# Math 316 HW1 Solutions

**1.2.1 a)** (Note: Having thought about the problem, I figured out what the key part of the argument was, and made it a lemma.)

**Lemma 1.** For $m \in \mathbb{Z}$, $m^2$ is a multiple of $3$ if and only if $m$ is a multiple of $3$. Similarly, $m^2$ is a multiple of $6$ if and only if $m$ is a multiple of $6$.

*Proof.* For any $m \in \mathbb{Z}$, either

- $m = 3k$, and $m^2 = 9k^2$, so both are a multiple of 3;
- $m = 3k+1$, so $m^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, and neither is a multiple of 3; or
- $m = 3k+2$, so $m^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, and neither is a multiple of 3.

Hence, by this case-by-case analysis, we see that $m^2$ is a multiple of $3$ if and only if $m$ is a multiple of 3.

Note that a number is a multiple of $6$ if and only if it is both a multiple of $2$ and a multiple of $3$. So $m$ is a multiple of $6$ if and only if $m$ is a multiple of both $2$ and $3$, if and only if $m^2$ is a multiple of both $2$ and $3$, if and only if $m^2$ is a multiple of $6$. $\square$

**Proposition 2.** The number $\sqrt{3}$ is irrational.

*Proof.* Suppose that $r = \frac{p}{q}$ is a rational number with $r^2 = 3$, and $q$ is minimal with this property. Then $p^2 = 3q^2$. So $p^2$ is a multiple of $3$. By the lemma, $p$ is a multiple of $3$, so $p = 3k$. Then $3k^2 = q^2$, so $q^2$ is a multiple of $3$. By the lemma, $q$ is a multiple of $3$. Dividing both $p$ and $q$ by $3$, we obtain another fractional representative for $r$. This contradicts the minimality of $q$. $\square$

Clearly this same argument works for $6$ replacing $3$, since the lemma still holds.

**b)** The lemma fails for $4$ replacing $3$, since $2^2 = 4$ but $2$ is not a multiple of $4$.

Note: You really should prove something like the lemma above, not just assert it. The point of this problem is to figure out what $2, 3, 6$ have in common that $4$ does not. The fact that $4$ does not have this property means that the property is not "obvious," so it should be proven when it holds.

**extra credit** If you decided to really figure out what matters under the hood, you might have come up with the following lemma.

**Lemma 3.** For $m \in \mathbb{Z}$ and $p$ a prime number, $m^2$ is a multiple of $p$ if and only if $m$ is a multiple of $p$. Similarly, if $q = p_1 \cdots p_d$ is a product of $d$ distinct primes, then $m^2$ is a multiple of $q$ if and only if $m$ is a multiple of $q$.

This lemma is even easier to prove than the proof above, because it just uses the definition of a prime number! I included the earlier proof because it is completely brute force and sometimes you don't need to be clever.

**1.2.4** There are many solutions to this problem, of course. Here is one. Note: I'm going to produce an infinite collection of disjoint subsets of $\mathbb{N}$, but they're not going to hit every number, so I'm going to add one more infinite set which is "everything else." For reasons of numbering, I'm going to call this "everything else" set $A_1$ (I can't call it $A_\infty$), meaning that my other sets will start indexing at $A_2, A_3, \ldots$.

Let $A_2 = \{1, 11, 21, \ldots\}$ be the set of numbers whose decimal expansion ends in a $1$. These are the numbers which have remainder $1$ after dividing by $10$.

Let $A_3 = \{10, 110, 210, \ldots\}$ be the set of numbers whose decimal expansion ends in $10$. These are the numbers which have remainder $10$ after dividing by $100$.

Let $A_4 = \{100, 1100, 2100, \ldots\}$ be the set of numbers whose decimal expansion ends in $100$. These are the numbers which have remainder $100$ after dividing by $1000$.

And so forth. (I would accept this, because the pattern is now obvious. If you want, you can also write the general formula: For $k \geq 2$, $A_k$ is the set of numbers which have remainder $10^{k-2}$ after dividing by $10^{k-1}$.)

Finally, let $A_1$ be all numbers where the last nonzero number in their decimal expansion is not $1$.

We claim that these sets are disjoint, and their union is $\mathbb{N}$. Each natural number $n$ is nonzero, so it has some last nonzero number in its decimal expansion. If this last number is not $1$, then $n \in A_1$. If this last number is $1$ and the number of zeroes is $k$, then $n \in A_{k+2}$. Each $n \in \mathbb{N}$ is therefore inside $A_\ell$ for a unique value of $\ell$, as desired.

**1.2.7 a)** We have $f(A) = [0, 4]$, and $f(B) = [1, 16]$. Thus $f(A) \cap f(B) = [1, 4]$. Meanwhile, $A \cap B = [1, 2]$ and $f(A \cap B) = [1, 4]$, which agrees with $f(A) \cap f(B)$. Similarly, $A \cup B = [0, 4]$ and $f(A \cup B) = [0, 16] = f(A) \cup f(B)$.

**b)** Let $A = [-1, 0]$ and $B = [0, 1]$. Then $f(A) = f(B) = f(A) \cap f(B) = [0, 1]$. Meanwhile, $A \cap B = \{0\}$ so $f(A \cap B) = \{0\}$.

**c)** Let $y \in g(A \cap B)$. Then there exists $x \in A \cap B$ such that $g(x) = y$. But $x \in A$ so that $y = g(x) \in g(A)$, and $x \in B$ so that $y = g(x) \in g(B)$. Thus $y \in g(A) \cap g(B)$. Since this is true for any element of $g(A \cap B)$, we have $g(A \cap B) \subset g(A) \cap g(B)$.

**d)**

**Proposition 4.** For any function $g$ and any subsets $A, B$ of the source of $g$, we have $g(A) \cup g(B) = g(A \cup B)$.

*Proof.* Suppose that $y \in g(A \cup B)$. Then there exists $x \in A \cup B$ with $g(x) = y$. Then either $x \in A$, so that $y \in g(A)$, or $x \in B$, so that $y \in g(B)$. Thus $y$ is either in $g(A)$ or $g(B)$, meaning that $y \in g(A) \cup g(B)$. Hence $g(A \cup B) \subset g(A) \cup g(B)$.

Conversely, let $y \in g(A) \cup g(B)$. Then either $y \in g(A)$, so there is $x \in A$ with $g(x) = y$, or $y \in g(B)$, so there is $x \in B$ with $g(x) = y$. Thus there is some $x$ in either $A$ or $B$ with $g(x) = y$, meaning that there is some $x$ in $A \cup B$ with $g(x) = y$. Thus $y \in g(A \cup B)$. Hence $g(A) \cup g(B) \subset g(A \cup B)$.

Since we have containment in both directions, $g(A) \cup g(B) = g(A \cup B)$. $\square$

Note: When trying to show an equality of sets, the easiest thing to do is show two containments. One can also try to have a series of "if and only if" statements, relating the conditions of being in each set. I'll do this instead in the next problem.

**1.2.9 a)** We have $f^{-1}(A) = [-2, 2]$ and $f^{-1}(B) = [-1, 1]$. (Note: we have $f^{-1}(B) = f^{-1}([0, 1])$.) We have

- $f^{-1}(A) = [-2, 2]$,

- $f^{-1}(B) = [-1, 1]$ (Note, this is also equal to $f^{-1}([0, 1])$),
- $A \cap B = [0, 1]$,
- $f^{-1}(A \cap B) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$,
- $A \cup B = [-1, 4]$,
- $f^{-1}(A \cup B) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$.

**b)** Fix a function $g\colon \mathbb{R} \to \mathbb{R}$, and subsets $A, B \subset \mathbb{R}$. For any $x \in \mathbb{R}$ we have

$$x \in g^{-1}(A) \cap g^{-1}(B) \iff g(x) \in A \text{ and } g(x) \in B \iff g(x) \in A \cap B \iff x \in g^{-1}(A \cap B).$$

Thus $g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B)$. We also have

$$x \in g^{-1}(A) \cup g^{-1}(B) \iff g(x) \in A \text{ or } g(x) \in B \iff g(x) \in A \cup B \iff x \in g^{-1}(A \cup B).$$

Thus $g^{-1}(A) \cup g^{-1}(B) = g^{-1}(A \cup B)$.

**1.2.11 a)** Negation: There exists a pair of real numbers $a < b$ such that, for all $n \in \mathbb{N}$, $a + 1/n \geq b$. (The claim is true, not the negation.)

**b)** Negation: For all $x > 0$, there exists an $n \in \mathbb{N}$ such that $x > 1/n$. (The negation is true, not the claim.)

**c)** Negation: There are two distinct real numbers with no rational numbers in between. (The claim is true, not the negation.)

**1.2.12 a)** We have $y_1 = 6 > -6$. So let us assume that $y_n > -6$. Then $y_{n+1} = (2y_n - 6)/3 > (2(-6) - 6)/3 = -18/3 = -6$. Thus the inductive step is proven.

**b)** Here is a non-inductive argument: $y_n - y_{n+1} = y_n - (2y_n - 6)/3 = (y_n + 6)/3 > 0$ since $y_n > -6$.

But what they want is an inductive argument. Note that $y_2 = 2 < y_1 = 6$, which handles the base case. Now suppose that $y_n < y_{n-1}$. Then $2y_n - 6 < 2y_{n-1} - 6$ and

$$\frac{2y_n - 6}{3} < \frac{2y_{n-1} - 6}{3}.$$

But this is exactly $y_{n+1} < y_n$.

Note: You could use EXACTLY this same argument, with $<$ reversed to $>$, to prove that the sequence is INCREASING! What will go wrong, of course, is the base case, since $y_2 < y_1$ and not the other way around. If we had set $y_1 = -10$, then one would compute that $y_2 > y_1$, and this argument would prove that the sequence was increasing! The base case is very important!!!

**1.2.13 a)**

*Proof.* We use induction on $n$. For $n = 1$ there is nothing to prove. The case $n = 2$ is assumed by the problem (this is Exercise 1.2.5). Suppose $n > 2$ and we know the result for any $k < n$. Let $B = A_1 \cup \cdots \cup A_{n-1}$. Then by the $k = n - 1$ case, we know that

$$B^c = A_1^c \cap \cdots \cap A_{n-1}^c. \tag{0.1}$$

Now
$$(A_1 \cup \cdots \cup A_{n-1} \cup A_n)^c = (B \cup A_n)^c = B^c \cap A_n^c = A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n^c$$
as desired. The first equality used the definition of $B$, the second equality used Exercise 1.2.5, and the third equality used (0.1). □

**b)** (Again, there are many good answers.) Let $B_i$ be the multiples of $i$ in $\mathbb{N}$. Then $\cap_{i=1}^n B_i$ contains $n!$, so it is nonempty. However, any element of $B_i$ is $\geq i$, so $\cap_{i=1}^\infty B_i$ must be empty, as otherwise $\mathbb{N}$ would have an upper bound.

**c)** For an element $x \in \mathbb{R}$, we have $x \in \bigcap_{i=1}^\infty A_i^c$ if and only if $x \notin A_i$ for all $i \in \mathbb{N}$. But this is true if and only if $x \notin \bigcup_{i=1}^\infty A_i$, which is true if and only if $x \in (\bigcup_{i=1}^\infty A_i)^c$.

**1.3.1 a)** A real number $s$ is an *infimum* for a set $A \subset \mathbb{R}$ if it meets the following two criteria:

- $s$ is a lower bound for $A$, i.e. $s \leq a$ for all $a \in A$;
- if $t$ is any lower bound for $A$, then $s \geq t$.

**1.3.2 a)** Let $B = \{3\}$. Then $\sup(B) = \inf(B) = 3$.

**b)** Any finite set $B$ has both a maximum and a minimum, so it contains its infimum and its supremum.

**c)** $(0, 1]$ has infimum $0$ and supremum $1$.