Lecture 10: main exercises

Exercise 10.1. Let $\mathscr{C}$ be a finite length abelian category. Prove that the classes of simple objects form a basis in $K_0(\mathscr{C})$. You may use the Jordan-Holder theorem, which asserts that any two composition series of $X \in \mathscr{C}$ have the same composition factors (up to isomorphism, and counted with multiplicity — in particular, any two composition series have the same length). Hint: The JH theorem implies that for any simple object S, there is a well-defined function $c_S : K_0(\mathscr{C}) \to \mathbb{Z}$ that takes $[X]$ to the multiplicity of S in a composition series for X.

Exercise 10.2. Suppose that $\mathscr{C}$ is equipped with a bounded t-structure $\mathscr{C}_t^{\leq 0}$. Show that there is an isomorphism $K_0(\mathscr{C}_t^{\heartsuit}) \to K_0(\mathscr{C})$ given by $[X] \mapsto [X]$. Describe the similarity between this argument and your argument from the previous exercise using the word "filtration".

Exercise 10.3. If $\mathscr{C}$ is a monoidal category, we say $X \in \mathscr{C}$ is right dualizable if there exists $X^R \in \mathscr{C}$ and maps $u : 1 \to X^R * X$, $c : X * X^R \to 1$ such that the compositions

$$X \xrightarrow{\mathrm{id} * u} X * X^R * X \xrightarrow{\mathrm{id} * c} X \qquad X^R \xrightarrow{\mathrm{id} * u} X^R * X * X^R \xrightarrow{\mathrm{id} * c} X^R$$

are the identity maps. The right dual X^R is unique up to isomorphism if it exists. Likewise, X is left dualizable if it is the right dual of another object X^L, and $\mathscr{C}$ is rigid if every object is both left and right dualizable.

Show that $\mathrm{Vect}^{G,fd}$, the monoidal category of finite-dimensional G-graded vector spaces under convolution, is rigid. Compute the automorphisms $\mathbb{Z}G \to \mathbb{Z}G$ given by $[X] \mapsto [X^R]$ and $[X] \mapsto [X^L]$ (more-precisely these are ring *anti*-automorphisms in general, i.e. they preserve addition but reverse multiplication).