Math 432/532 (Diff. Top.), Winter 2016
HW 8

Starred problems are for 532 students, and are extra credit for 432 students (not worth as much as a usual problem). 532 students must LaTeX their solutions.

1. G+P Chapter 2.4, exercise 1, 3, 6, 10, 11, 12* 13*

2. (*) (A modified version of problem 15 from your book.) Let $X \subset Y$ be diffeomorphic to S^1 and $Z \subset Y$ be diffeomorphic to $S^1 \coprod S^1$, and suppose that there is a submanifold with boundary $(W, \partial W) \subset Y$ diffeomorphic to the pair of pants, such that $\partial W = X \coprod Z$. See Figure 2-17 on the bottom of p84 for an illustration. Prove that for any $C \subset Y$ of dimension $\dim Y - 1$, one has $I_2(X, C) = I_2(Z, C)$.

3. A topological space X is *simply-connected* if every continuous function $S^1 \to X$ is homotopic to a constant map. For manifolds, this is equivalent to every smooth function $S^1 \to X$ being smoothly homotopic to a constant map; you may assume this fact.
   (a) Show that S^n is simply-connected for $n > 1$. (This was basically solved in a previous homework assignment.)
   (b) Show that S^1 is not simply-connected, using intersection theory.
   (c) Show that a map $S^1 \to X$ is homotopic to a constant map if and only if it extends to a map $D^2 \to X$. (You're welcome to work in continuous-land instead of smooth-land for this question.)
   (d) Show that $X \times Y$ is simply-connected if and only if X and Y are simply-connected.

4. Find an embedding $S^1 = X \to M$, where M is the Mobius band, such that $I_2(X, X) = 0$ but X is not homotopic to a constant map.

5. (*) Let us define a submanifold $X_{(m,n)}$ of the torus $Y = S^1 \times S^1$, for each pair of integers $1 \leq m, n$. In the end, if m and n are relatively prime, then $X_{(m,n)}$ will be an embedding of S^1. If not, it will be an embedding of a disjoint number of copies of S^1, equal to the gcd of m and n. When these are further embedded into $\mathbb{R}^3$, one obtains what are called *torus links*, or *torus knots* when (m, n) are relatively prime. Wikipedia has some pictures. The last picture on the Wikipedia page, "the (3,4) knot on the unwrapped torus surface," is roughly what I'm describing below, but theirs is slightly shifted.

   It will be most convenient to think of Y as a quotient of the square $[0,1] \times [0,1] \subset \mathbb{R}^2$, identifying the left wall with the right wall, and the top wall with the bottom wall. Place markers at the coordinates $(\frac{k}{m}, 0)$ and $(\frac{k}{m}, 1)$ on the top and bottom walls, for $0 \leq k \leq m$, and place markers at coordinates $(0, \frac{\ell}{n})$ and $(1, \frac{\ell}{n})$ on the right and left walls, for $0 \leq \ell \leq n$. Now, through every marker, draw a line with slope $\frac{m}{n}$ inside the square - note that these lines only meet the walls at the markers. The image of these lines in the quotient space Y is what we denote $X_{(m,n)}$.
   (a) When is $X_{(m,n)}$ transverse to $X_{(s,t)}$?
   (b) Data-gathering: compute the number of points in $X_{(2,3)} \cap X_{(3,2)}$. Repeat this for $X_{(2,3)} \cap X_{(m,n)}$ for all $1 \leq m, n \leq 5$.
   (c) Write down a formula for $I_2(X_{(m,n)}, X_{(s,t)})$. You need not prove it. Explain why your formula works when $X_{(m,n)}$ and $X_{(s,t)}$ are not transverse.