0. Introduction

This work was motivated by the following two problems from the classical representation theory. (Both problems make sense for an arbitrary complex semisimple Lie algebra but since we shall deal only with the $A_r$ case, we formulate them in this generality).

1. Construct a “good” basis in every irreducible finite-dimensional $sl_{r+1}$-module $V_\lambda$, which “materializes” the Littlewood-Richardson rule. A precise formulation of this problem was given in [3]; we shall explain it in more detail a bit later.

2. Construct a basis in every polynomial representation of $GL_{r+1}$, such that the maximal element $w_0$ of the Weyl group $S_{r+1}$ (considered as an element of $GL_{r+1}$) acts on this basis by a permutation (up to a sign), and explicitly compute this permutation. This problem is motivated by recent work by John Stembridge [10] and was brought to our attention by his talk at the Jerusalem Combinatorics Conference, May 1993.

We show that the solution to both problems is given by the same basis, obtained by the specialization $q=1$ of Lusztig’s canonical basis for the modules over the Drinfeld-Jimbo $q$-deformation $U_r$ of $U(sl_{r+1})$. More precisely, we work with the basis dual to Lusztig’s, and our main technical tool is the machinery of strings developed in [4]. The solution to Problem 1 appears below as Corollary 6.2, and the solution to Problem 2 is given by Proposition 8.8 and Corollary 8.9.

Now let us describe our results and their relationship with the preceding work in more detail. We start with Problem 1. The concept of “good bases” was introduced independently by K.Baclawski [1] and by I.M.Gelfand-A.Zelevinsky [6]. Technically speaking, in every irreducible $sl_{r+1}$-module $V_\lambda$ there is a family of subspaces $V_\lambda(\beta;\nu)$ (their definition and its motivation can be found, e.g., in [3], [4] or in Section 6 below); a basis $B_\lambda$ in $V_\lambda$ is good if $B_\lambda \cap V_\lambda(\beta;\nu)$ is a basis in $V_\lambda(\beta;\nu)$ for every subspace $V_\lambda(\beta;\nu)$ in this family. As explained in [4], the results by G.Lusztig and M.Kashiwara allow us to construct good bases in the following way. First, each $V_\lambda$ can be obtained by the specialization $q=1$ from the corresponding irreducible $U_r$-module, which, with some abuse of notation, we shall denote by the same symbol $V_\lambda$. The concept of good bases generalizes to the $U_r$-modules, and a good basis in the $U_r$-module $V_\lambda$ specializes to a good basis in the corresponding $sl_{r+1}$-module. Now consider the algebra $\mathcal{A} = \mathcal{A}_r$ over the field of rational functions $\mathbb{Q}(q)$,
which is a $q$-deformation of the ring of regular functions on the maximal unipotent subgroup $N_+ \subset SL_{r+1}$. Every $U_r$-module $V_\lambda$ has a canonical realization as a subspace of $A$. There exists a basis $B$ in $A$ (for instance, the dual of Lusztig’s canonical basis) such that $B_\lambda = B \cap V_\lambda$ is a good basis in $V_\lambda$ for all $\lambda$. This basis $B$ and the corresponding bases $B_\lambda$ are the main objects of study in the present paper.

The basis $B$ is naturally labeled by the so-called $A_r$-partitions. By an $A_r$-partition we mean a family of non-negative integers $d = (d_{ij})_{(i,j) \in I_r}$, where the index set $I_r$ consists of pairs of integers $(i, j)$ such that $1 \leq i \leq j \leq r$. (The set $I_r$ is in a natural bijection with the set of positive roots of type $A_r$, so we can think of $A_r$-partitions as partitions of weights into the sum of positive roots.) A labeling of $B$ by $A_r$-partitions was already used by G.Lusztig (see, e.g., [8], Chapter 42). We use a different approach which allows us to obtain much more explicit results. As an example, let us discuss the above-mentioned “materialization” of the Littlewood-Richardson rule. It is known (see e.g., [3]) that for every three highest weights $\lambda, \mu, \nu$ the multiplicity of $V_\mu$ in the tensor product $V_\lambda \otimes V_\nu$ is equal to the dimension of the subspace $V_\lambda(\mu - \nu; \nu) \subset V_\lambda$. So this multiplicity is equal to the number of $A_r$-partitions $d$ such that the corresponding basis vector $b_d \in B$ lies in the subspace $V_\lambda(\mu - \nu; \nu) \subset V_\lambda \subset A$. Corollary 6.2 below describes explicitly all such $d$, thus providing a combinatorial expression for the multiplicity of $V_\mu$ in $V_\lambda \otimes V_\nu$. This expression was earlier obtained in [3], where it was shown to be equivalent to the classical Littlewood-Richardson rule.

As for Problem 2 above, we put it in a more general context of studying various symmetries of the bases $B_\lambda$. There are several such symmetries, and we compute their action explicitly in terms of $A_r$-partitions. Our main technical tool is the study of certain continuous piecewise-linear transformations (“transition maps”) acting in the space of $A_r$-partitions. This is what we mean by “piecewise-linear combinatorics” appearing in the title of this paper. Such combinatorics appeared already in Lusztig’s work; we believe that it constitutes a natural combinatorial framework for the representation theory of quantum groups. This framework complements in a nice way the traditional machinery of Young tableaux.

Our solution of Problem 2 is a good illustration of the interaction between piecewise-linear and traditional combinatorics. First we compute in terms of $A_r$-partitions the action on $B_\lambda$ of a certain involution $\eta_\lambda : V_\lambda \rightarrow V_\lambda$ that specializes at $q = 1$ to the action of $w_0$ (Theorem 7.2 below). Then we translate this description from the language of $A_r$-partitions to that of Young tableaux (actually, we use an equivalent language of Gelfand-Tsetlin patterns that suits better for our purposes). The final result (Theorem 8.2) is that under a natural parametrization of $B_\lambda$ by Young tableaux of shape $\lambda$, the involution $\eta_\lambda$ acts by the well-known Schützenberger involution. The combinatorial implications of this results are explored by J.Stembridge in [11].
The material is organized as follows. For the convenience of the reader we collect in Sections 1 and 2 the necessary material from [4]; all the facts we need about the structure of the Drinfeld-Jimbo algebra $U_r$ and its irreducible modules $V_\lambda$ are presented in Section 5. In Section 3 we compute the action on the basis $B$ of the natural group of symmetries isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In Section 4 we compute explicitly the so-called exponents of the vectors from $B$; this calculation is crucial for our derivation of a “piecewise-linear Littlewood-Richardson rule.” The rule itself appears in Section 6. In Section 7 we describe the “twist” of the basis vectors from $B_\lambda$ under the action of three natural involutive automorphisms of $U_r$ (these automorphisms together with the identity automorphism form another group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$). One of these twists is the involution $\eta_\lambda$ mentioned above. Finally, in Section 8 we describe the relationship between $A_r$-partitions and Young tableaux; as a corollary, we show that $w_0$ acts on the canonical basis in every irreducible polynomial $GL_{r+1}$-module by means of the Schützenberger involution.

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1. The algebras $A$ and $U_+$

We fix a positive integer $r$ and let $A$ denote the associative algebra with unit over the field of rational functions $\mathbb{Q}(q)$ generated by the elements $x_1,\ldots,x_r$ subject to the relations:

$$x_i x_j = x_j x_i \text{ for } |i - j| > 1,$$

$$x_i^2 x_j - (q + q^{-1}) x_i x_j x_i + x_j x_i^2 = 0 \text{ for } |i - j| = 1.\quad (1.2)$$

This is a quantum deformation (or q-deformation) of the algebra of polynomial functions on the group of upper unitriangular $(r+1) \times (r+1)$ matrices.

We also consider an isomorphic copy $U_+$ of $A$ which is generated by $E_1,\ldots,E_r$ satisfying the same relations as the $x_i$. This is a q-deformation of the universal enveloping algebra of the Lie algebra of nilpotent upper triangular $(r+1) \times (r+1)$ matrices.

Both algebras are graded by the semigroup $Q_+$ generated by simple roots $\alpha_1,\ldots,\alpha_r$ of the root system of type $A_r$: we have $\deg x_i = \deg E_i = \alpha_i$. The homogeneous components of degree $\gamma$ will be denoted by $A(\gamma)$ and $U_+(\gamma)$.

These algebras are naturally dual to each other (as graded spaces), according to the following two propositions from [4].

**Proposition 1.1.** ([4], Proposition 1.1.) There exists a unique action $(E, x) \mapsto E(x)$ of the algebra $U_+$ on $A$ satisfying the following properties:
(a) (Homogeneity) If $E \in U_+(\alpha), x \in \mathcal{A}(\gamma)$ then $E(x) \in \mathcal{A}(\gamma - \alpha)$.

(b) (Leibniz formula)
$$E_i(xy) = E_i(x)y + q^{-\langle \gamma, \alpha_i \rangle}xE_i(y)$$
for $x \in \mathcal{A}(\gamma), y \in \mathcal{A}$.

(c) (Normalization) $E_i(x_j) = \delta_{ij}$ for $i, j = 1, \ldots, r$.

**Proposition 1.2.** ([4], Proposition 1.2.)

(a) If $\gamma \in Q_+ \setminus \{0\}$, and $x$ is a non-zero element of $\mathcal{A}(\gamma)$ then $E_i(x) \neq 0$ for some $i = 1, \ldots, r$.

(b) For every $\gamma \in Q_+$ the mapping $(E, x) \mapsto E(x)$ defines a non-degenerate pairing
$$U_+(\gamma) \times \mathcal{A}(\gamma) \to \mathcal{A}(0) = \mathbb{Q}(q).$$

Here $(\gamma, \alpha)$ in the Leibniz formula is the usual scalar product on the weight space, so that $\| (\alpha_i, \alpha_j) \|$ is the Cartan matrix of type $A_r$.

**2. The canonical basis in $\mathcal{A}$ and its string parametrizations**

We recall from [4] that there is a distinguished basis $B$ in $\mathcal{A}$ which is dual to Lusztig’s canonical basis in $U_+$. This basis is a string basis in the terminology of [4].

It is shown in [4] that the elements of $B$ can be labeled by certain integral sequences of length $\frac{r(r+1)}{2}$ called strings. The string parametrization is associated to every reduced decomposition of $w_0$, the maximal element in the Weyl group. Let us reproduce the definition of strings from [4], Section 2. Let $x$ be a non-zero homogeneous element of $\mathcal{A}$. For each $i = 1, \ldots, r$ we set

$$l_i(x) = \max \{ l \in \mathbb{Z}_+ : E_i^l(x) \neq 0 \};$$

(2.1)

we call $l_1(x), l_2(x), \ldots, l_r(x)$ the exponents of $x$. We shall use the following notation:

$$E_i^{(\text{top})}(x) := E_i^{(l_i(x))}(x)$$

(here $E_i^{(l)}$ stands for the divided power, see [4]). Let $i = (i_1, i_2, \ldots, i_m)$ be a sequence of indices from $\{1, 2, \ldots, r\}$ such that no two consecutive indices are equal to each other. We associate to $x$ and $i$ a nonnegative integer vector $a(i; x) = (a_1, \ldots, a_m)$ defined by

$$a_k = l_{i_k}(E_{i_{k-1}}^{(\text{top})}E_{i_{k-2}}^{(\text{top})} \cdots E_{i_1}^{(\text{top})}(x)).$$

We call $a(i; x)$ the string of $x$ in direction $i$. We abbreviate

$$E_i^{(\text{top})}(x) = E_{i_m}^{(\text{top})}E_{i_{m-1}}^{(\text{top})} \cdots E_{i_1}^{(\text{top})}(x).$$

Note that $E_i^{(\text{top})}(x)$ is a non-zero homogeneous element of $\mathcal{A}$ of degree $\deg(x) - \sum_k a_k \alpha_{i_k}$.

Let $W = S_{r+1}$ be the Weyl group of type $A_r$. For each $w \in W$ we denote by $R(w)$ the set of all reduced decompositions of $w$, i.e., the set of sequences $i = (i_1, i_2, \ldots, i_l)$ of the minimal possible length $l = l(w)$ such that $w$ is equal to the product of simple reflections $s_{i_1}s_{i_2} \cdots s_{i_l}$. We are particularly interested in the reduced decompositions of $w_0$, the maximal element of $W$. We denote $m = l(w_0) = r(r + 1)/2$. 4
Proposition 2.1. ([4], Theorems 2.3, 2.4.) For every $i = (i_1, i_2, \ldots, i_m) \in R(w_0)$ we have $E_i^{(top)}(b) = 1$ for all $b \in B$. Furthermore, the correspondence $b \mapsto a(i; b)$ is a bijection between $B$ and the semigroup $C_Z(i)$ of all integral points of some polyhedral convex cone $C(i) \subset R^m$.

Proposition 2.2. ([4], Theorem 2.2.) For every $i, i' \in R(w_0)$ there is a piecewise-linear automorphism $i'_T_1 : R^m \to R^m$ preserving the lattice $Z^m$ and such that $a(i'; b) = i'_T_1(a(i; b))$ for $b \in B$.

The transition maps $i'_T_1$ can be computed as follows. It is well-known that any two reduced decompositions of $w_0$ can be transformed into each other by a sequence of elementary transformations of two kinds:

\begin{align}
(i_1, i, j, i_2) &\mapsto (i_1, j, i, i_2) \text{ for } |i - j| > 1, \quad (2.2) \\
(i_1, i, j, i, i_2) &\mapsto (i_1, j, i, j, i_2) \text{ for } |i - j| = 1, \quad (2.3)
\end{align}

Proposition 2.3. ([4], Theorem 2.7.) Suppose $i = i_0, i_1, \ldots, i_p = i'$ are reduced decompositions of $w_0$ such that for every $t = 1, \ldots, p$ the transition from $i_{t-1}$ to $i_t$ is of the form (2.2) or (2.3).

(a) We have

$$i'_T_1 = i_p \circ \cdots \circ i_2 \circ T_1.$$ 

(b) If the transition from $i_{t-1}$ to $i_t$ is of the form (2.2) with $i, j$ occupying positions $k, k+1$, then $i'_T_1$ leaves all the components of a string $a$ except $a_k, a_{k+1}$ unchanged and changes $(a_k, a_{k+1})$ to $(a_{k+1}, a_k)$.

(c) If the transition from $i_{t-1}$ to $i_t$ is of the form (2.3) with $i, j, i$ occupying positions $k, k+1, k+2$, then $i'_T_1$ leaves all the components of a string $a$ except $a_k, a_{k+1}, a_{k+2}$ unchanged and changes $(a_k, a_{k+1}, a_{k+2})$ to

$$(\max (a_{k+2}, a_{k+1} - a_k), a_k + a_{k+2}, \min (a_k, a_{k+1} - a_{k+2})).$$

The most important for us will be the following reduced decomposition of $w_0$:

$$i(1) = (1; 2, 1; 3, 2, 1; \ldots; r, r - 1, \ldots, 1).$$

We abbreviate $\Gamma := C_Z(i(1))$. This semigroup has the following explicit description.

Proposition 2.4. ([4], Theorem 2.5.) The semigroup $\Gamma$ is the set of all integral sequences

$$(a_{11}; a_{22}, a_{12}; a_{33}, a_{23}, a_{13}; \ldots; a_{rr}, \ldots, a_{1r})$$

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such that $a_{jj} \geq a_{j-1,j} \geq \cdots \geq a_{1j} \geq 0$ for all $j = 1, \ldots, r$.

Note that we have chosen a double indexation for the strings from $\Gamma$. Let $I = I_r = \{(i,j) : 1 \leq i \leq j \leq r\}$ be the index set for our numeration. It can be identified with the set of positive roots of type $A_r$ via

$$(i,j) \mapsto \alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j.$$  \hspace{1cm} (2.4)

Let $\mathbb{Z}^I$ be the lattice of families $(d_{ij})_{(i,j) \in I}$ of integers indexed by $I$, and let $\mathbb{Z}^I_+ \subset \mathbb{Z}^I$ be the semigroup formed by all families $(d_{ij})$ of non-negative integers. We call elements of $\mathbb{Z}^I_+$ $A_r$-partitions (in [7] they were called multisegments). We define a map $\partial : \Gamma \to \mathbb{Z}^I_+$ by

$$\partial(a)_{ij} = a_{ij} - a_{i-1,j}.$$  \hspace{1cm} (2.5)

For $x \in A$ we abbreviate $\partial(x) := \partial(a(i(1);x))$. The following proposition is straightforward.

**Proposition 2.5.** The map $\partial$ is a semigroup isomorphism between $\Gamma$ and $\mathbb{Z}^I_+$. Thus, the mapping $b \mapsto \partial(b)$ is a bijection between $B$ and $\mathbb{Z}^I_+$. Furthermore, if $x$ is a homogeneous element of $A$ and $(d_{ij}) = \partial(x)$ then the degree of $x$ is equal to $\sum_{(i,j) \in I} d_{ij} \alpha_{ij}$.

According to Proposition 2.5, the elements of $B$ can be labeled by $A_r$-partitions $d \in \mathbb{Z}^I_+$. For $d \in \mathbb{Z}^I_+$ we shall denote by $b_d$ the element of $B$ with $\partial(b_d) = d$.

Clearly, the inverse bijection $\partial^{-1} : \mathbb{Z}^I_+ \to \Gamma$ is given by

$$(\partial^{-1}(d))_{ij} = d_{ij} + d_{2j} + \cdots + d_{ij}.$$  \hspace{1cm} (2.6)

3. The action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $B$

Let $x \mapsto x^*$ be the antiautomorphism of $A$ such that $x_i^* = x_j$ for $i = 1, \ldots, r$. Let $x \mapsto \hat{x}$ be the antiautomorphism of $A$ such that $\hat{x}_i = x_{r+1-i}$ for $i = 1, \ldots, r$. Clearly, both maps are involutions and commute with each other. Thus four transformations $\text{Id}, x \mapsto x^*, x \mapsto \hat{x}, x \mapsto \hat{x}^*$ form a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Moreover, it follows from the results of Lusztig (see [8], Section 14.4) that all these transformations preserve $B$. So they give an action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $B$. Using the parametrization of $B$ by $\mathbb{Z}^I_+$ described above, we obtain the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}^I_+$. Three non-trivial elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{Z}^I_+$ by the mutually commuting involutions $d \mapsto d^*, d \mapsto \hat{d}, d \mapsto \hat{d}^*$ defined by

$$b_d^* = b_d^+, \quad \hat{d} = b_d^+, \quad \hat{d}^* = b_d^*.$$  \hspace{1cm} (3.1)

We shall describe these involutions quite explicitly. We start with $d \mapsto \hat{d}$, which turns out to be a permutation of the components $d_{ij}$. We shall use the notation $\hat{i} := r + 1 - i$.  

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Proposition 3.1. We have \( \hat{d}_{ij} = d_{ji}^{\gamma} \) for all \( d \in \mathbb{Z}_+^I \).

Proof. For \( d = (d_{ij}) \in \mathbb{Z}_+^I \) we denote temporarily by \( d^i \) the \( A_i \)-partition given by \( d_{ij}^i = d_{ji}^{\gamma} \); so we have to prove that \( \hat{d} = d^i \).

We shall use the Poincaré-Birkhoff-Witt type bases in \( A \) constructed in [4]. For each \( (i, j) \in I \) we set

\[
x_{ij} = [x_i, \ldots [x_{j-2}, [x_{j-1}, x_j]] \ldots],
\]

(3.2)

where \([x, y] \) for \( x \in A(\gamma), y \in A(\gamma') \) is the \( q \)-commutator defined by

\[
[x, y] = \frac{xy - q(\gamma, \gamma') yx}{q - q^{-1}}
\]

(3.3)

(see [4], (1.5)); note that in [4] \( x_{ij} \) was denoted by \( t_{i,j+1} \). The elements \( x_{ij} \) satisfy the following commutation relations which are special cases of [4], Proposition 3.11:

\[
[x_{ij}, x_{i', j'}] = 0 \text{ for } i < i' \leq j' \leq j,
\]

(3.4)

\[
[x_{ij}, x_{i, j+1, k}] = x_{ik} \text{ for } i \leq j < k.
\]

(3.5)

Iterating (3.5), we get

\[
x_{ij} = [\ldots [[x_i, x_{i+1}], x_{i+2}], \ldots, x_j].
\]

(3.6)

For \( d = (d_{ij}) \in \mathbb{Z}_+^I \) we set

\[
x^d = x_{11}^{d_{11}} x_{12}^{d_{12}} x_{22}^{d_{22}} \ldots x_{1r}^{d_{1r}} x_{2r}^{d_{2r}} \ldots x_{rr}^{d_{rr}}.
\]

(3.7)

Note that the order of factors here differs from that used in the definition of \( t^d \) in [4], Section 3, which was just the order in Proposition 2.4 above; however, (3.4) implies that the products taken in these two orders differ from each other only by a multiple which is a power of \( q \). The advantage of the present order is clear from the following result.

Lemma 3.2. We have \( \hat{x}^d = x^d \) for all \( d \in \mathbb{Z}_+^I \).

Proof. First of all, applying the antiautomorphism \( x \mapsto \hat{x} \) to both sides of (3.2) and using (3.6) we see that \( \hat{x}_{ij} = x_{ji}^{\gamma} \). It follows that \( \hat{x}^d \) is the product of the same factors as \( x^d \) but taken in the different order:

\[
\hat{x}^d = x_{11}^{d_{11}} x_{12}^{d_{12}} \ldots x_{1r}^{d_{1r}} x_{22}^{d_{22}} \ldots x_{2r}^{d_{2r}} \ldots x_{rr}^{d_{rr}}.
\]

(3.8)

The orders in (3.7) and (3.8) differ as follows: the term with \( x_{ij} \) precedes the term with \( x_{i'j'} \) in (3.7) but goes after it in (3.8) if and only if \( i' < i \leq j < j' \). In view of (3.4), such terms commute with each other (the \( q \)-commutator in this case coincides with the ordinary commutator since the degrees of \( x_{ij} \) and \( x_{i'j'} \) are orthogonal to each other). Thus,
by interchanging some commuting terms in the product in (3.8) we can put the factors in the same order as in (3.7). This proves our lemma.

Consider the following linear order on \( Z^I_+ \): we say that \( d' < d \) if and only if \( \partial^{-1}(d') \) precedes \( \partial^{-1}(d) \) in the lexicographic order. Then the results in [4] imply that the expansion of \( x^d \) in the basis \( B \) has the following form:

\[
x^d = q^{n(d)}b_d + \sum_{d' < d} c_{dd'}b_{d'},
\]

(3.9)

where \( n(d) \) is some integer. (To prove (3.9) we notice that \( \partial(x^d) = d \), which follows from [4], (3.5) and (3.8), and then apply [4], Proposition 4.1.)

Now we can complete the proof of Proposition 3.1. Applying the map \( x \mapsto \hat{x} \) to both sides of (3.9) and using Lemma 3.2, we get

\[
x^d \hat{=} q^{n(d)}b_{\hat{d}} + \sum_{d' < d} c_{dd'}b_{\hat{d}'}.
\]

(3.10)

Comparing (3.10) with the expansion of \( x^d \hat{=} \) given by (3.9), we conclude that \( \hat{d} \preceq d \hat{=} \). In other words, we have \( \hat{d} \preceq d \) for all \( d \in Z^I_+ \). But the map \( d \mapsto \hat{d} \) is a weight-preserving bijection of the set \( Z^I_+ \) with itself. Restricting this bijection to every weight component of \( Z^I_+ \), we obtain a bijection of a finite linearly ordered set with itself such that \( \hat{d} \preceq d \) for all \( d \). Clearly, such a bijection is the identity map, so we have \( \hat{d} \hat{=} d \) for all \( d \). Hence \( \hat{d} \hat{=} d \). Proposition 3.1 is proved.

The involution \( d \mapsto \hat{d}^* \) can be expressed in terms of the transition operators \( \iota T_1 \) introduced in Section 2. For every sequence \( \mathbf{i} \) of indices from \( [1, r] \) let \( \hat{i}^* \) denote the sequence obtained from \( \mathbf{i} \) by replacing each term \( i \) by \( \hat{i} = r + 1 - i \) (leaving the terms in the same order). Clearly, if \( \mathbf{i} \in R(w_0) \) then \( \hat{i}^* \in R(w_0) \); in particular, \( \mathbf{i}(1)^* \in R(w_0) \). Following [4], we denote \( \mathbf{i}(1)^* \) by \( \mathbf{i}(r) \), so

\[
\mathbf{i}(r) = (r; r - 1, r; \ldots; 1, 2, \ldots, r).
\]

(3.11)

Proposition 3.3. We have

\[
\hat{d}^* = (\partial \circ \mathbf{i}(r) T_1(1) \circ \partial^{-1})(d)
\]

for all \( d \in Z^I_+ \).

Proof. We start with the following observation. Let \( E \mapsto E^* \) and \( E \mapsto \hat{E} \) be the antiautomorphisms of \( U_+ \) defined in the same way as the antiautomorphisms \( x \mapsto x^* \) and \( x \mapsto \hat{x} \) of \( \mathcal{A} \), that is, \( E_i^* = E_i \), \( \hat{E}_i^* = \hat{E}_i \) for \( i = 1, \ldots, r \). Then for every two homogeneous elements \( E \in U_+ \), \( x \in \mathcal{A} \) of the same degree we have

\[
E(x) = E^* (x^*) = \hat{E}(\hat{x}) = \hat{E}^* (\hat{x}^*).
\]

(3.12)
The first equality in (3.12) is proved in [4], Proposition 3.10; the other equalities are proved in exactly the same way.

Now since both maps \( x \mapsto \hat{x}^* \) and \( E \mapsto \hat{E}^* \) are automorphisms, the last equality in (3.12) and the definition of strings imply that

\[
a(\hat{i}^*; \hat{x}^*) = a(i; x) \tag{3.13}
\]

for all \( i \in R(w_0) \) and homogeneous \( x \in A \). Applying (3.13) to \( i = i(r) \) and \( b = b_d \) for \( d \in Z_+^l \), we obtain our proposition.

**Remark.** It can be shown that the involution \( d \mapsto d^* \) coincides with the *multisegment duality* \( \zeta \), studied recently in [7]. The main result of [7] is an explicit formula for \( \zeta \).

Comparing this with Propositions 3.1 and 3.3 yields an explicit formula for \( i(r)T_i(1) \), which should be helpful for understanding the linearity domains of this piecewise-linear map.

### 4. The exponents

In this section we compute the *exponents* (see (2.1) above) of the basis vectors from \( B \).

**Theorem 4.1.** For every \( d = (d_{ij}) \in Z_+^l \) the exponents of the corresponding basis vector \( b_d \in B \) are given by

\[
l_j(b_d) = \max_{1 \leq i \leq r} \left( \sum_{h=1}^{i} d_{h,j} - \sum_{h=1}^{i-1} d_{h,j-1} \right) \tag{4.1}
\]

for \( j = 1, \ldots, r \).

**Proof.** Let \( a = (a_{11}; a_{22}, a_{12}; \ldots; a_{rr}, \ldots, a_{1r}) \) be a string from \( \Gamma \). By slight abuse of notation, we shall write \( l_j(a) \) for \( l_j(b_{\emptyset(a)}) \). Taking into account (2.6), we see that (4.1) is equivalent to

\[
l_j(a) = \max \left( a_{1j}, a_{2j} - a_{1,j-1}, a_{3j} - a_{2,j-1}, \ldots, a_{jj} - a_{j-1,j-1} \right). \tag{4.2}
\]

By the definition of strings, \( l_j(a) \) is the first component of the string \( iT_i(1)(a) \) for any \( i \in R(w_0) \) which starts with \( j \). So our strategy in proving (4.2) will be to choose some \( i \in R(w_0) \) starting with \( j \), and to compute the first component of \( iT_i(1)(a) \) by using Proposition 2.3. In doing this, we can assume without loss of generality that \( j = r \) (because in computing \( l_j(a) \) we can just ignore the components \( a_{ik} \) of \( a \) with \( k > j \)).

For every sequence \( i = (i_1, \ldots, i_t) \) of indices let \( i^* = (i_t, \ldots, i_1) \) denote the same sequence written in the reverse order. Clearly, if \( i \in R(w_0) \) then \( i^* \in R(w_0) \) as well.
Lemma 4.2. The transformation $i(1)^* T_{i(1)}$ can be decomposed into a composition of several transformations of the type described in Proposition 2.3 (b). For every string $a = (a_{11}; a_{22}, a_{12}; \ldots; a_{rr}, \ldots, a_{1r})$ we have
\[
i(1)^* T_{i(1)}(a) = (a_{11}, a_{22}, \ldots, a_{rr}; a_{12}, a_{23}, \ldots, a_{r-1,r}; \ldots; a_{1,r-1}, a_{2r}; a_{1r}).
\] (4.3)

Proof of Lemma 4.2. We shall use the following notation from [4], Section 3: for $i < j$ let $i, j$ stand for the sequence $i, i+1, \ldots, j$, and $j, i$ stand for the sequence $j, j-1, \ldots, i$. In this notation, we have
\[
i(1) = (1, 2, 1, \ldots, r, 1), \quad i(1)^* = (1, r, 1, r-1, \ldots, 2, 1).
\]

Let us write $i(1)$ as $i(1) = (i'(1), r, 1)$, where $i'(1)$ is the element of the same kind as $i(1)$ but with $r$ replaced by $r - 1$. In order to transform $i(1)$ into $i(1)^*$, we first ignore the last group $r, 1$ in $i(1)$ and transform $i(1)$ to $(i'(1)^*, r, 1)$. Using induction on $r$, we can assume that this can be done by a chain of transformations of type (2.2), and that the corresponding transformation of strings is
\[
(i'(1)^*, r, 1) T_{i(1)}(a) = (a_{11}, a_{22}, \ldots, a_{r-1,r-1}; a_{12}, a_{23}, \ldots, a_{r-2,r-1}; \ldots; a_{1,r-1};
\]
\[a_{rr}, a_{r-1,r}, \ldots, a_{1r}).
\] (4.4)

It remains to transform
\[
(i'(1)^*, r, 1) = (1, r - 1, 1, r - 2, \ldots, 2, 1, r, 1)
\]

into $i(1)^*$. To do this, we move the term $r$ of the last group $r, 1$ to the left, interchanging it with its left neighbors until it arrives at the end of the first group $1, r - 1$. Then we do the same thing with the terms $r - 1, r - 2, \ldots, 2$ of the last group, moving each of them to the end of the corresponding group $1, r - 2, 1, r - 3, \ldots, 1$. This sequence of moves transforms the string in (4.4) into that in the right hand side of (4.3), which completes the proof of Lemma 4.2.

Now we can finish the proof of Theorem 4.1. Let us write $i(1)^*$ as $i(1)^* = (1, r, i'(1)^*)$ and transform it into $(1, r, i'(1))$ by using the transformation $i'(1)^* \mapsto i'(1)$ inverse to that in Lemma 4.2 (with $r$ replaced by $r - 1$). Computing the inverse transformation to that given by (4.3) we see that $(1, r, i'(1)) T_{i(1)^*}$ takes the string in (4.3) to the string $(a_{11}, a_{22}, \ldots, a_{rr}; a')$, where
\[
a' = (a_{12}; a_{23}, a_{13}; \ldots; a_{r-1,r}, a_{r-2,r}, \ldots, a_{1r}).
\]
Now let us recall that our goal is to prove (4.2). Using induction on $r$ we can assume that (4.2) holds for $j=r-1$ and $a$ replaced by $a'$. This gives

$$l_{r-1}(a') = \max (a_{1r}, a_{2r} - a_{1r-1}, a_{3r} - a_{2r-1}, \ldots, a_{r-1, r} - a_{r-2, r-1}). \tag{4.5}$$

Therefore we can find a reduced decomposition of $w_0$ of the form $(\overline{1}, r, i')$ such that $i'$ begins with $r-1$, and the string $(\overline{1}, r, i') T_{(\overline{1}, r, i')(1)}(a_{11}, a_{22}, \ldots, a_{rr}, l_{r-1}(a'))$ begins with $a_{11}, a_{22}, \ldots, a_{rr}, l_{r-1}(a')$. To conclude the proof, we concentrate on the first $r+1$ terms of $(\overline{1}, r, i')$, which are $1, 2, \ldots, r-1, r, r-1$. We transform this sequence into $r, 1, 2, \ldots, r-1, r$ by first performing the operation $(r-1, r, r-1) \to (r, r-1, r)$ of type (2.3) and then moving the first term $r$ to the left interchanging it with $r-2, r-3, \ldots, 1$. Using Proposition 2.3 (c), (b) we see that these operations transform the string $(\overline{1}, r, i') T_{i(1)}$ which begins with $a_{11}, a_{22}, \ldots, a_{rr}, l_{r-1}(a')$ into the string whose first term is $\max (l_{r-1}(a'), a_{rr} - a_{r-1,r-1})$. By definition, this first term is equal to $l_i(a)$, which implies (4.2) since $l_{r-1}(a')$ is given by (4.5). This completes the proof of (4.2) and hence of Theorem 4.1.

Combining Theorem 4.1 with the results in Section 3, we obtain a formula for the exponents $l_i(b_d^i)$ in terms of the $A_r$-partition $d$.

**Corollary 4.3.** For every $d = (d_{ij}) \in Z_+^r$ the exponents of the basis vector $b_d^i \in B$ are given by

$$l_i(b_d^i) = \max_{1 \leq j \leq r} \left( \sum_{k=j}^r d_{ik} - \sum_{k=j+1}^r d_{i+1,k} \right) \tag{4.6}$$

for $i = 1, \ldots, r$.

**Proof.** It follows from (3.13) that

$$l_i(\hat{x}^i) = l_i(x) \tag{4.7}$$

for $i = 1, \ldots, r$ and any homogeneous $x \in A$. Therefore, we have $l_i(b_d^i) = l_i(\hat{b_d}^i) = l_i(b_d^\sigma)$. We see that $l_i(b_d^i)$ is given by (4.1) with $j$ replaced by $\hat{i}$ and $d$ replaced by $\hat{d}$. Substituting into (4.1) the expressions for the components of $\hat{d}$ given by Proposition 3.1, we obtain (4.6).

5. The $q$-analog of $sl_{r+1}$ and its irreducible modules

According to Drinfeld and Jimbo, the $q$-analog of the universal enveloping algebra of $sl_{r+1}$ is the $\mathbb{Q}(q)$-algebra with unit $U_r$ generated by the elements $F_i, K_i, K_i^{-1}, E_i$ for $i = 1, \ldots, r$ subject to the following relations:

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1; \tag{5.1}$$

$$K_iF_j = q^{-(\alpha_i, \alpha_j)} F_jK_i, \quad K_iE_j = q^{(\alpha_i, \alpha_j)} E_jK_i \text{ for all } i, j. \tag{5.2}$$
\[ E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}; \quad (5.3) \]

\[ F_i F_j = F_j F_i, \quad E_i E_j = E_j E_i \text{ for } |i - j| > 1; \quad (5.4) \]

\[ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0, \quad E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \text{ for } |i - j| = 1. \quad (5.5) \]

The algebra \( U_+ \) introduced above, can and will be identified with the \( Q(q) \)-subalgebra of \( U_r \) generated by all \( E_i \) and 1.

Let us recall some well-known properties of finite-dimensional \( U_r \)-modules. All the proofs can be found in [8].

Let \( P \) be the weight lattice of the root system of type \( A_r \), that is,

\[ P = \{ \beta \in \mathbb{Q}_Q : (\beta, \alpha) \in \mathbb{Z} \text{ for all } \alpha \in \mathbb{Q} \}, \]

where \( Q \) is the root lattice and \( Q_Q = Q \otimes \mathbb{Q} \). Let \( P_+ \subset P \) be the semigroup of dominant weights, that is, the semigroup generated by fundamental weights \( \omega_1, \ldots, \omega_r \), where we have \((\omega_i, \alpha_j) = \delta_{ij}\). For a \( U_r \)-module \( V \) the weight component \( V(\beta) \) of weight \( \beta \in P \) is defined as

\[ V(\beta) = \{ v \in V : K_i(v) = q^{(\beta, \alpha_i)} v, \ i = 1, \ldots, r \}. \quad (5.6) \]

It is known that every finite-dimensional \( U_r \)-module \( V \) is diagonalizable, i.e., is the (direct) sum of its weight components. Furthermore, every such module is completely reducible, and the classification of irreducible modules coincides with that for \( sl_{r+1} \)-modules (see [8], Chapter 6). Thus, to every \( \lambda \in P_+ \) there corresponds an irreducible finite-dimensional \( U_r \)-module \( V_\lambda \) with highest weight \( \lambda \) (so the component \( V_\lambda(\lambda) \) is one-dimensional and is annihilated by all the \( E_i \)). The modules \( V_\lambda \) are non-isomorphic and exhaust all irreducible finite-dimensional \( U_r \)-modules.

Now we fix a weight \( \lambda = l_1 \omega_1 + \cdots + l_r \omega_r \in P_+ \). We shall use an explicit realization of \( V_\lambda \) as a subspace of \( \mathcal{A} \) given by the following proposition.

**Proposition 5.1.**

(a) The action of \( U_+ \) on \( \mathcal{A} \) given by Proposition 1.1 extends to the action of the whole algebra \( U_r \) on \( \mathcal{A} \) which is given by

\[ K_i(x) = q^{l_i - (\gamma, \alpha_i)} x, \quad (5.7) \]

\[ F_i(x) = \frac{q^{l_i} x x_i - q^{(\gamma, \alpha_i) - l_i} x_i x}{q - q^{-1}} \quad (5.8) \]

for \( x \in \mathcal{A}(\gamma), \ i = 1, \ldots, r. \)
(b) The elements \( x \in A \) such that \( E_i^{(l_i+1)}(x^*) = 0 \) for \( i = 1, \ldots, r \) form a \( U_r \)-submodule of the module \( A \) under the action in (a) (here \( x \mapsto x^* \) is the antiautomorphism introduced in Section 3). This submodule is isomorphic to \( V_{\lambda} \).

This proposition can be deduced from the results in [8] in the following way. In [8], the irreducible module \( V_{\lambda} \) is realized as a quotient of the Verma module \( M_{\lambda} \) (see [8], 3.4.5 and Propositions 3.5.6, 6.3.4, 6.3.5). A direct check shows that the \( U_r \)-module \( A \) described in (a) is obtained from \( M_{\lambda} \) by passing to the dual module and twisting it by the involutive automorphism \( \varphi \) of \( U_r \) given by \( \varphi(E_i) = F_i, \varphi(F_i) = E_i, \varphi(K_i) = K_i^{-1} \) (for the definition of the twisting see [8], 3.4.4 or Section 7 below). The same operation of passing to the dual and twisting by \( \varphi \) transforms the quotient \( V_{\lambda} \) of \( M_{\lambda} \) to the submodule of \( A \) described in (b). It remains to observe that this operation transforms \( V_{\lambda} \) to a module isomorphic to itself. We leave the details of this argument to the reader.

By some abuse of notation, we shall write

\[
V_{\lambda} = \{ x \in A : E_i^{(l_i+1)}(x^*) = 0 \text{ for } i = 1, \ldots, r \},
\]

with the understanding that the action of \( U_r \) on \( V_{\lambda} \) is that in Proposition 5.1 (a). Note that the weight components of \( V_{\lambda} \) are given by

\[
V_{\lambda}(\beta) = V_{\lambda} \cap A(\lambda - \beta).
\]

In particular, the highest vector of \( V_{\lambda} \) is just \( 1 \in A(0) \).

6. The canonical basis in \( V_{\lambda} \)

We retain the notation of the previous sections. So \( V_{\lambda} \) is an irreducible \( U_r \)-module with the highest weight \( \lambda = l_1 \omega_1 + \cdots + l_r \omega_r \in P_+ \). We set \( B_{\lambda} = B \cap V_{\lambda} \), where \( B \) is the canonical basis in \( A \). We recall from Section 2 that the vectors from \( B \) are labeled by \( A_r \)-partitions \( d = (d_{ij}) \in \mathbb{Z}_{++}^r \). Combining (5.9) and (4.6), we get

\[
B_{\lambda} = \{ b_d : \sum_{k=j}^{r} d_{ik} - \sum_{k=j+1}^{r} d_{i+1,k} \leq l_i \text{ for } 1 \leq i \leq j \leq r \}.
\]

For every \( \beta \in P \) and \( \nu = n_1 \omega_1 + \cdots + n_r \omega_r \in P_+ \) we set

\[
V_{\lambda}(\beta; \nu) = \{ x \in V_{\lambda}(\beta) : E_i^{(n_i+1)}(x) = 0 \text{ for } i = 1, \ldots, r \},
\]

where \( V_{\lambda}(\beta) \) is the component of weight \( \beta \) in \( V_{\lambda} \).
Proposition 6.1. For every \(\lambda, \nu \in P_+\), \(\beta \in P\) the set \(B_\lambda \cap V_\lambda(\beta; \nu)\) is a basis in \(V_\lambda(\beta; \nu)\).

Proof. As in [4], (4.1), for every \(\gamma \in Q_+\) and \(\nu \in P_+\) we set

\[
A(\gamma; \nu) = \{x \in A(\gamma) : E_i^{(n_i+1)}(x) = 0 \text{ for } i = 1, \ldots, r\}.
\]

By [4], Proposition 4.6, each subspace \(A(\gamma; \nu) \subset A\) is spanned by its intersection with \(B\). Since \(B\) is invariant under the antiautomorphism \(x \mapsto x^*\), every subspace \(A(\gamma; \nu)^*\) is also spanned by its intersection with \(B\). This implies our statement since, in view of (5.9) and (5.10), we have

\[
V_\lambda(\beta; \nu) = A(\lambda - \beta; \nu) \cap A(\lambda - \beta; \lambda)^*.
\]

In view of Proposition 2.5 and (4.1), we can reformulate Proposition 6.1 as follows.

Corollary 6.2. The space \(V_\lambda(\beta; \nu)\) has as a basis the set of elements \(b_d\), where \(d\) runs over all \(A_r\)-partitions satisfying three conditions:

\[
\sum_{i,j} d_{ij} \alpha_{ij} = \lambda - \beta, \quad (6.3)
\]

\[
\sum_{k=j}^r d_{ik} - \sum_{k=j+1}^r d_{i+1,k} \leq l_i \text{ for } 1 \leq i \leq j \leq r. \quad (6.4)
\]

\[
\sum_{h=1}^i d_{hj} - \sum_{h=1}^{i-1} d_{h,j-1} \leq n_j \text{ for } 1 \leq i \leq j \leq r. \quad (6.5)
\]

Corollary 6.2 allows us to “materialize” the Littlewood- Richardson rule along the lines of [3]. To do this we notice that the specialization \(q = 1\) makes \(V_\lambda\) into an irreducible \(sl_{r+1}\)-module with highest weight \(\lambda\), which we will (with some abuse of notation) denote also by \(V_\lambda\). More precisely, it is known (see [8], Chapter 22) that all the matrix entries of the operators \(E_i\) and \(F_i\) (acting on \(V_\lambda\)) in the basis \(B_\lambda\) are rational functions in \(q\) regular at \(q = 1\). Therefore, we can define the operators \(e_i\) and \(f_i\) by specializing the matrices of \(E_i\) and \(F_i\) at \(q = 1\), and let \(h_i\) act on each weight component \(V_\lambda(\beta)\) by multiplication by \((\beta, \alpha_i)\). Then the relations (5.1)-(5.5) imply that the \(e_i, f_i\) and \(h_i\) satisfy the commutation relations among the Cartan generators of \(sl_{r+1}\). This makes the \(C\)-space with the basis \(B_\lambda\) an \(sl_{r+1}\)-module.

Under this specialization, the subspace \(V_\lambda(\beta; \nu) \subset V_\lambda\) (or rather its \(C\)-form) becomes the space of vectors of weight \(\beta\) in \(V_\lambda\) annihilated by the operators \(e_i^{n_i+1}, i = 1, \ldots, r\). It is well-known that the dimension of this space is equal to the multiplicity of the irreducible \(sl_{r+1}\)-module \(V_{\nu+\beta}\) in the tensor product \(V_\lambda \otimes V_\nu\) (see [3]). Thus, Corollary 6.2 implies that this multiplicity is equal to the number of \(A_r\)-partitions \(d\) satisfying (6.3)-(6.5). This
statement was established in [3] in a combinatorial way, essentially by showing that it is equivalent to the classical Littlewood-Richardson rule. Proposition 6.1 and Corollary 6.2 provide us with a representation-theoretic proof of this result.

Note also that the map $x \mapsto x^*$ induces an isomorphism of vector spaces $V_\lambda(\lambda - \gamma; \nu)$ and $V_\nu(\nu - \gamma; \lambda)$ for every $\lambda, \nu \in P_+, \gamma \in P$. Thus, the involution $d \mapsto d^*$ on $A_r$-partitions provides a bijective proof of the fact that our expressions for the multiplicity of $V_{\lambda + \nu - \gamma}$ in $V_\lambda \otimes V_\nu$ and the multiplicity of $V_{\lambda + \nu - \gamma}$ in $V_\nu \otimes V_\lambda$ give the same answer.

7. Twisting $B_\lambda$ by the automorphisms of $U_r$

The commutation relations (5.1)-(5.5) imply that there exist three $Q(q)$-linear involutive automorphisms $\varphi, \psi$ and $\eta$ of the algebra $U_r$ acting on the generators as follows:

\begin{align}
\varphi(E_i) &= F_i, \quad \varphi(F_i) = E_i, \quad \varphi(K_i) = K_i^{-1}, \quad (7.1) \\
\psi(E_i) &= E_i, \quad \psi(F_i) = F_i, \quad \psi(K_i) = K_i; \quad (7.2) \\
\eta(E_i) &= F_i, \quad \eta(F_i) = E_i, \quad \eta(K_i) = K_i^{-1}, \quad (7.3)
\end{align}

where $\hat{i} = r + 1 - i$. Clearly, these three automorphisms together with the identity automorphism form a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It turns out that each of these automorphisms induces some transformation (“twist”) of the bases $B_\lambda$; in this section we shall compute this twist explicitly.

The general setup is as follows. If $V$ is a module over an associative algebra $U$ and $\sigma$ is an automorphism of $U$, then the twisted $U$-module $^\sigma V$ is the same vector space $V$ but with the new action $u * v = \sigma^{-1}(u)v$, $u \in U$, $v \in V$. Clearly, $^\sigma V = ^\sigma (^\tau V)$ for every two automorphisms $\sigma, \tau$ of $U$. Furthermore, if $V$ is a simple $U$-module then so is $^\sigma V$. In particular, if $U = U_r$ and $V = V_\lambda$ then $^\sigma V_\lambda$ is isomorphic to $V_{\sigma(\lambda)}$ for some highest weight $\sigma(\lambda)$. Thus there exists an isomorphism of vector spaces $\sigma_\lambda : V_\lambda \to V_{\sigma(\lambda)}$ such that

$$\sigma_\lambda(uv) = \sigma(u)\sigma_\lambda(v), \quad u \in U_r, \quad v \in V_\lambda.$$ 

Clearly, $\sigma_\lambda$ is unique up to a scalar multiple. It follows that the operator $\sigma_{\tau(\lambda)}\tau_\lambda$ is proportional to $(\sigma\tau)_\lambda$ for every two automorphisms $\sigma$ and $\tau$ of $U_r$.

Returning to our situation, it follows at once from (7.1) - (7.3) that

$$\varphi(\lambda) = \psi(\lambda) = -w_0(\lambda), \quad \eta(\lambda) = \lambda, \quad (7.4)$$

where $w_0$ is the maximal element of the Weyl group. We recall that every module $V_\lambda$ is canonically realized as a subspace in $A$, so that the highest vector in $V_\lambda$ is 1. We denote by $b^{\text{low}}_\lambda$ the lowest weight vector in $V_\lambda$, normalized by the condition that it lies in $B_\lambda = B \cap V_\lambda$.

Now we normalize each of the maps $\varphi_\lambda, \psi_\lambda$ and $\eta_\lambda$ by the requirement that

$$\varphi_\lambda(1) = b^{\text{low}}_{-w_0(\lambda)}, \quad \psi_\lambda(1) = 1, \quad \eta_\lambda(1) = b^{\text{low}}_\lambda \quad (7.5)$$

(of course, we also set $\text{Id}_\lambda$ to be the identity map of $V_\lambda$).
Proposition 7.1.

(a) Each of the maps $\varphi_\lambda$ and $\psi_\lambda$ sends $B_\lambda$ to $B_{-w_0(\lambda)}$, while $\eta_\lambda$ sends $B_\lambda$ to $B_\lambda$.

(b) For every two (not necessarily distinct) elements $\sigma, \tau$ of the group $\{\text{Id}, \varphi, \psi, \eta\}$ we have $(\sigma \tau)_\lambda = \sigma_{(\tau \lambda)} (\tau)_{\lambda}$.

Part (a) of the proposition is proved in [8], Proposition 21.1.2. Part (b) follows from (a) and the fact that $\sigma_{(\tau \lambda)} (\tau)_{\lambda}$ are always proportional to each other.

Using Proposition 7.1 (a), we shall write $\sigma_\lambda(b_d) = b_{\sigma_\lambda(d)}$ for all $\sigma \in \{\text{Id}, \varphi, \psi, \eta\}$ and all $b_d \in B_\lambda$. So the map $d \mapsto \sigma_\lambda(d)$ on $A_r$-partitions is well-defined for $d$ such that $b_d \in B_\lambda$, that is, for $d$ satisfying (6.4).

Theorem 7.2. The mappings $\varphi_\lambda$, $\psi_\lambda$ and $\eta_\lambda$ act on $A_r$-partitions as follows:

$$
\varphi_\lambda(d)_{j+1-i,j} = l_i - \left( \sum_{k=j}^r d_{i,k} - \sum_{k=j+1}^r d_{i+1,k} \right)
$$

(7.6)

$$
\psi_\lambda(d)_{r+1-i,j+1-i} = d_{i,j}^r
$$

(7.7)

$$
\eta_\lambda(d)_{j+1-i,r+1-i} = l_j - \left( \sum_{h=1}^i d_{h,j}^r - \sum_{h=1}^{i-1} d_{h+1,j}^r \right)
$$

(7.8)

for all $i, j$ such that $1 \leq i \leq j \leq r$.

Proof. First of all, let us show that (7.8) follows from (7.6) and (7.7). Indeed, in view of Proposition 7.1 (b) and (7.4), we have $\eta_\lambda(d) = \varphi_{-w_0(\lambda)}\psi_\lambda(d)$, so (7.6) and (7.7) imply

$$
\eta_\lambda(d)_{j+1-i,r+1-i} = \varphi_{-w_0(\lambda)}\psi_\lambda(d)_{r+1-i,j+1-i} = l_j - \left( \sum_{k=i}^r \psi_\lambda(d)_{j,k} - \sum_{k=i+1}^r \psi_\lambda(d)_{j+1,k} \right)
$$

$$
= l_j - \left( \sum_{k=i}^r d_{k,j}^r - \sum_{k=i+1}^r d_{k,j+1}^r \right)
$$

Substituting $h = \hat{k}$ in the last summation yields (7.8).

To prove (7.7), it is enough to show that the map $\psi_\lambda : V_\lambda \rightarrow V_{-w_0(\lambda)}$ is just the restriction to $V_\lambda$ of the automorphism $x \mapsto \hat{x}^*$ of the algebra $A$ (see Proposition 3.1 above). Remembering the definitions and the fact that $x \mapsto \hat{x}^*$ preserves $B$, it is easy to see that this map sends $b_\lambda^{\text{low}}$ to $b_{-w_0(\lambda)}^{\text{low}}$ and has the property $\hat{x}^* u = \psi_\lambda(u)\hat{x}^*$ for $u \in U_+, x \in A$. It follows that $\psi_\lambda(x) = \hat{x}^*$, which implies (7.7).

It remains to prove (7.6). Fix $b = b_d \in B_\lambda \cap V_\lambda(\beta)$, and consider the strings

$$
a = (a_{11}; a_{22}, a_{12}; \ldots; a_{rr}, \ldots, a_{1r}) = a(1; b),
$$

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By the definition (2.5), we have

\[ d_{ij} = a_{ij} - a_{i-1,j}, \quad \varphi_\lambda(d)_{j+1-i,j} = a_{j+1-i,j} - a_{j-i,j}, \quad (7.9) \]

so it will be enough for us to express the string \( a^- \) through \( a \).

For \( j = 1, \ldots, r \) we consider \( U_j \) as the subalgebra of \( U_r \) generated by the elements \( E_i, F_i \) and \( K_i^{\pm1} \) for \( i = 1, \ldots, j \). We denote by \( i(1, j) \) the initial subword \((1; 1, 2; 1, \ldots; j, j - 1, \ldots, 1)\) of \( i(1) \) (so \( i(1) \) itself is now denoted \( i(1,r) \)). Clearly, \( i(1,j) \in R(w_0(j)) \) is a reduced decomposition of the element \( w_0(j) \in W \) which can be identified with the maximal element of the Weyl group of \( U_j \).

We set

\[ b(j) = E_1^{(a_{1j})} E_2^{(a_{2j})} \cdots E_j^{(a_{jj})} \cdots E_1^{(a_{12})} E_2^{(a_{22})} E_1^{(a_{11})}(b); \]

in the notation of Section 2, we have \( b(j) = E_{i(1,j)}^{(top)}(b) \). We also set

\[ b^-(j) = F_1^{(a_{1j}^-)} F_2^{(a_{2j}^-)} \cdots F_j^{(a_{jj}^-)} \cdots F_1^{(a_{12}^-)} F_2^{(a_{22}^-)} F_1^{(a_{11}^-)}(b). \quad (7.10) \]

In view of (7.1), we have

\[ b^-(j) = \varphi_\lambda^{-1}(E_{i(1,j)}^{(top)}(\varphi_\lambda(b))). \quad (7.11) \]

It follows that we can also write \( b^-(j) = F_{i(1,j)}^{(top)}(b) \), with the obvious meaning that each power of the type \( F_i^{(a_i)} \) appearing in (7.10) is the maximal possible power of \( F_i \) which still produces a non-zero vector.

We shall show that for each \( j \) the vectors \( b(j) \) and \( b^-(j) \) are, respectively, the highest and lowest weight vectors of the same irreducible \( U_j \)-submodule in \( V_\lambda \). We need the following results from [4] which we reproduce here for the convenience of the reader.

**Lemma 7.3.** ([4], Theorem 2.2, Proposition 6.1). Let \( b \in B, w \in W \) and \( i \in R(w) \). Then the element \( E_i^{(top)}(b) \) belongs to \( B \), depends only on \( w \) (not on the choice of a reduced decomposition of \( w \)), and is annihilated by all \( E_j \) such that \( l(ws_i) < l(w) \).

Lemma 7.3 implies that \( E_i b(j) = 0 \) for \( i = 1, \ldots, j \), so \( b(j) \) is a highest weight vector of some irreducible \( U_j \)-submodule of \( V_\lambda \). Using (7.11), we see also that \( b^-(j) \) is a lowest weight vector of some irreducible \( U_j \)-submodule of \( V_\lambda \). To show that \( b(j) \) and \( b^-(j) \) generate the same \( U_j \)-submodule, it is enough to see that \( b(j) = E_{i(1,j)}^{(top)}(b^-(j)) \). In other words, we have to check the equality

\[ E_{i(1,j)}^{(top)} \circ F_{i(1,j)}^{(top)} = E_{i(1,j)}^{(top)}, \quad (7.12) \]

where both sides are considered as operators acting on \( B_\lambda \). To prove (7.12) we first notice that \( E_i^{(top)} \circ F_i^{(top)} = E_i^{(top)} \) for \( i = 1, \ldots, r \); this follows from the commutation relations.
(5.2), (5.3) between \( E_i, F_i \) and \( K_i \) by a standard argument from the representation theory of \( \text{sl}_2 \). Next we prove that
\[
E_{i(1,j)}^{(\text{top})} \circ F_i^{(\text{top})} = E_{i(1,j)}^{(\text{top})}
\]
for \( i = 1, \ldots, j \). To see this, we notice that there exists a reduced decomposition of \( w_0(j) \) starting with \( i \), i.e., having the form \((i, i')\); in view of Lemma 7.3, we have
\[
E_{i(1,j)}^{(\text{top})} = E_{(i,i')}^{(\text{top})} = E_{i'}^{(\text{top})} \circ E_i^{(\text{top})},
\]
so
\[
E_{i(1,j)}^{(\text{top})} \circ F_i^{(\text{top})} = E_{i'}^{(\text{top})} \circ E_i^{(\text{top})} \circ F_i^{(\text{top})} = E_{i'}^{(\text{top})} \circ F_i^{(\text{top})} = E_{i(1,j)}^{(\text{top})},
\]
proving (7.13). Finally, (7.13) obviously implies (7.12) since \( E_{i(1,j)}^{(\text{top})} \) “swallows” all the factors \( F_i^{(\text{top})} \) occurring in \( F_i^{(\text{top})} \).

Since \( b(j) \) and \( b^- (j) \) are the highest and lowest weight vectors of the same irreducible \( U_j \)-module, their weights with respect to \( U_j \) are obtained from each other by the action of \( w_0(j) \). To write down this statement explicitly, we need some notation. Let \( P(j) \) be the weight lattice for \( U_j \) (so the lattice \( P \) is \( P(r) \)). To distinguish the fundamental weights and simple roots of \( U_j \) from those of \( U_r \), we shall denote them by \( \omega'_1, \omega'_2, \ldots, \omega'_j \) and \( \alpha'_1, \alpha'_2, \ldots, \alpha'_j \). Clearly, the natural projection \( p_j : P \to P(j) \) acts as follows:
\[
p_j(\omega_i) = \begin{cases} 
\omega'_i & \text{if } 1 \leq i \leq j \\
0 & \text{if } j < i \leq r.
\end{cases}
\]

We also have
\[
w_0(j)(\omega'_i) = -\omega'_{j+1-i}, \quad w_0(j)(\alpha'_i) = -\alpha'_{j+1-i} \quad \text{for } i = 1, \ldots, j.
\]

Clearly, the \( U_j \)-weight of \( b(j) \) is equal to
\[
p_j(\beta) + \sum_{i=1}^{j} \left( \sum_{k=i}^{j} a_{ik} \right) \cdot \alpha'_i;
\]
similarly, the \( U_j \)-weight of \( b^- (j) \) is equal to
\[
p_j(\beta) - \sum_{i=1}^{j} \left( \sum_{k=i}^{j} a_{-ik} \right) \cdot \alpha'_i.
\]

Therefore, we have the equality
\[
p_j(\beta) + \sum_{i=1}^{j} \left( \sum_{k=i}^{j} a_{ik} \right) \cdot \alpha'_i = w_0(j)[p_j(\beta) - \sum_{i=1}^{j} \left( \sum_{k=i}^{j} a_{-ik} \right) \cdot \alpha'_i].
\]
\[ w_0(j)p_j(\beta) + \sum_{i=1}^{j} \left( \sum_{k=j+1-i}^{j} a_{j+1-i,k}^- \right) \cdot \alpha_i'. \]  

(7.16)

Taking the scalar product of both sides of (7.16) with \( \omega'_i \) for \( i = 1, \ldots, j \), we obtain

\[ \sum_{k=j+1-i}^{j} a_{j+1-i,k}^- = \sum_{k=i}^{j} a_{ik} + c_{ij}(\beta), \]

(7.17)

where

\[ c_{ij}(\beta) = (p_j(\beta), \omega'_i) - (w_0(j)p_j(\beta), \omega'_i). \]

(7.18)

The rest of the proof is a formal calculation deducing (7.6) from (7.17), (7.18).

First, we rewrite (7.18) in a more convenient form. Using (7.15) and the fact that \( w_0(j) \) preserves the scalar product in \( P(j) \), we obtain

\[ c_{ij}(\beta) = (p_j(\beta), \omega'_i + \omega'_{j+1-i}). \]

An easy check shows that

\[ \omega'_i + \omega'_{j+1-i} = \sum_{h=1}^{i} (\alpha'_h + \alpha'_{h+1} + \cdots + \alpha'_{h+j-i}). \]

Using (7.14), we can rewrite (7.18) as

\[ c_{ij}(\beta) = (\beta, \sum_{h=1}^{i} (\alpha_h + \alpha_{h+1} + \cdots + \alpha_{h+j-i})). \]

(7.19)

Now we subtract from (7.17) the similar equality obtained from it by replacing \((i, j)\) with \((i - 1, j - 1)\). Using (7.19), we can rewrite the resulting equality as follows:

\[ a_{j+1-i,j}^- = \sum_{k=i}^{j} a_{ik} - \sum_{k=i-1}^{j-1} a_{i-1,k} + c_{ij}(\beta) - c_{i-1,j-1}(\beta) \]

\[ = a_{ij} - a_{i-1,i-1} + \sum_{k=i}^{j-1} d_{ik} + (\beta, \sum_{k=i}^{j} \alpha_k). \]

(7.20)

Subtracting from (7.20) the similar inequality obtained from it by replacing \((i, j)\) with \((i + 1, j)\), and taking into account (7.9), we obtain

\[ \varphi_\lambda(d)_{j+1-i,j} = a_{ii} - a_{i-1,i-1} + \sum_{k=i}^{j-1} d_{ik} - \sum_{k=i+1}^{j} d_{i+1,k} + (\beta, \alpha_i). \]

(7.21)
The final step is to compute \((\beta, \alpha_i)\). In view of (5.10), we have

\[
\beta = \lambda - \sum_{i=1}^{r} \left( \sum_{k=i}^{r} a_{ik} \right) \cdot \alpha_i,
\]

which implies

\[
(\beta, \alpha_i) = l_i + \sum_{k=i-1}^{r} a_{i-1,k} - 2 \sum_{k=i}^{r} a_{ik} + \sum_{k=i+1}^{r} a_{i+1,k}
\]

\[
= l_i + a_{i-1,i-1} - a_{ii} - \sum_{k=i}^{r} d_{ik} + \sum_{k=i+1}^{r} d_{i+1,k}.
\]

Substituting this expression into (7.21) and performing the obvious cancellation we obtain (7.6). Theorem 7.2 is proved.

8. The Schützenberger involution

In this section we give a combinatorial description of the involution \(\eta_\lambda\) in terms of Young tableaux and Gelfand-Tsetlin patterns. Here \(\lambda = l_1 \omega_1 + \cdots + l_r \omega_r\) is a fixed highest weight for \(sl_{r+1}\). We associate with \(\lambda\) a partition \(\Lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1} \geq 0)\) of length \(\leq r+1\), where \(\lambda_{r+1}\) is an arbitrary non-negative integer, and \(l_i = \lambda_i - \lambda_{i+1}\) for \(i = 1, \ldots, r\). Let us recall some well-known combinatorial definitions. We identify \(\Lambda\) with its diagram (denoted by the same letter)

\[
\Lambda = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq i \leq r+1, 1 \leq j \leq \lambda_i\}.
\]

By an \(A_r\)-tableau of shape \(\Lambda\) we shall mean a map \(\tau : \Lambda \rightarrow [1, r+1]\) satisfying the conditions

\[
\tau(i, j + 1) \geq \tau(i, j), \quad \tau(i + 1, j) \geq \tau(i, j)
\]

for all \((i, j) \in \Lambda\); here and in the sequel we adopt the convention that \(\tau(i, j) = +\infty\) for \(i > r + 1, j \geq 1\) or \(1 \leq i \leq r + 1, j > \lambda_i\). (In the combinatorial literature the tableaux satisfying (8.1) are often called semistandard.)

The weight of an \(A_r\)-tableau \(\tau\) is an integral vector \(\beta = (\beta_1, \ldots, \beta_{r+1})\) defined by \(\beta_i = \# \tau^{-1}(i)\). We denote by the same letter \(\beta\) the \(sl_{r+1}\)-weight \(\sum_{i=1}^{r} (\beta_i - \beta_{i+1}) \omega_i\).

The language of \(A_r\)-tableaux is equivalent to that of Gelfand-Tsetlin patterns, which will be more convenient for us. By a GT-pattern of highest weight \(\Lambda\) we mean an array of integers \(\pi = (\pi_{ij})_{1 \leq i \leq j \leq r+1}\) such that \(\pi_{i, r+1} = \lambda_i\) for \(i = 1, \ldots, r+1\) and

\[
\pi_{i,j+1} \geq \pi_{ij} \geq \pi_{i+1,j+1}
\]
for $1 \leq i \leq j \leq r$. Such a pattern is displayed as a triangular array

$$
\pi = \begin{pmatrix}
\pi_{14} & \pi_{24} & \pi_{34} & \pi_{44} \\
\pi_{13} & \pi_{23} & \pi_{33} \\
\pi_{12} & \pi_{22} \\
\pi_{11}
\end{pmatrix}.
$$

Clearly, the numbers in each row are weakly decreasing: $\pi_{1j} \geq \pi_{2j} \geq \cdots \geq \pi_{jj}$. The weight of a GT-pattern $\pi$ is an integral vector $\beta = (\beta_1, \ldots, \beta_{r+1})$ defined by

$$
\beta_1 + \beta_2 + \cdots + \beta_j = \pi_{1j} + \pi_{2j} + \cdots + \pi_{jj}
$$

for $j = 1, \ldots, r+1$.

Let $YT_\Lambda$ denote the set of all $A_r$-tableaux of shape $\Lambda$, and $GT_\Lambda$ denote the set of all GT-patterns of highest weight $\Lambda$. It is well-known that the sets $YT_\Lambda$ and $GT_\Lambda$ can be identified with each other by the following weight-preserving bijection $\tau \mapsto \pi(\tau) = (\pi_{ij})$:

$$
\pi_{ij} = \# \{ s : 1 \leq s \leq \lambda_i, \tau(i,s) \leq j \}; \quad (8.3)
$$

indeed, it is easy to see that (8.3) transforms the conditions (8.1) to (8.2).

Now we introduce the Schützenberger involution $\eta : GT_\Lambda \rightarrow GT_\Lambda$. It was first defined in [9] by means of a beautiful combinatorial algorithm sometimes called the evacuation. We shall use an equivalent definition due to E.Gansner [5]. It is given in terms of the so-called Bender-Knuth involutive operators $t_1, t_2, \ldots, t_r$ acting on $A_r$-tableaux. Translating them into the language of GT-patterns, we arrive at the following definition. For $j = 1, \ldots, r$ and $\pi = (\pi_{ij}) \in GT_\Lambda$ we define the pattern $t_j(\pi) \in GT_\Lambda$ by

$$
t_j(\pi)_{ik} = \pi_{ik} \text{ for } k \neq j, \quad t_j(\pi)_{ij} = \min(\pi_{i,j+1}, \pi_{i-1,j-1}) + \max(\pi_{i+1,j+1}, \pi_{i,j-1}) - \pi_{ij}. \quad (8.4)
$$

(In view of (8.2), if we fix all the components $\pi_{ik}$ with $k \neq j$ then each $\pi_{ij}$ takes the values in the segment $[\max(\pi_{i,j+1}, \pi_{i-1,j-1}), \min(\pi_{i,j+1}, \pi_{i-1,j-1})]$; the transformation (8.4) is just the reflection of $\pi_{ij}$ in the midpoint of this segment.) Clearly, each $t_j$ is an involutive map $GT_\Lambda \rightarrow GT_\Lambda$.

As shown in [5], the Schützenberger involution $\eta : GT_\Lambda \rightarrow GT_\Lambda$ can be defined by

$$
\eta := (t_1 \cdots t_r)(t_1 \cdots t_{r-1}) \cdots (t_1 t_2) t_1; \quad \text{(8.5)}
$$

the fact that $\eta$ is an involution follows readily from the obvious relations

$$
t_j^2 = 1, \quad t_i t_j = t_j t_i \text{ for } |i-j| > 1.
$$
(The group generated by the piecewise-linear automorphisms $t_1, \ldots, t_r$ was studied in [2].)

We define the mapping $\partial : GT_\Lambda \rightarrow \mathbb{Z}_+^I$ by the formula

$$\partial(\pi)_{ij} = \pi_{i,j+1} - \pi_{ij} \quad (1 \leq i \leq j \leq r). \quad (8.6)$$

Comparing (8.6) and (8.3) we see that

$$\partial(\pi(\tau))_{ij} = \# \{s : 1 \leq s \leq \lambda_i, \tau(i,s) = j + 1\} \quad (8.7)$$

for any $\tau \in YT_\Lambda$; in the notation of [7], we have $\partial(\pi(\tau)) = d^{(1)}(\tau)$.

The following proposition is an easy consequence of (8.6) and (8.7) combined with (6.1) and (6.3) (cf. [7], (4.4)).

**Proposition 8.1.** The map $\partial$ is a bijection of $GT_\Lambda$ with the set $\{d \in \mathbb{Z}_+^I : b_d \in B_\Lambda\}$. In other words, the canonical basis $B_\Lambda$ in $V_\Lambda$ can be parametrized by GT-patterns of highest weight $\Lambda$ via $\pi \mapsto b_{\partial(\pi)}$. Furthermore, if $\pi \in GT_\Lambda$ has weight $\beta$ then $b_{\partial(\pi)} \in V_\Lambda(\beta)$, i.e., the parametrization $\pi \mapsto b_{\partial(\pi)}$ is weight-preserving.

Now we can state the main result of this section.

**Theorem 8.2.** For any $\pi \in GT_\Lambda$ we have

$$\eta_\Lambda(b_{\partial(\pi)}) = b_{\partial(\eta(\pi))}. \quad (8.8)$$

In other words, under the parametrization of basis vectors by GT-patterns, the involution $\eta_\Lambda$ acts on patterns as the Schützenberger involution.

In order to deduce Theorem 8.2 from Theorem 7.2, we shall translate the operations $t_j$ into the language of $A_r$-partitions. We define the maps $R_1, R_2, \ldots, R_r : \mathbb{Z}_+^I \rightarrow \mathbb{Z}_+^I$ in the following way. For $d = (d_{ij}) \in \mathbb{Z}_+^I$ and $j = 1, \ldots, r$ we define $R_j(d)$ by

$$R_j(d)_{ik} = d_{ik} \quad \text{for } k \neq j, j - 1;$$

$$R_j(d)_{ij} = \min(d_{ij}, d_{i-1,j-1}) + [d_{i,j-1} - d_{i+1,j}]_+; \quad (8.9)$$

$$R_j(d)_{i,j-1} = \min(d_{ij}, d_{i-1,j-1}) + [d_{i+1,j} - d_{i,j-1}]_+,$$

where $[x]_+ = \max(0, x)$. Here we use the convention that $d_{0,j-1}$ that can appear under the minimum sign in (8.9) is $+\infty$, and $[d_{j,j-1} - d_{j+1,j}]_+ = 0$. In particular, $R_1$ is just the identity map.

Note that each $R_j$ is invertible, and the inverse map $R_j^{-1}$ is given by similar formulas

$$R_j^{-1}(d)_{ik} = d_{ik} \quad \text{for } k \neq j, j - 1;$$

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\[ R_j^{-1}(d)_{ij} = \min (d_{ij}, d_{i,j-1}) + [d_{i-1,j-1} - d_{i-1,j}]_+; \]  
\[ R_j^{-1}(d)_{i-1,j-1} = \min (d_{ij}, d_{i,j-1}) + [d_{i-1,j} - d_{i-1,j-1}]_+ \]

(with the conventions similar to the above). The check that the maps defined by (8.10) and (8.9) are indeed inverse to each other, is straightforward (it uses the obvious identity \([x]_+ - [-x]_+ = x\)).

The most important for us will be the following composition of the maps \(R_j\), which is reminiscent of the definition (8.5) of the Schützenberger involution \(\eta\):

\[ \rho := (R_1 \cdots R_r)(R_1 \cdots R_{r-1}) \cdots (R_1R_2)R_1. \]  

We shall express both \(\eta\) and the involution \(d \mapsto d^*\) in terms of the map \(\rho\). For this we need another map \(\partial' : GT_\Lambda \to \mathbb{Z}_+ I\), whose definition is similar to (8.6):

\[ \partial'((\pi))_{ij} = \pi_{ij} - \pi_{i+1,j+1} \ (1 \leq i \leq j \leq r). \]  

Comparing (8.12) and (8.3) we see that

\[ \partial'((\pi(\tau)))_{ij} = \# \{ s : 1 \leq s \leq \lambda_i, \tau(i,s) \leq j, \tau(i+1,s) \geq j + 2 \} \]  

for any \(\tau \in YT_\Lambda\); in the notation of [7], we have \(\partial'((\pi(\tau))) = d^{(2)}(\tau)\).

A direct calculation shows that for every \(\pi \in GT_\Lambda\) the \(A_r\)-partitions \(d^{(1)} = \partial(\pi)\) and \(d^{(2)} = \partial'(\pi)\) are related as follows (cf. [7], (4.5), (4.6)):

\[ d^{(2)}_{ij} = l_i - \left( \sum_{k=j}^r d^{(1)}_{i,k} - \sum_{k=j+1}^r d^{(1)}_{i+1,k} \right); \]  
\[ d^{(1)}_{j+1-i,r+1-i} = l_j - \left( \sum_{h=1}^i d^{(2)}_{j+1-h,r+1-h} - \sum_{h=1}^{i-1} d^{(2)}_{j-h,r+1-h} \right).\]  

**Theorem 8.3.**

(a) The composition of \(\partial'\) with the Schützenberger involution \(\eta : GT_\Lambda \to GT_\Lambda\) is equal to

\[ \partial' \eta = \rho \partial. \]  

(b) For every \(d \in \mathbb{Z}_+\), the \(A_r\)-partitions \(\rho(d)\) and \(d^*\) are related as follows:

\[ \rho(d)_{j+1-i,r+1-i} = d^{*}_{ij}. \]  

**Remark.** We have already mentioned that \(d \mapsto d^*\) coincides with the involution \(\zeta\) from [7]. In fact, Theorem 8.3 is equivalent to Theorem 4.4 from [7], with \(\zeta\) replaced by \(d \mapsto d^*\).
Before proving Theorem 8.3, let us show that together with Theorem 7.2 it implies Theorem 8.2. Let \( \pi \in GT_\Lambda \), and \( d = \partial(\pi) \). By (8.16), \( \partial'(\eta(\pi)) = \rho(d) \); so (8.17) gives

\[
\partial'(\eta(\pi))_{j+1-i,r+1-i} = d_{ij}.'
\]

Substituting this expression into the right hand side of (8.15), we obtain

\[
\partial(\eta(\pi))_{j+1-i,r+1-i} = l_j - \left( \sum_{h=1}^{i} d_{hj}^* - \sum_{h=1}^{i-1} d_{h,j-1}^* \right).
\]

Comparing this with (7.8), we see that \( \eta(\lambda(b)) = b\partial(\eta(\pi)) \), which proves Theorem 8.2.

**Proof of Theorem 8.3 (a).** We include \( \partial \) and \( \partial' \) into a family of mappings \( \partial^\varepsilon : GT_\Lambda \to Z^I_+ \), where \( \varepsilon \) is a sign vector \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \), each \( \varepsilon_j \) being either + or -.

For \( 1 \leq i \leq j \leq r \) we set

\[
\partial^\varepsilon(\pi)_{ij} = \begin{cases} 
\partial'(\pi)_{ij} & \text{if } \varepsilon_j = +, \\
\partial(\pi)_{ij} & \text{if } \varepsilon_j = -.
\end{cases}
\] (8.18)

In particular, we have \( \partial^{(\varepsilon_{-1},\ldots,-)} = \partial, \partial^{(+,\ldots,+,+)} = \partial' \).

If a sign vector \( \varepsilon \) is such that \( \varepsilon_j = -, \varepsilon_{j-1} = + \) then we denote by \( s_j(\varepsilon) \) the sign vector obtained from \( \varepsilon \) by switching \( \varepsilon_j \) to + and \( \varepsilon_{j-1} \) to - (in particular, \( s_1(\varepsilon) \) makes sense if \( \varepsilon_1 = - \) and is then obtained from \( \varepsilon \) by switching this - to +).

**Lemma 8.4.** If a sign vector \( \varepsilon \) is such that \( s_j(\varepsilon) \) makes sense then we have

\[
\partial^{s_j(\varepsilon)} t_j = R_j \partial^\varepsilon.
\] (8.19)

**Proof.** Let \( \pi = (\pi_{ij}) \in GT_\Lambda \), and \( d = \partial^\varepsilon(\pi) \), so that

\[
d_{ij} = \pi_{i,j+1} - \pi_{ij}, \quad d_{i,j-1} = \pi_{i,j-1} - \pi_{i+1,j}.
\] (8.20)

It follows that

\[
\pi_{i,j-1} - \pi_{i+1,j+1} = d_{i,j-1} - d_{i+1,j}.
\] (8.21)

Using (8.4), (8.20), (8.21) and (8.9), we get

\[
\partial^{s_j(\varepsilon)} t_j(\pi)_{ij} = t_j(\pi)_{ij} - t_j(\pi)_{i+1,j+1}
\]

\[
= \min(\pi_{i,j+1}, \pi_{i-1,j-1}) + \max(\pi_{i+1,j+1}, \pi_{i,j-1}) - \pi_{ij} - \pi_{i+1,j+1}
\]

\[
= \min(\pi_{i,j+1} - \pi_{ij}, \pi_{i-1,j-1} - \pi_{ij}) + [\pi_{i,j-1} - \pi_{i+1,j+1}] +
\]

\[
= \min(d_{ij}, d_{i-1,j-1}) + [d_{i,j-1} - d_{i+1,j}] = R_j(d)_{ij}.
\]

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The equality $\partial^{s_j(\varepsilon)} t_j(\pi)_{i,j-1} = R_j(d)_{i,j-1}$ is proved in exactly the same way. Finally, for $k \neq j, j - 1$ we have

$$\partial^{s_j(\varepsilon)} t_j(\pi)_{ik} = \partial^{s_j(\varepsilon)} (\pi)_{ik} = d_{ik} = R_j(d)_{ik}.$$  

This completes the proof of the lemma.

To complete the proof of Theorem 8.3 (a), we notice that the sign vector $(+, +, \ldots, +)$ can be obtained from $(-, -, \ldots, -)$ by the following chain of transformations:

$$(+, +, \ldots, +) = (s_1 \cdots s_r)(s_1 \cdots s_{r-1}) \cdots (s_1 s_2)s_1(-, -, \ldots, -). \quad (8.22)$$

Using this chain of transformations and applying Lemma 8.4 in each step, we obtain

$$(R_1 \cdots R_r)(R_1 \cdots R_{r-1}) \cdots (R_1 R_2)R_1 \partial^{(-, - \cdots, -)}$$

$$= \partial^{(+, + \cdots, +)}(t_1 \cdot \cdots \cdot t_r)(t_1 \cdot \cdots \cdot t_{r-1}) \cdots (t_1 t_2)t_1,$$

which proves (8.16).

**Proof of Theorem 8.3 (b).** In view of Proposition 3.3, the involution $d \mapsto d^*$ is closely related to the transition operation $i(r)T_i(1)$ on strings. We start with studying the transition from $i(1)$ to $i(r)$ in more detail. To do this, we include $i(1)$ and $i(r)$ into a family of reduced decompositions $i(\varepsilon) \in R(w_0)$, labeled by the same sign vectors $\varepsilon$ as in part (a) above. We shall write a reduced decomposition $i \in R(w_0)$ as $i = (i^{(1)}; i^{(2)}; \ldots; i^{(r)})$, where each $i^{(j)}$ is a sequence of indices of length $j$. We shall also use the notation $\vec{i}, \vec{j}$ introduced in the proof of Lemma 4.2 above. In this notation, if $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ is a sign vector then $i(\varepsilon) \in R(w_0)$ is uniquely determined by the following properties:

1. Each $i(\varepsilon)^{(j)}$ has the form $\vec{k}, k + j - 1$ for some $k$ if $\varepsilon_j = +$, and $\vec{k}, k - j + 1$ for some $k$ if $\varepsilon_j = -$ (in particular, $i(\varepsilon)^{(r)} = \vec{r}, \vec{r}$ if $\varepsilon_r = +$, and $i(\varepsilon)^{(r)} = \vec{r}, \vec{r}$ if $\varepsilon_r = -$).

2. Let $j \geq 2$. If $i(\varepsilon)^{(j)} = \vec{k}, k + j - 1$ (i.e., $\varepsilon_j = +$) then we have

$$\varepsilon_{j-1} = + \quad \Rightarrow \quad i(\varepsilon)^{(j-1)} = \vec{k+1}, k + j - 1,$$

$$\varepsilon_{j-1} = - \quad \Rightarrow \quad i(\varepsilon)^{(j-1)} = \vec{k+1}, k + 1.$$  

If $i(\varepsilon)^{(j)} = \vec{k}, k - j + 1$ (i.e., $\varepsilon_j = -$) then we have

$$\varepsilon_{j-1} = + \quad \Rightarrow \quad i(\varepsilon)^{(j-1)} = \vec{k-1}, k - j + 1,$$

$$\varepsilon_{j-1} = - \quad \Rightarrow \quad i(\varepsilon)^{(j-1)} = \vec{k-1}, k - j + 1.$$
In particular, we have $i(1) = i(-, -, \ldots, -)$, $i(r) = i(+, +, \ldots, +)$. It is easy to check that all $i(\varepsilon)$ are indeed reduced decompositions of $w_0$ (for instance, using induction on $r$). Clearly, the map $\varepsilon \mapsto i(\varepsilon)$ is two-to-one: if $\varepsilon$ and $\varepsilon'$ differ only in the first component then $i(\varepsilon) = i(\varepsilon')$.

Suppose $i = (i^{(1)}; i^{(2)}; \ldots; i^{(r)}) \in R(w_0)$ is such that for some $j$ the pair $(i^{(j-1)}; i^{(j)})$ has the form $(k - j + 1, k - 1; k, k - j + 1)$. We denote by $s_j(i)$ the sequence obtained from $i$ by replacing $(i^{(j-1)}; i^{(j)})$ with $(k, k - j + 2; k - j + 1, k)$. It is clear that $s_j(i) \in R(w_0)$ (this follows from the fact that both $(k - j + 1, k - 1; k, k - j + 1)$ and $(k, k - j + 2; k - j + 1, k)$ are reduced decompositions of the same element in $S_{r+1}$). The operation $i \mapsto s_j(i)$ is consistent with the operation $\varepsilon \mapsto s_j(\varepsilon)$ on sign vectors introduced above: the definitions imply at once that

$$s_j(i(\varepsilon)) = i(s_j(\varepsilon)), \quad \text{(8.23)}$$

whenever $s_j(\varepsilon)$ makes sense (we use the convention that $s_1$ is the identity operation on reduced decompositions). Comparing (8.22) and (8.23) we see that

$$i(r) = (s_1 \cdots s_r)(s_1 \cdots s_{r-1}) \cdots (s_1 s_2) s_1(i(1)). \quad \text{(8.24)}$$

Our next step is to lift the chain of transformations (8.24) to the level of strings. We shall write vectors from $\mathbb{R}^m$ as $x = (x^{(1)}; \ldots; x^{(r)})$, where each $x^{(j)}$ belongs to $\mathbb{R}^j$.

**Proposition 8.5.** If $i \in R(w_0)$ is such that $s_j(i)$ makes sense then the transition map $s_j(i)T_1 : \mathbb{R}^m \rightarrow \mathbb{R}^m$ depends only on $j$, not on the choice of $i$. This map leaves unchanged all the components $x^{(k)}$ of $x \in \mathbb{R}^m$ except $x^{(j)}$ and $x^{(j-1)}$; furthermore, the $(j-1)$-st and $j$-th components of $s_j(i)T_1(x)$ depend only on $x^{(j)}$ and $x^{(j-1)}$.

**Proof.** The proposition follows at once from the description of transition maps given by Proposition 2.3. Indeed, $s_j(i)$ is obtained from $i = (i^{(1)}; \ldots; i^{(r)})$ by transforming $(i^{(j-1)}; i^{(j)}) = (k - j + 1, k - 1; k, k - j + 1)$ to $(k, k - j + 2; k - j + 1, k)$. In order to compute $s_j(i)T_1$ we have to decompose the transformation $(k - j + 1, k - 1; k, k - j + 1) \mapsto (k, k - j + 2; k - j + 1, k)$ into a sequence of transformations of type (2.2) and (2.3), and then compose the corresponding mappings in Proposition 2.3 (b), (c). Clearly, this composition does not depend on $k$ and has all the properties claimed in Proposition 8.5.

In view of Proposition 8.5, we shall denote the map $s_j(i)T_1 : \mathbb{R}^m \rightarrow \mathbb{R}^m$ simply by $T_j$; in particular, $T_1$ is the identity map. Combining Proposition 2.3 (a) and (8.24), we obtain

$$i(r)T_{i(1)} = (T_1 \cdots T_r)(T_1 \cdots T_{r-1}) \cdots (T_1 T_2)T_1. \quad \text{(8.25)}$$

The last step in the proof of Theorem 8.3 (b) is to relate the maps $T_j$ on strings with the maps $R_j$ on $A_r$-partitions. We recall that the semigroup $\Gamma \subset \mathbb{R}^m$ consists of $x = (x^{(1)}; \ldots; x^{(r)})$ such that for $j = 1, \ldots, r$ we have $x^{(j)} \in \mathbb{Z}_+^r$ and the components of
$x^{(j)}$ are weakly decreasing: $x_1^{(j)} \geq x_2^{(j)} \geq \cdots \geq x_j^{(j)} \geq 0$ (see Proposition 2.4). For every sign vector $\varepsilon$ we define the map $\partial^\varepsilon : \Gamma \rightarrow \mathbb{Z}_+^I$ by

$$
\partial^\varepsilon(x)_{ij} = \begin{cases} 
  x_i^{(j)} - x_{i+1}^{(j)} & \text{if } \varepsilon_j = +; \\
  x_{j+1-i}^{(j)} - x_{j+2-i}^{(j)} & \text{if } \varepsilon_j = -,
\end{cases}
$$

with the convention $x_{j+1}^{(j)} = 0$. Clearly, all $\partial^\varepsilon$ are semigroup isomorphisms. In particular, $\partial^{(-,-,\ldots,-)}$ is the map $\partial$ in (2.5).

The following lemma is completely analogous to Lemma 8.4.

**Lemma 8.6.** If a sign vector $\varepsilon$ is such that $s_j(\varepsilon)$ makes sense then we have

$$
\partial^{s_j(\varepsilon)} T_j = R_j \partial^\varepsilon
$$

(as mappings $\Gamma \rightarrow \mathbb{Z}_+^L$).

**Proof.** An easy calculation using the strategy described in the proof of Proposition 8.5, shows that $T_j$ acts on the components $x^{(j)}$ and $x^{(j-1)}$ in the following way:

$$
T_j(x)^{(j)}_i = \min \left( x_{i-1}^{(j-1)}, x_i^{(j-1)} + x_j^{(j)} - x_{j+2-i}^{(j)} \right),
$$

$$
T_j(x)^{(j-1)}_i = \max \left( x_{i+1}^{(j)}, x_i^{(j)} + x_{j+1-i}^{(j-1)} - x_j^{(j-1)} \right).
$$

(In fact, a part of this calculation was already done in the end of the proof of Theorem 4.1.)

Now (8.27) follows from (8.28), (8.26) and (8.9) by a straightforward calculation completely analogous to that in the proof of Lemma 8.4. We leave the details to the reader.

Now we can complete the proof of Theorem 8.3 (b). As in the proof of part (a), using repeatedly Lemma 8.6, we deduce from (8.25) and (8.22) that

$$
\partial^{(+,+,\ldots,+)} T_i^{(1)} = \rho \partial.
$$

Comparing (8.29) with Proposition 3.3, we see that for every $d \in \mathbb{Z}_+^I$ the $A_r$-partitions $\hat{d}^s$ and $\rho(d)$ are related by

$$
\hat{d}^s_{ij} = \rho(d)_{j+1-i,j}.
$$

It follows that

$$
d^s_{ij} = \hat{d}^s_{r+1-j,r+1-i} = \rho(d)_{j+1-i,r+1-i},
$$

which is the identity (8.17). Theorems 8.3 and 8.2 are proved.

**Remark 8.7.** The above calculations allow us to relate our transition maps $\nu^T_i$ with Lusztig’s piecewise-linear automorphisms $R_i^\nu : \mathbb{R}^m \rightarrow \mathbb{R}^m$ (the definition of Lusztig’s maps
can be found, e.g., in [8], 42.1.3, 42.2.1). Namely, let us define the linear automorphism \( \delta : \mathbb{R}^m \to \mathbb{R}^m \) by the formula

\[
\delta(x) = x_{j+1-i} - x_{j+2-i}
\]

(so \( \delta(x) \) is obtained from the vector \( d = \partial(x) \in \mathbb{R}^I \) by arranging its components \( d_{ij} \) in the order \( (d_{11}, d_{12}, d_{22}, \ldots, d_{1r}, \ldots, d_{rr}); \) cf. (8.26)). Then an easy calculation using Lemma 8.6 shows that for any two sign vectors \( \varepsilon, \varepsilon' \) we have

\[
R_i^{\varepsilon'}(\varepsilon) = \delta \circ i(\varepsilon') T_i(\varepsilon) \circ \delta^{-1} \quad \text{(8.30)}
\]

This formula shows in fact that the reduced decompositions \( i(\varepsilon) \) are rather special. It is almost obvious from the definitions that there exists a unique family of piecewise-linear automorphisms \( \delta_i : \mathbb{R}^m \to \mathbb{R}^m \) (\( i \in R(w_0) \)) such that \( \delta_i(1) = \delta \) and

\[
R_i^{\varepsilon} = \delta_i \circ T \circ \delta^{-1}
\]

for all \( i, i' \in R(w_0) \). In this notation, (8.30) means that \( \delta_i(\varepsilon) = \delta \) for every sign vector \( \varepsilon \). In general, the maps \( \delta_i \) are not even linear; the simplest example is given by \( i = (1, 3, 2, 1, 3, 2) \) for \( r = 3 \). Finding an explicit formula for \( \delta_1 \) is an interesting problem in piecewise-linear combinatorics.

Our last result is an application of Theorem 8.2 to the classical representation theory obtained by the specialization \( q = 1 \). Having in mind potential combinatorial applications, we prefer to speak about polynomial representations of the group \( GL_{r+1} \). Let \( V_\Lambda \) be the polynomial representation of \( GL_{r+1} \) corresponding to a partition \( \Lambda \). As an \( sl_{r+1} \)-module, \( V_\Lambda \) is just an irreducible module \( V_\Lambda \). Therefore, as explained in the end of Section 6, we can view \( B_\Lambda \) as a basis in \( V_\Lambda \). On the other hand, an element \( w_0 \in S_{r+1} \) can be viewed as an element of \( GL_{r+1} \), so it acts on \( V_\Lambda \).

**Proposition 8.8.** The action of \( w_0 \) on \( V_\Lambda = V_\lambda \) is equal to \( \varepsilon(\Lambda) \eta_\lambda \), where \( \varepsilon(\Lambda) = \pm 1 \). Therefore, \( w_0(b_{\eta(\pi)}) = \varepsilon(\Lambda)b_{\eta(\eta(\pi))} \) for all \( \pi \in GT_\Lambda \) (see (8.8)).

**Proof.** Obviously, the action of \( w_0 \) on \( V_\Lambda \) is compatible with the \( sl_{r+1} \)-action in the following sense:

\[
w_0(fx) = (w_0fw_0^{-1})(w_0(x)) \quad \text{(8.31)}
\]

for all \( f \in sl_{r+1}, x \in V_\Lambda \). In view of (7.3), the automorphism \( f \mapsto w_0fw_0^{-1} \) of \( sl_{r+1} \) coincides with the one obtained from the automorphism \( \eta \) of \( U_r \) by the specialization \( q = 1 \). Comparing (8.31) and (7.5), we see that \( w_0 = \varepsilon(\Lambda) \eta_\lambda \), where \( \varepsilon(\Lambda) \) is the coefficient of proportionality between \( w_0(1) \) and \( b_\lambda^{low} \). Since both \( w_0 \) and \( \eta_\lambda \) are involutions, it follows that \( \varepsilon(\Lambda) = \pm 1 \), and we are done.

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Corollary 8.9. The number of $A_r$-tableaux of shape $\Lambda$ invariant under the Schützenberger involution $\eta$, is equal to $|\text{tr}(w_0, V_\Lambda)|$.

The results in this section have interesting combinatorial consequences explored in [11].

References