Exercise 3.6. When dealing with Koszul algebras A which are not polynomial rings, it is more natural to construct resolutions of the form $K^\bullet = M \otimes A$, where M is not a ring, but a **module** over the Koszul dual algebra. This exercise explores that phenomenon when A is the exterior algebra in one variable.

Let $\Lambda^\vee = \mathbb{C}[\theta]/(\theta^2)$, where θ has weight -1 and degree 1. Let T be the one-dimensional $\Lambda^\vee$-module $\Lambda^\vee/(\theta)$.

Let $R = \mathbb{C}[x]$ where x has weight 1 and degree 0.

(a) Let $M = \mathrm{Hom}^\bullet_{\mathbb{C}}(R, \mathbb{C})$ denote the graded dual of M, the span of all homogeneous vector space maps $R \to \mathbb{C}$ of various degrees. A basis for M is $\{\varphi_k\}_{k \geq 0}$, where $\varphi_k(x^\ell) = \delta_{k\ell}$. The degree of φ_0 is zero.

There is a natural action of R on M, where $(r \cdot \psi)(s) = \psi(rs)$ for $r, s \in R$ and $\psi \in M$. Compute the action of $x \in R$ on the basis of M. In what degree should we place φ_k in order that M be a graded module?

(b) Is M finitely-generated? Can you identify M as a quotient module of $\mathbb{C}[x, x^{-1}]$?

(c) Find a free resolution of T as a $\Lambda^\vee$-module, having the form $K^\bullet = M \otimes \Lambda^\vee$. Describe the differential using the action of R on M.