Lecture 11: main exercises

Exercise 11.1. Recall the sheaves

$$\mathcal{P}_{k,\ell} := \mathcal{O}_{\mathsf{Gr}^k_{GL_n}} \otimes \det(L_0/L)^{\ell}[\frac{1}{2}k(n-k)]$$

as well as the convolution varieties

$$\mathsf{Gr}^{(k_1,k_2)}_{GL_n} := \{L_2 \overset{k_2}{\subset} L_1 \overset{k_1}{\subset} L_0 : tL_0 \subset L_1, tL_1 \subset L_2\}.$$

(a) Show that $\mathcal{P}_{1,\ell_1} * \mathcal{P}_{1,\ell_2} \cong \mathcal{P}_{1,\ell_2} * \mathcal{P}_{1,\ell_1}$ whenever $|\ell_1 - \ell_2| \leq 1$.

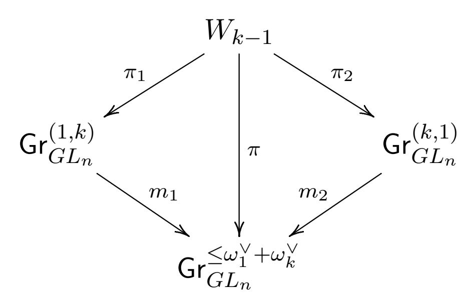
(b) Consider the following space

$$W_k := \{L_2 \overset{1}{\subset} L_1 \overset{k}{\subset} L_1' \overset{1}{\subset} L_0 : tL_0 \subset L_1, tL_1' \subset L_2\}$$

where $k \leq n-2$. Denote by $D \subset W_k$ the locus where $\{tL_0 \subset L_2\}$. Describe the ideal sheaf $\mathscr{I}_D$ in terms of line bundles.

(c) Show that $\mathcal{P}_{1,\ell_1} * \mathcal{P}_{k,\ell_2} \cong \mathcal{P}_{k,\ell_2} * \mathcal{P}_{1,\ell_1}$ whenever $|\ell_1 - \ell_2| \leq 1$.

It may help to use the following commutative diagram



where the maps π_1 and π_2 are given by forgetting L_1' and L_1 respectively.

Lecture 11: additional exercises

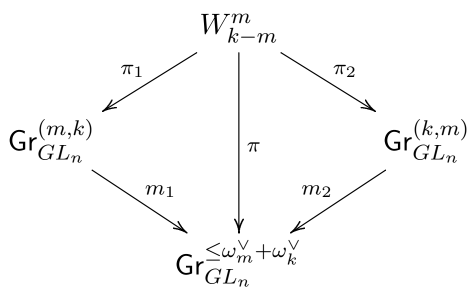
Exercise 11.2. (a) Consider the following space

$$W_k^m := \{L_2 \overset{m}{\subset} L_1 \overset{k}{\subset} L_1' \overset{m}{\subset} L_0 : tL_0 \subset L_1, tL_1' \subset L_2\}$$

where $k \leq n - 2m$. Denote by $D \subset W_k^m$ the locus where $t : L_0/L_1' \to L_1/L_2$ is *not* an isomorphism. Describe the ideal sheaf $\mathscr{I}_D$ in terms of line bundles.

(b) Show that $\mathcal{P}_{m,\ell_1} * \mathcal{P}_{k,\ell_2} \cong \mathcal{P}_{k,\ell_2} * \mathcal{P}_{m,\ell_1}$ whenever $|\ell_1 - \ell_2| \leq 1$.

Notice that you can assume $m \leq k$. It may then help to use the following commutative diagram



where the maps π_1 and π_2 are given by forgetting L_1' and L_1 respectively.