Math 316 (Fund. of analysis), Winter 2018
Quiz 1

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Name:

Q1		5
Q2		7
Q2		6
Q3		20
Q4		12
Total		50

General notes:

1. Terminology: Supremum = least upper bound. Injective = 1-to-1 = into. Surjective = onto.
2. Justification: On an exam, by default, you should justify your statements. If an example or counterexample is the best justification, you should provide one. Otherwise, you should provide a brief conceptual explanation (**one or two sentences**).
   - To justify that "Every closed interval contains 0" is false, you should provide a counterexample, such as $[1, 2]$.
   - To justify that "Some closed interval contains 0" is true, you should provide an example, such as $[-1, 1]$.
   - To justify that "Every closed interval contains some nonzero number" is true, you should explain. "For any interval $[a, b]$, either $a \neq 0$ or $b \neq 0$."
3. If I want a complete proof, I will say "Prove that ..."
4. If I don't need any justification, I will say "no justification necessary." No justification is needed for a definition.

1. **(5 pts)** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and $t \in \mathbb{R}$. State the *sequential* definition for continuity of f at t.

Theorem 0.1. Let (a_n) be a sequence with $a_n \geq 0$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence of partial sums is bounded.

Theorem 0.2. Let (a_n) and (b_n) be sequences satisfying $a_n \geq b_n \geq 0$ for all $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} a_n$ is convergent then the series $\sum_{n=1}^{\infty} b_n$ is convergent.

2. **(7 pts)** *DO ONE:* Prove Theorem 0.1, or prove Theorem 0.2 assuming Theorem 0.1.

3. **(6 pts)** For each of the functions defined below: is it continuous at 0? What is the functional limit at 0, or is it undefined? No justification required.

   (a) $f(x) = x$ if $x \in \mathbb{Q}$, and $f(x) = -x$ if $x \notin \mathbb{Q}$.

   (b) $g(x) = x + 1$ if $x \geq 0$, and $g(x) = -x$ if $x < 0$.

   (c) $h(x) = x + 1$ if $x \neq 0$, and $h(0) = -3$.

4. **(12 pts)** Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$. Using the ϵ-δ definition of continuity, prove that f is continuous everywhere.

5. **(20 pts)** *DO FOUR OUT OF FIVE.* For each of the following statements, is it true or false? Justification is required.

   (a) If $\sum_{n\in\mathbb{N}} x_n$ converges absolutely, and (y_n) is a bounded sequence, then $\sum_{n\in\mathbb{N}} x_n y_n$ converges.

   (b) The series $1 - \frac{1}{1^2} + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{3^2} + \dots$ converges.

   (c) If $\sum_{n\in\mathbb{N}} x_n$ converges, then $\sum_{n\in\mathbb{N}} (-1)^n x_n$ converges.

   (d) The following two series have the same limit.

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots, \qquad 1 + \frac{1}{3^2} - \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{4^2} + \dots$$

   (e) For $f, g\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$, if f is continuous at x and g is not continuous at x, then $f \cdot g$ is not continuous at x.

You're done! If you want extra credit, you can do ONE of the following things:

- (3 pts) The other half of Q2,
- (3 pts) The remaining question of Q5,
- (5 pts) does the series $\sum_{n \in \mathbb{N}} \frac{1}{n \log(n)}$ converge or diverge? Prove it.
- (5 pts) Use the ϵ-δ definition of continuity to prove that $f(x) = \frac{x-1}{x}$ is continuous at x, for any $x > 0$.