Lecture 4: main exercises

Exercise 4.1. Given $M^\bullet \in \mathrm{Ch}_R$, recall that we write $\tau^{\leq n}(M^\bullet)$ for the complex

$$\cdots \to M^{n-2} \xrightarrow{d^{n-2}} M^{n-1} \xrightarrow{d^{n-1}} Z^n \to 0 \to \cdots ,$$

where $Z^n := \ker d^n \subset M^n$. Show that the cokernel in Ch_R of the natural map $\tau^{\leq n-1}(M^\bullet) \to \tau^{\leq n}(M^\bullet)$ is quasi-isomorphic to $H^n(M^\bullet)[-n]$.

(Granting that $\mathrm{Ch}_R \to \mathrm{DMod}_R$ takes short exact sequences to exact triangles, this proves that we have an exact triangle $\tau^{\leq n-1}(M^\bullet) \to \tau^{\leq n}(M^\bullet) \to H^n(M^\bullet)[-n]$ in DMod_R).

Exercise 4.2. Let $R = \mathbb{C}[x]$ and $R^! = \mathbb{C}[\varepsilon]/(\varepsilon^2)$, where x has weight 1 and ε has weight -1. Recall that the Koszul t-structure on $\mathrm{DMod}_R^{\mathrm{gr,fg}}$ refers to the image of the standard t-structure on $\mathrm{DMod}_{R^!}^{\mathrm{gr,fg}}$ under the equivalence

$$K := \mathrm{sh} \circ \mathrm{RHom}_R(R/(x), -) : \mathrm{DMod}_R^{\mathrm{gr,fg}} \xrightarrow{\sim} \mathrm{DMod}_{R^!}^{\mathrm{gr,fg}} .$$

One can show that the indecomposable objects of $\mathrm{DMod}_R^{\mathrm{gr,fg}}$ are exactly the objects of the form $R[k]\langle \ell \rangle$ or $R/(x^n)[k]\langle \ell \rangle$.

(a) Determine which of these objects are in the heart of the Koszul t-structure. Hint: inspecting $R^!$, we see that any such object is of the form $R^!\langle n \rangle$ or $R^!/(\varepsilon)\langle n \rangle$ as a graded $R^!$-module. Now use the computations in the previous lecture and show that $K(M[n]\langle n \rangle) \cong (K(M))\langle n \rangle$.

(b) Identify a relationship between the short exact sequence

$$0 \to R\langle 1 \rangle \xrightarrow{x} R \to R/(x) \to 0$$

in $\mathrm{Mod}_R^{\mathrm{gr}}$ and the short exact sequence

$$0 \to R^!/(\varepsilon)\langle -1 \rangle \xrightarrow{\varepsilon} R^! \to R^!/(\varepsilon) \to 0$$

in $\mathrm{Mod}_{R^!}^{\mathrm{gr}}$. Hint: use the word triangle.

(c) An object X is *formal* with respect to a t-structure if it is isomorphic to $\oplus_n H_t^n(X)[-n]$. Determine which, if any, indecomposable objects of $\mathrm{DMod}_R^{\mathrm{gr,fg}}$ are **not** formal with respect to the Koszul t-structure.

(d) Show that there exist nonisomorphic objects in $\mathrm{DMod}_R^{\mathrm{gr,fg}}$ with the same Koszul cohomology. Show that there do not exist nonisomorphic objects with the same standard cohomology.

Lecture 4: additional exercises

Exercise 4.3. Suppose we are given a t-structure on $\mathscr{C} = \mathrm{DMod}_R$ and an exact triangle $X \to Y \to Z$. Using rotation of triangles and the fact that $H_t^n : \mathscr{C} \to \mathscr{C}_t^\heartsuit$ is a cohomological functor for all n, show that there is a long exact sequence

$$\cdots \to H_t^{n-1}(Z) \to H_t^n(X) \to H_t^n(Y) \to H_t^n(Z) \to H_t^{n+1}(X) \to \cdots$$

in t-cohomology.

Exercise 4.4. Let R and $R^!$ be as in Exercise 4.2.

(a) Compute $\mathrm{sh}(\mathrm{Hom}_R^\bullet(K^\bullet, R/(x^n))) \in \mathrm{Ch}_{R^!}^{\mathrm{gr}}$. Start with $n = 2$.

(b) Write down the long exact sequence in Koszul cohomology associated to the exact triangle $R/(x^{n-1})\langle 1\rangle \to R/(x^n) \to R/(x)$. Start with $n = 2$.