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CHAPTER 7.2

Mathematics in fourteenth-century theology

Mark Thakkar

All would-be historians of medieval mathematics must ask themselves where to look for their subject matter. One obvious place to start would be in works with promising titles; approaching the Latin fourteenth century in this way, one might investigate Bradwardine's *Arithmetica speculativa*, *Geometria speculativa*, and *De proportionibus velocitatum in motibus*, Swineshead's *Liber calculationum*, Oresme's *De proportionibus proportionum*, and so on.[1] But this method, for all its initial merits, has limited scope. This chapter explores a less obvious source of material: commentaries on a theological textbook called the *Sententiae in quattuor libris distinctae*, 'Sentences divided into four books'.

The 'Sentences', a compilation of authoritative opinions from the Church Fathers and later theologians, was put together in the 1150s by Peter Lombard, a master at the cathedral school of Notre Dame.[2] Its originality lay solely in its

1. There is a brief *dramatis personae* at the end of this chapter; basic biobibliographical information on almost all of these characters can be found in Gracia and Noone (2003). On obviously mathematical works like those just mentioned, see the chapters by Mahoney (145–178) and Murdoch and Sylla (206–264) in Lindberg (1978), and the new studies in Biard and Rommevaux (2008).

2. Lombard divided the *Sententiae* into short chapters but in the 1220s it was divided thematically into larger sections called *distinctiones* 'distinctions' (Lombard 1971, I 137–144). The Latin text is edited in Lombard (1971); Books I and II are now translated in Lombard (2007; 2008); for an overview, see Rosemann (2004). On the *sententiae* genre, see Teeuwen (2003, 336–339). On the commentary tradition, see the studies in Evans (2002); for its development into the fourteenth century, see Friedman (2002b).

selective arrangement of extant material, but its importance for the history of Western thought can scarcely be overstated. It is not simply that it became an enormously popular textbook, or that it earned its author a portrayal as one of Beatrice's crowning lights in *Paradiso* X (106–108). It is rather that in the thirteenth century it was increasingly used by theologians as a matrix for their own lectures, giving rise to a prolific commentary tradition that lasted for over three hundred years.

Still, the reader might be forgiven for thinking that little of interest to historians of mathematics could possibly be found in commentaries, however original, on a theological textbook. It would be as well to address such misgivings with some preliminary remarks on the context in which such works were produced.[3]

First, theology students—secular ones, at least—were required to hold the wide-ranging degree of Master of Arts, which took around seven years to obtain. By the time they proceeded to the 'higher' faculty of theology, therefore, they were already trained in, among other things, the arts of logic, arithmetic, geometry, and astronomy. This was not a period of narrow specialization; indeed, the modern emphasis on interdisciplinary studies pales beside what has aptly been called the 'unitary character' of education in the medieval university (Murdoch 1975; Asztalos 1992; Marenbon 2007, 205–328).

Second, theology was regarded as the pinnacle of intellectual enquiry, and attracted many of the sharpest minds; commentaries on the 'Sentences' are certainly some of the meatiest intellectual products of the day. The arts faculty, by contrast, was regarded as inferior, not least because of its propaedeutic role. The Parisian arts master John Buridan, one of very few notable scholastics never to have moved on to theology, suggested that another factor was 'the wealth of those who profess in the other faculties' (Zupko 2003, 143, 338).[4] By this he could have meant either that those faculties gave greater financial reward and so were more attractive, or that their members were rich and ipso facto highly regarded. Either way it would not have been odd for men of a mathematical bent to become theologians.

Third, the remit of theology was broader than one might expect. The four books of the 'Sentences' dealt respectively with God, creation, the Incarnation, and the sacraments. Fourteenth-century commentaries tended to be weighted in favour of the first two books, and commentaries on Book II, in particular, involved matters which we would not now think of as theological, such as motion, perception, cosmology, and astrology (Murdoch 1975, 277–279; Grant 2001, 264–280).

Consequently, the 'Sentences' came to be used in the fourteenth century as a springboard for discussion of all kinds of topics. An extreme example is Roger

3. For an excellent introduction to the medieval intellectual world, see Grant (2001).

4. Quare autem nostra facultas sit infima? Potest dici quod hoc est propter divitias eorum qui alias profitent.

Roseth's *Lectura*, written probably in Oxford in around 1335, which bears no resemblance in structure, style, or content to Lombard's work. Roseth instead used five theological questions as pegs on which to hang discussions of, inter alia, the universality of logic, the relationship between a whole and its parts, and infinity and the continuum (Roseth 2005; Hallamaa 1998). His lectures were so divorced from the roots of the tradition that part of the first question even circulated as a separate treatise, *De maximo et minimo*, which certainly fulfils the naive search criterion suggested above.

This phenomenon became so widespread that in 1346 Pope Clement VI wrote a letter of complaint to the masters and scholars of Paris. Most theologians, he said, were ignoring the Bible and the writings of saints and other church authorities in order to waste time on 'philosophical questions, subtle disputations, suspect opinions and various strange doctrines' (Denifle and Chatelain 1889–97, II 588–589, §1125).[5] The tone of the letter is vaguely threatening: if the warning is not heeded, 'we will no doubt think of another remedy'. Twenty years later, the new Parisian university statutes included the following (Asztalos 1992, 434):

> Those reading the 'Sentences' should not treat logical or philosophical questions or topics, except insofar as the text of the 'Sentences' demands or the solutions to arguments require; but they should pose and treat questions of speculative or moral theology that are relevant to the distinctions. Also, those reading the 'Sentences' should read the text in order, and expound it for the utility of the audience. (Denifle and Chatelain 1889–97, III 144, §1319)

The fourteenth-century context, then, was more conducive than one might have thought to the inclusion of technical material in theological commentaries. Still, it is hard to imagine where mathematics might have found a foothold. Without further ado, let us look at some examples.

Mathematics in theology

In distinction 24 of the first book of the 'Sentences', Lombard posed some questions about the ever-mysterious Trinity. What, for instance, is signified by the number three when we say that God is three persons? Here is Lombard's answer:

> When we say three persons, by the term three we do not posit a numerical quantity in God or any diversity, but we signify that our meaning is to be directed to none other than Father and Son and Holy Spirit, so that the meaning of the statement is this: There

5. Plerique quoque theologi, quod deflendum est amarius, de textu Biblie, originalibus et dictis sanctorum ac doctorum expositionibus [...] non curantes, philosophicis questionibus et aliis curiosis disputationibus et suspectis oppinionibus doctrinisque peregrinis et variis se involvunt, non verentes in illis expendere dies suos [...] Alias autem, nisi nostris monitis hujusmodi utique salubribus vobisque multum expedientibus non obtemperaveritis cum effectu [...] cogeremur proculdubio de alio, sicut videremus expediens, remedio providere.

are three persons, or Father and Son and Holy Spirit are three, that is, neither the Father alone, nor the Son alone, nor the Father and Son alone are in the divinity, but also the Holy Spirit, and no one else than these. Similarly, it is not only this or that person who is there, or this one and that one, but this, that, and the other, and no one else. And Augustine sufficiently shows that this is the sense in which we must understand this, when he says that by that term 'the intention was not to signify diversity, but to deny singleness.' (Lombard 2007, 132)

Lombard went on to give a similarly unilluminating commentary on the phrase 'two persons'. Now, if we look up this same distinction in some fourteenth-century commentaries, we find something very different. Here the question is 'whether the Trinity is a true number', and the consensus seems to be that we must first ask what numbers are. The Franciscan William of Ockham, revising his Oxford lectures for publication in the early 1320s,[6] devotes thirty pages to this latter question, with no mention of the Trinity, before resolving the theological issue in one page. Using Ockham as a source, his confrère Adam Wodeham, lecturing at a seminary in Norwich, likewise devotes sixty-two pages to the general question and only three to its theological application. The figures for the Augustinian Gregory of Rimini are twenty and two respectively.[7]

One contentious issue was whether numbers had real existence outside the mind. The difficulty was that if they did, the existence of any two numbers would guarantee the existence of infinitely many objects, which most scholastics found metaphysically abhorrent. The proof has a distinctly mathematical flavour: given a pair of sticks and a pair of stones, we would ipso facto also have a pair of pairs, making three pairs all told; but this triple of pairs would itself be an object, so we would have four objects; and so on ad infinitum. The problem was resolved either by allowing numbers only mental existence, or by denying that they existed separately from the objects that they numbered.

The contrast with Lombard's discussion—brief, unquestionably theological, and devoid of mathematical interest even in the broadest sense—is astonishing. Nor is this an isolated instance. Later in the 'Sentences', Lombard wrote about God's omnipotence, asking such questions as whether He can sin, lie, walk, die, do something that He has not foreseen, and so on. Commenting on this passage, Gregory of Rimini instead asks 'whether God, through His infinite power, can produce an actually infinite effect' (I.42–44.4 in Gregory 1979–87, III 438–481).[8] This occasions over forty pages of discussion, during which he argues that God

6. Lectures were sometimes recorded by students in a set of notes called a *reportatio*. A lecturer could rework a *reportatio* into a more polished *ordinatio*, published at the university stationers. On university publication in Paris and Oxford, see Bataillon, Guyot, and Rouse (1988) and Parkes (1992).

7. These figures, intended only to give a rough and ready comparison, are based on the pagination of the critical editions of I.24.2: Ockham (1979, 90–121); Wodeham (1990, 346–411); Gregory (1979–87, III 34–58).

8. Utrum deus per suam infinitam potentiam possit producere effectum aliquem actu infinitum.

can indeed create an infinite multitude, an infinite magnitude, and an infinitely intense quality.

Gregory appeals in each case to the division of an interval into proportional parts, that is, parts that diminish successively by a fixed proportion. (A modern mathematician might think of this in terms of geometric series with common ratio $1/n$, such as $1/3 + 1/9 + 1/27 + \ldots = 1/2$.) For instance, if God creates an angel at the start of each successive proportional part of an hour—one at the start, one after say half an hour, one after three quarters of an hour, and so on—then by the end of the hour he will have created an infinite multitude of angels (Gregory 1979–87, III 443:3–12). This is a clever line for Gregory to take. He himself is perfectly happy to say that a continuum contains an actual infinity of parts, and that the existence of an infinite multitude is not absurd, either of which makes the question trivial.[9] But he knows that his serious opponents might allow only a 'potential' infinity, so that although further increase is always possible, infinite increase can never be completed. His construction of a 'supertask', a task consisting of an infinite number of accelerated subtasks (Thomson 1954–5), neatly sidesteps this objection. All must agree that, no matter how fast God works, the stars move still and the clock will strike.

Speaking of angels, in 'Sentences' II.2.iv Lombard asked where they were created; his answer was that they were created in the highest heaven, the empyrean, and not in the firmament. Gregory asks instead *utrum angelus sit in loco indivisibili aut divisibili* 'whether an angel is in an indivisible or a divisible place', prompting the general question *an magnitudo componatur ex indivisibilibus* 'whether a magnitude is composed of indivisibles', to which he devotes fifty-three pages (II.2.2 in Gregory 1979–87, IV 277–339; see also Cross 1998; Sylla 2005). His answer is negative: a magnitude is composed of, as one might put it, magnitudes all the way down. He gives surprisingly short shrift to the thesis of composition from infinitely many indivisibles, arguing erroneously that infinitely many indivisibles would yield an infinite magnitude. He is keener to discredit the 'more commonly' held thesis of composition from finitely many indivisibles, which he does with a barrage of nine mathematical and four physical arguments.

Gregory's mathematical arguments use simple geometrical constructions to deduce absurdities from the atomist thesis. The first, for instance, runs as follows. Draw a line of six points. Construct on this base an isosceles triangle with two sides of fifteen points, and draw lines from one side to the other, joining the thirteen pairs of opposite points. These lines must shorten towards the apex, so since the base consists of only six points, they soon become smaller than a point, which is *ex hypothesi* impossible (Gregory 1979–87, IV 279).

9. Gregory also gives the quicker answers to which his position entitles him (1979–87, III 441:19–28; 443:13–27). On the distinction between actual and potential infinity see, for example, Dewender (1999, 286–287).

Gregory was by no means alone in using Lombard's angelology as a pretext for a geometrical refutation of atomism. The tradition seems to have begun in the first few years of the century with John Duns Scotus's Oxford lectures, in which he asked *utrum angelus possit moveri de loco ad locum motu continuo* 'whether an angel can move from place to place in a continuous motion' (II.2.2.5 in Scotus 1973, 292–300; see also Murdoch 1962, 24–30; 1982, 579; Trifogli 2004). Indeed, two of Gregory's arguments are explicitly adapted from Scotus ('*Doctor subtilis*'), and three more are borrowed from Wodeham ('*unus doctor*').[10]

Geometry also gave rise, in the mid-fourteenth century, to some peculiar arguments concerning the relative perfection of different species. The source here was *Elements* III.16, where Euclid says that the curvilinear angle between a tangent and the circumference of a circle (the angle of contingence) is less than any acute rectilinear angle, while the remaining angle between the circumference and the diameter perpendicular to the tangent (the angle of the semicircle) is greater than any acute rectilinear angle.[11] In his Parisian 'Sentences' lectures of 1348–9 the Cistercian Peter Ceffons (Fig. 7.2.1) used this proposition, together with the idea that these angles could be increased or decreased by varying the size of the circle, to derive nineteen corollaries on the proportional excess of certain types of angles over others (Murdoch 1969, 238–246; 1982, 580–582). These results could be applied to 'theological' problems of the following sort: a man and an ass are both infinitely inferior to God, but a man, although of finite perfection, is infinitely superior to an ass.

A more obviously theological problem is that of human free will and divine judgement, but even this was not immune from mathematical intrusion. In his 'Sentences' commentary of 1331–3, the English Dominican Robert Holcot raised a difficulty based, like Gregory's divine supertask, on the proportional parts of an hour. Holcot did not specify a proportion, but let us take it to be a half. Now suppose that a man is meritorious over the space of half an hour, sinful over the next fifteen minutes, meritorious over the next seven and a half minutes, and so on, and suppose that he dies at the end of the hour. Then God cannot reward or punish him, because there was no final instant of his life that would determine whether he died a bad man or a good man.[12] Holcot followed this up with eight similar arguments based on the continuum (Murdoch 1975, 327 n101).

My final example of a theological problem that attracted mathematical speculation is the question of the eternity of the world. Theologians were obviously

10. Renowned scholastics acquired honorific titles like 'the Subtle Doctor' (Scotus) and 'the Venerable Inceptor' (Ockham, also known as 'the More Than Subtle Doctor'), but contemporary authors were usually alluded to indirectly as 'one doctor' or 'some people'. On fourteenth-century citation practices, see Schabel (2005).

11. The proposition is numbered III.15 in some editions of Euclid (Murdoch 1963, 248–249).

12. Holcot's scenario is similar, though ultimately not identical, to that of the 'Thomson's lamp' paradox, in which a lamp is switched on and off with increasing rapidity (Thomson 1954–5).

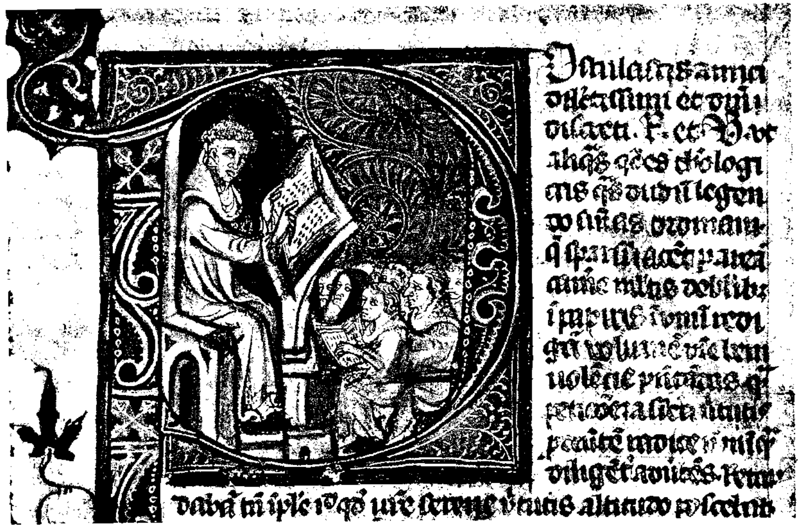

Figure 7.2.1 Peter Ceffons lecturing on the 'Sentences', from the sole manuscript that preserves his unedited commentary (Troyes BM 62 f. 1, c 1354). By permission of Médiathèque de l'Agglomération Troyenne, photo by Pascal Jacquinot

committed to the fact that the world had a beginning in time, but it was disputed whether this could be proved using reason alone. In his Parisian 'Sentences' commentary of the early 1250s, the Franciscan theologian Bonaventure (canonized in 1482) compiled a battery of six arguments to demonstrate that the notion of an eternal world was incoherent. Here I will mention only two. The first was that each passing day adds to the past revolutions of the heavens, and moreover the revolutions of the moon are twelve times as numerous as those of the sun; but one cannot add to or exceed the infinite because there is nothing greater than it. The fifth was that, given the permanence of species and the immortality of the soul, an eternal world would contain infinitely many rational souls; but it is impossible for infinitely many things to exist at the same time (II.1.1.i.2 in Bonaventure 1885, 20–22; Byrne 1964).[13]

Bonaventure's fifth argument explains why the subsequent debate was often conducted in terms of multitudes of souls, but the first one is more interesting for our purposes. In the fourteenth century two lines of response were developed.

13. These arguments ultimately came from the sixth-century Christian John Philoponus via the Islamic world, though the specific example of souls was introduced by the twelfth-century Muslim al-Ghazali (Davidson 1987, 117–134; Sorabji 1983, 214–226).

One was to deny, as Galileo was more famously to do in *Two new sciences* (1989, 40–41), that terms like 'equal to' and 'greater than' were applicable to the infinite. The other was to try to explain how these terms, and the terms 'part' and 'whole', behaved when they were applied to the infinite (Murdoch 1982, 569–573; Dales 1990; Friedman 2002a).

These, then, are some of the mathematical topics that one finds discussed in fourteenth-century theological works. It is hard to get a real sense of the territory from such an overview, though, so let us look more closely at two theologians writing in the early 1340s who disagreed on the question of infinite multitudes.

Infinite multitudes: Thomas Bradwardine

Thomas Bradwardine is known to historians of mathematics as one of the Oxford calculators, a group of technically-minded thinkers associated with Merton College in the second quarter of the fourteenth century.[14] It is in this context that we find him praising mathematics in his *Tractatus de continuo* 'Treatise on the continuum' as 'the revelatrix of all pure truth, which knows every hidden secret and bears the key to all subtle letters' (Murdoch 1969, 216 n1),[15] and quoting Boethius' remark from the *Institutio arithmetica* that 'whoever neglects mathematical studies has clearly lost all knowledge of philosophy' (Bradwardine 1961, 64).[16]

Bradwardine is also known for his later work as a theologian, and for holding the position of Archbishop of Canterbury for a month before succumbing to the Black Death in 1349. Unfortunately, his 'Sentences' commentary, which would have been written in around 1332, has not come down to us. Instead, we shall look at a theological work that has a substantial thematic overlap with such commentaries: his sprawling magnum opus of 1344, *De causa Dei contra Pelagium et de virtute causarum*, 'In defence of God against Pelagius, and on the power of causes', which he dedicated *ad suos Mertonenses* 'to his Mertonians' (Bradwardine 1618; Dolnikowski 1995).

The *De causa Dei* is essentially a polemic on divine freedom. Pelagius, a British monk active at the turn of the fifth century, had denied original sin and argued against Augustine that men were responsible for their own salvation (Pelikan 1971, 308–318). Bradwardine, perceiving such heretical tendencies among his contemporaries, took up the cudgels, stressing God's freedom to bestow grace wherever He saw fit. Bradwardine's stance on predestination was famous enough

14. On the calculators, see Snedegar (2006), North (2000), Sylla (1982), Kaye (1998, 163–199); on the Merton connection, see Martin and Highfield (1997, 52–62); on Bradwardine, see Leff (2004), Dolnikowski (2005).

15. Ipsa est enim revelatrix omnis veritatis sincere, et novit omne secretum absconditum, ac omnium litterarum subtilium clavem gerit.

16. testante Boethio, primo *Arithmeticae* suae: Quisquis scientias mathematicales praetermiserit, constat eum omnem philosophiae perdidisse doctrinam.

to be mentioned in the *Canterbury tales*.[17] Later, its resonance with Calvinism must account for the pedigree of the first printed edition: commissioned by the Archbishop of Canterbury, George Abbot, it was edited by the mathematical scholar and royal courtier Sir Henry Savile and printed in an unusual format at crippling expense by the King's Printer (Weisheipl 1968, 192; Vernon 2004; Wakely and Rees 2005, 484–487).

Savile warns the reader in his introduction that Bradwardine, 'since he was a first-rate mathematician, did not shrink from that art even in treating theological matters' (Bradwardine 1618).[18] Indeed, the *De causa Dei* is presented in so peculiarly Euclidean a style, proceeding from postulates to theorems and corollaries, that it has been described as having 'characteristics of a *Theologiae christianae principia mathematica* [mathematical principles of Christian theology]' (Molland 1978, 113; Sbrozi 1990). The deductive method cannot go very far unaided in such matters, though, as Savile observes: 'if in the lemmas and propositions he has not been able to attain such mathematical precision throughout, the reader will remember to impute this not to the author but to the subject matter of which he treats' (Bradwardine 1618).[19]

The subject matter of the *De causa Dei* turns out to be broader than its title suggests. In the fortieth and final corollary of the first chapter, Bradwardine fulminates at length against the Aristotelian doctrine of the eternity of the world, using what he calls *rationes quasi mathematicae* 'quasi-mathematical arguments' to deduce paradoxical consequences from the existence of actual infinities (Bradwardine 1618, 119–145). I will look at only a limited selection of Bradwardine's many arguments; some of the others are translated into French in Biard and Celeyrette (2005, 183–196).

Suppose we have an infinite multitude *A* of souls and an infinite multitude *B* of bodies, both arranged consecutively. Now 'let the souls be distributed [...] in this way: the first soul to the first body, the second to the second, and so on; when the distribution is complete, each soul will have a unique body, and each body a unique soul. So these [multitudes] jointly and severally correspond equally to one another' (Bradwardine 1618, 122A).[20] So far so good. But now instead:

> let the first soul be given to the first body, the second to the third (or the tenth, or to one as distant as you please from the first), and the third soul to the body as distant from the

17. In the 'Nun's Priest's tale', where Chaucer rhymed 'Bradwardyn' with 'Augustyn' (lines 475–476).

18. ut huius libri genius Lectori melius innotescat, non abs re fuerit pauca praemonuisse: Primo *Thomam* nostrum, cum summus esset Mathematicus, ut ex praecedentibus apparet, etiam in Theologicis tractandis non recessisse ab arte.

19. Quod si in lemmatibus, & propositionibus non semper ἀκρίβειαν illam Mathematicam potuit usquequaque assequi, meminerit Lector non id Auctori imputandum, sed subiectae, quam tractat, materiae.

20. distribuantur animae per Dei omnipotentiam, vel per imaginationem hoc modo: Prima, primo corpori; secunda, secundo; et ita deinceps, qua distributione completa quaelibet anima unicum corpus habebit, et quodlibet corpus unicam animam. Haec igitur singillatim atque coniunctim mutuo sibi aequaliter correspondent.

second ensouled body as the latter is to the first, and so on until the whole distribution is completed in this way. This done, either all the individual souls have been distributed to bodies, or there are some souls left over. If all the individual souls have been distributed to such bodies, the whole multitude *A* jointly and severally corresponds equally to that part of *B*, and vice versa.[21] If any soul is left over, then since there are only finitely many between it and the first, the bodies already taken from the multitude B are the same in number and finite; so the whole multitude *B*—which was supposed to be infinite—is likewise finite.[22] (Bradwardine 1618, 122A)

Thinking of it another way, Bradwardine argues that we could instead assign a thousand souls to each body in turn, which leaves us with a similar problem. If we run out of souls, then *A* was finite after all, contrary to the supposition. But if on the other hand the distribution can be completed, then:

to every unit of *B* there correspond a thousand units of *A*—nay, even ten thousand, a hundred thousand, a thousand thousand, or as large a finite number as you like, as long as it is distributed to the former in the above manner [...] From all this it follows, clear as day, that multitude *A* is enough to ensoul multitude *B*, and double *B*, and four times *B*, and so on without end.[23] (Bradwardine 1618, 122A–B, 122D–E)

Bradwardine's first complaint is metaphysical: such sheer superfluity 'in no way befits God most wise, [...] does not fit with nature, and is detested by all philosophers' (Bradwardine 1618, 123A–B).[24] But he also takes issue with infinite multitudes from a mathematical point of view:

Many people in many ways have their hands full responding to arguments like this, for they are not even ashamed to deny that 'every whole is greater than its part', or to concede that a whole is equal to its part; so that if *A* is the whole infinite multitude of all souls, *B* just one of them, and C the whole remaining multitude, they say that *A* is not greater than C but equal to it—which, consequently, they must also have to say about any two infinite amounts compared to one another. But does not Euclid in book I of his *Elements* suppose it as a principle immediately known to anyone that 'every whole is greater than its part', [...] which all mathematicians and natural philosophers will unanimously acknowledge? And to which it seems anyone's mind, upon knowing the terms, freely consents; and which seems to be evident from the meanings of the terms? Surely one thing is greater than another if it contains it and more, or another amount beyond or outside it? Whose mind says otherwise?[25] (Bradwardine 1618, 132D)

The complaint that Bradwardine voices so strongly here is a natural one, and not even the modern mathematical theory of the infinite has entirely silenced it. Nonetheless, supporters of actual infinity did find ways of answering it. One particularly notable response was that of Gregory of Rimini, to whom we now turn.

Infinite multitudes: Gregory of Rimini

Gregory of Rimini was a powerful and careful thinker whose influence—especially on the topic of predestination, on which he held a view not unlike Bradwardine's—was felt right through to the seventeenth century. A member of the order of the Hermits of St Augustine, he studied theology at Paris in the 1320s before teaching in Bologna, Padua, and Perugia in the 1330s. His return to Paris in around 1342 to lecture on the 'Sentences' is now recognized as a crucial link in the transmission of novel ideas from Oxford to Paris by way of Italy. He was unanimously elected the Augustinians' Prior General in 1357, a year before his death.[26]

Gregory quoted Bradwardine's *De causa Dei* on two occasions in his lectures on Book II of the 'Sentences', as Savile proudly notes in his introduction, but sadly for our purposes neither was in the context of infinity.[27] In fact, the dating of the two works is so close, and the *De causa Dei* so long (just shy of nine hundred folio pages in Savile's edition), that Gregory may not have read the

21. As a modern mathematician might put it, there is a bijection between $\{a_i\}$ and $\{b_{ki}\}$ for any positive integer k.

22. detur prima anima primo corpori, secunda tertio, vel decimo, vel quantum volueris distanti a primo; et tertia anima corpori tantum distanti a secundo corpore animato, quantum illud a primo, et ita deinceps donec tota distributio huiusmodi compleatur. Quo facto vel singulae et omnes animae sunt huiusmodi corporibus distributae, vel sunt aliquae remanentes; Si singulae et omnes sunt corporibus talibus distributae, tota *A* multitudo illi parti *B* divisim et coniunctim correspondet aequaliter et e contra. Si aliqua anima remanet, cum ab illa ad primam sint tantum finitae, et omnia talia corpora praeaccepta *B* multitudinis sunt totidem et finita; quare et tota *B* multitudo similiter est finita, quae posita fuerat infinita.

23. dentur primo corpori mille animae, et secundo totidem, et deinceps quamdiu multitudo sufficit animarum. Vel ergo distributio ista alicubi desinet, vel ad singula et omnia corpora se extendet: Si alicubi desinet, cum inter illum locum seu corpus loci illius, et primum corpus sint corpora finita tantummodo, erunt et totidem; quare et finiti tantummodo millenarii omnium animarum, et tota *A* multitudo finita, quae posita fuerat infinita. Si autem distributio illa ad omnia et singula corpora se extendit, cuilibet unitati *B* correspondet unus millenarius unitatum *A*, imo et decem, et centum, et mille millenarii, et quantuscunque numerus finitus volueris, si tantus distribuatur in primis modo praedicto [...] Ex his quoque ulterius luculenter infertur, quod *A* multitudo sufficit animare *B* multitudinem, et duplam, et quadruplam, et deinceps sine termino, sine statu.

24. ut quid ibi superfluunt animae infinitae, et infinities infinitae, ut patet perspicue ex praemissis? [...] ut quid ergo ibi superfluunt corpora infinita, et infinities infinita, sicut ex prioribus clare patet? Hoc Deum sapientissimum nusquam decet, [...] hoc natura non convenit, hoc omnes Philosophi detestantur.

25. Ad hoc autem et huiusmodi multi multipliciter satagunt respondere, quidem namque non verecundantur negare, *Omne totum esse maius sua parte*, neque concedere totum esse aequale suae parti; ut si *A* sit tota multitudo infinita omnium animarum, *B* vero una earum, *C* autem tota residua multitudo, dicunt quod *A* non est maior *C* sed aequalis, quod et dicunt, sicut et habent necessario dicere consequenter, de quibuslibet duobus quantis infinitis ad invicem comparatis. Sed nonne Euclides I. elementorum suorum supponit istud principium tanquam per se notum cuilibet, Omne totum est maius sua parte, [...] quod et omnes Mathematici atque naturales Philosophi concorditer profitentur? cui et videtur cuiuslibet animus sponte notis terminis consentire; quod et videtur *clarere ex significationibus terminorum*. Nonne illud est maius alio, quod continet illud et amplius, seu aliud quantum ultra vel extra? cuius animus contradicit?

26. Schabel (2007); on transmission, see Schabel (1998); on Gregory's 'Sentences' commentary, see Bermon (2002); on his views on predestination, and their relation to Bradwardine's, see Halverson (1998).

27. Gregory (1979–87, VI) quotes Bradwardine on free will and grace (131), and on sinful intentions (298–300).

passages on infinity before dealing with the same subject in his own lectures on Book I.[28]

In any case, Bradwardine's complaint that every whole is greater than its part was surely a common one, and Gregory tackles it head on. If this Euclidean maxim is supposed to be evident from the meanings of the terms, we must be clear about what those terms mean. What is a whole and what is a part, and what is it for one to be greater than the other? Gregory distinguishes two ways of answering both questions:

> I respond to the argument by making a distinction about 'whole' and 'part', for these can be taken in two ways, that is, generally and properly. (1) In the first way, everything that includes a thing—that is, everything which is a thing plus something else besides that thing and anything of that thing—is called a whole with respect to that thing; and everything included in this way is called a part of the thing that includes it. (2) In the second way, something is called a whole if it includes a thing in the first way and also includes more of a given amount than the included thing does (*includit tanti tot quot non includit inclusum*); conversely, such an included thing, not including as many of a given amount as the including thing (*non includens tot tanti quot includens*), is called a part of it.[29] (Gregory 1979–87, III 457:37–458:6)

In the context of multitudes, the general sense of 'part' clearly corresponds to the modern notion of a proper subset, and in this sense one infinite multitude can indeed be part of another. For instance, says Gregory, 'the multitude of proportional parts of one half of a continuum is a part of the multitude of parts of the whole continuum' (Gregory 1979–87, III 458:11–15).[30]

The additional condition for the proper sense is harder to understand. Gregory expands on it as follows: a proper whole includes 'more of a given amount (*tanti*)—that is, more of a particular quantity, for instance more pairs or triples—than the included multitude does' (Gregory 1979–87, III 458:16–19).[31] The pairs and triples here seem intended merely to indicate different ways of enumerating a multitude,

28. Two caveats are in order here. First, neither dating is certain. In particular, Bradwardine mentions at one point (1618, 559B) that he is writing in Oxford, leading one historian to argue that a major part of *De causa Dei* must have been written before 1335, the date of Bradwardine's move from Oxford to London (Oberman 1978, 88 n20). Second, ideas can of course circulate without being available in writing.

29. Secundo respondeo ad rationem distinguendo de toto et parte, nam haec dupliciter sumi possunt, scilicet communiter et proprie. Primo modo omne, quod includit aliquid, id est quod est aliquid et aliud praeter illud aliquid et quodlibet illius, dicitur totum ad illud; et omne sic inclusum dicitur pars includentis. Secundo modo dicitur totum illud, quod includit aliquid primo modo et includit tanti tot quot non includit inclusum, et econverso tale inclusum non includens tot tanti quot includens dicitur pars eius.

30. Et hoc modo una multitudo infinita potest esse pars alterius, sicut multitudo partium proportionalium unius medietatis continui est pars multitudinis partium totius continui, nam multitudo totius est omnes partes, sive omnia quorum quodlibet est pars, unius medietatis, et omnes partes alterius medietatis, quae sunt totaliter aliae ab illis. [I have altered the editors' punctuation a little. MT]

31. Secundo modo omnis multitudo includens aliam modo iam dicto et includens tanti tot, id est tot determinatae quantitatis, verbi gratia tot binarios vel tot ternarios, quot non includit multitudo inclusa, est totum respectu illius et illa econverso pars dicitur huius.

perhaps to allow the proper sense (like the general sense) to apply to wholes and parts that are not themselves multitudes. Understood in this way, a proper part is one that does not include as many units as the whole that includes it. In the proper sense, then, 'no infinite multitude is a whole or a part with respect to an infinite multitude, because none includes so many of a given amount without the other including as many' (Gregory 1979–87, III 458:19–21).[32]

Gregory's 'proper' sense is an odd way of understanding the terms 'whole' and 'part', but its additional condition has a more natural counterpart in his distinction between two senses of 'greater' and 'smaller':

> Secondly, I make a distinction about 'greater' and 'smaller', although there would be no need if it were not that some people use them improperly. (i) For in one way they are taken properly, and in this way a multitude is called greater if it contains a given amount more times, and smaller if it contains it fewer times; or in another way, which comes to the same thing, that is called greater which contains one more times or [contains] more units, and that is called smaller which contains [one] fewer times or [contains] fewer [units]. (ii) In another way they are taken improperly, and in this way every multitude which includes all the units of another multitude and some other units apart from them is called greater than it, even if the former does not include more than the latter; and in this way to be a greater multitude than another is none other than to include it and to be a whole with respect to it, taking 'whole' in the first way.[33] (Gregory 1979–87, III 458:26–35)

The second sense of 'greater' and 'smaller' that Gregory identifies does indeed seem improper. In this sense, 'one infinite is greater than another, just as it is also a whole with respect to the other, taking "whole" in the first way' (Gregory 1979–87, III 458:37–459:1).[34] In the proper sense, however, 'greater and smaller are not said of infinites with respect to each other, but only of finites, or of infinites with respect to finites and vice versa' (Gregory 1979–87, III 458:35–37).[35]

Having established these definitions, Gregory explores the connections between them. Of course, anything that is a proper whole or a proper part is also

32. Et hoc modo nulla multitudo infinita est totum aut pars respectu multitudinis infinitae, quia nulla tot tanti includit quin tot tanti alia includat.

33. Secundo distinguo hos terminos 'maius' et 'minus', quamvis non oporteret nisi propter aliquos improprie illis utentes: Nam uno modo sumuntur proprie, et sic multitudo dicitur maior, quae tantundem pluries continet, illa vero minor, quae paucies; sive alio modo, et venit in idem, illa dicitur maior, quae pluries continet unum vel plures unitates, illa vero minor, quae paucies seu pauciores. Alio modo sumuntur improprie, et sic omnis multitudo, quae includit unitates omnes alterius multitudinis et quasdam alias unitates ab illis, dicitur maior illa, esto quod non includat plures quam illa; et hoc modo esse maiorem multitudinem alia non est aliud quam includere illam et esse totum respectu illius, primo modo sumendo totum.

34. Secundo vero modo unum infinitum est maius alio, sicut etiam est totum ad illud primo modo sumendo totum.

35. Primo modo maius et minus non dicuntur de infinitis ad invicem, sed de finitis tantum vel de infinitis respectu finitorum et econverso.

a general whole or a general part, but the converse does not hold. The more interesting comparison is between the two senses of 'greater' and 'smaller':

> Not everything which contains more units than another contains those which the other contains, just as a group of ten men living in Rome includes more units than a group of six men living in Paris, but it does not include those [units]; and therefore not everything which is greater in the first way is greater in the second way. And not everything greater in the second way is greater in the first way, as is clear from one infinite multitude with respect to another infinite [multitude] which it includes.[36] (Gregory 1979–87, III 459:5–10)

It is clear from this that Gregory's two senses here correspond to the modern notions of (i) the size or 'cardinality' of a set, and (ii) the inclusion of a proper subset within a set.[37]

Finally, Gregory turns to the objection, which he has stated in the form of a dilemma: 'if there were an infinite multitude, either a part would not be smaller than its whole, or one infinite would be smaller than another' (Gregory 1979–87, III 459:13–14).[38] His response depends on how the terms are taken, and we may summarize his subsequent treatment of three of the four possibilities as follows:

(1.i) An infinite proper subset would indeed not have a lower cardinality than the set of which it is a part, but this is only to be expected; after all, it would not contain fewer things. (Here Gregory defuses Bradwardine's objection by showing the Euclidean maxim to be violated in a benign way; surely one multitude cannot be greater than another if it does not contain more things.)

(1.ii) An infinite set would indeed, as a proper subset, be 'smaller' in the improper sense than another infinite set; but the one infinite would not exceed the other, 'for nothing is properly said to be exceeded by another unless because it does not contain as many of a given amount as the other, which is not true of any infinite' (Gregory 1979–87, III 459:30–31).[39]

(2.i) An infinite proper subset is not a 'part' in the proper sense, that is, a part of lower cardinality. 'And this is the only sense in which it is absurd (*inconveniens*) to concede that a part is no smaller than its whole or that an infinite is smaller than another infinite' (Gregory 1979–87, III 459:35–36).[40]

36. Nam non omne, quod continet plures unitates quam aliud, continet illas, quas continet illud aliud, sicut denarius hominum existentium Romae plures unitates includit quam senarius existentium Parisius, non tamen includit illas; et ideo non omne, quod est maius primo modo, est maius secundo modo, item nec omne maius secundo modo est maius primo modo, sicut patet de multitudine una infinita respectu alterius infinitae, quam includit.

37. We need not be squeamish about using the terminology of set theory for the purposes of exposition; the essence of these notions was not an invention of the nineteenth century.

38. si esset aliqua multitudo infinita, vel pars non esset minor toto, vel unum infinitum esset minus alio.

39. nam nihil proprie dicitur excedi ab alio, nisi quod non continet tanti tot quot aliud; quod de nullo infinito est verum.

40. Et hoc modo tantum est inconveniens concedere partem non esse minorem toto aut infinitum esse minus esse infinito.

Surprisingly, Gregory does not deal with the fourth combination, (2.ii). But here he would again say, as in (2.i), that an infinite proper subset is not a 'part' in the proper sense, so that a fortiori it is not an example of a part that is smaller than its whole; and he would again say, as in (1.ii), that, as a proper subset, it would be 'smaller' in the improper sense.

Conclusion: the historiography of medieval mathematics

Historians of mathematics have traditionally said little about the scholastics, and what they have said has tended to be dismissive. There are, it must be admitted, whole areas of mathematics in which this judgement appears to be sound—algebra, for instance. But even a brief look at fourteenth-century debates over infinity should quash the notion that the scholastics either failed to notice the apparent paradoxes involved or simply put them aside. However, despite a surge in scholarly literature on the topic over the past forty years, this notion remains surprisingly widespread. Where not explicitly stated, it is often implicit in the following potted history: the Greeks abhorred the actual infinite, the medievals agreed, Galileo noticed that the integers could be paired off with their squares, Bolzano noticed the full extent of the phenomenon, and finally of course there was Cantor.[41]

Why has this misapprehension been so persistent? One undeniable factor is the deep-rooted conviction that the medieval period was one of pedantic stagnation; to see this, one need only look up 'medieval' or 'scholastic' in a dictionary.[42] But this cannot be the whole story, because even sympathetic writers have traditionally despaired of scholastic views on infinity; Bolzano, in his *Paradoxien des Unendlichen* 'Paradoxes of the infinite', claimed that the relationship between infinite sets and their proper subsets had previously been overlooked (Bolzano 1919, §20, 27; 2004, 16),[43] while Cantor, summarizing the history of the topic, wrote that 'as is well known, throughout the Middle Ages "infinitum actu non datur" [there is no actual infinite] was treated in all the scholastics as an incontrovertible proposition taken from Aristotle' (Cantor 1932, §4, 173–174).[44]

41. Any overview that fails to mention Gregory of Rimini is likely to give a similar story; in this regard, Zellini (2004) and Moore (2001) are commendable, though Moore (2001, 54) misrepresents Gregory's position on continua. The surge in scholarly literature is almost entirely due to the industry of John Murdoch.

42. For the roots of this prejudice in Renaissance self-congratulation, and its subsequent development, see Grant (2001, 283–355); a popular caricature of the debate on continua is discussed in Sylla (2005).

43. die man aber bisher zum Nachteil für die Erkenntnis mancher wichtigen Wahrheiten der Metaphysik sowohl als Physik und Mathematik übersehen hat.

44. Bekanntlich findet sich im Mittelalter durchgehends bei allen Scholastikern das 'infinitum actu non datur' als unumstößlicher, von Aristoteles hergenommer Satz vertreten.

Cantor's attachment to the scholastics is explored in Dauben (1979, 271–299) and the rather discursive Thiele (2005).

A second factor is the difficulty of gaining access to the relevant texts. For a long time they were available only in manuscripts or early printed editions, which demand far more time, patience, and training than can be expected of a casual researcher.[45] To make matters worse, medieval scholarship used to focus on the thirteenth century, which was thought to represent the zenith of scholasticism.[46] The past thirty or forty years have, however, seen critical editions of several important fourteenth-century 'Sentences' commentaries and an accompanying profusion of scholarly literature within the (inevitably and appropriately) broad field of medieval philosophy (Evans 2002). Much manuscript material remains to be edited, of course, but the textual situation is far happier than it once was, allowing a new appraisal of the quality of fourteenth-century thought.

Perhaps, though, the trouble lies also in the methodological question raised at the start of this chapter; from what we have seen above, it is entirely possible that some of the best mathematical brains of the time have wrong-footed historians of mathematics by working as theologians. For who, investigating treatments of the relationship between infinite sets and their proper subsets, would have thought to look in Thomas Bradwardine's polemic on divine freedom, or in Gregory of Rimini's commentary on a theological textbook? If there is some truth in this diagnosis, it may be worth repeating something that John Murdoch suggested to historians of science over thirty years ago:

> in terms of the subject involved, the historian's search for accomplishments of significance should not be guided by the resemblance of the subject to some feature within modern science. Thus, I would submit that a good deal more of substance, of importance and of interest can be found, for example, in the medieval analysis of the motion of angels than in whatever astronomy occurs in Easter tables, or in the examination of the question of whether or not the infinite past time up to today is greater than the infinite past time up to yesterday than in the geometry of star polygons. One would discover more of importance because we would learn more of the whole tenor of late medieval thought. (Murdoch 1974, 73)

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45. Williams (2003) gives a sympathetic overview of the textual issues facing medieval historians.
46. Two towering exceptions are Pierre Duhem (1956, 1985) and Annelise Maier (1964).

Dramatis personae

Peter Lombard (c 1100–1160), Paris
Bonaventure of Bagnoregio (c 1217–1274), Paris; Franciscan
John Duns Scotus (c 1266–1308), Oxford and Paris; Franciscan
William of Ockham (c 1285–1347), Oxford and London; Franciscan
Francis of Marchia (c 1290–1344+), Paris and Avignon; Franciscan
Robert Holcot (c 1290–1349), Oxford, London, and Northampton; Dominican
John Buridan (c 1300–c 1360), Paris; secular cleric
Thomas Bradwardine (c 1300–1349), Oxford and London; secular cleric
Adam of Wodeham (d 1358), London, Norwich, and Oxford; Franciscan
Roger Roseth (fl c 1335), Oxford; Franciscan
Gregory of Rimini (c 1300–1358), Bologna, Padua, Perugia, Paris; Augustinian
Peter Ceffons (fl 1348–1349), Paris; Cistercian
Richard Swineshead (fl 1340–1355), Oxford
Nicole Oresme (c 1320–1382), Paris

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CHAPTER 7.3

Mathematics, music, and experiment in late seventeenth-century England

Benjamin Wardhaugh

The Scientific Revolution saw many subjects given new scrutiny, with attempts to use mathematical, mechanical, or experimental modes of explanation to gain understanding of them. One of those subjects was music. Already a tradition of mathematical study of musical intervals stretched back through the middle ages to ancient Greece, where the emphasis had been on ratios of the lengths of strings that formed particular musical intervals. In the seventeenth century there were new mathematical techniques and new kinds of mechanical explanation that could be applied instead (Wardhaugh 2006). There were also new experiments and experimental instruments. In this chapter I will discuss those instruments, in the particular context of late seventeenth-century England.

The Royal Society, founded in 1660, provided a meeting-place for diverse approaches to music, and was a potential source of legitimization for the few studies of music that incorporated experiments. I will discuss below some of the musical experiments performed by the Society: they included the use of a very long string to find the absolute frequency of musical vibrations; the use of a short string to display relationships between the lengths and tensions of strings and their musical pitch; the use of a vibrating glass to display patterns of standing waves; the use of a toothed wheel to demonstrate the effect of particular ratios of frequency; and finally an experimental musical performance using specially