Lecture 17: main exercises

Exercise 17.1. For any $M^\bullet \in \mathrm{Ch}_R$, let $\overline{M}^\bullet$ denote the complex with $\overline{M}^k \cong M^k \oplus M^{k+1}$, and with differential given in matrix form by

$$\begin{pmatrix} d & 1 \\ 0 & -d \end{pmatrix}.$$

(a) Show that this indeed squares to zero, and that $\overline{M}^\bullet$ is quasi-isomorphic to the zero complex (hence becomes isomorphic to zero in DMod_R). To do this, it suffices to find a *chain homotopy* from the identity map to the zero map. That is, a degree -1 map $h : \overline{M}^\bullet \to \overline{M}^\bullet$ such that $h \circ d + d \circ h = \mathrm{id} - 0 = \mathrm{id}$. Hint: matrix form is convenient.

(b) Show that there is a short exact sequence $0 \xrightarrow{i} M^\bullet \to \overline{M}^\bullet \to M[1]^\bullet \to 0$ in Ch_R (here it is conventient to define the differential on $M[1]^\bullet$ as minus the differential on $M^\bullet$, though the resulting complexes are isomorphic).

Remark 17.4. Granting that $\mathrm{Ch}_R \to \mathrm{DMod}_R$ takes short exact sequences to exact triangles, this shows that for any $X \in \mathrm{DMod}_R$, the cofiber of $X \to 0$ is indeed represented by taking the shift of any complex representing X.

Exercise 17.2. If $X, Y \in \mathscr{C}$ are objects in a stable ∞-category $\mathscr{C}$, there are homotopy equivalences of (base)pointed spaces:

$$\mathrm{Map}_{\mathscr{C}}(X[1], Y) \cong \Omega \mathrm{Map}_{\mathscr{C}}(X, Y) \cong \mathrm{Map}_{\mathscr{C}}(X, Y[-1]).$$

Here $\Omega Z = \mathrm{Map}_{\mathrm{Spc}_*}(S^1, Z) = \mathrm{fib}(\mathrm{pt} \to Z)$ is the *loop space* of a pointed space Z.

(a) Compare the equivalences listed above to the natural isomorphisms

$$\mathrm{Hom}_R(\mathrm{cok}(X \xrightarrow{f} M), Y) \cong \ker(\mathrm{Hom}_R(M, Y) \xrightarrow{\circ f} \mathrm{Hom}_R(X, Y))$$
$$\mathrm{Hom}_R(X, \ker(M \xrightarrow{f} Y)) \cong \ker(\mathrm{Hom}_R(X, M) \xrightarrow{f \circ} \mathrm{Hom}_R(X, Y))$$

of abelian groups, where $X, Y, M \in \mathrm{Mod}_R$ and f is a module homomorphism (these follow from the universal properties of kernels/cokernels). What plays the role of M?

(b) Show that $\mathrm{Hom}_{\mathscr{C}}(X, Y) := \pi_0 \mathrm{Map}_{\mathscr{C}}(X, Y)$ has the structure of an abelian group for any $X, Y \in \mathscr{C}$. Use the fact that for any pointed space Z and any $n \geq 2$, the set $\pi_n Z := \pi_0 \Omega^n Z$ has the structure of an abelian group (here Ω^n means "apply Ω n times").

(c) Suppose $\mathscr{C}$ has a t-structure. We defined $Y \in \mathscr{C}^{\geq 1}$ to mean $\mathrm{Hom}_{\mathscr{C}}(X, Y) = 0$ for all $X \in \mathscr{C}^{\leq 0}$ (again we really mean $\mathrm{Hom}_{\mathscr{C}}(X, Y) := \pi_0 \mathrm{Map}_{\mathscr{C}}(X, Y)$). Show that this is equivalent to the a priori stronger condition that $\mathrm{Map}_{\mathscr{C}}(X, Y)$ is contractible for all $X \in \mathscr{C}^{\leq 0}$ (which is equivalent to the condition that $\pi_n \mathrm{Map}_{\mathscr{C}}(X, Y) = 0$ for all $X \in \mathscr{C}^{\leq 0}$ and $n \geq 0$).

Lecture 17: additional exercises

Exercise 17.3. In the setting of Exercise 8.2(c), show that if $Y \in \mathscr{C}^{\heartsuit}$, then $\mathrm{Map}_{\mathscr{C}}(X, Y)$ has contractible connected components for all $X \in \mathscr{C}^{\leq 0}$. In particular, $\mathrm{Map}_{\mathscr{C}}(X, Y)$ is homotopy equivalent to a space with the discrete topology for all $X, Y \in \mathscr{C}^{\heartsuit}$. This can be interpreted as saying that even though $\mathscr{C}$ is an ∞-category, the subcategory $\mathscr{C}^{\heartsuit}$ is still an ordinary category (up to the relevant notion of equivalence).

Exercise 17.4. Let $f : M^\bullet \to N^\bullet$ be a morphism in Ch_R. Describe the cokernel $C(f)$ of the map $(f, i) : M^\bullet \to N^\bullet \oplus \overline{M}^\bullet$ explicitly (this is called the *mapping cone* of f, though there exist different sign conventions for it). Note that (a) (f, i) is injective since i is, and that (b) $N^\bullet \oplus \overline{M}^\bullet$ is quasi-isomorphic to $N^\bullet$ since $\overline{M}^\bullet$ is quasi-isomorphic to zero. In particular, the short exact sequence

$$0 \to M^\bullet \to N^\bullet \oplus \overline{M}^\bullet \to C(f) \to 0$$

in Ch_R tells us there is an exact triangle

$$M^\bullet \to N^\bullet \to C(f)$$

in DMod_R, so $C(f)$ gives us a way of computing cofibers in DMod_R.