines a correct mapping on $\operatorname{gr} A$.

To verify that $\sigma$ is a homomorphism on $g r A$ take monomials $u:=x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}, v:=$ $x_{1}^{k_{1}} \cdots x_{n}^{k_{n}} \in A$. Due to Theorem 3.39 there exists a unique monomial $x_{1}^{l_{1}} \cdots x_{n}^{l_{n}} \in \mathbf{B}$ such that $\nu\left(x_{1}^{l_{1}} \cdots x_{n}^{l_{n}}\right)=\nu(u v)$. Hence there exists $\alpha \in \mathbb{k}^{*}$ such that $\nu\left(\alpha x_{1}^{l_{1}} \cdots x_{n}^{l_{n}}-\right.$ $u v)<\nu(u v)$ due to Theorem 3.39. Therefore

$$
\sigma(u v)=\mu\left(\alpha x_{1}^{l_{1}} \cdots x_{n}^{l_{n}}\right)=\alpha \gamma_{1}^{l_{1}} \cdots \gamma_{n}^{l_{n}} \nu(u v)=\gamma_{1}^{i_{1}+k_{1}} \cdots \gamma_{n}^{i_{n}+k_{n}} \nu(u) \nu(v)=\sigma(u) \sigma(v)
$$

(we use the product notation for the monoid operation).
Finally, one can check that $\sigma$ is an isomorphism.
Corollary 3.54. Let $A$ be $a \mathbb{k}$-domain of a dimension $d$ over a radically closed field $\mathbb{k}$, and $\nu$ be an injective valuation of $A^{*}$ onto a well-ordered finitely-generated monoid. Then the variety $\operatorname{Spec}(\operatorname{gr}(A, \nu))$ is toric of dimension $d$.
Example 3.55. Consider a domain $A:=\mathbb{k}[x, y] /\left(x^{6}-2 y^{4}-1\right)$. Take a common line $H$ from the tropical variety $\operatorname{Trop}\left(x^{6}-2 y^{4}-1\right)$ defined by the equation $2 i+3 j=6$ (cf. Theorem 3.21) and a corresponding map $\nu:\left\{x^{i} y^{j}: i, j \geq 0\right\} \rightarrow \mathbb{Z}_{\geq 0}^{2}$ for which
$\nu(x)=2, \nu(y)=3$. We obtain that the graded algebra $\operatorname{gr}(A, \nu)$ has a zero divisor iff the polynomial $x^{6}-2 y^{4}$ is reducible over $\mathbb{k}$. In the latter case $\nu$ does not provide a valuation on $A \backslash\{0\}$ (see Theorem 3.1), and the variety $\operatorname{Spec}(g r(A, \nu))$ is reducible. Otherwise, if $x^{6}-2 y^{4}$ is irreducible over $\mathbb{k}$ then $\nu$ provides a valuation on $A^{*}$ being not injective since $\nu\left(x^{3}\right)=\nu\left(y^{2}\right)=6$, so $\operatorname{dim}\left(A_{\nu \leq 6} / A_{\nu<6}\right)=2$.

Example 3.56. We give an example of a non-commutative algebra admitting an injective valuation onto $\mathbb{Z}_{\geq 0}^{2}$ such that its graded algebra (with respect to the valuation) is non-commutative (unlike Proposition 3.53). Denote by $A_{q}$ a quantum $\mathbb{k}(q)$-algebra generated by $x, y$ satisfying a relation $x y=q^{2} y x$. Then $\nu(x)=(0,1), \nu(y)=(1,0)$ defines an injective valuation $\nu: A_{q} \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{2}$. The graded algebra $g r A_{q}$ has an adapted basis $\left\{b(m, n):=q^{-m n} x^{m} y^{n}: m, n \geq \overline{0}\right\}$ with a multiplication table

$$
b(m, n) b\left(m_{1}, n_{1}\right)=q^{m n_{1}-m_{1} n} b\left(m+m_{1}, n+n_{1}\right) .
$$

Thus, $g r A_{q}$ is a twisted monoidal algebra.
In contrast to $A_{q}$, Weyl algebra generated by $x, y$ satisfying a relation $x y=y x+1$ admitting an injective valuation defined by $\nu(x)=(1,0), \nu(y)=(0,1)$ onto $\mathbb{Z}_{\geq 0}^{2}$, has a graded algebra isomorphic to the polynomial ring $\mathbb{k}[x, y]$.

Let $A=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] / J$ be a $\mathbb{k}$-algebra with $\operatorname{dim} A=1$ and $\nu: A^{*} \rightarrow C \subset \mathbb{Z}_{\geq 0}$ be an injective valuation (cf. Corollary 3.47). Then there exist non-negative integers $r_{1}, \ldots, r_{n}$ such that $\nu\left(x_{1}^{i_{n}} \cdots x_{n}^{i_{n}}\right) \preceq \nu\left(x_{1}^{j_{1}} \cdots x_{n}^{j_{n}}\right)$ iff $i_{1} r_{1}+\cdots+i_{n} r_{n} \leq j_{1} r_{1}+\cdots+j_{n} r_{n}$ (cf. Theorem 3.50).

Take $t \in \mathbb{k}^{*}$, and for any polynomial $g \in \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ with a monomial $x_{1}^{i_{n}} \cdots x_{n}^{i_{n}}$ of the highest valuation $\nu\left(x_{1}^{i_{n}} \cdots x_{n}^{i_{n}}\right)=c$ among the monomials of $g$, replace $x_{i}$ by $x_{i} t^{-r_{i}}, 1 \leq i \leq n$. The resulting polynomial has the form $\left(g_{0}+t g_{1}\right) t^{i_{1} r_{1}-\cdots-i_{n} r_{n}}$ where $g_{0}$ coincides with the sum of the monomials of $g$ having their valuation equal $c$, while every monomial in $g_{1}$ has the valuation less than $c$. The resulting ideal we denote by $J_{t} \subset \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ and denote $A_{t}:=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] / J_{t}$.

Thus, one can view the family $A_{t}, t \in \mathbb{k}^{*}$ as a deformation of $A_{0}$ being a binomial algebra isomorphic to $\mathbb{k} C$ (see Proposition 3.53 ) when the field $\mathbb{k}$ is radically closed. Summarizing, we have established the following proposition.

Proposition 3.57. Let $A=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] / J$ be $a \mathbb{k}$-algebra with $\operatorname{dim} A=1$ and $\nu: A \backslash\{0\} \rightarrow C \subset \mathbb{Z}_{\geq 0}$ be an injective valuation. Then there exist non-negative integers $r_{1}, \ldots, r_{n}$ such that for any $t \in \mathbb{k}^{*}$ the transformation $x_{i} \rightarrow x_{i} t^{-r_{i}}, 1 \leq i \leq n$ provides an ideal $J_{t} \subset \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ and an algebra $A_{t}:=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] / J_{t}$. Obviously, $A_{1}=A$. The associated graded algebra gr $A_{t} \simeq \mathbb{k} C, t \in \mathbb{k}^{*}$ (see Proposition 3.53) when $\mathbb{k}$ is radically closed. One can view the family $A_{t}, t \in \mathbb{k}^{*}$ as a deformation of $A_{0}$ being a binomial algebra isomorphic to $\mathbb{k} C$ (see Proposition 3.53) when the field $\mathbb{k}$ is radically closed.

Conjecture 3.58. For any domain $A$ with injective well-ordered valuation $\nu$ there is a family $A_{t}$ such that $A_{1}=A$ and $A_{0}=g r A$ with respect to the filtration on $A$ induced by $\nu$ (as in Proposition 3.57).

Problem 3.59. Let $E$ be a locally nilpotent derivation of a domain $A$ (see Lemma 4.9). Classify all subalgebras $B$ of $A$ such that $\lambda_{E}(B \backslash\{0\}) \cup\{0\}$ is a subalgebra of $A$.

Problem 3.60. Classify all injective decorated valuations $(\nu, \lambda)$ on $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ (see Definition 4.5).

Problem 3.61. Given an injective valuation $\nu: \mathbb{k}\left[x_{1}, \ldots, x_{m}\right] \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{m}$, is it true that $C_{\nu}$ is a finitely generated submonoid of $\mathbb{Z}_{\geq 0}^{m}$ ?

Given a submonoid $M$ of $\mathbb{Z}^{m}$, denote $\bar{M}:=\left(\mathbb{R}_{\geq 0} \otimes M\right) \cap \mathbb{Z}^{m}$ and refer to it as the saturation of $M$. By definition, $\bar{M}$ is a submonoid of $\mathbb{Z}^{m}$ and $M$ is a submonoid of $\bar{M}$. We say that $M$ is saturated if $\bar{M}=M$.

Problem 3.62. Suppose that $\nu$ is a saturated injective valuation $A \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{m}$. Is it true that $\operatorname{Spec} A$ is smooth or rational?

Problem 3.63. If $A \subset \mathbb{k}\left[x_{1}, \ldots, x_{m}\right]$ and $\operatorname{dim} A=d<m$. Can $A$ be embedded into $\mathbb{k}\left[y_{1}, \ldots, y_{d}\right]$ ?
Problem 3.64. Describe all injective valuation on $\mathbb{k}\left[x_{1}, \ldots, x_{m}\right]$ into $\mathbb{Z}_{\geq 0}^{m}$ whose valuation monoid is finitely generated but not saturated.
3.8. Algorithm testing a family of generators of a valuation monoid. Let $\left(f_{1}, \ldots, f_{m}\right): \mathbb{K}^{n} \rightarrow \mathbb{k}^{m}, m \leq n$ be a dominant polynomial map, i.e. the polynomials $f_{1}, \ldots, f_{m} \in \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ are algebraically independent (cf. Example 3.90). This provides an injective homomorphism $\mathbb{k}\left[y_{1}, \ldots, y_{m}\right] \hookrightarrow \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$, and thereby an injective valuation $\nu: \mathbb{k}\left[y_{1}, \ldots, y_{m}\right] \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{n}$ (we fix some injective valuation on $\left.\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] \backslash\{0\}\right)$ due to Lemma 3.13 .

Problem 3.65. Is the valuation monoid $\nu\left(\mathbb{k}\left[y_{1}, \ldots, y_{m}\right] \backslash\{0\}\right) \subset \mathbb{Z}_{\geq 0}^{n}$ finitely-generated?
The goal of this subsection is to prove the following proposition.
Proposition 3.66. Let polynomials $f_{1}, \ldots, f_{m} \in \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$, $m \leq n$ define an injective homomorphism $\mathbb{k}\left[y_{1}, \ldots, y_{m}\right] \hookrightarrow \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$. Given a computable injective valuation on $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] \backslash\{0\}$ to $\mathbb{Z}_{\geq 0}^{n}$ (e.g., lex or deglex), this provides an injective valuation $\nu: \mathbb{k}\left[y_{1}, \ldots, y_{m}\right] \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{n}$.

There is an algorithm which given generators $g_{1}, \ldots, g_{p} \in \mathbb{k}\left[y_{1}, \ldots, y_{m}\right]$ of $\mathbb{k}\left[y_{1}, \ldots, y_{m}\right]$ tests whether the elements

$$
\nu\left(g_{1}\right), \ldots, \nu\left(g_{p}\right) \in C:=\nu\left(\mathbb{k}\left[y_{1}, \ldots, y_{m}\right] \backslash\{0\}\right) \subset \mathbb{Z}_{\geq 0}^{n}
$$

generate the valuation monoid $C$. If not then the algorithm yields an element $c \in$ $C \backslash \mathbb{Z}_{\geq 0}\left\{\nu\left(g_{1}\right), \ldots, \nu\left(g_{p}\right)\right\}$.

Proof. We have $\mathbb{k}\left[y_{1}, \ldots, y_{m}\right]=\mathbb{k}\left[g_{1}, \ldots, g_{p}\right] / I$ for a suitable ideal $I$. Consider a (non-strict) linear ordering $\prec$ on monomials in $g_{1}, \ldots, g_{p}$ according to $\nu$. Denote by $H \subset \mathbb{R}^{p}$ a (rational) $(p-m)$-dimensional plane such that $\nu\left(g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}\right)=\nu\left(g_{1}^{j_{1}} \cdots g_{p}^{j_{p}}\right)$ iff $\left(i_{1}-j_{1}, \ldots, i_{p}-j_{p}\right) \in H$ (cf. the proof of Theorem 3.39), in other words, $\left(i_{1}, \ldots, i_{p}\right) \preceq\left(j_{1}, \ldots, j_{p}\right) \preceq\left(i_{1}, \ldots, i_{p}\right)$ (slightly abusing the notations we identify a monomial $g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}$ with the vector $\left(i_{1}, \ldots, i_{p}\right)$ ). Define a linear ordering $\triangleleft$ on monomials in $g_{1}, \ldots, g_{p}$ as follows: $g_{1}^{i_{1}} \cdots g_{p}^{i_{p}} \triangleleft g_{1}^{j_{1}} \cdots g_{p}^{j_{p}}$ iff either $g_{1}^{i_{1}} \cdots g_{p}^{i_{p}} \prec g_{1}^{j_{1}} \cdots g_{p}^{j_{p}}$ or $\nu\left(g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}\right)=\nu\left(g_{1}^{j_{1}} \cdots g_{p}^{j_{p}}\right)$ and the vector $\left(i_{1}, \ldots, i_{p}\right)$ is less than $\left(j_{1}, \ldots, j_{p}\right)$ in deglex (again cf. the proof of Theorem 3.39).

The algorithm constructs a Gröbner basis of $I$ with respect to $\triangleleft$. Denote by $G \subset \mathbb{Z}_{\geq 0}^{p}$ the complement to the monomial ideal of leading monomials of the Gröbner basis. $\overline{\text { Then }} G$ is a finite (disjoint) union of sets of the form

$$
u+\left\{\left(u_{1}, \ldots, u_{p}\right): u_{i} \vdash_{i} 0\right\}
$$

where each $\vdash_{i}, 1 \leq i \leq p$ is either $=$ or $\geq$.
Consider a plane $H_{0} \subset \mathbb{R}^{p}$ parallel to $H$ which has a common point with $\mathbb{Z}_{\geq 0}^{p}$. Then $H_{0} \cap G \neq \emptyset$. Indeed, otherwise for any point $g_{1}^{i_{1}} \cdots g_{p}^{i_{p}} \in H_{0}$ we get (taking into account properties of Gröbner bases) that

$$
g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}=\sum_{L} \alpha_{L} g^{L}, \alpha_{L} \in \mathbb{k}^{*}, \nu\left(g^{L}\right) \prec \nu\left(g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}\right) \text { for each } L,
$$

which contradicts to the subadditivity of the valuation.
First, assume that for every $H_{0}$ parallel to $H$ such that $H_{0} \cap \mathbb{Z}_{\geq 0}^{p} \neq \emptyset$ it holds $\mid G \cap$ $H_{0} \mid=1$. Then we fall in the conditions of Theorem 3.21, and thus $\nu\left(g_{1}\right), \ldots, \nu\left(g_{p}\right)$ constitute a family of generators of $\nu\left(\mathbb{k}\left[y_{1}, \ldots, y_{m}\right] \backslash\{0\}\right)=\mathbb{Z}_{\geq 0}^{p} / H_{\mathbb{Z}}$.

Now on the contrary, assume that $\left|G \cap H_{0}\right| \geq 2$ for some $H_{0}$ parallel to $H$. In this case the algorithm can find a pair of different monomials $g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}, g_{1}^{j_{1}} \cdots g_{p}^{j_{p}} \in G \cap H_{0}$ for some $H_{0}$ invoking integer linear programming. If the algorithm fails, it means that $\left|G \cap H_{0}\right|=1$ for all $H_{0}$ (see the previous case). Since the valuation $\nu$ is injective, there exists $\alpha \in \mathbb{k}^{*}$ such that

$$
c_{0}:=\nu\left(g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}-\alpha g_{1}^{j_{1}} \cdots g_{p}^{j_{p}}\right) \prec \nu\left(g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}\right)=\nu\left(g_{1}^{j_{1}} \cdots g_{p}^{j_{p}}\right)
$$

If $\mathbb{Z}_{\geq 0}^{p} \ni c_{0} \notin \mathbb{Z}_{\geq 0}\left\{\nu\left(g_{1}\right), \ldots, \nu\left(g_{p}\right)\right\}$ then $c:=c_{0}$ meets the requirements of the Proposition.

Otherwise, if $c_{0} \in \mathbb{Z}_{\geq 0}\left\{\nu\left(g_{1}\right), \ldots, \nu\left(g_{p}\right)\right\}$ there exists a monomial $g_{1}^{l_{1}} \cdots g_{p}^{l_{p}}$ such that $\nu\left(g_{1}^{l_{1}} \cdots g_{p}^{l_{p}}\right)=c_{0}$. Again due to the injectivity there exists $\beta \in \mathbb{k}^{*}$ for which it holds

$$
\nu\left(g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}-\alpha g_{1}^{j_{1}} \cdots g_{p}^{j_{p}}-\beta g_{1}^{l_{1}} \cdots g_{p}^{l_{p}}\right) \prec c_{0} .
$$

Observe that $g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}-\alpha g_{1}^{j_{1}} \cdots g_{p}^{j_{p}}-\beta g_{1}^{l_{1}} \cdots g_{p}^{l_{p}} \neq 0$, because otherwise this would contradict to that $g_{1}^{i_{1}} \cdots g_{p}^{i_{p}}, g_{1}^{j_{1}} \cdots g_{p}^{j_{p}} \in G$. Continuing in a similar way, the algorithm eventually arrives at a required element $c \in C \backslash \mathbb{Z}_{\geq 0}\left\{\nu\left(g_{1}\right), \ldots, \nu\left(g_{p}\right)\right\}$ since $C$ is well-ordered.

Remark 3.67. In the proof of the latter Proposition we used $f_{1}, \ldots, f_{m}$ only to be able to compute $\nu(g)$ for $g \in \mathbb{k}\left[y_{1}, \ldots, y_{m}\right]$. In fact, one could stick with an arbitrary computable injective valuation.
3.9. The space of injective valuations on a domain. For a $\mathbb{k}$-domain $A$ we consider the space $V:=V(A)$ of all injective valuations $\nu: A \backslash\{0\} \rightarrow \mathbb{R}_{\geq 0}$. Given a basis $\mathbf{B}$ of $A$ endowed with a linear order $\prec$ one can consider a set $V_{\mathbf{B}, \prec}$ of mappings $\nu: \mathbf{B} \rightarrow \mathbb{R}_{\geq 0}$ such that for any $b, b_{0} \in \mathbf{B}$ a relation $b \prec b_{0}$ implies $\nu(b)<\nu\left(b_{0}\right)$, and for any $b_{1}, \bar{b}_{2} \in \mathbf{B}$ for which

$$
b_{1} b_{2}=\alpha_{b_{0}} b_{0}+\sum_{b \in \mathbf{B}} \alpha_{b} b, \alpha_{b_{0}}, \alpha_{b} \in \mathbb{k}, b \prec b_{0}
$$

it holds $\nu\left(b_{0}\right)=\nu\left(b_{1}\right)+\nu\left(b_{2}\right)$. Then $\nu$ induces an injective valuation on $A$ with an adapted basis $\mathbf{B}$ : namely, for any $a=\alpha_{b_{0}} b_{0}+\sum_{b \in \mathbf{B}} \alpha_{b} b, b \prec b_{0}$ we define $\nu(a):=$ $\nu\left(b_{0}\right)$. One can define a topology on $V$ with open basic sets $V_{\mathbf{B}, \prec}$ for all $\mathbf{B}, \prec$.

Question. Is $V$ connected?
Recall (see the proof of Theorem 3.50) that for any valuation $\nu: A \backslash\{0\} \rightarrow C$ onto a well-ordered finitely-generated monoid $C$ one can find generators $X_{1}, \ldots, X_{m}$ of $A$ and elements $w_{1}, \ldots, w_{m} \in \mathbb{R}_{\geq 0}[\varepsilon]$ such that $\nu\left(X_{1}\right), \ldots, \nu\left(X_{m}\right)$ generate $C$, and the order on monomials $i_{1} \nu\left(X_{1}\right)+\cdots+i_{m} \nu\left(X_{m}\right)$ is determined by $w_{1} i_{1}+\cdots+$ $w_{m} i_{m}$. Choosing a basis $\mathbf{B}$ among monomials of the form $X_{1}^{i_{1}} \cdots X_{m}^{i_{m}}$ and defining $\nu\left(X_{1}^{i_{1}} \cdots X_{m}^{i_{m}}\right):=w_{1} i_{1}+\cdots+w_{m} i_{m}$, one obtains that $\nu \in V$, provided that $w_{1}, \ldots, w_{m} \in \mathbb{R}_{\geq 0}$.
3.10. Injective well-ordered valuations of 2-dimensional algebras. Let $A:=$ $\mathbb{k}[x, y, z] /(f)$ be a 2 -dimensional algebra where $f \in \mathbb{k}[x, y, z]$. Consider a valuation $\nu: A \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{2}$ studied in Theorem 3.21. Then $\nu$ is induced by an edge $e$ of the Newton polytope $N(f) \subset \mathbb{R}_{\geq 0}^{3}$ and a 2-dimensional plane $Q \subset \mathbb{R}^{3}$ containing $e$. Note that in the notations of Theorem $3.21 H$ is the line passing through $e$, and $Q \in \operatorname{Trop}(f)$. Moreover, the proof of Theorem 3.21 and Remark 3.20 imply that if $\nu$ is injective then the endpoints of $e$ (up to a permutation of the coordinates $x, y, z$ ) are $(p, 0,0),(0, q, r)$ where $p, q, r \in \mathbb{Z}_{\geq 0}$ have no common divisor.

Lemma 3.68. If two edges $e_{1}, e_{2}$ of the Newton polytope $N(f) \subset \mathbb{R}_{\geq 0}^{3}$ induce injective valuations then $e_{1}, e_{2}$ have a common vertex located on a coordinate line.

Proof. We say that a point $v \in N(f)$ belongs to a roof of $N(f)$ if on a ray emanating from the origin $(0,0,0)$ and passing through $v$, the latter is the last point from $N(f)$ on the ray. Then $e_{1}, e_{2}$ lie on the roof.

Consider a 2-dimensional plane $Q_{1} \subset \mathbb{R}^{3}$ which contains $e_{1}$ and supports $N(f)$. The projection of the roof to $Q_{1}$ by means of the rays contains the projections of $e_{1}$ and of $e_{2}$. If $e_{1}, e_{2}$ had no common vertex located on a coordinate line then the projections of $e_{1}, e_{2}$ would have a common point being internal in the projection of either $e_{1}$ or $e_{2}$. The obtained contradiction completes the proof.
Corollary 3.69. All the edges of the Newton polytope $N(f)$ inducing injective valuations either
i) have a common vertex located on a coordinate line or
ii) form a triangle with its vertices on the coordinate lines, and in this case the roof of $N(f)$ consists of this triangle.

Remark 3.70. i) Observe that in case i) of Corollary 3.69 when all the edges have a common vertex $(p, 0,0)$, a common adapted basis of all the injective valuations induced by the edges, is $\left\{x^{i} y^{j} z^{k}: 0 \leq i<p, 0 \leq j, k<\infty\right\}$ (cf. the proof of Theorem 3.21, Remark 3.20 and Theorem 3.39).
ii) One can verify that in case ii) of Corollary 3.69 three injective valuations do not possess a common adapted basis.

For example, let $f:=z+x^{2}+y^{3}$. Then $A:=\mathbb{k}[x, y, z] /(f) \simeq \mathbb{k}[x, y]$. Denote by $\nu_{x}$ the injective valuation induced by the edge $\left(z, y^{3}\right)$ with respect to lex ordering in which $y>x$, and an adapted basis $\left\{y^{i} x^{j}: 0 \leq i, j<\infty\right\}$. Denote by $\nu_{y}$ the
injective valuation induced by the edge $\left(z, x^{2}\right)$ with respect to lex ordering in which $x>y$, and an adapted basis $\left\{x^{i} y^{j}: 0 \leq i, j<\infty\right\}$. Finally, denote by $\nu_{z}$ the injective valuation induced by the edge $\left(y^{3}, x^{2}\right)$ with respect to lex ordering in which $x, y>z$, and $\nu_{z}(x)=(3,0), \nu_{z}(y)=(2,0), \nu_{z}(z)=(0,1)$. An adapted basis of $\nu_{z}$ is $\left\{y^{i} z^{j}, x y^{i} z^{j}: 0 \leq i, j<\infty\right\}$ (cf. Examples 3.6, 3.38).

One can compute all three JHb (see Theorem 4.24):

$$
\begin{gathered}
\mathbf{K}_{\nu_{z}, \nu_{y}}(2 j, i)=(2 i, j), \mathbf{K}_{\nu_{z}, \nu_{y}}(2 j+1, i)=(2 i+3, j) ; \\
\mathbf{K}_{\nu_{z}, \nu_{x}}(3 j, i)=(3 i, j), \mathbf{K}_{\nu_{z}, \nu_{x}}(3 j+1, i)=(3 i+2, j), \mathbf{K}_{\nu_{z}, \nu_{x}}(3 j+2, i)=(3 i+4, j) ; \\
\mathbf{K}_{\nu_{x}, \nu_{y}}(j, i)=(i, j) .
\end{gathered}
$$

One can generalize Corollary 3.69 (in one direction) to hypersurfaces of arbitrary dimensions.

Remark 3.71. Let $A:=\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] /(f)$ where $f \in \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$. An edge $e$ of the Newton polytope $N(f) \subset \mathbb{R}_{\geq 0}^{n}$ induces an injective valuation $\nu$ on $A \backslash\{0\}$ iff the endpoints of $e$ are $v=\left(v_{1}, \ldots, v_{n}\right), u=\left(u_{1}, \ldots, u_{n}\right) \in \mathbb{Z}_{\geq 0}^{n}$ such that $\min \left\{v_{i}, u_{i}\right\}=$ $0,1 \leq i \leq n$, and the integers $\max \left\{v_{i}, u_{i}\right\}, 1 \leq i \leq n$ have no common divisor (cf. Theorem 3.21 and Remark 3.20). Denote by $H \subset \mathbb{R}^{n}$ the line containing $e$.

Denote by $\prec$ a linear ordering of the valuation cone (being a subset of $\mathbb{Z}^{n-1}$ ) of $\nu$. Following the proof of Theorem 3.39 ii) one can extend $\prec$ to a linear ordering $q$ on $\mathbb{Z}_{\geq 0}^{n}$ in two different ways according to an ordering (direction) in $H$. Then according to one of these two choices of $q$ either $u$ or $v$ becomes a leading monomial of $f$ with respect to $q$. Any of these two choices of $q$ provides an adapted basis of $A$ with respect to $\nu$ being a complement of a principal monomial ideal in $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ (see Theorem 3.39).

If edges $e_{1}, \ldots, e_{s}$ of $N(f)$ induce injective valuations $\nu_{1}, \ldots, \nu_{s}$, respectively, of $A^{*}$ and have a common vertex in $N(f)$ then $\nu_{1}, \ldots, \nu_{s}$ possess a common adapted basis being a complement of a principal monomial ideal in $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$.

Example 3.72. Now we give an example of a pair of injective valuations on a 3dimensional algebra $A:=\mathbb{k}[x, y, z, t] /\left(f:=x^{2}+y^{3}+z^{5}+t^{7}\right)$ to $\mathbb{Z}_{\geq 0}^{3}$ endowed with the lexicographical ordering. Each pair of monomials of $f$ provides an injective valuation of $A \backslash\{0\}$ (see Theorem 3.21 and Proposition 3.25). In particular, a pair of monomials $x^{2}, y^{3}$ provides an injective valuation $\nu_{1}$ for which it holds

$$
\nu_{1}(x)=(3,0,0), \nu_{1}(y)=(2,0,0), \nu_{1}(z)=(0,1,0), \nu_{1}(t)=(0,0,1)
$$

In its turn, a pair of monomials $z^{5}, t^{7}$ provides an injective valuation $\nu_{2}$ for which it holds

$$
\nu_{2}(z)=(7,0,0), \nu_{2}(t)=(5,0,0), \nu_{2}(x)=(0,1,0), \nu_{2}(y)=(0,0,1)
$$

Denote $w:=x^{2}+y^{3}$. Then $\nu_{1}, \nu_{2}$ have a common adapted basis

$$
\left\{y^{i} t^{j} w^{l} x^{k} z^{m}: i, j, l \geq 0,0 \leq k \leq 1,0 \leq m \leq 4\right\}
$$

and $\nu_{1}(w)=\nu_{1}\left(-z^{5}-t^{7}\right)=(0,5,0), \nu_{2}(w)=(0,2,0)$.

One can generalize this construction to a polynomial of the form $f:=\sum_{1 \leq i \leq n} x_{i}^{q_{i}}$ where $q_{i}, 1 \leq i \leq n$ are pairwise relatively prime. Each pair of monomials of $f$ provides an injective valuation of $\mathbb{k}\left[x_{1} \ldots, x_{n}\right] /(f)^{*}$ into $\mathbb{Z}_{\geq 0}^{n-1}$.

### 3.11. Enumerating injective well-ordered valuations of a hypersurface of a prime degree at a main variable.

Remark 3.73. Let a polynomial $f=y^{d}+f_{1} \in \mathbb{k}\left[y, x_{1}, \ldots, x_{n}\right]$ be normalized with respect to $y$, i.e. $\operatorname{deg}_{y}\left(f_{1}\right)<d$ (one can reduce to this situation invoking Noether normalization). Denote $A:=\mathbb{k}\left[y, x_{1}, \ldots, x_{n}\right] /(f)$. We say that an edge of Newton polytope $N_{f} \subset \mathbb{R}^{n+1}$ is long if its endpoints are $(d, 0, \ldots, 0),\left(0, i_{1}, \ldots, i_{n}\right)$ and $G C D\left(d, i_{1}, \ldots, i_{n}\right)=1$.

Then the domain $A \backslash\{0\}$ admits an injective well-ordered valuation into $\mathbb{Q}_{\geq 0}^{n}$ with an adapted basis

$$
\left\{y^{k} x_{1}^{j_{1}} \cdots x_{n}^{j_{n}}: 0 \leq k<d, 0 \leq j_{1}, \ldots, j_{n}<\infty\right\}
$$

and

$$
\nu\left(x_{l}\right)=e_{l}, 1 \leq l \leq n, \nu(y)=\frac{i_{1} e_{1}+\cdots+i_{n} e_{n}}{d}
$$

where $e_{l}=(0, \ldots, 0,1,0, \ldots, 0) \in \mathbb{Z}^{n}, 1 \leq l \leq n$ is an ort vector (cf. Proposition 3.8).
Observe that for any valuation (not necessary injective or well-ordered) on $A \backslash$ $\{0\}$ for which $\nu\left(x_{l}\right)=e_{l}, 1 \leq l \leq n$ there exists an edge of $N_{f}$ with endpoints $\left(d_{1}, j_{1}, \ldots, j_{n}\right),\left(d_{2}, k_{1}, \ldots, k_{n}\right)$ such that

$$
\begin{equation*}
\nu(y)=\frac{\left(k_{1}-j_{1}\right) e_{1}+\cdots+\left(k_{n}-j_{n}\right) e_{n}}{d_{1}-d_{2}} . \tag{3.9}
\end{equation*}
$$

As a more general setting than in Remark 3.73 we assume that a domain $A$ is a finite integral extension of $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ of a rank $d$. We study injective well-ordered valuations $\nu$ on $A \backslash\{0\}$ into $\mathbb{Q}_{\geq 0}^{n}$ for which $\nu\left(x_{l}\right)=e_{l}, 1 \leq l \leq n$.

The following construction is valid for a valuation $\nu$ not necessary injective or wellordered. Denote by $G(\nu) \subset \mathbb{Q}^{n} / \mathbb{Z}^{n}$ the image $\nu(A \backslash\{0\}) / \mathbb{Z}^{n}$. Note that $G(\nu)$ is an abelian semigroup. Moreover, for every element $a \in A \backslash\{0\}$ there exists a polynomial $h \in \mathbb{k}\left[z, x_{1}, \ldots, x_{n}\right]$ such that $h(a)=0$ and $\operatorname{deg}_{z}(h) \leq d$. As in Remark 3.73 there is an edge of Newton polytope $N_{h} \subset \mathbb{R}^{n+1}$ with endpoints $\left(d_{1}, j_{1}, \ldots, j_{n}\right),\left(d_{2}, k_{1}, \ldots, k_{n}\right)$ such that

$$
\nu(a)=\frac{\left(k_{1}-j_{1}\right) e_{1}+\cdots+\left(k_{n}-j_{n}\right) e_{n}}{d_{1}-d_{2}}
$$

(see (3.9)). Hence $\left(d_{1}-d_{2}\right) \nu(a)$ is the unit element in $G(\nu)$, thus $G(\nu)$ is a group. Moreover,

$$
G(\nu) \subset \frac{\mathbb{Z}^{n}}{L C M\{1, \ldots, d\}} / \mathbb{Z}^{n}
$$

in particular, $G(\nu)$ is finite. We call $G(\nu)$ the group of the valuation.
Lemma 3.74. Let a domain $A$ be a free $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$-module of a rank $d$. Let $\nu$ be a well-ordered valuation of $A \backslash\{0\}$ being an extension into $\mathbb{Q}_{\geq 0}^{n}$ of a valuation on $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right] \backslash\{0\}$ with $\nu\left(x_{l}\right)=e_{l}, 1 \leq l \leq n$. Then for the size $s$ of the group of the valuation $G(\nu)$ holds
i) $s \leq d$;
ii) if $\nu$ is injective and archimedian then $s=d$.

Proof. i) Let $b_{1}, \ldots, b_{d} \in A$ be a free $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$-basis of $A$. Pick $t_{1}, \ldots, t_{s} \in$ $A \backslash\{0\}$ such that $\nu\left(t_{l}\right)-\nu\left(t_{j}\right) \notin \mathbb{Z}^{n}$ for $1 \leq l \neq j \leq s$. Express $t_{j}=\sum_{1 \leq i \leq d} h_{j, i} b_{i}$ for appropriate polynomials $h_{j, i} \in \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$. Denote

$$
M_{N}:=\left\{x_{1}^{l_{1}} \cdots x_{n}^{l_{n}}: 0 \leq l_{1}+\cdots+l_{n} \leq N\right\}, W_{N}:=t_{1} M_{N}+\cdots+t_{s} M_{N}
$$

for an integer $N$. Then $\operatorname{dim}_{\mathbb{k}} W_{N}=s\left|M_{N}\right|$ due to the valuation property. On the other hand,

$$
W_{N} \subset b_{1} M_{N+c}+\cdots+b_{d} M_{N+c}
$$

for a suitable constant $c \in \mathbb{Z}_{\geq 0}$. Therefore, considering sufficiently big $N$, we obtain that $s \leq d$.
ii) Denote

$$
V_{N}:=\left\{\left(i_{1}, \ldots, i_{n}\right) \in \mathbb{Z}_{\geq 0}^{n}: \nu\left(i_{1}, \ldots, i_{n}\right) \leq N\right\}
$$

(we identify the archimedian valuation with defining it linear form). Then $\left|V_{N}\right| \sim$ $c_{0} \cdot N^{n}$ for an appropriate $0<c_{0} \in \mathbb{R}$. Observe that $\nu$ on $\mathbb{Q}_{\geq 0}^{n}$ is defined by the same linear form as $\nu$ is.

Denote by $g_{1}, \ldots, g_{s}$ the unique representatives of the elements of $G(\nu)$ in the cube $[0,1)^{n}$. Then

$$
\nu\left(b_{1} \cdot V_{N}+\cdots+b_{d} \cdot V_{N}\right) \subset g_{1} \cdot V_{N+c} \bigsqcup \cdots \bigsqcup g_{s} \cdot V_{N+c}
$$

for a suitable constant $c$. Since $\nu$ is injective this implies that $s \geq d$ taking into account that $\operatorname{dim}\left(b_{1} \cdot V_{N}+\cdots+b_{d} \cdot V_{N}\right)=d\left|V_{N}\right|$.

Remark 3.75. One can literally extend Lemma 3.74 to a domain $A \supset \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ such that $\mathbb{k}\left(x_{1}, \ldots, x_{n}\right)$-dimension of $A \otimes_{\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]} \mathbb{k}\left(x_{1}, \ldots, x_{n}\right)$ equals $d$.

It would be interesting to clarify whether Lemma 3.74 and Remark 3.75 are true for not necessary archimedian valuations.
Corollary 3.76. Under the conditions of Lemma 3.74 or Remark 3.75 in case of a square-free d the group $G(\nu)$ is cyclic of size d and every its generating element has a form $\frac{i_{1} e_{1}+\cdots+i_{n} e_{n}}{d}$ where $G C D\left(d, i_{1}, \ldots, i_{n}\right)=1$.

Now let $A:=\mathbb{k}\left[y, x_{1}, \ldots, x_{n}\right] /(f)$ be as in Remark 3.73 and $d$ be a prime. Our goal is to design an algorithm which enumerates all injective well-ordered archimedian valuations on $A \backslash\{0\}$ into $\mathbb{Q}_{\geq 0}^{n}$ (with $\nu\left(x_{l}\right)=e_{l}, 1 \leq l \leq n$, cf. Corollary 3.76).

The algorithm produces a finite tree $T$ by recursion. Some leaves of $T$ correspond to injective well-ordered valuations on $A \backslash\{0\}$. As a base of recursion a root of $T$ is produced.

As a recursive hypothesis assume that at a vertex $v$ of a depth $s$ of $T$ a constructible set $U_{v} \subset \mathbb{k}^{s}$ and a set of monomials $m_{1}, \ldots, m_{s} \in \mathbb{Z}_{\geq 0}^{n}$ are produced (we identify monomials in the variables $x_{1}, \ldots, x_{n}$ with $\mathbb{Z}_{\geq 0}^{n}$ ). We suppose that $m_{s}$ does not belong to the monomial ideal generated by $m_{1}, \ldots, m_{s-1}$. In addition, the algorithm produces a polynomial $f_{v}=y^{d}+f_{v, 1} \in \mathbb{k}\left[y, x_{1}, \ldots, x_{n}, z_{1}, \ldots, z_{s}\right]$ (where $\operatorname{deg}_{y}\left(f_{v, 1}\right)<$ d) such that for any point $\left(\alpha_{1}, \ldots, \alpha_{s}\right) \in U_{v}$ it holds

$$
\begin{equation*}
f_{v}\left(y-\alpha_{1} m_{1}-\cdots-\alpha_{s} m_{s}, x_{1}, \ldots, x_{n}, \alpha_{1}, \ldots, \alpha_{s}\right)=0 \tag{3.10}
\end{equation*}
$$

and that Newton polytopes $N_{f_{v}} \subset \mathbb{R}^{n+1}$ are the same for all the points $\left(\alpha_{1}, \ldots, \alpha_{s}\right) \in$ $U_{v}$.

Now we proceed to the description of the recursive step of the algorithm. First assume that Newton polytope $N_{f_{v}}$ has a long edge with endpoints $(d, 0, \ldots, 0),\left(0, i_{1}, \ldots, i_{n}\right)$, denote by $g\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{k}\left[z_{1}, \ldots, z_{n}\right]$ the coefficient of $f_{v}$ at the monomial $x_{1}^{i_{1}} \cdots x_{n}^{i_{n}}$. The algorithm verifies (invoking linear programming) whether there exists a linear archimedian ordering $\succ$ (compatible with addition) on $\mathbb{Q}_{\geq 0}^{n}$ such that $m_{1} \succ \cdots \succ$ $m_{s} \succ \frac{i_{1} e_{1}+\cdots+i_{n} e_{n}}{d}$ (otherwise, the algorithm ignores the long edge under consideration). If such an ordering does exist then Remark 3.73 provides an injective well-ordered valuation $\nu$ on $A^{*}$ such that

$$
\nu\left(y-\alpha_{1} m_{1}-\cdots-\alpha_{s} m_{s}\right)=\frac{i_{1} e_{1}+\cdots+i_{n} e_{n}}{d}
$$

Thus, as an adapted basis of $A$ with respect to $\nu$ one can take

$$
\left\{\left(y-\alpha_{1} m_{1}-\cdots-\alpha_{s} m_{s}\right)^{k} \cdot x_{1}^{j_{1}} \cdots x_{n}^{j_{n}}\right\}, 0 \leq k<d, 0 \leq j_{1}, \ldots, j_{n}<\infty
$$

The algorithm produces a vertex being a son of $v$ and a leaf in $T$ which corresponds to $\nu$.

Now we consider a not long edge of $N_{f_{v}}$ with endpoints $\left(d_{1}, j_{1}, \ldots, j_{n}\right),\left(d_{2}, k_{1}, \ldots, k_{n}\right)$ (obviously, $0 \leq d_{1} \neq d_{2} \leq d$ ). Denote

$$
m_{s+1}:=\frac{\left(k_{1}-j_{1}\right) e_{1}+\cdots+\left(k_{n}-j_{n}\right) e_{n}}{d_{1}-d_{2}}
$$

(cf. (3.9), (3.10)), provided that $m_{s+1} \in \mathbb{Z}_{\geq 0}^{n}$. Observe that if $m_{s+1} \notin \mathbb{Z}_{\geq 0}^{n}$ then for no element $a \in A$ it holds $\nu(a)=m_{s+1}$ for an injective well-ordered archimedian valuation $\nu$ because of Corollary 3.76.

The algorithm verifies (invoking linear programming) whether there exists a linear archimedian ordering $\succ$ on $\mathbb{Z}_{>0}^{n}$ such that $m_{1} \succ \cdots \succ m_{s} \succ m_{s+1}$ (otherwise, the algorithm ignores the edge of $N_{f_{v}}$ under consideration). The latter is necessary because the algorithm looks for an injective well-ordered valuation $\nu$ such that $\nu(y-$ $\left.\alpha_{1} m_{1}-\cdots-\alpha_{s} m_{s}\right)=\nu\left(m_{s+1}\right)\left(=m_{s+1}\right)$. Note that in particular, the existence of a suitable linear ordering $\succ$ implies that $m_{s+1}$ does not belong to the monomial ideal generated by $m_{1}, \ldots, m_{s}$.

The algorithm calculates a polynomial $g \in \mathbb{k}\left[y, x_{1}, \ldots, x_{n}, z_{1}, \ldots, z_{s}, z_{s+1}\right]$ such that

$$
g\left(y-\alpha_{1} m_{1}-\cdots-\alpha_{s} m_{s}-z_{s+1} m_{s+1}, x_{1}, \ldots, x_{n}, \alpha_{1}, \ldots, \alpha_{s}, z_{s+1}\right)=0
$$

for any point $\left(\alpha_{1}, \ldots, \alpha_{s}\right) \in U_{v}$ (cf. (3.10)). Note that it still holds $g=y^{d}+g_{1}$ where $\operatorname{deg}_{y}\left(g_{1}\right)<d$. For different values of $z_{s+1}$ in $\mathbb{k}$ there is a finite number of possible shapes of Newton polytopes $N_{g} \subset \mathbb{R}^{n+1}$ (which are determined by their vertices). For each fixed shape the algorithm produces a vertex $w$ being a son of $v$ in $T$ together with a constructible set $U_{w} \subset \mathbb{k}^{s+1}$ assuring the fixed shape. We put a polynomial $f_{w}:=g$. This completes the description of the recursive step of the algorithm.

Observe that the tree $T$ is finite since along every its path the monomial ideal generated by $m_{1}, \ldots, m_{s}$ strictly increases and therefore, the path is finite due to Hilbert's Idealbasissatz (also we make use of König's Lemma). Summarizing, we have established the following proposition.

Proposition 3.77. There is an algorithm which for a polynomial $f=y^{d}+f_{1} \in$ $\mathbb{k}\left[y, x_{1}, \ldots, x_{n}\right]$ with a prime $d$ where $\operatorname{deg}_{y}\left(f_{1}\right)<d$, enumerates all injective wellordered archimedian valuations on $\left(\mathbb{k}\left[y, x_{1}, \ldots, x_{n}\right] /(f)\right) \backslash\{0\}$.

It would be interesting to generalize the latter proposition to arbitrary affine algebras. The next example demonstrates that it is not possible to generalize it directly even to domains being free $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$-modules of composite ranks.

Example 3.78. A domain $A=\mathbb{k}\left[x^{1 / 2}, y^{1 / 2}\right]$ is 4-dimensional free $\mathbb{k}[x, y]$-module. Then $\nu\left(x^{1 / 2}\right)=e_{1} / 2, \nu\left(y^{1 / 2}\right)=e_{2} / 2$ and the group $G(\nu)$ is isomorphic to $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$. Note that an element $z:=x^{1 / 2}+y^{1 / 2}$ has a minimal polynomial of degree 4 , namely $z^{4}-2(x+y) z^{2}+x^{2}+y^{2}-4 x y=0$ whose Newton polytope has no long edge (cf. Remark 3.73).

Remark 3.79. Let a domain $A$ be a $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$-module, admitting an injective well-ordered valuation $\nu$ satisfying the conditions of Remark 3.75 such that $d$ is square-free. Due to Corollary 3.76 the group $G(\nu)$ is cyclic of the size $d$. Pick $y \in A$ such that $\nu(y) / \mathbb{Z}^{n}$ is a generator of $G(\nu)$. Denote by $f \in \mathbb{k}\left[x_{1}, \ldots, x_{n}, y\right]$ the minimal polynomial of $y$ over $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$. Then $\operatorname{deg}_{y} f=d$ since Newton polytope of $f$ contains an edge with a denominator equal to $d$, and $f=q y^{d}+\cdots, q \in$ $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$. The domain $A_{0}:=\mathbb{k}\left[x_{1}, \ldots, x_{n}, q y\right]$ is a free $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$-module with a basis $1, q y, \ldots,(q y)^{d-1}$, and Newton polytope of the minimal polynomial $q^{d-1} f$ of $q y$ contains a long edge. According to Remark 3.73 this long edge provides on $A_{0} \backslash\{0\}$ the valuation coinciding with the restriction of $\nu$.

Furthermore, this restriction is extended uniquely to $A \backslash\{0\}$. Namely, for any $a \in A \backslash\{0\}$ there exists $p \in \mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ such that $p a \in A_{0} \backslash\{0\}$, therefore $\nu(a)=$ $\nu(p a)-\nu(p)$.
3.12. Convexity of the extended Jordan-Hölder bijection for valuations in an archimedian monoid. Let $\nu: A \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{n}$ be an injective valuation in a monoid with archimedian linear order (cf. Remark 3.51). We say that $\nu$ is finitary if the ordering $\prec$ on vectors $v=\left(v_{1}, \ldots, v_{n}\right) \in \mathbb{Z}_{\geq 0}^{n}$ is determined by a linear function $\alpha(v):=\alpha_{1} v_{1}+\cdots+\alpha_{n} v_{n}$ where positive reals $\alpha_{1}, \ldots, \alpha_{n}$ are $\mathbb{Q}$-linearly independent. Then the ordering $\prec$ is isomorphic to $\mathbb{Z}_{\geq 0}$, i.e. the monoid $\mathbb{Z}_{>0}^{n}$ is archimedian. For instance, lex is not finitary. In particular, for any $c \in \nu(A \backslash\{0\})$ the $\mathbb{k}$-linear space $A_{\leq c}:=\{a \in A \backslash\{0\}: \nu(a) \preceq c\} \cup\{0\}$ is finite-dimensional.

Let $\nu_{0}: A \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{n}$ be another injective valuation (not necessary archimedain). Denote the valuation cones $C:=\nu(A \backslash\{0\}), C_{0}:=\nu_{0}(A \backslash\{0\})$. Consider a convex cone $C_{0}^{(\mathbb{Q})}:=C_{0} \otimes \mathbb{Q} \geq 0$ and denote

$$
S_{0}:=C_{0}^{(\mathbb{Q})} \cap\left\{\left(v_{1}, \ldots, v_{n}\right) \in \mathbb{Q}_{\geq 0}^{n}: v_{1}+\cdots+v_{n}=1\right\} .
$$

The following map

$$
\mathbf{K}\left(c_{0}\right):=\min _{\prec}\left\{\nu\left(\nu_{0}^{-1}\left(c_{0}\right)\right)\right\} \in C
$$

is a bijection $\mathbf{K}: C_{0} \widetilde{\rightarrow} C$ called generalized JHb (see Theorem 4.24). Denote a function $\overline{\mathbf{K}}:=\alpha \circ \mathbf{K}: C_{0} \rightarrow \mathbb{R}_{\geq 0}$. One can define $\overline{\mathbf{K}}$ on $S_{0}$ (so, on rational points) as follows. For a point $u:=\left(u_{1}, \ldots, u_{n}\right) \in S_{0}$ we have $\overline{\mathbf{K}}((p+q) u) \leq \overline{\mathbf{K}}(p u)+\overline{\mathbf{K}}(q u)$, provided that $p u, q u \in C_{0}$, since the subadditivity $\overline{\mathbf{K}}\left(w_{1}+w_{2}\right) \leq \overline{\mathbf{K}}\left(w_{1}\right)+\overline{\mathbf{K}}\left(w_{2}\right)$
holds for any elements $w_{1}, w_{2} \in C_{0}$. Therefore, due to Fekete's subadditivity lemma [33] there exists the limit

$$
\underline{\mathbf{K}}(u):=\lim _{p \rightarrow \infty, p u \in C_{0}} \frac{\overline{\mathbf{K}}(p u)}{p} .
$$

Proposition 3.80. Function $\underline{\mathbf{K}}: S_{0} \rightarrow \mathbb{R}_{\geq 0}$ is convex.
Proof. Consider a convex combination $w=\sum_{i} \lambda_{i} w_{i}$ of points $w, w_{i} \in S_{0}$ where $0<\lambda_{i} \in \mathbb{Q}, \sum_{i} \lambda_{i}=1$ for all $i$. Then for a suitable $0<s \in \mathbb{Z}$ it holds $s w, s w_{i} \in$ $C_{0}$ for all $i$. Denote by $q$ the common denominator of all $\lambda_{i}$. Hence $\overline{\mathbf{K}}(p q s w) \leq$ $\sum_{i} \overline{\mathbf{K}}\left(p q s \lambda_{i} w_{i}\right)$ for any $p \in \mathbb{Z}_{\geq 0}$ because of the subadditivity of $\overline{\mathbf{K}}$. Dividing the both sides of the latter inequality by $p q s$ and tending $p$ to infinity, we conclude that $\underline{\mathbf{K}}(w) \leq \sum_{i} \lambda_{i} \underline{\mathbf{K}}\left(w_{i}\right)$, which completes the proof.

Remark 3.81. We have taken an ordering $\prec$ to be archimedian since otherwise one can't assure an inequality after tending to a limit. For example, in lex ordering each element of a sequence $(1-1 / p, 1) \in \mathbb{Q}_{\geq 0}^{2}$ is less than $(1,0)$, while their limit against $p$ is not.

Corollary 3.82. One can extend $\underline{\mathbf{K}}$ (from rational points) to real points in the interior $\operatorname{int}\left(S_{0} \otimes \mathbb{R}_{\geq 0}\right)$ being a (continuous) convex function.

Proof. $\underline{\mathbf{K}}$ is locally Lipschitz in $\operatorname{int}\left(S_{0}\right)$ (cf. e. g. [28]), hence it can be (uniquely) extended to a continuous function (moreover, locally Lipschitz) on $\operatorname{int}\left(S_{0} \otimes \mathbb{R}_{\geq 0}\right)$ which is also convex.
3.13. Jordan-Hölder bijections for valuations of an algebra. We say that a map $\mathbf{K}: P \rightarrow Q$ of ordered partial semigroups is sub-multiplicative if it satisfies the following:
$\mathbf{K}(u \circ v) \preceq \mathbf{K}(u) \circ \mathbf{K}(v)$ whenever $u \circ v$ and $\mathbf{K}(u) \circ \mathbf{K}(v)$ are defined in $P$ and in $Q$, respectively.
Proposition 3.83. Let $\nu: A \backslash\{0\} \rightarrow P, \nu_{1}: A \backslash\{0\} \rightarrow P^{\prime}$ be a pair of injective valuations of an algebra $A$ to partial semigroups $P$ and $P^{\prime}$, respectively. Then JHb $\mathbf{K}_{\nu^{\prime}, \nu}$ from $\nu(A \backslash\{0\})$ to $\nu^{\prime}(A \backslash\{0\})$ is sub-multiplicative.

Proof. For any elements $u, v \in P$ for which $u \circ v$ and $\mathbf{K}(u) \circ \mathbf{K}(v)$ are defined take $a, b \in A \backslash\{0\}$ such that $\nu(a)=u, \nu(b)=v$ and $\nu^{\prime}(a)=\mathbf{K}(u), \nu^{\prime}(b)=\mathbf{K}(v)$. Then $\nu(a b)=u \circ v$ and $\mathbf{K}(u \circ v) \preceq \nu^{\prime}(a b)=\mathbf{K}(u) \circ \mathbf{K}(v)$.

Remark 3.84. We expect that the converse also holds. If $\mathbf{K}$ is a sub-multiplicative bijection $P \widetilde{\rightarrow} Q$ of partial semigroups such that $\mathbf{K}^{-1}$ is sub-multiplicative as well, then there exist injective valuations $\nu: A \backslash\{0\} \rightarrow P, \nu^{\prime}: A \backslash\{0\} \rightarrow P^{\prime}$ of an appropriate algebra $A$ such that $\mathbf{K}=\mathbf{K}_{\nu, \nu^{\prime}}$.

Example 3.85. Consider a partial semigroup $P$ endowed with two different orders. They provide two injective valuations $\nu_{1}, \nu_{2}: \mathbb{k} P \backslash\{0\} \rightarrow P$. Then $\{[u]: u \in P\} \subset$ $\mathbb{k} P$ (Definition 2.28) is a common adapted basis for $\nu_{1}, \nu_{2}$, and the JH bijection $\mathbf{K}_{\nu_{1}, \nu_{2}}$ is identity map $I d_{P}$.

Proposition 3.86. Let $P$ be a set with two (partial) operations $(a, b) \mapsto a \circ b$ and $(a, b) \mapsto a \bullet b$ so that $P$ has partial semigroup structures $P$ 。 and $P$. respectively. Define a new operation

$$
a b:=a \circ b+a \bullet b
$$

on the vector space $\mathbb{k} P$ (with the convention if a summand is not defined it is replaced by zero) and denote this algebra by $A_{\circ}, \bullet$.
(a) $A_{\circ, \bullet}$ is associative iff $\circ$ and $\bullet$ are mutually associative:

$$
(a \circ b) \bullet c=a \bullet(b \circ c),(a \bullet b) \circ c=a \circ(b \bullet c)
$$

for all $a, b, c \in P$. We say that $(a \circ b) \bullet c$ is defined if both $a \circ b$ and $(a \circ b) \bullet c$ are defined (here we assume that $(a \circ b) \bullet c$ and $a \bullet(b \circ c)$ are defined or not defined simultaneously (as well as $(a \bullet b) \circ c$ and $a \circ(b \bullet c)$ ).
(b) Suppose additionally that both $P_{\circ}$ and $P_{\bullet}$ are ordered with $\preceq^{\circ}$ and $\preceq^{\bullet}$, respectively, and it holds

$$
a \circ b \preceq^{\bullet} a \bullet b \preceq^{\circ} a \circ b,
$$

provided that $a \circ b$ and $a \bullet b$ are defined (cf. Proposition 3.83). Then the assignment $[a] \mapsto a$ define injective valuations $\nu_{\circ}$ and $\nu_{\bullet}$ on $A_{\circ}^{*} \bullet$ to $P_{\circ}$ and to $P_{\bullet}$, respectively Moreover, identity map $P \mapsto P$ is the corresponding JH bijection, and $[P] \subset A_{\circ, \bullet}$ is a common adapted basis of valuations $\nu_{\circ}$ and $\nu_{\bullet}$.

Proof. Prove (a). Indeed,

$$
\begin{aligned}
& (a b) c=(a \circ b+a \bullet b) c=(a \circ b) c+(a \bullet b) c \\
= & (a \circ b) \circ c+(a \circ b) \bullet c+(a \bullet b) \circ c+(a \bullet b) \bullet c
\end{aligned}
$$

On the other hand,

$$
\begin{gathered}
a(b c)=a(b \circ c+b \bullet c) \\
=a \circ(b \circ c)+a \bullet(b \circ c)+a \circ(b \bullet c)+a \bullet(b \bullet c)
\end{gathered}
$$

This gives associativity because $(a \circ b) \bullet c=a \bullet(b \circ c)$ and $(a \bullet b) \circ c=a \circ(b \bullet c)$.
(b) We claim that $\nu_{\bullet}$ is a valuation. Take $a, b \in P \subset A_{\circ} \bullet$ such that $a \bullet b$ is defined. If $a \circ b \prec \bullet a \bullet b$ (or $a \circ b$ is not defined) then $\nu_{\bullet}(a b)=a \bullet b=\nu_{\bullet}(a) \bullet \nu_{\bullet}(b)$. Otherwise, if $a \circ b=a \bullet b$ then $a b=2 a \bullet b$, and again we get that $\nu_{\bullet}(a b)=a \bullet b=\nu_{\bullet}(a) \bullet \nu_{\bullet}(b)$. The claim is proved.

In a similar manner we establish that $\nu_{\mathrm{o}}$ is a valuation as well.
We mention that an issue of whether a sum of two associative products form again an associative product similar to (a) is widely studied (see, e.g. [29]), while not in the context of semigroup algebras.

Remark 3.87. Let $A$ be a $\mathbb{k}$-algebra with a basis $B$ equipped with a linear order $\prec$. For $b_{1}, b_{2} \in B$ define $b_{1} \circ b_{2} \in B$ to be the highest (with respect to $\prec$ ) element in the decomposition in $B$ of $b_{1} b_{2}$, provided that $b_{1} b_{2} \neq 0$, otherwise $b_{1} \circ b_{2}$ is not defined. Assume that the following properties are fulfilled:
i) if $b_{1} \prec b_{2}$ then $b_{1} \circ b_{0} \preceq b_{2} \circ b_{0}$, provided that $b_{1} \circ b_{0}, b_{2} \circ b_{0}$ are defined (respectively, $b_{0} \circ b_{1} \preceq b_{0} \circ b_{2}$, provided that $b_{0} \circ b_{1}, b_{0} \circ b_{2}$ are defined);
ii) $\left(b_{1} \circ b_{2}\right) \circ b_{3}=b_{1} \circ\left(b_{2} \circ b_{3}\right)$, moreover, $b_{1} \circ b_{2},\left(b_{1} \circ b_{2}\right) \circ b_{3}$ are defined iff $b_{2} \circ b_{3}, b_{1} \circ\left(b_{2} \circ b_{3}\right)$ are defined, $b_{1}, b_{2}, b_{3} \in B$.

Then $(B, \circ)$ is a partial semigroup. For any $a \in A \backslash\{0\}$ consider its decomposition $a=\lambda b+\cdots, \lambda \in \mathbb{K}^{*}$ in $B$ where $b \in B$ is the highest element of $B$ in this decomposition, we define $\nu(a):=b$. Then $\nu: A \backslash\{0\} \rightarrow B$ is an injective valuation.

Conversely, having an injective valuation $\nu: A \backslash\{0\} \rightarrow P$ and an adapted basis $B$ one defines (as above) the (partial) operation $\circ$ on $B$ such that the partial semigroup ( $B, \circ$ ) is isomorphic to $P$.

Note that for two injective valuations $\nu, \nu^{\prime}$ on $A \backslash\{0\}$ the images $\nu(A \backslash\{0\})$ and $\nu^{\prime}(A \backslash\{0\})$ are not necessarily isomorphic as (partial) semigroups. It was demonstrated in Remark 3.70, we give here more examples.

Example 3.88. Let $\varphi$ and $\psi$ are injective homomorphisms $\mathbb{k}\left[z_{1}, z_{2}\right] \rightarrow \mathbb{k}\left[t_{1}^{ \pm 1}, t_{2}^{ \pm 1}\right]$ given respectively by: $\varphi\left(z_{1}\right)=t_{1}, \varphi\left(z_{2}\right)=t_{2}, \psi\left(z_{1}\right)=t_{1}, \psi\left(z_{2}\right)=t_{1}^{2}+t_{2}$. Clearly, $C_{\varphi}=C_{\psi}=\mathbb{Z}_{\geq 0}^{2}$. One can easily see that the set

$$
\mathbf{B}=\left\{b_{\mathbf{d}}^{\varepsilon}=z_{1}^{\varepsilon} z_{2}^{d_{1}}\left(z_{2}-z_{1}^{2}\right)^{d_{2}} \mid \mathbf{d}=\left(d_{1}, d_{2}\right) \in \mathbb{Z}_{\geq 0}^{2}, \quad \varepsilon \in\{0,1\}\right\}
$$

is a basis of $\mathbb{k}\left[z_{1}, z_{2}\right]$ adapted to both $\nu_{\varphi}$ and $\nu_{\psi}$ (i.e., is an JH-basis in $\mathbb{k}\left[z_{1}, z_{2}\right]$ ) and

$$
\nu_{\varphi}\left(b_{\mathbf{d}}^{\varepsilon}\right)=\left(\varepsilon+2 d_{2}, d_{1}\right), \nu_{\psi}\left(b_{\mathbf{d}}^{\varepsilon}\right)=\left(\varepsilon+2 d_{1}, d_{2}\right)
$$

for all $\mathbf{d}=\left(d_{1}, d_{2}\right) \in \mathbb{Z}_{\geq 0}^{2}, \varepsilon \in\{0,1\}$. Therefore,

$$
\mathbf{K}_{\varphi, \psi}\left(a_{1}, a_{2}\right)=\left(2 a_{2}+\left(a_{1}\right)_{2},\left\lfloor\frac{a_{1}}{2}\right\rfloor\right)
$$

for all $a_{1}, a_{2} \in \mathbb{Z}_{\geq 0}$. where $(a)_{2}=a-2\left\lfloor\frac{a}{2}\right\rfloor$ is the parity of $a$. Moreover, $\mathbf{K}_{\varphi, \psi}$ is an involution on $\mathbb{Z}_{\geq 0}^{2}$.

Example 3.89. Let $\varphi$ be an endomorphism of $A=\mathbb{C}\left[t_{11}, t_{12}, t_{21}, t_{22}\right]$ given by $\varphi\left(\begin{array}{cc}t_{11} & t_{12} \\ t_{21} & t_{22}\end{array}\right)=\left(\begin{array}{cc}t_{11} t_{12} t_{21} & t_{11} t_{12} \\ t_{11} t_{21} & t_{11}+t_{22}\end{array}\right)$. One can show that $\varphi$ is injective, therefore, $\nu_{0} \circ \varphi$ is a well-defined injective valuation $A^{*} \rightarrow \mathbb{Z}_{\geq 0}^{4}$ (here $\nu_{0}$ denotes the tautological valuation, see subsection 3.4). A common adapted basis of injective valuations $\nu_{0}$ and $\nu_{0} \circ \varphi$ is

$$
\mathbf{B}=\left\{x_{11}^{m_{1}} x_{12}^{m_{2}} x_{21}^{m_{3}} x_{22}^{m_{4}}\left(x_{11} x_{22}-x_{12} x_{21}\right)^{m_{5}}: \text { all } m_{i} \in \mathbb{Z}_{\geq 0}, \min \left(m_{1}, m_{4}\right)=0\right\}
$$

Then $\left(\nu_{0} \circ \varphi\right)\left(A^{*}\right)$ is a submonoid of $\nu_{0}\left(A^{*}\right)=\mathbb{Z}_{\geq 0}^{4}$ given by $\left\{\left(\begin{array}{ll}c_{11} & c_{12} \\ c_{21} & c_{22}\end{array}\right): c_{11} \geq\right.$ $\left.\max \left(c_{12}, c_{21}\right), \min \left(c_{12}, c_{21}\right) \geq c_{2,2} \geq 0\right\}$, and $\mathbf{K}_{\nu_{0} \circ \varphi, \nu_{0}}$ and $\mathbf{K}_{\nu_{0}, \nu_{0} \circ \varphi}$ are given, respectively, by

$$
\begin{aligned}
& \mathbf{K}_{\nu_{0} \circ \varphi, \nu_{0}}(\mathbf{d})=\left(\begin{array}{cc}
\max \left(d_{11}, d_{22}\right)+d_{12}+d_{21} & d_{11}+d_{12} \\
d_{11}+d_{21} & \min \left(d_{11}, d_{22}\right)
\end{array}\right), \mathbf{d}=\left(\begin{array}{ll}
d_{11} & d_{12} \\
d_{21} & d_{22}
\end{array}\right) \in \nu_{0}\left(A^{*}\right) \\
& \mathbf{K}_{\nu_{0}, \nu_{0} \circ \varphi}(\mathbf{c})=\left(\begin{array}{cc}
\max \left(c_{22}, c_{12}+c_{21}-c_{11}\right) & \min \left(c_{11}-c_{21}, c_{12}-c_{22}\right) \\
\min \left(c_{11}-c_{12}, c_{21}-c_{22}\right) & \max \left(c_{22}, c_{11}+2 c_{22}-c_{12}-c_{21}\right)
\end{array}\right), \mathbf{c} \in \nu_{0} \circ \varphi\left(A^{*}\right) .
\end{aligned}
$$

Example 3.90. (String valuations and JH bijections on polynomials in 3 variables)
Let $E_{1}, E_{2}$ be (locally nilpotent) derivations of $\mathbb{k}\left[x_{1}, x_{2}, x_{3}\right]$ given by

$$
E_{1}=\frac{\partial}{\partial x_{1}}+x_{2} \frac{\partial}{\partial x_{3}}, E_{2}=\frac{\partial}{\partial x_{2}}+x_{1} \frac{\partial}{\partial x_{3}} .
$$

Clearly, the Lie algebra generated by $E_{1}, E_{2}$ is isomorphic to the $3 \times 3$ nilpotent matrices because of the defining Serre relations $\left[E_{i},\left[E_{i}, E_{3-i}\right]\right]=0$ for $i=1,2$.

Let $\mathbf{E}_{i}=\left(E_{i}, E_{3-i}, E_{i}\right)$ for $i=1,2$, in the notation of Section 4.2 and abbreviate $\nu_{i}:=\nu_{\mathbf{E}_{i}}, i=1,2$. One can show that $\nu_{i}=\nu_{0} \circ \varphi_{i}$, where $\nu_{0}: \mathbb{k}\left[t_{1}, t_{2}, t_{3}\right] \backslash\{0\}=$ $\mathbb{k}_{\mathbb{Z}}^{3}{ }_{\geq 0} \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{3}$ is the standard (tautological) valuation and $\varphi_{1}, \varphi_{2}: \mathbb{k}\left[x_{1}, x_{2}, x_{3}\right] \hookrightarrow$ $\mathbb{k}\left[t_{1}, t_{2}, t_{3}\right]$ are injective homomorphisms given respectively by

$$
\varphi_{1}\left(x_{1}, x_{2}, x_{3}\right)=\left(t_{1}+t_{3}, t_{2}, t_{1} t_{2}\right), \varphi_{2}\left(x_{1}, x_{2}, x_{3}\right)=\left(t_{2}, t_{1}+t_{3}, t_{2} t_{3}\right)
$$

It is easy to see that the basis $x^{d}=x_{1}^{d_{1}} x_{2}^{d_{2}} x_{3}^{d_{3}} d=\left(d_{1}, d_{2}, d_{3}\right) \in \mathbb{Z}_{\geq 0}^{3}$ is adapted to $\nu_{2}$ and the basis $\tilde{x}^{d}=x_{1}^{d_{1}} x_{2}^{d_{2}} x_{4}^{d_{3}}, d=\left(d_{1}, d_{2}, d_{3}\right) \in \mathbb{Z}_{\geq 0}^{3}$, where $x_{4}:=x_{1} x_{2}-x_{3}$, is adapted to $\nu_{1}$.

Actually, they have a common adapted basis. Let $b_{\mathbf{m}}=x_{1}^{m_{1}} x_{2}^{m_{2}} x_{3}^{m_{3}} x_{4}^{m_{4}}$ for $\mathbf{m}=$ $\left(m_{1}, m_{2}, m_{3}, m_{4}\right) \in \mathbb{Z}_{\geq 0}^{4}$, and let

$$
\mathbf{M}=\left\{\left(m_{1}, m_{2}, m_{3}, m_{4}\right) \in \mathbb{Z}_{\geq 0}^{4}: \min \left(m_{1}, m_{2}\right)=0\right\}
$$

Clearly, the relation $x_{1} x_{2}=x_{3}+x_{4}$ implies that the set $\mathbf{B}:=\left\{b_{\mathbf{m}}, \mathbf{m} \in \mathbf{M}\right\}$ is a basis of $\mathbb{k}\left[x_{1}, x_{2}, x_{3}\right]$. We claim that $\mathbf{B}$ is adapted to both $\nu_{1}$ and $\nu_{2}$.

Indeed, $E_{i}\left(x_{j}\right)=\delta_{i, j}$ and $E_{i}\left(x_{j+2}\right)=\delta_{i, j} x_{3-i}$ for $i, j=1,2$ and

- $\nu_{E_{i}}\left(b_{\mathbf{m}}\right)=m_{i}+m_{i+2}, \lambda_{E_{i}}\left(b_{\mathbf{m}}\right)=x_{3-i}^{m_{3-i}+m_{2+i}} x_{5-i}^{m_{5-i}}$.
- $\nu_{E_{3-i}}\left(\lambda_{E_{i}}\left(b_{\mathbf{m}}\right)\right)=m_{3-i}+m_{2+i}+m_{5-i}, \lambda_{\left(E_{i}, E_{3-i}\right)}\left(b_{\mathbf{m}}\right)=\lambda_{E_{3-i}}\left(\lambda_{E_{i}}\left(b_{\mathbf{m}}\right)\right)=x_{i}^{m_{5-i}}$.
- $\nu_{E_{i}}\left(\lambda_{\left(E_{i}, E_{3-i}\right)}\left(b_{\mathbf{m}}\right)\right)=m_{5-i}, \lambda_{\mathbf{E}_{i}}\left(b_{\mathbf{m}}\right)=\lambda_{E_{i}}\left(x_{i}^{m_{5-i}}\right)=1$.

Therefore,

$$
\nu_{i}\left(b_{\mathbf{m}}\right)=\left(m_{i}+m_{i+2}, m_{3-i}+m_{i+2}+m_{5-i}, m_{5-i}\right)
$$

and $C_{\varphi_{i}}=C_{\nu_{\mathrm{E}_{i}}}=C$ for $i=1,2$, where

$$
C=\left\{\left(a_{1}, a_{2}, a_{3}\right) \in \mathbb{Z}_{\geq 0}^{3}: a_{2} \geq a_{3}\right\}
$$

Taking into account that $\varphi_{i}^{*}\left(b_{\mathbf{m}}\right)=\left(t_{1}+t_{3}\right)^{m_{i}} t_{2}^{m_{3-i}}\left(t_{1} t_{2}\right)^{m_{2+i}}\left(t_{2} t_{3}\right)^{m_{5-i}}$, we obtain

$$
\nu_{i}\left(b_{\mathbf{m}}\right)=\nu_{0}\left(\varphi_{i}^{*}\left(b_{\mathbf{m}}\right)\right)=\left(m_{i}+m_{i+2}, m_{3-i}+m_{i+2}+m_{5-i}, m_{5-i}\right)=\nu_{i}\left(b_{\mathbf{m}}\right)
$$

The JH bijection $K_{\nu_{\mathbf{E}_{2}}, \nu_{\mathbf{E}_{1}}}: C \rightarrow C$ is given by

$$
\left(a_{1}, a_{2}, a_{3}\right) \mapsto\left(\max \left(a_{3}, a_{2}-a_{1}\right), a_{1}+a_{3}, \min \left(a_{1}, a_{2}-a_{3}\right)\right)
$$

Finally, define injective homomorphisms $\psi_{i}: \mathbb{k}\left[x_{1}, x_{2}, x_{3}\right] \hookrightarrow \mathbb{k}\left[t_{1}, t_{2}, t_{3}\right], i=1,2$ by

$$
\psi_{1}\left(x_{1}, x_{2}, x_{3}\right)=\left(t_{1}, t_{3}, t_{2}\right), \psi_{2}\left(x_{1}, x_{2}, x_{3}\right)=\left(t_{2}, t_{1} t_{2}-t_{3}, t_{1}\right)
$$

and abbreviate $\nu_{i}^{\prime}:=\nu \circ \psi_{i}$
It is easy to see that $\left.\nu_{( }^{\prime} b_{\mathbf{m}}\right)=\left(m_{i}+m_{5-i}, m_{i+2}, m_{3-i}+m_{5-i}\right)$ for $i=1,2$ and all $\mathbf{m} \in \mathbf{M}$. Therefore, $\mathbf{B}$ is adapted to both $\psi_{1}$ and $\psi_{2}$ as well and $C_{\nu_{1}^{\prime}}=C_{\nu_{2}^{\prime}}=\mathbb{Z}_{\geq 0}^{3}$. The JH bijection $K_{\nu_{2}^{\prime}, \nu_{1}^{\prime}}: \mathbb{Z}_{\geq 0}^{3} \leftrightarrows \mathbb{Z}_{\geq 0}^{3}$ is given by

$$
\left(d_{1}, d_{2}, d_{3}\right) \mapsto\left(d_{2}+\max \left(0, d_{3}-d_{1}\right), \min \left(d_{1}, d_{3}\right), d_{2}+\max \left(0, d_{1}-d_{3}\right)\right)
$$

Also, $K_{\nu_{i}, \nu_{i}^{\prime}}: \mathbb{Z}_{\geq 0}^{3} \rightarrow C$ is given by

$$
\left(d_{1}, d_{2}, d_{3}\right) \mapsto\left(d_{2}+\max \left(0, d_{1}-d_{3}\right), d_{2}+d_{3}, \min \left(d_{1}, d_{3}\right)\right)
$$

and $K_{\nu_{3-i}, \nu_{i}^{\prime}}: \mathbb{Z}_{\geq 0}^{3} \widetilde{\rightarrow} C$ is given by $\left(d_{1}, d_{2}, d_{3}\right) \mapsto\left(d_{3}, d_{1}+d_{2}, d_{2}\right)$.

Example 3.91. Now we study a non-commutative analog of Example 3.90. Denote by $\mathbb{k}\left\langle t_{1}, t_{2}, t_{3}\right\rangle$ the free algebra endowed with a well-ordering $\prec$ on monomials defined as follows. If a monomial $m_{1}$ is shorter than a monomial $m_{2}$ then $m_{1} \prec m_{2}$. Otherwise, if their lengths coincide then $\prec$ is determined by lex with respect to $t_{3} \prec t_{2} \prec t_{1}$.

Consider homomorphisms

$$
\begin{gathered}
\varphi_{i}: A:=\mathbb{k}\left\langle x_{1}, x_{2}, x_{3}\right\rangle \rightarrow \mathbb{k}\left\langle t_{1}, t_{2}, t_{3}\right\rangle, i=1,2 \\
\varphi_{1}\left(x_{1}\right)=t_{1}+t_{3}, \varphi_{1}\left(x_{2}\right)=t_{2}, \varphi_{1}\left(x_{3}\right)=t_{1} t_{2} ; \varphi_{2}\left(x_{1}\right)=t_{2}, \varphi_{2}\left(x_{2}\right)=t_{1}+t_{3}, \varphi_{2}\left(x_{3}\right)=t_{2} t_{3} .
\end{gathered}
$$

Denote $m:=x_{1} x_{2}-x_{3}$. We claim that a set $\mathbf{B} \subset A$ of monomials in $x_{1}, x_{2}, x_{3}, m$ without submonomials $x_{1} x_{2}$ constitutes a common adapted basis of $A$ with respect to valuations $\nu_{\varphi_{1}}, \nu_{\varphi_{2}}$.

First, $\mathbf{B}$ spans $A$ since in any monomial in $x_{1}, x_{2}, x_{3}$ one can replace each occurrence of submonomial $x_{1} x_{2}$ by $m+x_{3}$.

Now we observe that given a monomial $b \in \mathbf{B}$ in order to compute the leading monomial of $\varphi_{1}(b) \in \mathbb{k}\left\langle t_{1}, t_{2}, t_{3}\right\rangle$ one has to replace each occurrence of $x_{1}, x_{2}, x_{3}$ and of $m$ as follows:

$$
x_{1} \rightarrow t_{1}, x_{2} \rightarrow t_{2}, x_{3} \rightarrow t_{1} t_{2}, m \rightarrow t_{3} t_{2}
$$

Note that these leading monomials are pairwise distinct for different monomials from B since each such monomial $T$ in $t_{1}, t_{2}, t_{3}$ can be uniquely represented in the following way. Between any pair of adjacent occurences in $T$ of submonomials of the form either $t_{1} t_{2}$ or $t_{3} t_{2}$ the submonomial of $T$ has a form $t_{2} \ldots t_{2} t_{1} \ldots t_{1}$. In other words, one can describe the set of leading monomials of $\varphi_{1}(\mathbf{B})$ as the monoid $C_{\varphi_{1}} \subset\left\langle t_{1}, t_{2}, t_{3}\right\rangle$ generated by $t_{1}, t_{2}, t_{3} t_{2}$.

This implies that the elements of $\mathbf{B}$ are linearly independent, so $\mathbf{B}$ is a basis of $A$, in addition that $\varphi_{1}$ is a monomorphism, and $\mathbf{B}$ is an adapted basis with respect to the valuation $\nu_{\varphi_{1}}$.

In a similar manner, the leading monomial of $\varphi_{2}(b)$ is obtained by means of the following replacements:

$$
x_{1} \rightarrow t_{2}, x_{2} \rightarrow t_{1}, x_{3} \rightarrow t_{2} t_{3}, m \rightarrow t_{2} t_{1} .
$$

Again, these leading monomials are distinct for different elements $b \in \mathbf{B}$. Any leading monomial $T_{2}$ can be uniquely represented as follows. Between an adjacent occurences of a pair of submonomials of the form either $t_{2} t_{3}$ or $t_{2} t_{1}$ the submonomial of $T_{2}$ coincides with $t_{1} \ldots t_{1} t_{2} \ldots t_{2}$. Hence $\varphi_{2}$ is also a monomorphism. Thus, $\mathbf{B}$ is a common adapted basis with respect to both valuations $\nu_{\varphi_{1}}, \nu_{\varphi_{2}}$. The set of leading monomials of $\varphi_{2}(\mathbf{B})$ equals the monoid $C_{\varphi_{2}} \subset\left\langle t_{1}, t_{2}, t_{3}\right\rangle$ generated by $t_{1}, t_{2}, t_{2} t_{3}$.

Note that unlike the commutative case (Example 3.90) it holds $C_{\varphi_{1}} \neq C_{\varphi_{2}}$ : for instance, $t_{3} t_{2} \in C_{\varphi_{1}} \backslash C_{\varphi_{2}}$, while $t_{2} t_{3} \in C_{\varphi_{2}} \backslash C_{\varphi_{1}}$. One obtains $\mathrm{JHb} \mathbf{K}_{\nu_{\varphi_{2}}, \nu_{\varphi_{1}}}$ as follows. In a monomial $T \in C_{\varphi_{1}}$ (see the notations above) we replace each occurrence of $t_{1} t_{2}$ by $t_{2} t_{3}$, respectively, each occurrence of $t_{3} t_{2}$ by $t_{2} t_{1}$, in addition, in a submonomial of the form $t_{2} \ldots t_{2} t_{1} \ldots t_{1}$ between a pair of adjacent occurences of either $t_{1} t_{2}$ or $t_{3} t_{2}$, we replace $t_{2}$ by $t_{1}$ and $t_{1}$ by $t_{2}$, thereby we get a submonomial $t_{1} \ldots t_{1} t_{2} \ldots t_{2}$. The resulting monomial is $\mathbf{K}_{\nu_{\varphi_{2}}, \nu_{\varphi_{1}}}(T) \in C_{\varphi_{2}}$.

Definition 3.92. Let $\nu_{\bullet}$ and $\nu_{\circ}$ be injective valuations on an algebra $A$. Suppose that a basis $\mathbf{B}$ is adapted to both valuations. This turns $\mathbf{B}$ into ordered partial semigroups ( $\mathbf{B}, \circ, \preceq^{\circ}$ ) and ( $\mathbf{B}, \bullet, \preceq^{\bullet}$ ), see Remark 3.87.

We say that $\nu_{\bullet}$ and $\nu_{0}$ are polar with respect to $\mathbf{B}$ if any $b^{\prime \prime}$ occurring in $b b^{\prime} \neq 0$ satisfies

$$
b \bullet b^{\prime} \preceq^{\circ} b^{\prime \prime}, b \circ b^{\prime} \preceq^{\bullet} b^{\prime \prime} .
$$

Remark 3.93. i) For the algebra $\mathbb{k}[x, y, z] /\left(z+x^{2}+y^{3}\right)$ constructed in Remark 3.70, each pair among its injective valuations $\nu_{x}, \nu_{y}, \nu_{z}$ is polar with respect to the produced common adapted basis. For instance, for the common basis $\left\{y^{i} z^{j} x^{k}: i, j \geq 0,0 \leq\right.$ $k \leq 1\}$ of $\nu_{y}, \nu_{z}$ it holds $x \cdot x=-y^{3}-z$, hence $\nu_{y}\left(x^{2}\right)=\nu_{y}(z) \succ \nu_{y}\left(y^{3}\right), \nu_{z}\left(x^{2}\right)=$ $\nu_{z}\left(y^{3}\right) \succ \nu_{z}(z)$.

In a similar way, for the algebra $\mathbb{k}[x, y, z, t] /\left(x^{2}+y^{3}+z^{5}+t^{7}\right)$ constructed in Example 3.72, its valuations $\nu_{1}, \nu_{2}$ are polar with respect to the produced their common adapted basis $\left\{y^{i} t^{j} w^{l} x^{k} z^{m}: i, j, l \geq 0,0 \leq k \leq 1,0 \leq m \leq 4\right\}$. Indeed, $x \cdot x=w-y^{3}, z^{2} \cdot z^{3}=z \cdot z^{4}=w-t^{7}$, and $\nu_{1}\left(x^{2}\right)=\nu_{1}(w) \succ \nu_{1}\left(y^{3}\right), \nu_{2}\left(x^{2}\right)=$ $\nu_{2}\left(y^{3}\right) \succ \nu_{2}(w), \nu_{1}\left(z^{5}\right)=\nu_{1}\left(t^{7}\right) \succ \nu_{1}(w), \nu_{2}\left(z^{5}\right)=\nu_{2}(w) \succ \nu_{2}\left(t^{7}\right)$, therefore the polar condition holds also for the decomposition in the basis of the products $x z^{2} \cdot x z^{3}=$ $x z \cdot x z^{4}=-w t^{7}+w^{2}+y^{3} t^{7}-y^{3} w$.
ii) Observe that the injective valuations produced in Examples 3.88, 3.89 are polar with respect to the basis $\mathbf{B}$. The same is true for the injective valuations $\nu_{0} \circ \varphi_{1}, \nu_{0} \circ \varphi_{2}$ with respect to the basis $\mathbf{B}$ produced in Example 3.90.

## 4. Appendix: Valuations of vector spaces and Jordan-Hölder BiJECTIONS

4.1. Valuations on vector spaces. Given a vector space $S$ over a field $\mathbb{k}$ and a totally ordered set $(C,<)$, following [19, 22, 23], we say that a map $\nu: S \backslash\{0\} \rightarrow C$ is a valuation if $\nu\left(\mathbb{k}^{\times} \cdot x\right)=\nu(x)$ for all nonzero $x \in S$ and $\nu(x+y) \leq \max (\nu(x), \nu(y))$ for $x+y \neq 0$ (this implies that $\nu(x+y)=\max (\nu(x), \nu(y))$ whenever $\nu(x) \neq \nu(y))$.

Denote by $C_{\nu}$ the image $\nu(S \backslash\{0\})$.
One can construct valuations on another vector space (resp. integral domain) $S^{\prime}$ by importing a given valuation $\nu$ on $S$ via any injective $\mathbb{k}$-linear map $f: S^{\prime} \hookrightarrow S$ (resp. an injective homomorphism of $\mathbb{k}$-algebras). Namely, the composition $\nu \circ f$ is a valuation on $S^{\prime}$.

Each valuation $\nu: S \backslash\{0\} \rightarrow C$ defines a filtration $S_{\leq}$of subspaces on $S$ via

$$
S_{\leq a}:=\{0\} \cup\{x \in S \backslash\{0\}: \nu(x) \leq a\}
$$

for $a \in C_{\nu}$ (if $S$ is an integral domain, this is a filtration on a $\mathbb{k}$-algebra). We also abbreviate $S_{<a}:=\sum_{a^{\prime}<a} S_{\leq a^{\prime}}$ and denote $S_{a}:=S_{\leq a} / S_{<a}$ for $a \in C$ ( $S_{a}$ is called in [19] the leaf at a).

Conversely, if $C$ is a well-order, then any increasing filtration $S_{\leq a}, a \in C$ of $S$ defines a (well-ordered) valuation $\nu: S \backslash\{0\} \rightarrow C$ via

$$
\nu(x)=\min \left\{a \in C: x \in S_{\leq a}\right\}
$$

for all $x \in S \backslash\{0\}$.

Following [4, 20], we say that $\mathbf{B} \subset S$ is adapted to a valuation $\nu: S \backslash\{0\} \rightarrow C$ if for each $a \in C$ the restriction to $\mathbf{B}_{a}=\{b \in \mathbf{B} \mid \nu(b)=a\}$ of the canonical projection $\pi_{a}: S_{\leq a} \rightarrow S_{a}$ is injective and the image $\pi_{a}\left(\mathbf{B}_{a}\right)$ is a basis of $S_{a}$.

If the filtration on $S$ is induced by a valuation $\nu: S \backslash\{0\} \rightarrow C$, we refer to adapted subsets of $S$ as $\nu$-adapted.

Remark 4.1. Such a (necessarily independent) subset $\mathbf{B}$ of $S$ is called valuationindependent in [22], [23], but we prefer terminology of [20, Definition 2]. If we denote gr $S:=\bigoplus_{a \in C} S_{a}$, then clearly any adapted subset of $S$ defines a basis of $g r S$.

We say that $\nu$ is locally finite iff there exists an isomorphism $f: S \widetilde{\rightarrow} g r S=\bigoplus_{a \in C} S_{a}$ of $\mathbb{k}$-vector spaces such that $f\left(S_{\leq a}\right)=\bigoplus_{a^{\prime} \leq a} S_{a^{\prime}}$ for each $a \in C$.

The following is immediate.
Lemma 4.2. Let $S$ be $a \mathbb{k}$-vector space and $\nu: S \backslash\{0\} \rightarrow C$ be any valuation. Then:
(a) For any basis $\underline{\mathbf{B}}$ of gr $S$ such that any $\underline{\mathbf{B}}_{a}:=S_{a} \cap \underline{\mathbf{B}}$ a basis of $S_{a}$ one can construct an adapted set $\mathbf{B}$ as follows.

$$
\mathbf{B}_{a}=\iota_{a}\left(\underline{\mathbf{B}}_{a}\right),
$$

where and $\iota_{a}: S_{a} \hookrightarrow S_{\leq a}$ is any simultaneous splitting of canonical projections $\pi_{a}: S_{\leq a} \rightarrow S_{a}$.
(b) $\nu$ is locally finite iff $S$ admits an adapted basis.
(c) If $C$ is a well-order, then $\nu$ is locally finite, moreover, any adapted subset of $S$ is a basis.

Example 4.3. Let $S=\mathbb{k}\left(\left(t^{-1}\right)\right)$ be the algebra of all formal Laurent series in $t^{-1}$ over $\mathfrak{k}$. Then setting $\nu(f)=n$ for each $f=\sum_{m=-\infty}^{n} a_{m} t^{m}$ with $a_{n} \neq 0$ defines a valuation $S \backslash\{0\} \rightarrow \mathbb{Z}$. This valuation is not locally finite, in particular, the subset $\mathbf{B}=\left\{t^{m}, m \in \mathbb{Z}\right\}$ is adapted to $\nu$, however, it is not a basis of $S$. In fact, there is no adapted bases in $S$. At the same time, the restriction of $\nu$ to the subalgebra $S_{0}=\mathbb{k}\left[t, t^{-1}\right]$ of Laurent polynomials is a locally-finite valuation on $S_{0}$.

It turns out that we can always propagate valuations to tensor products without assuming that they are well-ordered.

Proposition 4.4. Let $S$ be $\mathbb{k}$-vector space and $\nu: S \backslash\{0\} \rightarrow C$ be a valuation. Then for any $\mathbb{k}$-vector space $S^{\prime}$ we have:
(a) There exists a unique a valuation $\nu^{S^{\prime}}: S \otimes S^{\prime} \backslash\{0\} \rightarrow C$ such that

$$
\begin{equation*}
\nu^{S^{\prime}}(x \otimes y)=\nu(x) \tag{4.1}
\end{equation*}
$$

for all $x \in S \backslash\{0\}, y \in S^{\prime} \backslash\{0\}$ so that the associated filtration on $S \otimes S^{\prime}$ is

$$
\left(S \otimes S^{\prime}\right)_{\leq a}=S_{\leq a} \otimes S^{\prime}
$$

for $a \in C$.
(b) For any valuation $\nu^{\prime}: S^{\prime} \backslash\{0\} \rightarrow C^{\prime}$ there exists a unique valuation $\nu \otimes \nu^{\prime}$ : $S \otimes S^{\prime} \backslash\{0\} \rightarrow C \times C^{\prime}$ such that

$$
\begin{equation*}
\left(\nu \otimes \nu^{\prime}\right)(x \otimes y)=\left(\nu(x), \nu^{\prime}(y)\right) \tag{4.2}
\end{equation*}
$$

for $x \in S \backslash 0, y \in S^{\prime} \backslash 0$ (where we equip $C \times C^{\prime}$ with the lexicographic ordering, i.e., ( $\left.a, a^{\prime}\right)<\left(\tilde{a}, \tilde{a}^{\prime}\right)$ whenever either $a<\tilde{a}$ or $\tilde{a}=a$ and $\left.a^{\prime}<\tilde{a}^{\prime}\right)$ so that the associated filtration on $S \otimes S^{\prime}$ is

$$
\left(S \otimes S^{\prime}\right)_{\leq\left(a, a^{\prime}\right)}=S_{\leq a} \otimes S_{\leq a^{\prime}}^{\prime}+S_{<a} \otimes S^{\prime}
$$

for $\left(a, a^{\prime}\right) \in C \times C^{\prime}$.
We prove Proposition 4.4 in Section 4.8.
By definition, $C_{\nu S^{\prime}}=C_{\nu}, \nu^{S^{\prime}}\left(s \otimes\left(S^{\prime} \backslash\{0\}\right)\right)=\nu(s)$ for any $s \in S \backslash\{0\}$ and $\nu^{S^{\prime}}\left((S \backslash\{0\}) \otimes s^{\prime}\right)=C_{\nu}$ for any $s^{\prime} \in S^{\prime} \backslash\{0\}$ in Proposition 4.4(a).
4.2. Decorated valuations. In this section we generalize valuations by taking into account their leading coefficients.

Definition 4.5. Given $\mathbb{k}$-vector spaces $S, S^{\prime}$ and a valuation $\nu: S \backslash\{0\} \rightarrow C$, we say a map $\lambda: S \backslash\{0\} \rightarrow S^{\prime} \backslash\{0\}$ is a leading coefficient of $\nu$ if $\lambda(c x)=c \lambda(x)$ for all $c \in \mathbb{k}, x \in S \backslash\{0\}$ and $\lambda(x+y)=\left\{\begin{array}{ll}\lambda(x)+\lambda(y) & \text { if } \nu(x)=\nu(y)=\nu(x+y) \\ \lambda(x) & \text { if } \nu(x)>\nu(y) \\ \lambda(y) & \text { if } \nu(x)<\nu(y)\end{array}\right.$ for any $x, y \in S$ such that $x+y \neq 0$. Sometimes we will refer to the pair $(\nu, \lambda)$ as a decorated valuation.

If both $S$ and $S^{\prime}$ are $\mathbb{k}$-algebras, we require additionally any leading coefficient $\lambda$ of $\nu$ to satisfy

$$
\begin{equation*}
\lambda(x y)=\lambda(x) \lambda(y) \tag{4.3}
\end{equation*}
$$

for all $x, y \in S \backslash\{0\}$ (i.e., $\lambda$ is a homomorphism of multiplicative semigroups).
We will sometimes refer to a leading coefficient $\lambda$ satisfying (4.3) as multiplicative.
The following are immediate.
Lemma 4.6. For any decorated valuation $(\nu, \lambda)$ on a vector space $S$ one has
(a) $\lambda\left(x_{1}+\cdots+x_{r}\right)=\sum_{\substack{j \in[1, r] ; \\ \nu\left(x_{j}\right)=\max \left(\nu\left(x_{1}\right), \ldots, \nu\left(x_{r}\right)\right)}} \lambda\left(x_{j}\right)$ whenever $x_{1}+\cdots+x_{r} \neq 0$ and
$\nu\left(x_{1}+\cdots+x_{r}\right)=\max \left(\nu\left(x_{1}\right), \ldots, \nu\left(x_{r}\right)\right)$.
(b) For any subspace $S_{0}$ of $S$ the restriction of $(\nu, \lambda)$ to $S_{0} \backslash\{0\}$ is a decorated valuation on $S_{0}$.
(c) For any injective linear map $f: S^{\prime} \hookrightarrow S^{\prime \prime}$ the pair $(\nu, f \circ \lambda)$ is a decorated valuation on $S$.

Lemma 4.7. Let $\nu: S \backslash\{0\} \rightarrow C$ and $\nu^{\prime}: S^{\prime} \backslash\{0\} \rightarrow C^{\prime}$ be valuations and $\lambda: S \backslash\{0\} \rightarrow S^{\prime} \backslash\{0\}$ be a leading coefficient of $\nu$. Then the assignments $x \mapsto$ $\left(\nu(x), \nu^{\prime}(\lambda(x))\right.$ define a valuation $\nu \times_{\lambda} \nu^{\prime}: S \backslash\{0\} \rightarrow C \times C^{\prime}$, with the lexicographic order on $C \times C^{\prime}$.

The following immediate result gives various characterizations of decorated valuations in terms of adapted bases.

Lemma 4.8. Let $S$ and $S^{\prime}$ be $\mathbb{k}$-vector spaces, $\nu: S \backslash\{0\} \rightarrow C$ be a valuation, and B be a basis of $S$ adapted to $\nu$. Then
(a) Any map $f: \mathbf{B} \rightarrow S^{\prime} \backslash\{0\}$ uniquely extends to a leading coefficient $\lambda=\lambda_{\mathbf{B}, f}$ : $S \backslash\{0\} \rightarrow S^{\prime} \backslash\{0\}$.
(b) The assignments $b \otimes s^{\prime} \mapsto s^{\prime}, b \in \mathbf{B}, s^{\prime} \in S^{\prime}$ define a leading coefficient $\lambda^{\mathbf{B}}: S \otimes S^{\prime} \backslash\{0\} \rightarrow S^{\prime}$ of the valuation $\nu^{S^{\prime}}$ (in the notation of Proposition 4.4(a)).
(c) For any leading coefficient $\lambda: S \backslash\{0\} \rightarrow S^{\prime} \backslash\{0\}$ of $\nu$ there exists a unique injective linear map $\delta=\delta_{\mathbf{B}, \lambda}: S \hookrightarrow S \otimes S^{\prime}$ such that $\lambda=\lambda^{\mathbf{B}} \circ \delta$ (in fact, $\delta$ is given by $\delta(b)=b \otimes \lambda(b)$ for all $b \in \mathbf{B})$.

The following provides an example of decorated valuations "in nature."
Lemma 4.9. Let $\mathbb{k}$ be of characteristic $0, S$ be $a \mathbb{k}$-vector space, and $E$ be a locally nilpotent linear map $S \rightarrow S$, i.e., for each nonzero $x \neq 0$ there is a unique number $\nu_{E}(x) \in \mathbb{Z}_{\geq 0}$ such that $E^{\nu_{E}(x)}(x) \neq 0$ and $E^{\nu_{E}(x)+1}(x)=0$. Then
(a) The assignments $x \mapsto \nu_{E}(x)$ define a valuation $\nu_{E}: S \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}$.
(b) The assignments $x \mapsto E^{\left(\nu_{E}(x)\right)}(x)$ define the leading coefficient $\lambda_{E}: S \backslash\{0\} \rightarrow$ $S \backslash\{0\}$, where we abbreviate $E^{(n)}:=\frac{1}{n!} E^{n}$, the $n$-th divided power.
(c) If $S$ is an integral domain over $\mathbb{k}$ and $E$ is a locally nilpotent derivation of $S$, then $\nu_{E}$ is an additive valuation on $S$ and $\lambda_{E}$ is its multiplicative leading coefficient.

More generally, let $\mathbf{E}=\left(E_{1}, \ldots, E_{m}\right)$ be a family of locally nilpotent linear maps $S \rightarrow S$. Define the map $\lambda_{\mathbf{E}}: S \backslash\{0\} \rightarrow S \backslash\{0\}$ by

$$
\lambda_{\mathbf{E}}:=\lambda_{E_{m}} \circ \cdots \circ \lambda_{E_{1}},
$$

where $\lambda_{E}: S \backslash\{0\} \rightarrow S \backslash\{0\}$ is as in Lemma 4.9 (with the convention $\lambda_{\emptyset}=I d_{S \backslash\{0\}}$ ).
Then define the map $\nu_{\mathbf{E}}: S \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{m}$ by

$$
\nu_{\mathbf{E}}(x)=\left(a_{1}, \ldots, a_{m}\right) \in \mathbb{Z}_{\geq 0}^{m},
$$

where $a_{k}=\nu_{E_{k}}\left(\lambda_{\left(E_{1}, \ldots, E_{k-1}\right)}(x)\right)$ for $k \in[m]$ (actually, $a_{1}=\nu_{E_{1}}(x)$ ).
The following is a generalization of Lemma 4.9.
Corollary 4.10. Let $\mathbb{k}$ be of characteristic $0, S$ be $a \mathbb{k}$-vector space. Then for any family $\mathbf{E}=\left(E_{1}, \ldots, E_{m}\right)$ of locally nilpotent linear maps $S \rightarrow S$ one has:
(a) $\nu_{\mathbf{E}}: S \backslash\{0\} \rightarrow \mathbb{Z}_{\geq 0}^{m}$ is a valuation and $\lambda_{\mathbf{E}}: S \backslash\{0\} \rightarrow S \backslash\{0\}$ is its leading coefficient.
(b) If $S$ is an integral domain over $\mathbb{k}$ and each $E_{k}$ is a locally nilpotent derivation of $S$, then $\nu_{\mathbf{E}}$ is additive and $\lambda_{\mathbf{E}}$ is multiplicative.

Remark 4.11. The decorated valuations ( $\nu_{\mathbf{E}}, \lambda_{\mathbf{E}}$ ) generalize string valuations and their leading coefficients introduced by Andrei Zelevinsky and the first author in [5].

Remark 4.12. In fact, all valuations $\nu_{\mathbf{E}}$ factor (and thus can be defined recursively) as in Lemma 4.7: $\nu_{\left(E_{1}, \ldots, E_{m}\right)}=\nu_{\left(E_{1}, \ldots, E_{k}\right)} \times_{\lambda_{\left(E_{1}, \ldots, E_{k}\right)}} \nu_{\left(E_{k+1}, \ldots, E_{m}\right)}$ for any $k \in[1, m-1]$.
4.3. Injective valuations. Our main focus is on the class of what we call injective valuations, i.e., locally finite valuations such that $S_{a}=S_{\leq a} / S_{<a}$ is one-dimensional for each $a \in C_{\nu}$ (such valuations were called valuations with one-dimensional leaves in [18]). Note, however, that the valuation on $\mathbb{k}\left(\left(t^{-1}\right)\right)$ in Example 4.3 has onedimensional leaves, but is not locally finite, hence not injective.

The following is an immediate consequence of Lemma 4.2(c).

Lemma 4.13. A well-ordered valuation $\nu: S \backslash\{0\} \rightarrow C$ is injective iff there exists a basis $\mathbf{B}$ of $S$ such that the restriction of $\nu$ to $\mathbf{B}$ is an injective map $\mathbf{B} \hookrightarrow C$.

As in Section 4.1, we refer to any basis $\mathbf{B}$ satisfying Lemma 4.13 as adapted to $\nu$ and denote by $\mathbf{A}_{\nu}$ the set of all bases of $S$ adapted to $\nu$ (in [22], [23] each $\mathbf{B} \in \mathbf{A}_{\nu}$ is referred to as a valuation basis).

One can easily show that for any basis $\mathbf{B}$ adapted to (an injective valuation) $\nu$ one has $\nu(\mathbf{B})=C_{\nu}$ and $S_{\leq a}=\bigoplus_{b \in \mathbf{B}: \nu(b) \leq a} \mathbb{k} \cdot b, S_{<a}=\bigoplus_{b \in \mathbf{B}: \nu(b)<a} \mathbb{k} \cdot b$ for all $a \in C_{\nu}$.

The following result establishes a convenient criterion of injectivity of a valuation.
Proposition 4.14 (Euclidean property). The following are equivalent for a given well-ordered valuation $\nu: S \backslash\{0\} \rightarrow C$.
(a) $\nu$ is injective.
(b) For any non-zero $x, y \in S$ such that $\nu(x)=\nu(y)$ and $x \notin \mathbb{k} \cdot y$ there exists ( $a$ unique) $c \in \mathbb{k}^{\times}$such that $\nu(x-c y)<\nu(x)$.

We prove Proposition 4.14 in Section 4.8.
Proposition 4.14 is well-known for finite-dimensional $S$ (see e.g., [19]), for infinitedimensional $S$ we could not find it in the literature.

Corollary 4.15. For a given well-ordered injective valuation $\nu: S \backslash\{0\} \rightarrow C$ any $\nu$-adapted set is an (adapted) basis of $S$.

Remark 4.16. We demonstrate that the conclusion of Corollary 4.15 is not valid without the assumption of well-orderness. Consider a space $S$ with a basis $\left\{e_{i}: 0 \leq\right.$ $i<\infty\}$ and an injective valuation $\nu: S \backslash\{0\} \rightarrow \mathbb{Z}_{\leq 0}$ such that $\nu\left(e_{i}\right)=-i$. Then a set $R:=\left\{e_{i}+e_{i+1}: 0 \leq i<\infty\right\}$ is adapted, while it is not a basis of $S$ : for instance, $e_{1}$ does not belong to the span of $R$.

We can build new injective valuations out of existing ones by the following immediate consequence of the injectivity criterion in Proposition 4.14(b).

Corollary 4.17. Let $\nu: S \backslash\{0\} \rightarrow C$ be a well-ordered injective valuation. Then for any subspace $\underline{S}$ of $S$ the restriction $\underline{\nu}:=\left.\nu\right|_{\underline{S} \backslash\{0\}}$ is an injective valuation on $\underline{S}$.

Remark 4.18. It is interesting whether an analog of Corollary 4.17 holds without assumption of well-orderness of $C$.

Given a valuation $\nu: S \backslash\{0\} \rightarrow C$, we say that a family $S_{i}, I \in I$ of subspaces of $S$ is $\nu$-compatible if $\nu\left(\bigcap_{i \in I} S_{i} \backslash\{0\}\right)=\bigcap_{i \in I} \nu\left(S_{i} \backslash\{0\}\right)$ (clearly, for any $\nu$ the left hand side is always a subset of the right hand side).

Proposition 4.19. Suppose that a valuation $\nu$ on a space $S$ is well-ordered injective. Then a family of subspaces $\left\{S_{i}, i \in I\right\}$ of $S$ is $\nu$-compatible iff there exists an adapted with respect to $\nu$ basis $\mathbf{B}$ in $S$ such that $\mathbf{B} \cap S_{i}$ is a basis in $S_{i}$ for each $i \in I$. In addition, in this case $\mathbf{B} \cap \bigcap_{j \in J} S_{j}$ is a basis in $\bigcap_{j \in J} S_{j}$ for each $J \subseteq I$ and $\nu\left(\left(\sum_{j \in J} S_{j}\right) \backslash\{0\}\right)=\bigcup_{j \in J} \nu\left(S_{j} \backslash\{0\}\right)$ for every subset $J \subseteq I$.

We prove Proposition 4.19 in Section 4.8.

Remark 4.20. If $\left\{S_{i}, i \in I\right\}$ is a $\nu$-compatible family, $|I|<\infty, \operatorname{dim}\left(S_{i}\right)<\infty, i \in I$ then for any subset $J \subseteq I$ it holds

$$
\operatorname{dim}\left(\sum_{j \in J} S_{j}\right)=\sum_{L \subseteq J}(-1)^{|L|+1} \operatorname{dim}\left(\bigcap_{l \in L} S_{l}\right)
$$

We can also construct injective valuations on the quotients as follows.
Proposition 4.21. Let $S$ be a $\mathbb{k}$-vector space and let $\nu: S \backslash\{0\} \rightarrow C$ is an (injective) valuation for some well-order $C$. Then for any subspace $J \subset S$ the assignments

$$
\nu^{\prime}(v+J):=\min \{\nu(v+J)\}
$$

for all non-zero $v+J \in S / J$ define an (injective) valuation $\nu^{\prime}: S / J \backslash\{0\} \rightarrow C$.
Remark 4.22. Note, however, that if $S$ is a commutative integral domain, $J$ a prime ideal, $C$ a monoid and $\nu(a b)=\nu(a)+\nu(b)$ for all $a, b \in S \backslash\{0\}$ in Proposition 4.21, then $\nu^{\prime}(a b+J) \leq \nu^{\prime}(a+J)+\nu^{\prime}(b+J)$ for all $a, b \in S \backslash J$ because of the inequality $\min \{\nu(X \cdot Y)\} \leq \min \{\nu(X)\}+\min \{\nu(Y)\}$ for any subsets $X, Y \subset S \backslash\{0\}$ (here $X \cdot Y$ is the $\mathbb{k}$-linear span of $\{x y \mid x \in X, y \in Y\})$.

It turns out that any valuation can be assembled out of injective ones as follows.
Proposition 4.23. Let $S$ be $a \mathbb{k}$-vector space and $\nu: S \backslash\{0\} \rightarrow C$ be a locally finite valuation (see Section 4.1). Then there are $\mathbb{k}$-vector spaces $\underline{S}$ and $S^{\prime}$, an injective valuation $\underline{\nu}: \underline{S} \backslash\{0\} \rightarrow C$ and $a \mathbb{k}$-linear embedding $\mathbf{j}: S \hookrightarrow \underline{S} \otimes S^{\prime}$ such that $C_{\underline{\nu}}=C_{\nu}$ and

$$
\nu=\underline{\nu}^{S^{\prime}} \circ \mathbf{j}
$$

in the notation (4.1).
We prove Proposition 4.23 in Section 4.8.
4.4. Jordan-Hölder bijections. For any valuations $\nu, \nu^{\prime}: S \backslash\{0\} \rightarrow C$ such that $\nu^{\prime}$ is well-ordered define a map $\mathbf{K}_{\nu^{\prime}, \nu}: C_{\nu} \rightarrow C_{\nu^{\prime}}$ by

$$
\begin{equation*}
\mathbf{K}_{\nu^{\prime}, \nu}(a)=\min \left\{\nu^{\prime}\left(\nu^{-1}(a)\right)\right\} \tag{4.4}
\end{equation*}
$$

for all $a \in C_{\nu}$.
Our first result provides an "industry" for establishing combinatorial bijections.
Theorem 4.24. For any well-ordered injective valuations $\nu$ and $\nu^{\prime}$ on $S$ the maps $\mathbf{K}_{\nu^{\prime}, \nu}: C_{\nu} \rightarrow C_{\nu^{\prime}}$ and $\mathbf{K}_{\nu, \nu^{\prime}}: C_{\nu^{\prime}} \rightarrow C_{\nu}$ are well-defined and mutually inverse bijections. Moreover, there exists a basis $\mathbf{B}_{\nu, \nu^{\prime}}$ of $S$ adapted to both $\nu$ and $\nu^{\prime}$ such that $\mathbf{K}_{\nu^{\prime}, \nu}(\nu(b))=\nu^{\prime}(b)$ for all $b \in \mathbf{B}_{\nu, \nu^{\prime}}$.

We prove Theorem 4.24 in Section 4.8.
We refer to $\mathbf{K}_{\nu^{\prime}, \nu}$ as Jordan-Hölder bijection (JHb) and call any basis $\mathbf{B}_{\nu, \nu^{\prime}}$ as an JH-basis.

Remark 4.25. In fact, Theorem 4.24 generalizes well-known facts that any two complete flags in $\mathbb{k}^{n}$ have a canonical relative position $w$, which is a permutation of $\{1, \ldots, n\}$, and admit a common basis. Namely, an injective valuation $\nu: S \backslash\{0\} \rightarrow C$ defines a complete flag $\mathcal{F}_{\nu}$ indexed by $C_{\nu}$ via $\left(\mathcal{F}_{\nu}\right)_{\leq a}=\{v \in S \backslash\{0\}: \nu(v) \leq a\}$, $a \in C_{\nu}$ (see Sections 4.1 and 4.3 for details). Conversely, any complete flag $\mathcal{F}$ on $S$ is of the form $\mathcal{F}_{\nu}$. If the indexing sets for flags $\mathcal{F}_{\nu}$ and $\mathcal{F}_{\nu^{\prime}}$ are well-ordered, then

Theorem 4.24 asserts that there exist a canonical relative position $\mathbf{K}_{\nu^{\prime}, \nu}$ of $\mathcal{F}_{\nu}$ and $\mathcal{F}_{\nu^{\prime}}$ and a common (JH) basis. This can be also reformulated in terms of generalized Jordan-Hölder correspondence developed by Abels in 1991, see, e.g., Section 2.3 of [8].

The following result is an immediate consequence of Theorem 4.24.
Corollary 4.26. In the assumptions of Theorem 4.24 the set $\mathbf{A}_{\nu} \cap \mathbf{A}_{\nu^{\prime}}$ is nonempty.
The following result is a reverse of Theorem 4.24, however, we do not assume that valuations are well-ordered.

Proposition 4.27. Let $\nu$ and $\nu^{\prime}$ be (not necessarily well-ordered) injective valuations on $S$ such that $\mathbf{A}_{\nu} \cap \mathbf{A}_{\nu^{\prime}}$ is nonempty. Then the assignments (4.4) define a bijection $\mathbf{K}_{\nu^{\prime}, \nu}: C_{\nu} \widetilde{\rightarrow} C_{\nu^{\prime}}$ so that $\nu^{\prime}(b)=\mathbf{K}_{\nu^{\prime}, \nu}(\nu(b))$ for any $\mathbf{B} \in \mathbf{A}_{\nu} \cap \mathbf{A}_{\nu^{\prime}}$ and all $b \in \mathbf{B}$.

We prove Proposition 4.27 in Section 4.8.
Example 4.28. Let $S=\mathbb{k}[t]$ and let $\nu, \nu^{\prime}: S \backslash\{0\} \rightarrow-\mathbb{Z}_{\geq 0}$ be valuations given by

$$
\nu\left(t^{k}\right)=\nu^{\prime}\left(t^{k}+1\right)=-k
$$

for $k \in \mathbb{Z}_{\geq 0}$. These valuations are obviously injective, and are adapted respectively to the bases $\mathbf{B}=\left\{t^{k}, k \in \mathbb{Z}_{\geq 0}\right\}, \mathbf{B}^{\prime}=\left\{1+t^{k}, k \in \mathbb{Z}_{\geq 0}\right\}$, however, $\nu\left(\mathbf{B}^{\prime}\right)=\nu^{\prime}(\mathbf{B})=\{0\}$.

Denote $\mathbf{B}^{\prime \prime}=\left\{t^{k}-t^{k+1}, k \in \mathbb{Z}_{\geq 0}\right\}$. Clearly, $\nu\left(\mathbf{B}^{\prime \prime}\right)=\nu^{\prime}\left(\mathbf{B}^{\prime \prime}\right)=-\mathbb{Z}_{\geq 0}$ because

$$
\nu\left(t^{k}-t^{k+1}\right)=-k, \nu^{\prime}\left(t^{k}-t^{k+1}\right)=\max \left(\nu^{\prime}\left(1+t^{k}\right), \nu^{\prime}\left(1+t^{k+1}\right)\right)=-k
$$

for $k \in \mathbb{Z}_{\geq 0}$. However $\mathbf{B}^{\prime \prime}$ is not a basis of $S$, moreover, $\mathbf{A}_{\nu} \cap \mathbf{A}_{\nu^{\prime}}=\emptyset$ In particular, Proposition 4.27 and Theorem 4.24 are not applicable to $\left(\nu, \nu^{\prime}\right)$ (note that $-\mathbb{Z}_{\geq 0}$ endowed with the natural order is not well-ordered) and thus $\mathbf{K}_{\nu^{\prime}, \nu}$ is undefined.

In some cases, we can obtain injective valuations by utilizing leading coefficients of valuations on their ambient spaces (see Section 4.2).

Proposition 4.29. Let $\nu: S \backslash\{0\} \rightarrow C$ be a well-ordered valuation and $\lambda: S \backslash\{0\} \rightarrow$ $S^{\prime} \backslash\{0\}$ be its leading coefficient. Let $S_{0}$ be a subspace of $S$ such that $\lambda\left(S_{0}\right)=\mathbb{k}^{\times} \cdot s^{\prime}$ for some $s^{\prime} \in S^{\prime}$. Then the restriction of $\nu$ to $S_{0}$ is an injective valuation on $S_{0}$.

Proof. Without loss of generality, we consider the case when $S_{0}=S, S^{\prime}=\mathbb{k}, s^{\prime}=1$. It suffices to verify the condition (b) of Proposition 4.14. Indeed, let $x, y \in S \backslash\{0\}$ be such that $y \notin \mathbb{k} x$ and $\nu(x)=\nu(y)$. Denote $c:=\frac{\lambda(x)}{\lambda(y)}$. Suppose, by contradiction, that $\nu(x-c y)=\nu(x)$. Then $\lambda(x-c y)=\lambda(x)+\lambda(-c y)=\lambda(x)-c \lambda(y)=0$, which is impossible.

The contradiction finishes the proof.
We can apply this result to integral domains as follows. Given a commutative integral domain $\mathcal{B}$ over $\mathbb{k}$ and a subalgebra $\mathcal{A}$, denote by $\mathcal{B}_{\mathcal{A}}$ the set of all $x \in \mathcal{B}$ such that $\mathcal{A} \cdot x \cap(\mathcal{A} \backslash\{0\})$ is nonempty. Clearly, $\mathcal{B}_{\mathcal{A}}$ is a subalgebra of $\mathcal{B}$ (we will sometimes refer to it as the localization of $\mathcal{A}$ in $\mathcal{B}$ ).

Theorem 4.30. Let $\mathcal{B}$ be an integral domain over $\mathbb{k}, M$ be a well-ordered monoid, $\nu$ be an additive valuation $\mathcal{B} \backslash\{0\} \rightarrow M$, and $\lambda: \mathcal{B} \rightarrow \mathcal{C}$ be its multiplicative leading
coefficient (here $\mathcal{C}$ is an integral domain over $\mathbb{k}$ ). Suppose that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ such that $\lambda(\mathcal{A} \backslash\{0\})=\mathbb{k}^{\times}$. Then
(a) $\lambda\left(\mathcal{B}_{\mathcal{A}} \backslash\{0\}\right)=\mathbb{k}^{\times}$.
(b) The restriction of $\nu$ to $\mathcal{B}_{\mathcal{A}}$ is an injective additive valuation $\mathcal{B}_{\mathcal{A}} \backslash\{0\} \rightarrow M$.

Proof. Indeed, let $b \in \mathcal{B}_{\mathcal{A}} \backslash\{0\}$. That is, $x b=y$ for some $x, y \in \mathcal{A} \backslash\{0\}$. Therefore,

$$
\lambda(y)=\lambda(x b)=\lambda(x) \lambda(b)
$$

since $\lambda$ is multiplicative. Hence $\lambda(y) \in \mathbb{k}^{\times}$because $\lambda(x), \lambda(y) \in \mathbb{k}^{\times}$by the assumption. This proves (a).

Part (b) follows from (a) and Proposition 4.29.
The theorem is proved.
4.5. Well-ordered submonoids of $\mathbb{Z}^{m}$. For $M \subset \mathbb{Z}^{m}$ and $k \in[m-1]$ denote by $M_{k}$ the image of $M$ under the standard projection $\left.\mathbb{Z}^{m} \rightarrow \mathbb{Z}^{k}\left(a_{1}, \ldots, a_{m}\right) \mapsto\left(a_{1}, \ldots, a_{k}\right)\right)$.

Proposition 4.31. Let $m \geq 1$ and $M \subset \mathbb{Z}^{m}$. Then the following are equivalent:
(a) $M$ is well-ordered with respect to the lexicographic order on $\mathbb{Z}^{m}$.
(b) For $k=0, \ldots, m-1$ there exist functions $f_{k}: M_{k} \rightarrow \mathbb{Z}$ such that:

$$
a_{1}+f_{0} \geq 0, a_{2}+f_{1}\left(a_{1}\right) \geq 0, a_{3}+f_{2}\left(a_{1}, a_{2}\right) \geq 0, \ldots, a_{m}+f_{m-1}\left(a_{1}, \ldots, a_{m-1}\right) \geq 0
$$

for all $a=\left(a_{1}, \ldots, a_{m}\right) \in M$.
If $M$ is a monoid one can additionally require in (b) that $f_{0}=f_{1}(0)=\cdots=$ $f_{m-1}(0, \ldots, 0)=0$.

Proof. First assume (a). For any $0 \leq k<m$ fix a point $\left(a_{1} \ldots, a_{k}\right) \in M_{k}$. There exists an integer $N$ such that $a_{k+1} \geq N$ for any $a_{k+1}$ such that $\left(a_{1}, \ldots, a_{k}, a_{k+1}\right) \in$ $M_{k+1}$. We put $f_{k}\left(a_{1}, \ldots, a_{k}\right):=-N$.

Conversely, assume (b) and that (a) is false. Then there exists an infinite decreasing sequence of elements of $M$. Therefore for a suitable maximal possible $0 \leq k<m$ all elements of the sequence starting with some point have the same prefix $a_{1}, \ldots, a_{k}$ for appropriate $\left(a_{1}, \ldots, a_{k}\right) \in M_{k}$. Since $a_{k+1}+f_{k}\left(a_{1}, \ldots, a_{k}\right) \geq 0$ we get a contradiction with the maximality of $k$.

When $M$ is a monoid and an element $a:=\left(0, \ldots, 0, a_{k}, \ldots, a_{m}\right) \in M$ where $a_{k} \neq 0$, it holds $a_{k}>0$ because otherwise $a>2 a>3 a>\ldots$. This implies the last statement of the proposition.

Example 4.32. Given $r \in \mathbb{Q}_{>0}$, then $M_{r}=\left\{\left(a_{1}, a_{2}\right) \in \mathbb{Z}^{2}: a_{1} \geq 0, a_{2}+r a_{1}^{2} \geq 0\right\}$ is a well-ordered submonoid of $\mathbb{Z}^{2}$.

We say that $g \in G L_{m}(\mathbb{Q})$ is tame if $g\left(e_{j}\right) \in e_{j}+\sum_{i=1}^{j-1} \mathbb{Q}_{\geq 0} \cdot e_{i}$ for $j \in[m]$, where $\left\{e_{1}, \ldots, e_{m}\right\}$ is the standard basis of $\mathbb{Z}^{m}$.

Corollary 4.33. A finitely generated submonoid $M \subset \mathbb{Z}^{m}$ is well-ordered (with respect to the lexicographic order on $\left.\mathbb{Z}^{m}\right)$ iff $M \subset g^{-1}\left(\mathbb{Z}_{\geq 0}^{m}\right)$ for some tame $g \in$ $G L_{m}(\mathbb{Z})$.
4.6. Tame valuations on the Laurent polynomial ring. In this section we will view each $\mathbb{R}^{n}$ as a totally ordered set with respect to the lexicographic ordering.

We say that a valuation $\nu: \mathbb{k}\left[x_{1}^{ \pm 1}, \ldots, x_{m}^{ \pm 1}\right] \backslash\{0\} \rightarrow \mathbb{R}^{n}$ is tame if it is completely determined by its values $\nu\left(x_{i}\right)=v_{i} \in \mathbb{R}^{n}, i=1, \ldots, n$. Clearly, $\nu$ is tame iff it is of the form $\nu_{\mathbf{v}}, \mathbf{v}=\left(v_{1}, \ldots, v_{m}\right) \in\left(\mathbb{R}^{n}\right)^{m}$ :

$$
\nu_{\mathbf{v}}\left(\sum_{d \in \mathbb{Z}^{m}} c_{d} x^{d}\right)=\max _{d \in \mathbb{Z}^{m}: c_{d} \neq 0}\left\{d_{1} v_{1}+\cdots d_{m} v_{m}\right\}
$$

The following is obvious.
Lemma 4.34. A tame valuation $\nu=\nu_{\mathbf{v}}$ is injective iff the vectors $v_{1}, \ldots, v_{m}$ are linearly independent (in particular, $n \geq m$ ).

Since the monomials form an adapted basis to a tame valuation one can apply Proposition 4.27 to any pair of tame valuations on the Laurent polynomial algebra $\mathbb{k}\left[x_{1}^{ \pm 1}, \ldots, x_{m}^{ \pm 1}\right]$ and get

Corollary 4.35. Any pair of injective tame valuations on $\mathbb{k}\left[x_{1}^{ \pm 1}, \ldots, x_{m}^{ \pm 1}\right]$ has an adapted basis.

A tame valuation provides a total well ordering of the monomials. [31] and Theorem 9 from [11] state that all the total well orderings of the monomials are exhausted by the tame ones.

One can view $\mathbf{v}$ as $n \times m$ matrix. Then lex ordering corresponds to the unit matrix, and deglex ordering corresponds to $m \times m$ matrix with ones on the diagonal and in the first row with zeroes at the rest entries.
4.7. Algorithms computing Jordan-Hölder bijections. Consider a pair $\nu, \nu^{\prime}$ : $S \backslash\{0\} \rightarrow(C,<)$ of injective well-ordered valuations. Assume that there are given algorithms mapping $C_{\nu}$ (respectively, $C_{\nu^{\prime}}$ ) to an adapted basis $\mathbf{B} \in \mathbf{A}_{\nu}$ (respectively, $\left.\mathbf{B}^{\prime} \in \mathbf{A}_{\nu^{\prime}}\right)$. Also assume that $\left(C_{\nu},<\right)$ is isomorphic to $\mathbb{Z}_{\geq 0}$ and there is given an algorithm exhibiting this isomorphism. Note that deglex on the polynomial ring fulfills the latter feature. Then one can compute $\mathrm{JHb} \mathbf{K}_{\nu^{\prime}, \nu}: C_{\nu} \rightarrow C_{\nu^{\prime}}^{\prime}$ and a common adapted basis from $\mathbf{A}_{\nu} \cap \mathbf{A}_{\nu^{\prime}}$.

Indeed, for any $a \in C_{\nu}$ the algorithm produces $b_{a} \in \mathbf{B}$ with $\nu\left(b_{a}\right)=a$ and all $b_{i} \in \mathbf{B}, i \in I$ such that $\nu\left(b_{i}\right)<a$. The algorithm expands each $b_{a}, b_{i}, i \in I$ in basis $\mathbf{B}^{\prime}$ and an element $b_{a}+\sum_{i \in I} c_{i} \cdot b_{i}$ with indeterminate coefficients $c_{i}, i \in I$

$$
b_{a}+\sum_{i \in I} c_{i} \cdot b_{i}=\sum_{1 \leq j \leq p} A_{j} \cdot b_{j}^{\prime}
$$

for suitable linear functions $A_{j}, 1 \leq j \leq p$ in $c_{i}, i \in I$ and $b_{j}^{\prime} \in \mathbf{B}^{\prime}, 1 \leq j \leq p$. Let $\nu^{\prime}\left(b_{1}^{\prime}\right)>^{\prime} \nu^{\prime}\left(b_{2}^{\prime}\right)>^{\prime} \cdots>^{\prime} \nu^{\prime}\left(b_{p}^{\prime}\right)$.

Consecutively, for $l=1,2, \ldots, p$ the algorithm tests whether a linear in $c_{i}, i \in I$ system $A_{1}=A_{2}=\cdots=A_{l-1}=0$ has a solution. Consider maximal $l$ satisfying the latter property. Then $\mathrm{JHb} \mathbf{K}_{\nu^{\prime}, \nu}(a)=\nu^{\prime}\left(b_{l}^{\prime}\right)$. Pick any solution $c_{i}, i \in I$ of the linear system $A_{1}=A_{2}=\cdots=A_{l-1}=0$. then

$$
\left\{b_{a}+\sum_{i \in I} c_{i} \cdot b_{i}\right\}_{a \in C_{\nu}}
$$

constitute a common adapted basis for $\mathbf{A}_{\nu} \cap \mathbf{A}_{\nu^{\prime}}$.
Just described algorithm computes $\mathrm{JHb} \mathbf{K}_{\nu^{\prime}, \nu}(a)$ for an arbitrary input $a \in C_{\nu}$ in a general case of a vector space. Since in case of a polynomial algebra JHb is more rigid than in general, one is able to design a partial algorithm for computing JHb and a common adapted basis for both $\nu, \nu^{\prime}$ in a finite form (we call this form a piece-wise monoidal representation), provided that the partial algorithm terminates. Moreover, the partial algorithm terminates iff JHb admits a piece-wise monoidal representation. Below we assume that $C_{\nu}=\mathbb{Z}_{\geq 0}^{m}$.

We accomplish the algorithm from the beginning of this subsection for computing $\mathrm{JHb} \mathbf{K}_{\nu^{\prime}, \nu}(a)$ step by step for increasing $a \in C_{\nu}$ by recursion. Thus, we assume as a recursive hypothesis that $\mathbf{K}_{\nu^{\prime}, \nu}(a)$ is already computed for all $a<a_{0}$ for some $a_{0}$. After each step the result of the algorithm can be given as the following piecewise monoidal representation. Polynomials $f_{1}, \ldots, f_{n} \in \mathbb{k}\left[x_{1}, \ldots, x_{m}\right]$ are given together with a partition of $\mathbb{R}_{\geq 0}^{m}$ into simplicial cones generated by vectors $a_{1}:=\nu\left(f_{1}\right), \ldots, a_{n}:=\nu\left(f_{n}\right) \in \mathbb{Z}_{\geq 0}^{m}$. Consider one (with some dimension $p \leq m$ ) of these cones generated by vectors $a_{i_{0}}, \ldots, a_{i_{p}}$ and denote by $M \subset \mathbb{Z}_{\geq 0}^{m}$ the monoid generated by vectors $a_{i_{0}}, \ldots, a_{i_{p}}$. In addition, to each integer point $a$ from the parallelotop $P=\left\{\alpha_{0} \cdot a_{i_{0}}+\cdots+\alpha_{p} \cdot a_{i_{p}}: 0 \leq \alpha_{0}, \ldots, \alpha_{p}<1\right\} \subset \mathbb{R}_{\geq 0}^{m}$ generated by $a_{i_{0}}, \ldots, a_{i_{p}}$ is attached a polynomial $f_{a} \in \mathbb{k}\left[x_{1}, \ldots, x_{m}\right]$ with $\nu\left(f_{a}\right)=a$. Then the monoid of all integer points from $\left(M \otimes \mathbb{R}_{\geq 0}\right) \cap \mathbb{Z}_{\geq 0}^{m}$ is a disjoint union of shifted monoids $M+a$ for all $a \in P \cap \mathbb{Z}_{\geq 0}^{m}$.

These data determine a basis $\mathbf{B}$ of $\mathbb{k}\left[x_{1}, \ldots, x_{m}\right]$ adapted for $\nu$. Namely, for any point $v=c_{0} \cdot a_{i_{0}}+\cdots c_{p} \cdot a_{i_{p}}+a \in M+a$ where $c_{0}, \ldots, c_{p} \in \mathbb{Z}_{\geq 0}$ put $b_{v}:=$ $f_{i_{0}}^{c_{0}} \cdots f_{i_{p}}^{c_{p}} \cdot f_{a} \in \mathbf{B}$, hence $\nu\left(b_{v}\right)=v$. Also we define map $\mathbf{K}: C_{\nu} \rightarrow C_{\nu^{\prime}}$ by $\mathbf{K}(v):=$ $\nu^{\prime}\left(b_{v}\right)=c_{0} \cdot \nu^{\prime}\left(f_{i_{0}}\right)+\cdots+c_{p} \cdot \nu^{\prime}\left(f_{i_{p}}\right)+\nu^{\prime}\left(f_{a}\right)$. Thereby, $\mathbf{K}$ is linear on each shifted monoid $M+a$. Clearly, $\mathbf{K}_{\nu^{\prime}, \nu} \leq^{\prime} \mathbf{K}$ holds point-wise.

Now we produce an algorithmic criterion whether the partial algorithm terminates at the current step of recursion. It terminates iff for every pair of distinct points $v, v_{0} \in \mathbb{Z}_{\geq 0}^{m}$ it holds $\nu^{\prime}\left(b_{v}\right) \neq \nu^{\prime}\left(b_{v_{0}}\right)$. The latter condition is equivalent to non-solvability of a suitable integer programming problem. If the partial algorithm terminates then $\mathbf{B}$ is a common adapted basis for both $\nu, \nu^{\prime}$ and $\mathbf{K}=\mathbf{K}_{\nu^{\prime}, \nu}$ (see Proposition 4.27).

Otherwise, if the partial algorithm does not terminate at the current recursive step, the algorithm described at the beginning of this subsection accomplishes the next step for computing JHb at a greater (wrt the ordering $<$ on $C_{\nu}$ ) point. Assume (for the sake of simplicity) that the algorithm at this step computes just $\mathbf{K}_{\nu^{\prime}, \nu}\left(a_{0}\right)$ and $f_{0} \in \mathbb{k}\left[x_{1}, \ldots, x_{m}\right]$ satisfying $\mathbf{K}_{\nu^{\prime}, \nu}\left(c_{0}\right)<^{\prime} \mathbf{K}\left(a_{0}\right)$ such that $\nu\left(f_{0}\right)=a_{0}$ and $\nu^{\prime}\left(f_{0}\right)=\mathbf{K}_{\nu^{\prime}, \nu}\left(a_{0}\right)$. Then at the current recursive step the partial algorithm adds $f_{0}$ to $f_{1}, \ldots, f_{n}$.

Let $a_{0}$ belong to a $p$-dimensional simplicial cone $T$ generated by vectors $a_{i_{0}}, \ldots, a_{i_{p}}$ for some $p \leq m$. The partition of $T$ into simplicial cones $T_{j}, 0 \leq j \leq p$ generated by $a_{i_{0}}, \ldots, a_{i_{j-1}}, a_{0}, a_{i_{j+1}}, \ldots, a_{i_{p}}$ induces the partition of $\mathbb{Z}_{\geq 0}^{m}$ into the union of shifted monoids (we keep from the previous recursive step the partitions of all the cones not containing $a_{0}$ ), and thereby, we get a piecewise monoidal representation after the current recursive step. To define (the modified after the current recursive step) $\mathbf{K}^{\prime}: C_{\nu} \rightarrow C_{\nu^{\prime}}$ on a shifted monoid $M_{j}^{\prime}+a$ where monoid $M_{j}^{\prime}$ is generated by
$a_{i_{0}}, \ldots, a_{i_{j-1}}, a_{0}, a_{i_{j+1}}, \ldots, a_{i_{p}}$, and an integer point $s$ belongs to the parallelotope generated by the same vectors $a_{i_{0}}, \ldots, a_{i_{j-1}}, a_{0}, a_{i_{j+1}}, \ldots, a_{i_{p}}$, we take polynomial $f_{a}:=b_{a} \in \mathbf{B}$ constructed at the previous recursive step.

This completes the description of a piecewise monoidal representation of $\mathbf{K}^{\prime}$ at the current recursive step and the design of the partial algorithm.

Proposition 4.36. The designed partial algorithm terminates and in this case yields a piece-wise monoidal representation of $\mathbf{K}_{\nu, \nu^{\prime}}$ (provided that $C_{\nu}=\mathbb{Z}_{\geq 0}^{m}$ ) together with a common adapted basis for both $\nu, \nu^{\prime}$ iff $\mathbf{K}_{\nu^{\prime}, \nu}$ admits a piece-wise monoidal representation (in particular, $\mathbf{K}_{\nu^{\prime}, \nu}$ is linear on each of the shifted monoids from the representation).

Proof. We have already shown that if the designed partial algorithm terminates then $\mathbf{K}=\mathbf{K}_{\nu^{\prime}, \nu}$.

Conversely, suppose that $\mathbf{K}_{\nu^{\prime}, \nu}$ admits a piece-wise monoidal representation with vectors $a_{1}, \ldots, a_{n} \in C_{\nu}$. After that the partial algorithm computes $\mathbf{K}_{\nu^{\prime}, \nu}(a)$ for $a \in\left\{a_{1}, \ldots, a_{n}\right\}$ and for all integer points $a$ belonging to the parallelotopes from the latter piece-wise monoidal representation generated by vectors $a_{1}, \ldots, a_{n}$, the resulting $\mathbf{K} \leq^{\prime} \mathbf{K}_{\nu^{\prime}, \nu}$ since $\mathbf{K}$ is determined by these values $\mathbf{K}_{\nu^{\prime}, \nu}(a)$. On the other hand, always holds $\mathbf{K} \geq^{\prime} \mathbf{K}_{\nu^{\prime}, \nu}$, therefore $\mathbf{K}=\mathbf{K}_{\nu^{\prime}, \nu}$ and the Proposition is proved.

It would be interesting to understand, whether the designed partial algorithm always terminates when say, $\nu$ is deglex valuation and $\nu^{\prime}=\nu_{\varphi}$ for any injective homomorphism. $\tau: \mathbb{k}\left[x_{1}, \ldots, x_{m}\right] \rightarrow \mathbb{k}\left[x_{1}, \ldots, x_{m}\right]$ ?

### 4.8. Proofs of results of Section 4.

Proof of Proposition 4.4. Prove (a). Let $S$ and $S^{\prime}$ be $\mathbb{k}$-vector spaces, for any nonzero $z \in S \otimes S^{\prime}$ denote by $V(z) \subset S$ the smallest (by inclusion) subspace of $S$ such that $z \in V(z) \otimes S^{\prime}$.

The following is obvious.
Lemma 4.37. For each nonzero $z \in S \otimes S^{\prime}$ one has:
(a) $V(z) \neq 0$ and $V\left(\mathbb{k}^{\times} \cdot z\right)=V(z)$,
(b) $V\left(z+z^{\prime}\right) \subseteq V(z)+V\left(z^{\prime}\right)$ for any nonzero $z^{\prime} \in S \otimes S^{\prime} \backslash\{0,-z\}$.
(c) $V(z)$ is finite dimensional, moreover, it is the $\mathbb{k}$-linear span of $\left\{x_{1}, \ldots, x_{m}\right\}$ for any expansion

$$
\begin{equation*}
z=x_{1} \otimes y_{1}+\ldots x_{m} \otimes y_{m} \tag{4.5}
\end{equation*}
$$

with minimal possible $m$ (such an $m$ was called rank of $z$ in [15]).
Furthermore, given a valuation $\nu: S \backslash\{0\} \rightarrow C$. Then, clearly, for any finitedimensional subspace $S_{0} \subset S$ the set

$$
\left\{\nu(x) \mid x \in S_{0} \backslash\{0\}\right\}
$$

is a finite subset of $C$; denote by $\nu\left(S_{0}\right)$ its maximal element.
Furthermore, in the notation of Lemma 4.37, for each nonzero $z \in S \otimes S^{\prime}$, denote

$$
\begin{equation*}
\nu^{S^{\prime}}(z):=\nu(V(z)) \tag{4.6}
\end{equation*}
$$

Clearly, $V(x \otimes y)=\mathbb{k} \cdot x$ for any nonzero $x \in S, y \in S^{\prime}$, hence $\nu^{S^{\prime}}(x \otimes y)=\nu(x)$, as in (4.1). This and Lemma 4.37 imply that the assignment $z \mapsto \nu^{S^{\prime}}(z)$ is the desired valuation on $S \otimes S^{\prime}$. This finishes the proof of Proposition 4.4(a).

Prove (b). For any nonzero $z \in S \otimes S^{\prime}$ denote by $\underline{V}(z)$ the smallest (by inclusion) subspace of $S_{\leq a} / S_{<a}, a=\nu^{S^{\prime}}(z)$ such that

$$
z+S_{<a} \in \underline{V}(z) \otimes S^{\prime}
$$

in $\left(S_{\leq a} / S_{<a}\right) \otimes S^{\prime}=\left(S_{\leq a} \otimes S^{\prime}\right) /\left(S_{<a} \otimes S^{\prime}\right)$ (that is, $\underline{V}(z)$ is the image of $V(z)$ under the canonical projection $\left.S_{\leq a} \rightarrow S_{\leq a} / S_{<a}\right)$. Furthermore, denote by $V^{\prime}(z)$ the smallest (by inclusion) subspace of $S^{\prime}$ such that

$$
z+S_{<a} \in \underline{V}(z) \otimes V^{\prime}(z)
$$

in $\left(S_{\leq a} / S_{<a}\right) \otimes S^{\prime}$, where $a=\nu^{S^{\prime}}(z)$.
The following is obvious.
Lemma 4.38. For each nonzero $z \in S \otimes S^{\prime}$, the subspace $V^{\prime}(z)$ is finite-dimensional. Moreover, $\operatorname{dim} \underline{V}(z)=\operatorname{dim} V^{\prime}(z)$ and for any expansion (4.5) with smallest possible $m$, one has

$$
\underline{V}(z)=\bigoplus_{i \in[1, m]: \nu\left(x_{i}\right)=a} \mathbb{k} \cdot\left(x_{i}+V(z)_{<a}\right), V^{\prime}(z)=\bigoplus_{i \in[1, m]: \nu\left(x_{i}\right)=a} \mathbb{k} \cdot y_{i}
$$

Furthermore for each nonzero $z \in S \otimes S^{\prime}$, denote

$$
\begin{equation*}
\left(\nu \otimes \nu^{\prime}\right)(z):=\left(\nu^{S^{\prime}}(z), \nu^{\prime}\left(V^{\prime}(z)\right)\right) \tag{4.7}
\end{equation*}
$$

Clearly, $V^{\prime}(x \otimes y)=\mathbb{k} \cdot y$ for any nonzero $x \in S, y \in S^{\prime}$. Since $\nu^{S^{\prime}}(x \otimes y)$, we obtain $\left(\nu \otimes \nu^{\prime}\right)(x \otimes y)=\left(\nu(x), \nu^{\prime}(y)\right)$, as in (4.2). This and Lemma 4.38 imply that the assignment $z \mapsto\left(\nu \otimes \nu^{\prime}\right)(z)$ is the desired valuation on $S \otimes S^{\prime}$. This finishes the proof of Proposition 4.4(b).

The proposition is proved.
Proof of Proposition 4.14. Prove (a) $=>$ (b). Indeed, let $x, y \in S \backslash\{0\}$ with $a=\nu(x)=\nu(y)$. Then $S_{\leq a}=\mathbb{k} x+S_{<a}=\mathbb{k} y+S_{<a}$ which implies that $x-c y \in S_{<a}$ for some (unique) nonzero scalar $c \in \mathbb{k}$. This proves the implication (a) $=>$ (b).

Prove $(\mathrm{b})=>(\mathrm{a})$. Choose $\mathbf{B} \in \tilde{\mathbf{A}}_{\nu}$ in the notation of Section 4.1. By Lemma 4.2(c), this is a basis of $S$ such that the restriction of $\nu$ to $\mathbf{B}$ is a surjective map $\mathbf{B} \rightarrow C_{\nu}$ and $\mathbf{B}_{<a}=S_{<a} \cap \mathbf{B}$ is a basis of $S_{<a}$ for all $a \in C_{\nu}$. It remains to establish injectivity of $\left.\nu\right|_{\mathbf{B}}$, which we will do by contradiction. Suppose $b, b^{\prime} \in \mathbf{B}$ be such that $b \neq b^{\prime}$, $\nu(b)=\nu\left(b^{\prime}\right)$. Then there exists $c \in \mathbb{k}^{\times}$such that $b^{\prime}-c b \in S_{<a}$, where $a=\nu(b)$, which implies that $b^{\prime}$ is a linear combination of elements of $\mathbf{B}$. The resulting contradiction proves that $\left.\nu\right|_{\mathbf{B}}: \mathbf{B} \rightarrow C_{\nu}$ is a bijection. In view of Lemma 4.13, this proves the implication (b) $=>$ (a).

The proposition is proved.
Proof of Proposition 4.19. Let B be an adapted basis of $S$ such that $\mathbf{B} \cap S_{i}$ is a basis in $S_{i}$ for each $i \in I$. Then $\mathbf{B} \cap S_{i}=\left\{b \in \mathbf{B}: \nu(b) \in \nu\left(S_{i} \backslash\{0\}\right)\right\}$ (cf. Corollary 4.17). Therefore, if for $b \in \mathbf{B}$ it holds $\nu(b) \in \bigcap_{j \in J} \nu\left(S_{j} \backslash\{0\}\right)$ for some $J \subseteq I$ then $b \in \bigcap_{j \in J} S_{j}$, hence $\nu(b) \in \nu\left(\bigcap_{j \in J} S_{j} \backslash\{0\}\right)$, this justifies that the family
$\left\{S_{i}, i \in I\right\}$ is $\nu$-compatible. Moreover, this implies that $\mathbf{B} \cap \bigcap_{j \in J} S_{j}=\{b \in \mathbf{B}:$ $\left.\nu(b) \in \nu\left(\bigcap_{j \in J} S_{j} \backslash\{0\}\right)\right\}$ is a basis of $\bigcap_{j \in J} S_{j}$.

In addition, $\sum_{j \in J} S_{j}$ is contained in the linear hull of the vectors $\{b \in \mathbf{B}: \nu(b) \in$ $\left.\bigcup_{j \in J} \nu\left(S_{j} \backslash\{0\}\right)\right\}$. Thus, $\nu\left(\left(\sum_{j \in J} S_{j}\right) \backslash\{0\}\right) \subseteq \bigcup_{j \in J} \nu\left(S_{j} \backslash\{0\}\right)$. The opposite inclusion is obvious.

Now conversely, assume that the family $\left\{S_{i}, i \in I\right\}$ is $\nu$-compatible. For each $c \in \nu(S \backslash\{0\})$ there exists a unique subset $J \subseteq I$ such that

$$
c \in \bigcap_{j \in J} \nu\left(S_{j} \backslash\{0\}\right) \backslash \bigcup_{l \notin J} \nu\left(S_{l} \backslash\{0\}\right) .
$$

The case $J=\emptyset$ means that $c \notin \bigcup_{i \in I} \nu\left(S_{i} \backslash\{0\}\right)$. Due to $\nu$-compatibility there exists a vector $b_{c} \in \bigcap_{j \in J} S_{j}$ such that $\nu\left(b_{c}\right)=c$ (the case $J=\emptyset$ means that $b_{c} \in S$ ). Observe that for any $c \in \bigcap_{j \in J} \nu\left(S_{j} \backslash\{0\}\right)$ it holds that the vector $b_{c} \in \bigcap_{j \in J} S_{j}$. Hence $\left\{b_{c}: c \in \bigcap_{j \in J} \nu\left(S_{j} \backslash\{0\}\right)\right\}$ is a basis of $\bigcap_{j \in J} S_{j}$ since $\nu$ is injective (cf. Corollary 4.17). Hence the basis $\mathbf{B}:=\left\{b_{c}: c \in \nu(S \backslash\{0\})\right\}$ is required.

Proof of Proposition 4.23. By Lemma 4.2(b), $\mathbf{A}_{\nu}$ is non-empty, so fix $\mathbf{B} \in \mathbf{A}_{\nu}$. Then $\mathbf{B}_{\leq a}:=S_{\leq a} \cap \mathbf{B}=\bigsqcup_{a^{\prime} \leq a} \mathbf{B}_{a^{\prime}}$ is a basis in $S_{\leq a}$, where $\mathbf{B}_{a^{\prime}}=\left\{b \in \mathbf{B}: \nu(b)=a^{\prime}\right\}$.

Furthermore, choose a well-ordering of each $\mathbf{B}_{a}$. Denote by $\mathbf{B}^{0} \subset \mathbf{B}$ the set which consists of all minimal (with respect to the chosen well-ordering) elements of all $\mathbf{B}_{a}$. By the construction, $\left|\mathbf{B}^{0} \cap \mathbf{B}_{a}\right|=1$ for all $a \in C_{\nu}$ and the restriction of $\nu$ to $\mathbf{B}^{0}$ is a bijection $\mathbf{B}^{0} \rightrightarrows \rightarrow C_{\nu}$.

Using transfinite induction, we repeat this procedure and obtain the following.
Lemma 4.39. For each $\mathbf{B} \in \mathbf{A}_{\nu}$ there is a well-ordered set $\mathbf{I}$ with the minimal element $\mathbf{0}$ such that

- $\mathbf{B}=\bigsqcup_{\mathbf{i} \in \mathbf{I}} \mathbf{B}^{\mathbf{i}}$.
- The restriction of $\nu$ to $\mathbf{B}^{\mathbf{i}}$ is an injective map $\mathbf{B}^{0} \hookrightarrow C_{\nu}$.
- $\nu\left(\mathbf{B}^{\mathbf{i}}\right) \subset \nu\left(\mathbf{B}^{\mathbf{i}^{\prime}}\right)$ if $\mathbf{i}^{\prime} \leq \mathbf{i}$ and $\nu\left(\mathbf{B}^{0}\right)=C_{\nu}$

Using this, we obtain an injective map $\underline{\mathbf{j}}: \mathbf{B} \hookrightarrow \mathbf{B}^{0} \times \mathbf{I}$ given by

$$
\begin{equation*}
\underline{\mathbf{j}}(b)=\left(b^{0}, \mathbf{i}\right) \tag{4.8}
\end{equation*}
$$

for each $b \in \mathbf{B}^{\mathbf{i}}$, where $b^{0}$ is the only element of $\mathbf{B}^{0}$ such that $\nu(b)=\nu\left(b^{0}\right)$.
Linearizing, we obtain an injective $\mathbb{k}$-linear map $\mathbf{j}=\mathbb{k} \underline{\mathbf{j}}: S \rightarrow \underline{S} \otimes S^{\prime}$, where $\underline{S}=\mathbb{k} \mathbf{B}^{0} \subset S$ and $S^{\prime}=\mathbb{k} \mathbf{I}$.

By the very construction, the restriction of $\nu$ to $\underline{S}$ is an injective valuation $\underline{\nu}$ : $\underline{S} \backslash\{0\}$. Also, for each nonzero $x \in S$ written as $x=\sum_{\mathbf{i} \in \mathbf{I}, b \in \mathbf{B}^{\mathbf{i}}} c_{b}^{\mathrm{i}} b^{\mathbf{i}}$ one has

$$
\mathbf{j}(x)=\sum_{\mathbf{i} \in \mathbf{I}, b \in \mathbf{B}^{\mathbf{i}}} c_{b}^{\mathbf{i}}\left(b^{0}, \mathbf{i}\right)
$$

in the notation (4.8).
In particular, $\nu(x)=\max _{\mathbf{i} \in I, b \in \mathbf{B}^{\mathbf{i}}: c_{b}^{\mathbf{j}} \neq 0}\{\nu(b)\}=\max _{\mathbf{i} \in I, b \in \mathbf{B}^{\mathbf{i}}: c_{b}^{\mathbf{i}} \neq 0}\left\{\nu\left(b^{0}\right)\right\}=\underline{\nu}^{S^{\prime}}(x)$.
The proposition is proved.

Proof of Theorem 4.24. Let $\nu$ and $\nu^{\prime}$ be injective valuations on $S$. Fix any $a \in C_{\nu}$. Then choose $x \in S \backslash\{0\}$ such that $\nu(x)=a$ and $\nu^{\prime}(x)=\min \left\{\nu^{\prime}\left(\nu^{-1}(a)\right)\right\}$ and $y \in S \backslash\{0\}$ such that $\nu(y)=\min \left\{\nu\left(\nu^{\prime-1}\left(\nu^{\prime}(x)\right)\right)\right\}$, hence $\nu^{\prime}(y)=\nu^{\prime}(x)$. By definition, $\nu^{\prime}(x)=\mathbf{K}_{\nu^{\prime}, \nu}(a)$ and $\nu(y)=\mathbf{K}_{\nu, \nu^{\prime}}\left(\nu^{\prime}(x)\right)=\mathbf{K}_{\nu, \nu^{\prime}}\left(\mathbf{K}_{\nu^{\prime}, \nu}(a)\right)$. Note that $x \in \nu^{\prime-1}\left(\nu^{\prime}(x)\right)$ hence $\nu(y) \leq \nu(x)$.

Using Proposition 4.14(b), choose $c \in \mathbb{k}$ such that $\nu^{\prime}(x-c y)<\nu^{\prime}(x)$. Thus

$$
\mathbf{K}_{\nu^{\prime}, \nu}(a)=\nu^{\prime}(x)>\nu^{\prime}(x-c y) .
$$

Since $\nu^{\prime}(x) \leq \nu^{\prime}(z)$ for all $z \in S \backslash\{0\}$ with $\nu(z)=\nu(x)$, this implies that $\nu(x-c y) \neq$ $\nu(x)$. In turn, this and the inequality $\nu(y) \leq \nu(x)$ imply that $\nu(y)=\nu(x)$.

This proves that $\mathbf{K}_{\nu, \nu^{\prime}}\left(\mathbf{K}_{\nu^{\prime}, \nu}(a)\right)=a$ for all $a \in C_{\nu}$, i.e., $\mathbf{K}_{\nu, \nu^{\prime}} \circ \mathbf{K}_{\nu^{\prime}, \nu}=I d_{C_{\nu}}$. Switching $\nu$ and $\nu^{\prime}$ in the above argument, we also obtain $\mathbf{K}_{\nu^{\prime}, \nu} \circ \mathbf{K}_{\nu, \nu^{\prime}}=I d_{C_{\nu^{\prime}}}$.

This proves the first assertion of the theorem.
Prove the second assertion now. For each $a \in C_{\nu}$ denote by $S_{a}$ the set of all $b \in S \backslash\{0\}$ such that $\nu(b)=a$. Furthermore, let $S_{a}^{\text {min }}$ be the set of all $b \in S_{a}$ such that $\nu^{\prime}(b)=\min \left\{\nu^{\prime}\left(b^{\prime}\right): b^{\prime} \in S_{a}\right\}$. Then, well-ordering of $\nu$ implies that $S_{a}^{\min }$ is nonempty and then $\nu^{\prime}(b)=\mathbf{K}_{\nu^{\prime}, \nu}(a)$ for each $b \in S_{a}^{\min }$ by (4.4). Finally, for each $a \in C_{\nu}$ choose a single element $b_{a} \in S_{a}^{\text {min }}$. Clearly, $\mathbf{B}=\left\{b_{a}: a \in C_{\nu}\right\}$ is adapted to $\nu$ because the restriction of $\nu$ to $\mathbf{B}$ is a bijection $\mathbf{B} \widetilde{\rightarrow} C_{\nu}\left(b_{a} \mapsto \nu\left(b_{a}\right)=a\right)$. Hence $\mathbf{B} \in \mathbf{A}_{\nu}$ is a basis of $S$ by Lemma $4.2(\mathrm{c})$. Finally, injectivity of $\mathbf{K}_{\nu^{\prime}, \nu}$ implies that $\mathbf{B}$ is adapted to $\nu^{\prime}$ because for any distinct $a, a^{\prime} \in C_{\nu}$ one has $\nu^{\prime}\left(b_{a}\right)=\mathbf{K}_{\nu^{\prime}, \nu}(a) \neq \mathbf{K}_{\nu^{\prime}, \nu}\left(a^{\prime}\right)=\nu^{\prime}\left(b_{a^{\prime}}\right)$.

The theorem is proved.
Proof of Proposition 4.27. Let $\mathbf{B} \in \mathbf{A}_{\nu} \cap \mathbf{A}_{\nu^{\prime}}$ and for each $d \in C_{\nu}$ denote by $b_{d}$ the only element of $\mathbf{B}$ with $\nu\left(b_{d}\right)=d$.

Clearly, for any $a \in C_{\nu}$ each $s \in \nu^{-1}(a)$ can be uniquely written as:

$$
s=c_{a} \cdot b_{a}+\sum_{\tilde{a}<a} c_{\tilde{a}} b_{\tilde{a}}
$$

where $c_{a} \neq 0$.
Therefore, $\nu^{\prime}(s) \geq \nu^{\prime}\left(b_{a}\right)$ for all $s \in \nu^{-1}(a)$, i.e., $\nu^{\prime}\left(\nu^{-1}(a)\right) \geq \nu^{\prime}\left(b_{a}\right)$ in $C^{\prime}$.
On the other hand, $b_{a} \in v^{-1}(a)$, therefore the minimum in $\min \left\{\nu^{\prime}\left(\nu^{-1}(a)\right)\right\}$ (see (4.4)) is attained and equals $\nu^{\prime}\left(b_{a}\right) \in \nu^{\prime}\left(\nu^{-1}(a)\right)$, i. e. $\mathbf{K}_{\nu^{\prime}, \nu}(a)=\nu^{\prime}\left(b_{a}\right)$ is defined. The proposition is proved.

## References

[1] A. Arbieto, C. Morales, Dynamics of partial actions, Coloquio Brasileiro de Matematica, 27, IMPA, 2009.
[2] A. Berenstein, D. Kazhdan, Geometric and unipotent crystals II: from geometric crystals to crystal bases, Contemp. Math., 433, Amer. Math. Soc., Providence, RI, 2007, pp. 13-88.
[3] A. Berenstein, Y. Li, Geometric Multiplicities, arXiv:1908. 11581.
[4] A. Berenstein, K. Schmidt, Factorizable Module Algebras, Int. Math. Res. Not., 2019 (21), pp. 6711-6764 (2019).
[5] A. Berenstein, A. Zelevinsky, String bases for quantum groups of type $A_{r}$, Adv. in Soviet Math. 16, Part 1 (1993), pp. 51-89.
[6] G. Bergman, Ordering coproducts of groups and semigroups, Journal of Algebra Volume 133, Issue 2, pp. 313-339 (1990).
[7] T. Blyth, Lattices and ordered algebraic structures, Springer, 2005.
[8] A. Borovik, I. Gelfand and N. White. Coxeter matroids. Birkhäuser, 2003.
[9] J. Brundan, Quiver Hecke algebras and categorification, EMS Ser. Congr. Rep., Zürich, 2013, pp. 103-133.
[10] A. Chistov, Polynomial complexity of the Newton-Puiseux algorithm, Lect. Notes Comput. Sci., 233, 1986, pp. 247-255.
[11] S. Chulkov, A. Khovanskii, Geometry of the Semigroup $Z_{(\geq 0)}^{n}$ and its Applications to Combinatorics, Algebra and Differential Equations, Springer-Verlag, Berlin-New York, 2016.
[12] D. Cox, J. Little, D. O'Shea, Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, Springer, 2006.
[13] G. Dantzig, M. Thapa, Linear Programming: 2: Theory and Extensions, Springer, 1997.
[14] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, Invent. Math., 21, 1973, pp. 287-301.
[15] D. Grigoriev, Relation between the rank and the multiplicative complexity of a bilinear form over a commutative Noetherian ring, J. Soviet Math., vol.17, 1981, p.1987-1998.
[16] D. Grigoriev, P. Milman, Nash desingularization for binomial varieties as Euclidean division in positive dimension. Polynomial complexity in dimension less than 3, Adv. Math., 231, 2012, pp. 3389-3428.
[17] N. Jacobson, Lectures in abstract algebra: theory of fields and Galois theory, Springer, 1964
[18] K. Kaveh, Crystal bases and Newton-Okounkov bodies, Duke Math. J., 164, 2015, pp. 24612506.
[19] K. Kaveh, A. Khovanskii, Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory, Ann. Math., 176, 2012, pp. 925-978.
[20] K. Kaveh, C. Manon, Khovanskii bases, higher rank valuations, and tropical geometry, SIAM J. Appl. Algebr. Geom., 3, 2019, pp. 292-336.
[21] A. Khovanskii, Sums of finite sets, orbits of commutative semigroups, and Hilbert function, Funct. Anal. Appl., 29, 1995, pp. 36-50.
[22] S. Kuhlmann, Valuation bases for extensions of valued vector spaces, Forum Math. 8 (1996), no. 6, pp. 723-735.
[23] F.-V. Kuhlmann, S. Kuhlmann, Ax-Kochen-Ershov principles for valued and ordered vector spaces, Ordered algebraic structures (Curacao, 1995), 237-259, Kluwer Acad. Publ., Dordrecht, 1997.
[24] H. Li, Grõbner bases in ring theory, World Scientific, 2011.
[25] D. Maclagan, B. Sturmfels, Introduction to tropical geometry, Springer, 2015.
[26] E. Miller, B. Sturmfels, Combinatorial commutative algebra, Springer, 2005.
[27] T. Mora, An introduction to commutative and noncommutative Gröbner bases, Theor. Comput. Sci., 134, 1994, pp. 131-173.
[28] B. T. Nguen, P. D. Khanh, Lipschitz Continuity of Convex Functions, Appl. Math. Optim., 84, 2021, pp. 1623-1640.
[29] A. Odesskii, V. Sokolov, Algebraic structures connected with pairs of compatible associative algebras, Intern. Math. Res. Notices, 2006, 2006.
[30] J.-E. Pin, A. Pinguet, P. Weil, Ordered categories and ordered semigroups, Commun. Algebra, 30, 2002, pp. 5651-5675.
[31] L. Robbiano, Term ordering on the polynomial ring, Lect. Notes Comput. Sci., 204, 1985, pp. 513-517.
[32] M. Satyanarayana, Positively Ordered Semigroups, Lecture Notes in Pure and Applied Mathematics, 42, 1979.
[33] E. Schlechter, Handbook of analysis and its foundations, Academic Press, 1997.
[34] B. Sturmfels, Gröbner bases and convex polytopes, AMS, 1996.
[35] B. van der Waerden, Algebra, vol. II, Springer, 1991.
[36] R. Walker, Algebraic curves, Springer, 1978.

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